Binomial Coefficients and Logarithms Activity - Solutions

Binomial Coefficients

Definition: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for all n ≥ k ≥ 0

Example 1: Calculate: $\binom{10}{3}$

Solution: $$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3!} = \frac{720}{6} = 120$$

Answer: 120

Example 2: How many 8-bit strings contain exactly 3 zeros?

Solution: We need to choose 3 positions out of 8 total positions for the zeros. The remaining 5 positions will contain ones.

Number of ways = $\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{336}{6} = 56$

Answer: 56 strings

Example 3: Use the definition to simplify each of the following:

a) $\binom{n}{0}$

Solution: $$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

Answer: 1

b) $\binom{n}{n}$

Solution: $$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{n!}{n! \cdot 1} = 1$$

Answer: 1

Example 4: Suppose a string of n characters contains only two possible characters: * and |. Write an expression for the number of n-bit strings that contain exactly 2 *'s. Simplify your expression to be written as a polynomial.

Solution: We need to choose 2 positions out of n positions for the * characters.

Number of strings = $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2 \cdot 1 \cdot (n-2)!} = \frac{n(n-1)}{2}$

Expanding: $\frac{n(n-1)}{2} = \frac{n^2 - n}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$

Answer: $\frac{1}{2}n^2 - \frac{1}{2}n$ or $\frac{n(n-1)}{2}$

Logarithms

Definition: $y = \log_b x \Leftrightarrow b^y = x$ **Notation**: We use $\lg x$ to represent $\log_2 x$.

Example 1: Use the definition to determine each of the following:

a) $\lg 32$

Solution: We need to find y such that $2^y = 32$ Since $32 = 2^5$, we have $\lg 32 = 5$

Answer: 5

b) $\lg 1024$

Solution: We need to find y such that $2^y = 1024$ Since $1024 = 2^{10}$, we have $\lg 1024 = 10$

Answer: 10

c) $\lg 1$

Solution: We need to find y such that $2^y = 1$ Since $1 = 2^0$, we have $\lg 1 = 0$

Answer: 0

d) $\log_3 81$

Solution: We need to find y such that $3^y = 81$ Since $81 = 3^4$, we have $\log_3 81 = 4$

Answer: 4

e) $\log 100,000$

Solution: We need to find y such that $10^y = 100,000$ Since $100,000 = 10^5$, we have $\log 100,000 = 5$

Answer: 5

Example 2: Use powers of 2 to calculate:

a) $\lfloor \lg 70 \rfloor$

Solution: We need the largest integer k such that $2^k \leq 70$ $2^6 = 64 \leq 70 < 128 = 2^7$ Therefore $\lfloor \lg 70 \rfloor = 6$

Answer: 6

b) $\lceil \lg 1100 \rceil$

Solution: We need the smallest integer k such that $2^k \geq 1100$ $2^{10} = 1024 < 1100 < 2048 = 2^{11}$ Therefore $\lceil \lg 1100 \rceil = 11$

Answer: 11

Example 3: Use a calculator and the change of base formula to calculate:

Change of Base Formula: $\log_a x = \frac{\log_c x}{\log_c a}$

a) $\log_3 687$

Solution: $$\log_3 687 = \frac{\log 687}{\log 3} = \frac{2.8370}{0.4771} \approx 5.95$$

Answer: ≈ 5.95

b) $\log_5 791$

Solution: $$\log_5 791 = \frac{\log 791}{\log 5} = \frac{2.8982}{0.6990} \approx 4.15$$

Answer: ≈ 4.15

Example 4: Use properties of logs to simplify each of the following:

Properties:

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a(x^r) = r\log_a x$

a) $\lg(8n)$

Solution: $$\lg(8n) = \lg(8) + \lg(n) = \lg(2^3) + \lg(n) = 3\lg(2) + \lg(n) = 3 \cdot 1 + \lg(n) = 3 + \lg(n)$$

Answer: $3 + \lg(n)$

b) $\lg(n^2)$

Solution: $$\lg(n^2) = 2\lg(n)$$

Answer: $2\lg(n)$

c) $\lg(n^{10})$

Solution: $$\lg(n^{10}) = 10\lg(n)$$

Answer: $10\lg(n)$

d) $\lg(\sqrt{n})$

Solution: $$\lg(\sqrt{n}) = \lg(n^{1/2}) = \frac{1}{2}\lg(n)$$

Answer: $\frac{1}{2}\lg(n)$

e) $\lg(n^n)$

Solution: $$\lg(n^n) = n\lg(n)$$

Answer: $n\lg(n)$

Example 5: Binary Search Algorithm Analysis

Problem: Determine the number of times the while loop is executed in the worst case for binarySearch algorithm.

Solution: In binary search, each iteration of the while loop reduces the search space by half. Starting with n elements:

- After 1 iteration: n/2 elements remain
- After 2 iterations: n/4 elements remain
- After 3 iterations: n/8 elements remain
- After k iterations: n/2^k elements remain

The loop continues until we have 1 or fewer elements to search, so we need: $$\frac{n}{2^k} \geq 1$$ $$n \geq 2^k$$ $$\lg(n) \geq k$$

Therefore, the maximum number of iterations is $k = \lfloor \lg(n) \rfloor + 1$.

Examples given:

- Search 3770 student records: $\lfloor \lg(3770) \rfloor + 1 = \lfloor 11.87 \rfloor + 1 = 11 + 1 = 12$ iterations
- Search 332,000,000 people: $\lfloor \lg(332,000,000) \rfloor + 1 = \lfloor 28.31 \rfloor + 1 = 28 + 1 = 29$ iterations

Answer: $\lfloor \lg(n) \rfloor + 1$ iterations in the worst case

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