Tensors, Topology and Manifolds

ᗯ$\hbar$Yβmaths

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1 Introduction

"Hello and welcome to this video about tensors. I am going to assume that you have already seen tensors in some shape or form, most likely in the context of Special or General Relativity. The tensors you will have seen have upper and lower indices,

$$e.g \quad V^{\mu}, g_{\mu\nu}, T^{\alpha\beta}, R^{\mu}{}_{\nu\sigma\rho}$$

and you should be familiar with how to contract indices using the summation convention.

$$X^{\mu}{}_{\nu} Y^{\nu}{}_{\sigma} = Z^{\mu}{}_{\sigma}$$

and that contracting with the metric raises and lowers indices"

$$g_{\mu\nu} V^{\nu} = V_{\mu}$$

"Whilst performing these operations using this rather abstract and compact index notation is perfectly ok, it hides an extra layer of structure that is often overlooked when studying tensors for the first time. In this video I will hope to uncover this extra layer of machinery, and we will see how this index notation emerges."

"You may be familiar with statements like:"

"A tensor is defined by how it transforms under coordinate transformations"

$$x^{\mu} \longrightarrow x^{\mu'} = x^{\mu'}(x^{\mu}) : T^{\mu\nu} \longrightarrow T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} T^{\mu\nu}$$

This 'definition' of a tensor relies heavily on coordinates; and in fact is only really a statement about the tensor components, as we shall see. This already implies we are dealing with tensors on a manifold[1] and we have decided to use a certain coordinate system and appropriate basis. Since this choice of coordinates (and hence basis) is completely arbitrary, the utility of tensors in this context comes from how they (their components) transform when we change coordinates, as we are free to do arbitrarily.

However, we want to seek a more general definition of a tensor; one that does not rely on coordinates, as we will only wish to add this structure later. We will begin by returning to a tensor you will hopefully be very familiar with; the vector *show arrows*. However, we will be abandoning this geometric realisation of vectors for now and we will just wish to define precisely what it means mathematically to be a vector, and we will spend some time getting comfortable with this idea as it is key to understanding and unlocking the rest of the structures we will create later on. However before we can talk properly about vector spaces we need to just quickly review two more very important ideas that we will need to rely heavily on, sets and maps. The next few sections will seem pretty lengthy and formal but just stick with me it will pay off to be as precise as possible now so that we can get very comfortable with these rather abstract definitions. Having a precise definition of a vector space through the language of sets and maps will be at the core of everything that follows on the road to defining a tensor.

[1] what exactly a manifold is will be detailed later, for now just think of an abstract geometric space

2 Sets and Maps

We will not need to rigorously define what it means to be a set $\{\,\}$, and can instead just rely on our intuition that a set is a collection of elements, and we have the relation $\in$ meaning "is an element of". Given two sets X, Y we can construct a map, or mapping, between the elements of the two sets. So given an element $x \in X$ we define the mapping assignment $x \mapsto y : y \in Y$ which simply takes an element from the domain set X and defines an element y in the co-domain set Y. This is most easily illustrated with a diagram. *show blob set diagram with discrete elements* We can have surjective, injective or bijective maps. We now wish to define the map for the whole domain, the usual notation for this is given a map, Λ, with domain X and co-domain Y the map is expressed as

$$\Lambda : X \longrightarrow Y$$

$$\forall x \in X : \exists y \in Y : x \mapsto y = \Lambda(x)$$

Don't be scared by this formal looking notation, I'll translate: For all elements in the domain set X, there exists an element y in the co-domain set Y which is the result of the mapping assignment, x maps to y under, the action of the map lambda.

This should all feel very familiar. Whilst this round bracket function style notation for the map is fine, I prefer an alternative notation for the map Λ

$$\langle \Lambda, \rangle : X \longrightarrow Y$$

$$\forall x \in X : \exists y \in Y : x \mapsto y = \langle \Lambda, x \rangle$$

this thing having two slots will turn out to be really handy later on.

3 Vector Spaces

Now that we are comfortable with sets and maps between them we are ready for the main dish. A vector space is simply a set, V, equipped with two operations (maps), vector addition and scalar multiplication. Vector addition is a map that take two elements from the vector space and returns a third element of the vector space.

$$\color{red}{+} : V \times V \longrightarrow V$$

$$\forall (a, b) \in V : \exists c \in V : (a, b) \mapsto c = a \color{red}{+} b$$

Scalar multiplication is why we need a field: scalar multiplication is a map that takes a scalar and a vector and returns another vector

$$\color{red}{\cdot} : \mathbb{R} \times V \longrightarrow V$$

This along with several others is one of the axioms both of these operations must satisfy *show axioms* in order for V to be a vector space. $\forall (a, b, c) \in V : \forall (\lambda, \gamma) \in K$

Vector Addition $\color{red}{+}$		Scalar Multiplication $\color{red}{\cdot}$	
Commutivity	$a \color{red}{+} b = b \color{red}{+} a$	Associativity	$\gamma \color{red}{\cdot} (\lambda \color{red}{\cdot} a) = (\gamma\lambda) \color{red}{\cdot} a$
Associativity	$(a \color{red}{+} b) \color{red}{+} c = a \color{red}{+} (b \color{red}{+} c)$	Distributivity	$(\lambda + \gamma) \color{red}{\cdot} a = \lambda \color{red}{\cdot} a \color{red}{+} \gamma \color{red}{\cdot} b$
Identity	$\exists 0 \in \mathcal{V} : a \color{red}{+} 0 = a$	Distributivity	$\lambda \color{red}{\cdot} (a \color{red}{+} b) = \lambda \color{red}{\cdot} a \color{red}{+} \lambda \color{red}{\cdot} b$
Inverse	$\exists (-a) \in \mathcal{V} : a \color{red}{+} (-a) = 0$	Identity	$\exists \mathbb{I} \in K : \mathbb{I} \color{red}{\cdot} a = a$

Now notice we have not yet mentioned anything about components or bases or been specific about what we actually mean these vectors are, but that's the point. Any set that you can show it's elements satisfy these axioms under the two operations $(+, \cdot)$ is a vector space, and hence it's elements are vectors. This is what we would call a realisation of a vector space, constructing a set that you can show has this particular 'algebraic structure'. A comfortable example is arrows in $\mathbb{R}$ as you can easily show that you can add and scalar multiply vectors in the required way.

4 Bases

Having been fairly abstract and general in our definition of a vector space we can now introduce a more concrete and familiar notion, that of a basis. Every (finite dimensional) vector space contains a (finite) subset of vectors $\mathcal{B} \subset V$ such that every vector in the vector space can be expressed as a linear combination of the basis vectors.

$$\forall v \in V : \exists v^i \in \mathbb{R} : \exists \underline{e}_i \in \mathcal{B} : v = v^i \cdot \underline{e}_i \ : \{i = 1, ..., d\}$$

The number of basis vectors is equal to the dimension of the vector space. Where $d = dim(V)$ is equal to the number of basis vectors *lmao*. That might sound like nonsense but in practice you prove that there exists a certain linearly independent subset of vectors in the vector space such that any vector in the vector space can be expressed as a linear combination. The number of basis vectors required to achieve this is then equal to the dimension of the vector space. Connecting this back to the sort of vector ideas you should be comfortable with, these v^i are the 'components' of the vector with respect to that particular basis vector $\underline{e}_i$. Now as you should hopefully be aware, the vector itself is an independent quantity, the choice of basis is potentially arbitrary and so the vector components can have many different representations depending on the choice of basis. How these components transform under changes of bases are the calculations you are most likely familiar with. *show vector with components in cart then polar coords* For what follows we will not specify a basis and just remain general in most cases. You might be wondering why we chose to place our indices in this way, it turns out this choice is also completely arbitrary so long as we remember to abide by the summation convention that one upper and one lower index of the same type are summed. We will see shortly the consequence of choosing to place our indices in this way, and if you are anticipating terminology like contra and covariant this will get mentioned shortly but then suitably ignored as it is clunky and not required.

5 Vector Realised as Linear Maps

Now that we have constructed our vector space $(V, +, \cdot)$ which I remind you again is just a set V with two operations under which the elements satisfy the vector space axioms. Now suppose there exists some other vector space $(W, +, \cdot)$ just another set with it's pair of vector operations. We could consider constructing a map between these two vector spaces, which is simply just an assignment for each element of V into W.

$$\Lambda : V \longrightarrow W$$

$$\forall v \in V : \exists w \in W : v \mapsto w : w = \langle \Lambda, v \rangle$$

Now if we require that this be a linear map, meaning that for any two vectors in V

$$\forall v_1, v_2 \in V : \langle \Lambda, v_1 + v_2 \rangle = \langle \Lambda, v_1 \rangle + \langle \Lambda, v_2 \rangle$$

and that for any scalar

$$\forall a \in \mathbb{R} : \langle \Lambda, a \cdot v \rangle = a \cdot \langle \Lambda, v \rangle$$

care must be taken to use the operation for the respective vector spaces before and after the map. We haven't yet been specific about how to actually perform this map, we are just requiring that it be a linear map. This requirement alone will allow us to explicitly define how the map acts on any vector when expressed in a particular basis as follows. We must first choose a basis for the domain space and then define how Λ acts on each basis vector $\langle \Lambda, \underline{e}_i \rangle = \underline{\epsilon}_i = \Lambda^j{}_i \underline{e}_j \in \mathcal{W}$. Then by the linearity of the map the result of mapping any vector can be computed:

$$\langle \Lambda, v^i \cdot \underline{e}_i \rangle = v^i \cdot \langle \Lambda, \underline{e}_i \rangle = v^i \cdot \underline{\epsilon}_i$$

but since $\langle \Lambda, v \rangle = w = w^j \underline{\epsilon}_j$ We will choose to denote by $\mathrm{Hom}(V, W)$ the set of all structure preserving maps between V and W. What does structure preserving mean? Simply that the map produces a co-domain that has the same algebraic structure as the domain. In the case of vector spaces the structure preserving maps (also called homomorphisms) are the linear maps. Since the maps are linear they can be added and scalar multiplied and you can show that the set of all maps $\mathrm{Hom}(V, W)$ between the two vector spaces is itself a vector space. This will be a key point moving forward so let's spend some time marinading in this idea. Since the set of all linear maps between two vector spaces is just a set, given the requirement that these be linear maps is enough to show that you can add them like this *point to vector add* and scalar multiply them like this *point to smult* and so the set of all of these objects is a vector space through the eyes of our earlier definition, and hence the elements, the linear maps $\langle \Lambda, \rangle$, are vectors.

6 The Dual Space

Now that we have constructed a vector space $(V, +, \cdot)$ defined over a field which we take to be $\mathbb{R}$ and basis $\underline{e}_\mu$ where we use greek indices now because they look better. Now it can be shown that the set of real numbers is itself a vector space, and so we could consider the set of all linear maps from our vector space into $\mathbb{R}$, which using our previous notation we could call $\mathrm{Hom}(V, \mathbb{R})$. We will denote this set of all linear maps between V and $\mathbb{R}$ by V^*, the dual space to V. As we showed previously this set of all linear maps is again a vector space, and so the elements of V^* are vectors, but they are also linear maps from V into $\mathbb{R}$. We will denote elements of this dual space using Greek letters, they are usually referred to as dual vectors or sometimes covectors. With the understanding that these dual vectors should act as linear maps, from V into $\mathbb{R}$ meaning that they will take a vector as an input and map it to a real number. Using our brackety notation for the map:

$$\forall v \in V : \exists \omega \in V^* : \exists a \in \mathbb{R} : \qquad \omega : V \longrightarrow \mathbb{R} : \langle \omega, v \rangle \mapsto a \in \mathbb{R}$$

$$\text{where} \qquad V^* = \mathrm{Hom}(V, \mathbb{R}) = \{ \langle \omega, \rangle : V \longrightarrow \mathbb{R} \} \text{ “set of all maps”}$$

Now again we have been general in our construction in this set of all possible maps, we will need to provide how the map acts on basis vectors in order to perform explicit calculations. To do this let's suppose that the dual space, as it's a vector space, has it's own basis ϵ^μ where the index is upstairs for reasons we will see shortly. Given this dual basis any dual vector can be expanded as a linear combination

$$\forall \omega_\mu \in \mathbb{R} : \omega = \omega_\mu \epsilon^\mu$$

now given any vector $v = v^\mu \underline{e}_\mu$ and with the realisation that the dual vector is a linear map we can insert these expansions into our bracket and expand it using the linearity of the map.

$$\langle \omega_\mu \epsilon^\mu, v^\nu \underline{e}_\nu \rangle = \omega_\mu v^\nu \langle \epsilon^\mu, \underline{e}_\nu \rangle$$

Now for reasons we will elaborate on later and might appear fairly arbitrary for now we will define how this map acts by asserting that $\langle \epsilon^\mu, \underline{e}_\nu \rangle = \delta^\mu{}_\nu$. Then the simple sum $\omega_\mu v^\nu \langle \epsilon^\mu, \underline{e}_\nu \rangle = \omega_\mu v^\nu \delta^\mu{}_\nu = \omega_\mu v^\mu$

7 Multi-linear Maps

Now that we've constructed a vector space and it's dual we have all the machinery we need to move on and generalise this concept to Tensors. I'll remind you what we have so far. We constructed a vector space, which is just a set $(V, +, \cdot)$ with the operations $+$ and $\cdot$. We then constructed the set of all linear maps $\mathrm{Hom}(V, \mathbb{R})$ between the vector space and the underlying field. The linearity of these maps also guarantees that this set also has a vector space structure, and we call this the dual vector space V^*. Now since we are dealing with finite dimensional vector spaces we saw that we can make the identification that the dual space of the dual space is simply just the original vector space, so we can understand vectors as being linear maps from V^* into $\mathbb{R}$. Now to motivate tensors I want you to consider the following space. $V \times V$, the Cartesian product of the vector space with itself. This cartesian product is just another set, but the elements of this set now are ordered pairs of vectors (a, b) with one coming from each vector space. Now we want to construct maps from this space into the underlying field, let's just pause for a moment to think about how we might possibly do this. We know how to map individual vectors into $\mathbb{R}$, we simply feed them into a dual vector. So we might expect that to map this pair we would somehow need a pair of dual vectors to feed them into. To do this we define a new object, the tensor T, formed using the tensor product of two dual vectors

$$T = \omega \otimes \sigma : (a, b) \mapsto \mathbb{R} \tag{1}$$

The T is now a single object, formed out of two dual vectors, that eats a pair of vectors and returns a real number. We understand how the map acts as follows, in both function and bracket style notation:

$$T(a, b) = \omega(a) \otimes \sigma(b) = \langle \omega, a \rangle \otimes \langle \sigma, b \rangle \tag{2}$$

Now we fully understand how to compute such an object, since each of these maps simply gives a real number and these are then multiplied by the tensor product. The tensor product symbol simply tells us how to string the individual maps together so that they can eat an ordered pair of vectors. The whole object now is called a multi-linear map, it is linear in each of the slots, meaning it must satisfy the following

$$T(\alpha a + \beta b, c) = \alpha T(a, c) + \beta T(b, c) \tag{3}$$

and similar for the second slot. It is now illuminating to expand this tensor in terms of a basis as follows. Given a basis for V, and the corresponding dual basis defined by $\langle \epsilon^i, e_j \rangle = \delta^i_j$ we can expand the map:

$$\langle \omega_\mu \epsilon^\mu, a^i e_i \rangle \otimes \langle \sigma_\nu \epsilon^\nu, b^j e_j \rangle = \omega_\mu a^i \langle \epsilon^\mu, e_i \rangle \otimes \sigma_\nu b^j \langle \epsilon^\nu, e_j \rangle = \omega_\mu a^i \delta^\mu_i \otimes \sigma_\nu b^j \delta^\nu_j = \omega_\mu a^\mu \otimes \sigma_\nu b^\nu \tag{4}$$

$$T = (\omega_\mu \epsilon^\mu) \otimes (\sigma_\nu \epsilon^\nu) \implies \omega_\mu \sigma_\nu \epsilon^\mu \otimes \epsilon^\nu = T_{\mu\nu} \epsilon^\mu \otimes \epsilon^\nu \tag{5}$$

The multi-linearity of the map, means that the whole object T itself behaves like a vector, meaning that tensors can be added and scalar multiplied just like ordinary vectors. This means that we can treat the set of all possible multilinear maps between $V \times V$ and R as a vector space. To illustrate this consider

$$T(a + b, c) = T(a, c) + T(b, c) \tag{6}$$

Since a full tensor is just a real number both LHS and RHS must be equal to some number γ. The sum on the RHS is then a sum of two distinct real numbers, $\alpha + \beta$, so the whole tensor behaves like a vector. If we now consider a specific tensor, i.e having some certain set of components w.r.t the basis

$$T_1 = \langle \omega_\mu \epsilon^\mu, \rangle \otimes \langle \sigma_\nu \epsilon^\nu, \rangle, T_2 = \langle \rho_\lambda \epsilon^\lambda, \rangle \otimes \langle \xi_\alpha \epsilon^\alpha, \rangle \tag{7}$$

$$T_1 + T_2 = [(\omega_\mu \sigma_\nu) \langle \epsilon^\mu, \rangle \otimes \langle \epsilon^\nu, \rangle] + \left[(\rho_\lambda \xi_\alpha) \langle \epsilon^\lambda, \rangle \otimes \langle \epsilon^\alpha, \rangle\right] \tag{8}$$

Now for simplicity assume the dimension is two, so that our indices run from 1 to 2.

$$T_1 = \left[(\omega_1\sigma_1)\langle\epsilon^1,\rangle\otimes\langle\epsilon^1,\rangle + (\omega_1\sigma_2)\langle\epsilon^1,\rangle\otimes\langle\epsilon^2,\rangle + (\omega_2\sigma_1)\langle\epsilon^2,\rangle\otimes\langle\epsilon^1,\rangle + (\omega_2\sigma_2)\langle\epsilon^2,\rangle\otimes\langle\epsilon^2,\rangle\right] \quad (9)$$

$$T_2 = \left[(\rho_1\xi_1)\langle\epsilon^1,\rangle\otimes\langle\epsilon^1,\rangle + (\rho_1\xi_2)\langle\epsilon^1,\rangle\otimes\langle\epsilon^2,\rangle + (\rho_2\xi_1)\langle\epsilon^2,\rangle\otimes\langle\epsilon^1,\rangle + (\rho_2\xi_2)\langle\epsilon^2,\rangle\otimes\langle\epsilon^2,\rangle\right] \quad (10)$$

Now we see that the basis part of these expressions are equivalent so we can factorise the components and relabel the indices on the second tensor

$$T_1 + T_2 = (\omega_\mu\sigma_\nu + \rho_\mu\xi_\nu)\langle\epsilon^\mu,\rangle\otimes\langle\epsilon^\nu,\rangle = T_3 \quad (11)$$

So we see the sum of two tensors reduces to a third tensor, with components $(\omega_\mu\sigma_\nu + \rho_\mu\xi_\nu)$.

With this in mind we can now introduce the tensor product space $T^p_q = \text{Hom}(V^* \underset{p}{\times} V^* \times V \underset{q}{\times} V, \mathbb{R})$

$$T^p_q = V \underset{p}{\otimes} V \otimes V^* \underset{q}{\otimes} V^* : V^* \underset{p}{\times} V^* \times V \underset{q}{\times} V \longrightarrow \mathbb{R} : (\omega_1, ..., \omega_p, v_1, ..., v_q) \mapsto \mathbb{R} \quad (12)$$

$$T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}\, e_{\mu_1} \underset{p}{\otimes} e_{\mu_p} \otimes \epsilon^{\nu_1} \underset{q}{\otimes} \epsilon^{\nu_q} \quad (13)$$

This generalises the notion of a vector and dual vector, where we now identify vectors as (1,0) tensors and dual vectors as (0,1) tensors. So vectors are just a particular type of tensor, but as we said previously, the multilinearity of the tensor map ensures that the tensor itself is a vector, confusing! But we understand the two terms contextually to mean different things, we understand a vector simply as an object that we can add and scalar multiply within some vector space, so we identify that the tensor product space is simply a vector space who's elements are the (p,q) tensors.

8 The Metric

So far we have made the distinction between vectors and dual vectors, however we have no way to relate a given vector to it's corresponding dual vector. The introduction of a metric gives us a way to associate a vector with it's corresponding dual vector. To do this we must introduce a new object on our vector space, a so called bilinear form $g(,) : V \times V \longrightarrow \mathbb{R}$. This is simply just a (0,2) tensor but with a few special properties. Namely that,

$$symmetry : g(u,v) = g(v,u) \quad (14)$$

$$positive\ \ definite : g(u,u) > 0 \quad \forall u \neq 0 \quad (15)$$

To understand the utility of this tensor we should expand it in a basis, being a (0,2) tensor this is simply

$$g = g_{\mu\nu}\epsilon^\mu \otimes \epsilon^\nu = g_{\mu\nu}\langle\epsilon^\mu,\rangle\otimes\langle\epsilon^\nu,\rangle \quad (16)$$

As we know, this is waiting to eat two vectors and return a real number. However, something interesting happens if we only feed it a single vector, which we write in terms of the vector basis

$$g(u,) = g_{\mu\nu}\langle\epsilon^\mu, v^i e_i\rangle \otimes \langle\epsilon^\nu,\rangle \quad (17)$$

now again using the rule that relates the dual basis to the vector basis this becomes

$$g(u,) = g_{\mu\nu}v^i\delta^\mu_i \otimes \langle\epsilon^\nu,\rangle = g_{\mu\nu}v^\mu\langle\epsilon^\nu,\rangle = v_\nu\langle\epsilon^\nu,\rangle \quad (18)$$

Hence we see after all the indices have contracted we are left with $v_\nu\epsilon^\nu$, so by feeding the (0,2) tensor a single vector we have created an object that is waiting to eat a single vector, i.e a dual vector. The metric has provided us with a way to map a vector into it's corresponding dual vector, the components of which are given by $g_{\mu\nu}v^\mu$. This operation is usually referred to as index raising and lowering, which we can now understand through the action of the metric on single vectors.

9 Topology

So far we have considered sets as simple collections of objects, we can define maps between these elements to give the set additional algebraic structure, but there is another type of structure we can give to a set known as a topology. We will simply state the definition then spend some time digesting it. Given a set, X, we define a topology, $\{\mathcal{T}_X\}$ on X which is a collection of subsets (a set whose elements are sets) of X which satisfy the following axioms

$$\emptyset, X \in \mathcal{T}_X \tag{19}$$

i) The set itself and the empty set are in the topology

$$\forall \tau \in \mathcal{T}_X : \bigcap_{\text{finite}\#} \tau \in \mathcal{T}_X \tag{20}$$

ii) A finite number of intersections of elements in the topology is also in the topology

$$\forall \tau \in \mathcal{T}_X ; \bigcup_{\text{infinite}\#} \tau \in \mathcal{T}_X \tag{21}$$

iii) A (potentially) infinite number of topology elements must have a union that is also in the topology

This is a pretty uninspiring definition, no coffee cups or doughnuts in sight, but whilst it appears to be fairly simple and mundane it actually holds an enormous amount of structure that we will begin to unpick now. The set taken together with its topology forms a topological space $(X, \mathcal{T}_X)$. There are many possible topologies one can define for a set X, the most trivial being the chaotic topology $\mathcal{T}_X = \{\emptyset, X\}$ or the discrete topology $\mathcal{T}_X = \{\emptyset, P(X)\}$, where P(X) is the power set of X (set of all possible subsets of X). The chaotic topology is the 'smallest' possible topology, while the discrete is the 'largest'; all other topologies will lie somewhere in-between these two extreme cases. The size here referring to the number of sets that the topology contains. Whilst these are perfectly valid topologies they are of little use in practice. What does a useful topology look like? As a first example we will consider the set $\mathbb{R} = \mathbb{R} \times \mathbb{R} : (x_1, ..., x_d)$ this set is viewed as the ordered d-tuple of real numbers. It is helpful to visualise this cartesian space as d real lines nailed together with a common origin (0) point. The elements of each real line are labelled here as x_i for the i-th real line. W.L.O.G consider the set $\mathbb{R}^2$ and visualise it as the 2D plane (x_1, x_2). As a bare set with no additional structure, the points of each real line can be arbitrarily re ordered and we still have the same overall set $\mathbb{R}^2$, a set has no way to distinguish between the orderings of it's elements $(\{1, 2, 3\} \equiv \{2, 1, 3\}.. + allperms$. However, giving this set the additional structure of a topology means this is no longer the case (except if the set has the chaotic topology, which just gives the behaviour of the bare set). We will frequently use the so called standard topology (for $\mathbb{R}$) which is defined as the following set:

$$\mathcal{U} \in \mathcal{T}_{\text{std}} : \left\{ y_i \in \mathcal{U} = B_r(x_i) : \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2} < r \right\} \tag{22}$$

that consists of the set of all *open balls*, with radius $r > 0$ around each point x_i (more about this shortly).

10 Continuity + Simple Topology example

A topology is the minimal amount of structure that a set needs in order to define the notion of continuity for maps between topological spaces. Let $(M, \mathcal{T}_M)$ and $(n, \mathcal{T}_n)$ be topological spaces, and f be a map between the two sets M, N.

$$f : M \longrightarrow N : m \in M \longmapsto f(m) \in N \tag{23}$$

The map is said to be continuous (with respect to the specific choice of topology on both spaces) if for all open sets in the target topology; the preimage of mapping this subset with f needs to be open in the domain topology to be continuous.

$$\forall \mathcal{V} \in \mathcal{T}_N : \text{preim}_f(V) \in \mathcal{T}_M \tag{24}$$

The preimage is simply the set of elements in the domain that map to a specific subset of the co-domain

$$\text{preim}_f(V) := \{m \in M | f(m) \in V\} \tag{25}$$

Now let's examine a simple topological space and check this definition for a simple map. Let $M, N = \{1, 2\}$ and the corresponding topologies be $\mathcal{T}_M = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}, \mathcal{T}_N = \{\varnothing, \{1, 2\}\}$. (The discrete and chaotic topologies for M and N respectively). Now consider if the map $f : M \longrightarrow N$

$$f(1) \mapsto 2 : f(2) \mapsto 1 \tag{26}$$

is continuous (w.r.t the choice of topologies). We need to check that the preimage of f is open (in the domain topology) for open sets in the codomain topology. For $\{\{1, 2\}\}$, ie the original domain of the map, the preimage is then simply $\{\{1, 2\}\}$, which is in the domain topology. The empty set just maps to the empty set which is in both topologies. Hence this map is continuous. Now consider the inverse map, ie

$$f^{-1} : N \longrightarrow M : f^{-1}(1) \mapsto 2, f^{-1}(2) \mapsto 1 \tag{27}$$

We want to check if this is continuous. Since we have to check for every open set in the codomain topology i.e $\mathcal{T}_M$, just consider $\{\{1\}\}$, its preimage would be $\{\{2\}\}$, but this is not in the domain topology$\mathcal{T}_N$, hence the inverse map is not continuous, due to the choice of topology.

11 Metric Spaces

A metric space is a topological space $(M, \mathcal{T}_M)$ on which we have defined a notion of 'distance' between points in the space. This distance is provided by defining a metric on the space. A metric is a symmetric bilinear form $d : M \times M \longrightarrow \mathbb{R}$ which satisfies:

$$i)\text{symmetry} : d(x, y) = d(y, x)$$
$$ii)\text{positive definite} \quad d(x, y) \geq 0 : 0 \text{ iff } x = y$$

If the space we are considering is $\mathbb{R}$ for example, then $d(x, y)$ can be defined as $|x - y|$. This metric can be used to define the standard topology on $\mathbb{R}$ by

$$\mathcal{U} \in \mathcal{T}_{\text{std}} : \{y \in \mathcal{U} = B_r(x) : d(x, y) < r\} \tag{28}$$

This topology has as it's elements the open balls (of all raddi $r > 0$) around each point $x \in \mathbb{R}$. Now if we consider starting with the set $\mathbb{R}^1$ where all the elements are in ascending order (for simplicity). Consider an open ball (of any size) around a point $x \in \mathbb{R}$. Now consider swapping a point (within the ball) with another point *outside* of the ball, producing a new set $\mathbb{R}'$. At the set level the two spaces are equivalent, however, they are no longer equivalent as topological spaces, since the open ball around our original point $x \in \mathbb{R}'$ is now a *different* ball to that around $x \in \mathbb{R}$, since it contains the swapped point. Hence we conclude that the two spaces must have a different topology and thus cannot be equivalent. Note we could have swapped two points within the ball and found the two sets would be equivalent, however since we can always find a ball that lies between the points we are swapping, we can always deduce that swapping two points must alter the topology. Introducing the standard topology to our set has effectively fixed the position of all the elements relative to eachother, since every point in a sense 'knows' about all the other points that lie within its open balls (neighbourhood). Attempting to move the elements around messes up this structure and the spaces are therefore distinct (also performing such a swap is a non-continuous action, since it requires a 'cut' in the topological space). However, since the metric we defined is arbitrary we can uniformly 'stretch' the real line and the two spaces will in general be topologically equivalent.

12 Topology Intro

In this video I want to informally introduce some topological concepts so we can begin to get a feel for what the subject is about. I will use terminology that won't be formally defined but the intuition should be made clear, then as we go on to formally define these concepts hopefully having these ideas in the back of your mind will be helpful.

The essential purpose of topology is to classify spaces by studying their topological invariants. We will formalise exactly what we mean by a topological space eventually, but for now let's just view it as an abstract object that we can somehow realise as a set. The example most widely considered would be a set of points corresponding to some object or shape, however the topological space could be entirely abstract and unintuitive. We will mostly want to stick to intuitive notion of 'space' so we will mostly consider geometrical objects as our examples of a topological space. So here are some simple examples of one dimensional topological spaces *draw circle, square, wiggly circle*

I used the word geometry previously. As geometric figures these pictures are distinct, however, we will see that topology actually knows nothing about geometry. Quantities like angles and distances are meaningless for topological spaces so through the eyes of topology these pictures are completely equivalent. This is an example of a homeomorphism, the notion that two topological spaces are "the same". By "the same", we mean that these topological spaces can be continuously deformed into one another without performing any illegal actions like cutting or gluing. More importantly, in performing these homeomorphisms we are preserving all of the topological invariants that these spaces possess, in fact, in order to be homeomorphic the two spaces must share all of the same topological invariants.

So what are topological invariants? A topological invariant is simply any sort of property that can be used to classify spaces, (by classify we mean to separate topological spaces into equivalence classes of spaces that share that invariant). In principle a topological invariant could be any property we can think of, and some are more useful than others.

I'll give some examples of invariants shortly, but for now I just want to note that there could in principle be an infinite number of different invariants we could possibly invent. This is unfortunate, as it means that given two top spaces there is no way to know if they ARE homeomorphic by simply comparing their invariants. Even if they share all the invariants you can think of, there is the possibility that they may differ on some yet un-thought of invariant. Only by comparing topological spaces that DO NOT share the same invariants is it possible to conclude that the two spaces ARE NOT homeomorphic. This is disappointing, as we would like a concrete way to conclude if two spaces are homeomorphic by simply comparing their invariants (other than constructing a specific homeomorphism between them). However, the ability to conclude they are not homeomorphic is still a powerful and useful tool.

Some examples of topological invariants would include the genus - number of 'holes' in a top space (draw tori). So we can conclude that all of these are distinct as topological spaces, it is impossible to construct a homeomorphism form a sphere to a torus for example.

Another example would be the Euler characteristic, which for polyhedra has a simple formula $\chi = V - E + F$. For more general objects like a surface, you have to divide the object into polygonal sections and calculate the combined Euler characteristic of the polygonal sections. Any convex polyhedra has $\chi = 2$, which also matches the Euler characteristic of a sphere - this among other things can be used to show that any convex polyhedra is homeomorphic to a sphere S^2.

How should we realise topological spaces as sets? As an easy example consider the circle S^1. We might realise this as a set of points

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

However, this definition already relies on additional structure that we do not want to define, i.e this would be an example of a circle which has been embedded into a higher two dimensional space. We want an intrinsic 1D definition of a circle. Let us consider a round circle now for simplicity (remember we can then generalise this to an arbitrary circle by constructing a homeomorphism). We want to only use one dimensional properties that are intrinsic to the circle in order to define the points of the topological space. The obvious choice would be the angular 'coordinate' θ, as this uniquely characterises every point on the circle. However, we need to be careful since when we travel around the circle through 2π we are returned to the original point. This means that we need to identify the points 0 and 2π as being the same point. We can do this by constructing a so called equivalence relation in the following way:

$$\{\theta \in \mathbb{R} \mid \theta \sim \theta + 2\pi\}$$

This simply means that a point theta is equivalent to the point theta + 2π. Essentially this gives us the interval $\theta = [0, 2\pi]$ where the endpoints are identified as being the same point, and this is how we define the topological space S^1. Notice now that it is no longer clear what the circle looks like (geometrically), it can be any object whose points can be labelled by some interval $[0, 2\pi]$ so long as the 0 and 2π points are identified. Intuitively then, the circle is the straight line segment with endpoints identified. This segment can then be stretched and deformed as much as we like so long as the endpoints remain identified and we don't cut or glue. Hence we have succeeded in finding an abstract way to characterise the topological space S^1 that is free from geometric properties. What we have defined is much more powerful, since any space homeomorphic to the circle can be realised as this set of points.

Now before moving on to more complex spaces I want to briefly mention a few more things about the cartesian product which was defined in the preliminaries video. The cartesian product of two sets

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

can be visualised in the following way. For simplicity let us use two intervals to be the sets $A = [a, b], B = [c, d]$. The cartesian product of these two sets is the set of all pairs with one element from each set. We can visualise this in the following way, consider taking the first interval A and to each point in the interval attach a copy of B. I.e at each point $a_i \in A$ we attach an entire copy of the set B, this is then the set of pairs $(a_i, [c, d])$ for all the elements in B. If we then repeat this procedure for every point in A we fill out this square which contains all the possible pairs of elements from A and B.

Now I want to use this idea to construct more complicated topological spaces. Consider the torus, T^2, which can be expressed as the cartesian product of two circles $T^2 = S^1 \times S^1$. To visualise this intuitively, consider drawing a circle (in three dimensions). Then at every point on that circle, construct another circle around that point, such that the two circles lie in orthogonal planes. If we then construct the same circle at every point on our larger circle we will fill out the surface of a torus. (draw cross sections). Now we can use our previous definitions of the circle and the cartesian product to realise the torus as a set. If we take one circle and form it's cartesian product with another circle, we know how to visualise this cartesian product as a set - we simply attach copies of one set to every point in the other set. We will fill out a square as before, however, we need to remember that each of our sets is a circle so the endpoints must be identified. Translating this to the cartesian product means that all the points on the bottom edge are identified with the top edge (endpoints of the second circle), and similarly all points on the side edges are to be identified. We are then left with this square with the top/bottom edges identified and side edges identified. To see how this is a torus consider our square to be made of ideal rubber. If we first glue the side edges together (we are allowed to glue in this circumstance since the points are identified when constructing the space, once the space is constructed no further gluing is allowed) we form a cylinder. Then if we stretch the cylinder so we can glue the two (circular) edges together we will form a torus. By physically constructing this object we have shown that the torus is homeomorphic to a square with opposite edges identified, or as a set the cartesian product of two circles.

Now we can use this simplified representation of the square with edges identified and extend this to other topological spaces. First if we consider just a square we have two possible sets of edges that can be identified. If we identify one edge with the opposite edge we would form a cylinder, I can just as equally draw this cylinder as an annulus in the plane as follows. Now consider what happens if we identify the points on this edge with the opposite point on the other edge. This would be taking the strip, performing a half twist and then gluing it so that the opposite points are identified. If I now try drawing this like the annulus we had before we see that all we need to do is join these edges to the opposite one. This now forms the mobius strip. The mobius strip is a two dimensional object, however it cannot be embedded into $\mathbb{R}^2$ without self intersection. This can be easily seen from the fact that we had to twist part of the annulus in a direction orthogonal to the plane in order to identify these edges with opposite orientation. If we imagine now our two (unconnected) boundary edges are constrained to move in the plane and are connected by flexible fibres that form the surface connecting them, there is no way for us to turn one of the edges (in the plane) so that it can be joined to the other edge with oposite orientation without the firbes intersecting themselves. This can also be seen by performing the half twist and how the fibres all intersect at a point. However, if the surface can extend into a third dimension we can separate these fibres and the surface can exist without self intersecting. This idea is useful to keep in mind as it is exactly the same reason as to why the Klein bottle cannot exist in 3D without self intersecting.

If we consider the mobius strip, we have left these two edges unidentified - they are in-fact the same edge since they form the boundary which is circular. Consider now identifying these two edges first, so that we form a cylinder. Now if we identify each circular boundary on the cylinder with the opposite orientation we will form the klein bottle. Essentially, we bring this part of the cylinder round as before as if we were forming a torus, but rather than rotating by a full 2π we only go half of the way around, and glue the circle to the other so their orientations are matching. It is clear in order to do this we will have to intersect the surface in some way, since this is a 2D surface embedded in 3D we have to extend into a fourth dimension in order to do this without self intersection.

The boundary is now a one dimensional circle and the surface formed is one sided. So the mobius strip has lots of interesting properties and we can explore a few of these now. The key property of the mobius strip is that it is non orientable.

13 Manifolds intro

Manifolds are an extremely useful type of topological space, to begin to understand how they are constructed it is helpful to consider a few light examples. We shall see that manifolds are often able to be visualised as geometric objects such as curves or surfaces (embedded in three dimensional real space). However, the notion of a manifold extends far beyond familiar geometric objects and can be used to describe many abstract topological spaces in a concrete fashion. The essential philosophy behind manifolds is that we have some topological space, which is an abstract set of elements, and we want to construct a way to concretely talk about the elements of this abstract set. The way we do this is to *cover* the topological space in what are knows as charts, or coordinate patches. As their name implies, charts are maps which effectively cover the space in coordinates, with these coordinates being elements of some d-dimensional real space. These charts allow us to realise the abstract elements of this topological space as concrete objects, namely d-tuples of real numbers.

This is usually expressed by saying that locally, a manifold is homeomorphic to (it can be continuously mapped into) $\mathbb{R}^d$, whilst globally the manifold might have a wildly different structure. As a simple example, consider the manifold S^2, (the surface of a sphere) which we can easily visualise when embedded (discussed later) into 3D real space. Now it should be noted that we are still not talking about the geometric properties of this space, it's only round because I have drawn it this way for simplicity. We know that topologically, any blobby surface could be considered to be equivalent to this sphere, only once further structure has been added (a metric connection) does this sphere have a specific 'shape'. For now we can ignore this fact and just consider the surface of the round sphere without loss of generality. This object is a manifold. The topological space is the set of all points corresponding to the points on the sphere (defined by some equation like $x^2 + y^2 = r^2$ say), and the topology is in this case inherited from the embedding space $\mathbb{R}^3$, namely the standard topology.

As a simple motivational example, consider the surface of the Earth (which is a sphere) to be our manifold, this should be realised as the set of all possible locations on the surface of the Earth. Now we know that we can produce (literal!) maps, that represent portions of the surface of the Earth. These maps are the charts which cover our manifold. A map is a two dimensional, flat image and can be understood to be a subset of $\mathbb{R}^2$. On this map we construct coordinates to label each point, simply by a set of two numbers (latitude and longitude in this case). The points on this map represents a particular location on our manifold (an abstract set of locations) by a concrete object, namely the coordinates of that location. The key point to realise now is that the map we have drawn is arbitrary, we can draw the map any way we like but the underlying manifold does not change. Furthermore, if we have two maps that cover the same region of the manifold, the maps should agree. If we consider a point that lies in the overlapped region of the manifold, the point in question will have a different set of coordinates in each chart, however, there should always be a definite way to *transition* between these charts. This can be understood in the following way. If we have constructed an entire set of charts that completely covers our manifold (in this case the surface of the Earth still) then we should be able to reconstruct the manifold simply by considering the union of all of these charts. If we produce an entire set of maps that covers every point on the Earth (an atlas) and they are sufficiently accurate; we should be able to tear out all the pages and glue them together (making sure to align overlapping regions) and reconstruct the spherical surface from the flat pages. This can only be done if the maps agree and are drawn accurately enough such that overlapping regions can be glued together in a smooth way. This will turn out to be a key property that these charts must satisfy for any given manifold, that overlapping charts must agree and there must exist a smooth way to transition between the coordinates in each chart.

14 Topological Manifolds

Now that we have briefly seen what makes a set into a topological space we can move on to consider an extremely useful class of topological spaces, topological manifolds. A d-dimensional topological manifold $(M, \mathcal{T}_M, \mathcal{A})$ is defined as follows:

$$\forall p \in M : \exists\, \mathcal{U}_p \in \mathcal{T}_M \text{ and } \exists \varphi_p : \mathcal{U}_p \longrightarrow \varphi_p(\mathcal{U}_p) \subset \mathbb{R}^d \tag{29}$$

"For all points p in the manifold there exists an open set around that point (that comes from the topology), and a map from that open set into an open subset of d-dimensional real space." The pair $(\mathcal{U}_p, \varphi_p)$ is known as a chart (with chart map φ) and the collection of all the charts is called the atlas $\mathcal{A}$. The union of all the charts should cover the manifold i.e

$$\bigcup_p \mathcal{U}_p = M \tag{30}$$

Charts are can be viewed as giving coordinates to the points in the manifold (sometimes referred to as coordinate patches). There are many possible charts one could define, which correspond to different choices of coordinates on our manifold. In order for this definition to be consistent we require that the charts must agree on regions where two charts overlap. Each chart will map the intersection to some open subset of $\mathbb{R}^d$, but we can go between the charts using the transition functions. If we have two charts that overlap, $\mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing$ we can map the overlapping region using either φ_a or φ_b, each of these will in general map to a different open subset of $\mathbb{R}^d$, however we can go between the open subsets using the transition functions

$$\psi_{ab} = \varphi_a \circ \varphi_b^{-1} : \varphi_b(\mathcal{U}_a \cap \mathcal{U}_b) \longrightarrow \varphi_a(\mathcal{U}_a \cap \mathcal{U}_b) \tag{31}$$

This function can be viewed as a change of coordinates. We are expressing the overlapping region of the manifold by some set of coordinates, but we are free to change these coordinates by adjusting how we map from $\mathcal{U} \longrightarrow \mathbb{R}^d$. The underlying space remains the same we are just using a different set of numbers to represent the space using coordinates.

A good analogy is to consider the surface of the Earth to be our manifold. We can produce (literal!) maps of local regions on the surface, these maps being flat two dimensional surfaces. Points on the surface of the Earth will correspond to some point on our map, and we could in general have several maps that cover the same set of points. If we produced an entire atlas of these maps, covering the entire surface of the Earth, we could then consider sewing these (flat) portions together in such a way that they reconstruct the original manifold. If we sew the maps together such that we join overlapping regions together, we should in theory reconstruct the spherical surface if the maps were sufficiently accurately constructed.

15 The Tangent Space

Now that we have discussed manifolds we are ready to begin exploring more complicated structures. The tangent space is a vector space that exists at every point in the manifold, and that arises naturally when constructing charts on the manifold. I will now sketch how we construct the tangent space schematically, before returning to it in greater detail once we have some additional machinery at our disposal (pullback).

To construct the tangent space at a point $p \in M$, we begin with the set of all possible smooth curves $\phi = x^\mu(\lambda) \in C_p^\infty(M)$, passing through that particular point, where x^μ refers to the coordinates of M. Such curves are known as integral curves and can be expressed in terms of a parameter, λ

$$x^\mu : \mathbb{R} \longrightarrow M \tag{32}$$

and the parameter λ, is chosen such that $x^\mu(0) = p$, i.e every curve passes through the origin point of the tangent space being constructed. The image of the curve $x^\mu(\lambda)$, is a subset of points tracing out a particular trajectory through the manifold. We could now consider defining a function on that subset of points (or restricting a function defined on M to the points on the curve). A vector is then realised as a linear operator, a derivative, that acts on such curves producing their tangents. For now, we will focus on developing the intuition needed for tangent vectors and return to discuss vectors in terms of integral curves and diffeomorphisms later.

Given a function, $f(x^\mu(\lambda))$, defined along the curve $x^\mu(\lambda)$ we now want to consider taking the parameter derivative of such a function (and evaluating it at the point in question). Each curve (just a multi-linear function) will have a parameter derivative along each of the coordinate directions x^μ, and this derivative is viewed as creating the tangent (velocity) vector to the curve. We therefore identify the set of all tangent vectors to all possible smooth curves through a point as the vector space known as the tangent space at that particular point T_pM. The derivative of f can be easily evaluated using the chain rule as follows

$$\frac{\partial}{\partial\lambda} f(x^\mu(\lambda)) = \frac{\partial x^\mu}{\partial\lambda}\frac{\partial f}{\partial x^\mu} \tag{33}$$

This can now be suggestively re-arranged to give us the form of our vector as a linear operator

$$\left(\frac{\partial x^\mu}{\partial\lambda}\frac{\partial}{\partial x^\mu}\right) f = \left(v^\mu \frac{\partial}{\partial x^\mu}\right) f = v(f) \tag{34}$$

where now we have identified the vector components, $v^\mu = \frac{\partial x^\mu}{\partial\lambda}$, as being the individual velocities (parameter derivatives) in each coordinate direction. We also identify the The basis for this space is given by the derivatives $\frac{\partial}{\partial x^\mu}$ where the x^μ refers to the μ-th coordinate direction (specified by the dimension of the manifold and the particular chart map in question). These partial derivatives can be easily seen to form a basis for the space since they are just linear maps (they can be added and scalar multiplied), and they are linearly independent so hence form a basis. Any vector in this tangent space can then be expressed as the linear combination

$$V = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \partial_\mu \tag{35}$$

It might seem abstract to think of a vector in this way, in particular the basis vector ∂_μ seems to be incomplete, a derivative needs an argument. However, this definition of a vector now aligns with our previous realisation that a vector is a linear map, it must act on *something*, the *something* in this case being a function that describes a curve through the manifold. Now we should be absolutely clear about what we are talking about, we are considering the algebraic structure generated by these derivatives that act (on smooth curves) at each point in the manifold, and we identify these derivatives as spanning a vector space called the tangent space. This then gives us a 'realisation' of a vector as a linear map, namely a derivative that acts on smooth functions in the manifold.

This basis is frequently referred to as the coordinate basis and as we shall now see is the origin of many bullshit statements like "A tensor is something that transforms like a tensor". This should rightly trigger you, to simply ask, WTF is a tensor? We have previously defined a tensor as a multilinear map, and we will see that this statement would be more accurate if it read "Tensor components transform under a change of coordinates, and this transformation resembles the chain rule when expressed in the coordinate basis".

Let us explore how a vector transforms under a change of coordinates. We know that coordinates refer to a specific chart, let this be $(\mathcal{U}, x^\mu)$. Here we label each individual chart map into $\mathbb{R}$ by an index μ, so we interpret these x^μ as being our coordinates on the chart. Now take a point $p \in \mathcal{U}$ and construct its tangent space T_pM. The basis for this space is given by the coordinate basis $\frac{\partial}{\partial x^\mu}$. So far so normal, but now suppose we have a different chart map, y^μ, on the same chart for simplicity. Both x^μ and y^μ are equally good choices of coordinates for the chart, so we expect to be able to transition between them and find an expression for the vector in terms of the new coordinate basis, $\frac{\partial}{\partial y^\mu}$. To see this we can simply write the vector in terms of each basis:

$$V = V^\mu(x)\frac{\partial}{\partial x^\mu} = V^\nu(y)\frac{\partial}{\partial y^\nu} \tag{36}$$

Now we need to find an expression that relates the basis in one coordinate system to the basis in the other, this is obtained simply using the chain rule (this operation has a more rigorous interpretation which we will introduce shortly)

$$\frac{\partial}{\partial x^\nu} = \frac{\partial y^\mu}{\partial x^\nu}\frac{\partial}{\partial y^\mu} \tag{37}$$

substitute this expression into (36) to obtain an expression for the vector components $V^\nu(y)$ in the new coordinate system:

$$V^\nu(y) = V^\mu(x)\frac{\partial y^\nu}{\partial x^\mu} \tag{38}$$

This method of obtaining the vector transformation law using the chain rule is often the most simple in practice, however this operation is more formally stated using the notion of push-forward, which we will discuss shortly.

16 Differential Forms

Differential forms, sometimes abbreviated to q-forms, can be realised as a dual object to vectors, and these dual objects live in the cotangent space. Formally, the cotangent space is defined as the dual space to the tangent space, such that any dual vector is a linear map from $T_pM \longrightarrow \mathbb{R}$. We then saw that any tensor can be expressed as a multilinear map from the appropriate cartesian product of vector/dual spaces. The exact same notions will apply here, and we will be able to construct arbitrary (p,q) tensors using our tangent and cotangent spaces. So briefly, I will just define a tensor (we will come back to this much more later).

To be completely precise, the cotangent space contains more objects than just differential forms (all (0,q) tensors, but for reasons that are beyond this video we are only really going to be interested in the (0,q) tensors which are anti-symmetric, such tensors are known as q-forms. I will begin by considering q-forms over $\mathbb{R}$, and the generalisation to any manifold follows by defining the q-form in the respective charts (just a copy of $\mathbb{R}$).

To begin defining q-forms I will need to introduce a fair bit of new machinery (the exterior calculus), whilst this can appear daunting at first, I encourage you to stick with it as we will eventually see how this exterior calculus is nothing but a generalisation of the more familiar vector calculus. The full picture can be hard to grasp the first time around, since you really need to have seen all of the construction before it starts to make sense, so stick with it and re-watch if you're confused!

To begin, I will simply introduce the notion of a zero form, simply a function.

The space $\Omega^q(p)$ is the vector space of all degree q forms at the point p and T_pM^* the cotangent space the meaning of the number q will become evident shortly. The space of one forms at a point is known as the cotangent space, T_pM^*, and is the dual vector space to the tangent space. The collection of all the cotangent spaces at every point in the manifold forms the cotangent bundle, TM^*. The notion of differential forms being dual to vectors will be explored shortly.

We will now explain how to construct degree q differential forms. The simplest form, a 0-form, is a function $f \in C^\infty(M)$. Hence we can identify the space of degree 0 forms $\Omega^0(M) = C^\infty(M)$. All higher degree forms can be constructed using an operator known as the exterior derivative. The exterior derivative,

$$\mathrm{d} : \Omega^q(M) \longrightarrow \Omega^{q+1}(M), \tag{39}$$

is defined such that it maps a degree $q-$form to a $(q+1)$-form. We will present the exact formula for calculating the exterior derivative shortly.

We can now use this to define the space of degree 1-forms, the cotangent space $\Omega^1(p) = T_pM^*$. The cotangent space is spanned by the basis elements dx^μ, where $\mu = 1, ..., \mathrm{dim}M$. These are to be understood as the exterior derivative of the coordinate 0-forms, x^μ. We can now define an arbitrary 1-form as

$$\omega = \omega_\mu \mathrm{d}x^\mu. \tag{40}$$

Now we have the basis for 1-forms we can fully define the exterior derivative by it's action on 0-forms, $f \in \Omega^0(M)$, as

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu, \tag{41}$$

which for the case $f = x^\mu$ reduces to $df = dx^\mu$. Hence we can see how the exterior derivative of a 0-form is everything we need to construct a 1-form, and indeed higher degree forms as we will now explore.

Now that we have 1-forms, we can define a new operation known as the exterior (or wedge) product. This product obeys the crucial property of anti-symmetry,

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu. \tag{42}$$

When constructing higher degree forms this anti-symmetry becomes important, however, it is also required for a more fundamental fact known as Poincare's lemma which we will discuss shortly. For example, degree two forms have the basis $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. However, basis elements such as $\mathrm{d}x^1 \wedge \mathrm{d}x^2$ and $\mathrm{d}x^2 \wedge \mathrm{d}x^1$ are equivalent due to anti-symmetry. We account for this overcounting of basis elements by imposing that the (implied) sum over μ and ν should only range over $\mu < \nu$. Recall that μ/ν run from 1 to $d = \mathrm{dim}M$. Consider for example the case of $d = 3$. The basis for $\Omega^2(M)$ is therefore

$$\{\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \mid \mu < \nu \leq 3\} = \{\mathrm{d}x^1 \wedge \mathrm{d}x^2, \mathrm{d}x^2 \wedge \mathrm{d}x^3, \mathrm{d}x^1 \wedge \mathrm{d}x^3\} \tag{43}$$

An arbitrary 2-form $\omega \in \Omega^2(M)$ can therefore be expressed as the linear combination

$$\omega = \omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \omega_{12}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \omega_{23}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \omega_{13}\mathrm{d}x^1 \wedge \mathrm{d}x^3 \tag{44}$$

where the sum over $\mu < \nu$ is implied. The components can be realised as the antisymmetric matrix in the following way

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \implies \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \tag{45}$$

where the off diagonal components vanish due to antisymmetry.

We can now state Poincare's lemma,that the second exterior derivative of an arbitrary degree form always vanishes, namely

$$\mathrm{d}(\mathrm{d}\omega) = 0 \tag{46}$$

We can prove this for 1-forms and the general case follows by extension. Consider

$$\mathrm{d}(\mathrm{d}f) = \mathrm{d}\left(\frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu\right) = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu}\mathrm{d}x^\nu \mathrm{d}x^\mu = -\frac{\partial^2 f}{\partial x^\nu \partial x^\mu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0 \tag{47}$$

which vanishes due to antisymmetry (partials commute).

17 The cotangent space

Previously when we defined vectors, we discovered that every vector space should have a dual, consisting of the space of all linear functionals (dual vectors) which map a vector in the space into $\mathbb{R}$. We have a similar notion here for tangent vectors, where we now identify the cotangent space as the dual to the tangent space. Technically, the cotangent space should consist of all $(0, q)$ tensors (symmetric + antisymmetric), but we usually just consider the space of forms and call this the cotangent space.

Since we know that dual vectors should be linear functionals (maps) from a vector into $\mathbb{R}$ we expect 1-forms to behave similarly. This is made explicit using the definition of the interior product, given $v \in T_pM$ and $\omega \in T_pM^*$:

$$\langle\ ,\ \rangle : T_pM \times T_pM^* \longrightarrow \mathbb{R} \tag{48}$$

$$i_\omega v = \omega(v) = \langle \omega, v\rangle \tag{49}$$

This should be read as the insertion of a 1-form into a vector (and is completely equivalent to the reverse operation, inserting a vector into a 1-form).

This is then evaluated by choosing a particular coordinate basis, and using the fundamental relation that the two bases must be dual, i.e

$$\left\langle \mathrm{d}x^\mu, \frac{\partial}{\partial x^\nu}\right\rangle = \delta^\mu_\nu \tag{50}$$

where $\mathrm{d}x^\mu$ is the (1-form) dual basis to the vector coordinate basis ∂_ν. This duality condition is core to everything that follows so we will frequently encounter it.

We can now define tensor fields on our manifold in exactly the same way as previous, namely that a tensor is a multilinear map from the

$$T^{\mu_p}{}_{\nu_q} : T_pM^* \underset{p}{\times} T_pM^* \times T_pM \underset{q}{\times} T_pM \longrightarrow \mathbb{R} \tag{51}$$

where the tensor is an element of the tensor product space

$$T \in \mathcal{T}^p_q = T_pM \underset{p}{\otimes} T_pM \otimes T_pM^* \underset{q}{\otimes} T_pM^* \tag{52}$$

Hence a tensor is simple a multi-linear map waiting to eat (p, q) (dual v., vector) respectively. Tensor fields can now be expressed using the appropriate bases, for example a $(0, 2)$ tensor would have the form

$$T = T_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{53}$$

18 The Metric Tensor (on a manifold)

As we defined previously, the metric is the $(0,2)$ tensor

19 Vector Calculus $(\Omega^q(\mathbb{R}^3))$

Here we will see how all of the machinery introduced in vector calculus is simply the special case of the exterior calculus over $\mathbb{R}^3$. Recall some vector calculus terminology, 'the gradient' of a function. This would have usually been written as

$$\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x^i} \tag{54}$$

Similarly, an object called the 'total derivative' or 'total differential'

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \tag{55}$$

Previously this may have been justified by talking about the infentesimal 'vectors' $\mathrm{d}x^i$ and taking the dot product $\nabla f \cdot \mathrm{d}x^i$. Shortly, we will understand this relation as the contraction of a vector (∇f) with a 1-form $(\mathrm{d}x^i)$.

The expression for the 'total derivative' can now be identified as the exterior derivative (Eq. 41) of a 0-form on $\mathbb{R}^3$. The corresponding 1-form, $\mathrm{d}f$ has components$\frac{\partial f}{\partial x^i}$ with respect to the coordinate 1-form basis $\mathrm{d}x^i$.

Similarly, we can identify that the gradient as the basis vector (field), evident from the following expression

$$\nabla f = \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} f + \frac{\partial}{\partial z} f \tag{56}$$

Expressions such as $\nabla f \cdot v$ are now identified as the action of the vector field vector field on the function f, i.e the directional derivative of f along v

$$\nabla f \cdot v = v(f) = v^i \frac{\partial f}{\partial x^i} \tag{57}$$

With these identifications made, we can now use the fact that vectors and 1-forms are dual objects, to identify the following relation:

$$\mathrm{d}f(v) = v(f) \tag{58}$$

This is easily seen in $\mathbb{R}^3$, using

$$\langle \mathrm{d}f, v \rangle = \left\langle \frac{\partial f}{\partial x^i}\mathrm{d}x^i, v^j \frac{\partial}{\partial x^j} \right\rangle = v^j \frac{\partial f}{\partial x^i} \left\langle \mathrm{d}x^i, \frac{\partial}{\partial x^j} \right\rangle = v^j \frac{\partial f}{\partial x^i} \delta^i_j = v^i \frac{\partial f}{\partial x^i} = v(f) \tag{59}$$

where we have used the dual basis relation

$$\left\langle \mathrm{d}x^i, \frac{\partial}{\partial x^j} \right\rangle = \delta^i_j \tag{60}$$

We should interpret this expression as the action of a 1-form (field) $\mathrm{d}f$ on a vector field v. It is now illuminating to consider this expression for the coordinate function $f = x^\mu$, such that

$$\begin{aligned}
\mathrm{d}x^\mu(v) &= v(x^i) \\
\langle \mathrm{d}x^i, v^j \frac{\partial}{\partial x^j} \rangle &= v^j \frac{\partial x^i}{\partial x^j} \\
v^j \left\langle \mathrm{d}x^i, \frac{\partial}{\partial x^j} \right\rangle &= v^j \delta^i_j \\
v^j \delta^i_j &= v^i
\end{aligned}$$

So, we see that contracting or inserting a vector into a basis one form simply yields the components of that vector. Fortunately, for the case of 1-forms in $\mathbb{R}^3$ there is a convenient geometrical visualisation 1-forms as normal surfaces. Consider the basis one forms $\mathrm{d}x^i$, three one forms one for each coordinate direction. Consider a single $\mathrm{d}x$ now, this can be visualised as the set of normal planes to the x axis. Any vector in $\mathbb{R}^3$ will have a certain component along the x axis, which can be obtained by 'measuring' how many of the normal planes are 'pierced' by this vector. This 'measuring' is the geometric interpretation of inserting the vector into the 1-form, and then given any vector you can measure it's components by this procedure. A more complicated 1-form will be some linear combination of some function multiplying these 'basis' normal surfaces - essentially spreading them out based on the function used. Consider the 1-form $x^2\mathrm{d}x$, intuitively, the normal planes at coordinate x are now at coordinate x^2, and so any vector measured along this 1-form would have components $\sqrt{x}$ (make sure to show why!)

We can get a feel for why this is useful from considering the 'dot product'. Previously, the operation would have been described as multiplying a row vector with a column, now we should identify that this operation is exactly the insertion of a vector into a 1-form ('row vectors' or transposed vectors in linear algebra). So therefore we should realise that the operation of taking the dot product of two vectors, actually involves turning one of the vectors into a 1-form (more later) and inserting the other vector.

20 Divergence

Now we can turn to other expressions from vector calculus. Consider the divergence, usually expressed as the dot product of the ∇ 'vector operator' with a function.