Boundary conditions and edge modes in gauge theories

MATH4050: MSc dissertation in Gravity, Particles and Fields

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September 11, 2020

This work explores the phenomena of edge modes (topological boundary conditions) first observed in gauge theories with non-trivial boundary. Following the methods used in Ref. [1] the boundary condition is implemented using the homotopy pullback of bulk and boundary groupoids. We then introduce edge modes in the (2+1) dimensional Einstein-Cartan theory, in an attempt to compensate for the degenerate boundary geometry found in the standard formulation. Using the edge modes to construct 'dressed' boundary fields we consider adding a dressed boundary term to the action, that under the variation is found to compensate for the degenerate boundary geometry. Furthermore, we derive the resulting constraints the dressed fields must satisfy on the boundary and verify they are consistent with cosmological half Minkowski and de Sitter spacetime vacuum solutions. For consistency in the de Sitter solution one is required to introduce a spacelike hypersurface boundary cosmological constant λ with a suitable dressed boundary cosmological constant term. The resulting augmented constraints are consistent in the de Sitter case for an identity Lorentz edge mode and obtain the boundary cosmological constant $\lambda^2 = \Lambda$.

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1 Introduction

This work examines the phenomena of 'edge modes', boundary conditions arising in gauge theories with a non-trivial boundary and introduces edge modes into the Einstein-Cartan theory in an attempt to resolve issues with the boundary geometry. We begin in section 2 with a brief summary of how gauge theories (non-abelian Yang-Mills theories) are formulated using a principal connection on a principal G-bundle, before introducing the groupoid perspective for gauge fields and transformations. We then introduce the phenomena of edge modes, topological boundary conditions arising in gauge (and gravity) theories with non-trivial boundary. The groupoid perspective allows us to formally construct the boundary condition using the homotopy pullback of bulk and boundary gauge groupoids, and identify the corresponding edge modes that are realised as gauge transformations (or diffeomorphisms in the gravity case). Such a homological construction is useful as it may be applied to more general 'gauge like' theories such as Einstein-Cartan gravity, that may be formulated using the connection and groupoid formalism.

Section 3 introduces the Einstein-Cartan formalism for general relativity (GR), which since is formulated using a (Cartan) connection has many features that resemble a gauge theory. We formalise the appropriate tools from Cartan geometry before discussing their applications in constructing the spacetime geometry. After presenting the standard EC formalism, section 4 considers the effects of introducing a non-trivial boundary and identifies an issue present in the standard theory for the boundary geometry; that it is implied to be degenerate. In an attempt to resolve this issue we introduce edge modes in the EC theory, by constructing the appropriate homotopy pullback. One finds that in the gravitational case one requires *two* edge modes (boundary conditions), namely a Lorentz (gauge) transformation and diffeomorphism on the boundary. Once the edge modes have been identified this allows us to construct 'dressed fields' using the edge modes, from which additional terms in the action can be constructed. By adding a suitable dressed boundary term we show that under the variation this term is sufficient to compensate for the boundary degeneracy implied previously. Furthermore, we obtain two additional constraints that must be satisfied by the dressed fields (and hence the edge modes) on the boundary.

Once such constraints have been obtained we verify the model in section 5 by evaluating the constraints for (2+1) dimensional cosmological solutions with boundary (Minkowski and de Sitter). We focus on the three dimensional case for simplicity, since one finds for a (1+1) dimensional timelike boundary hypersurface one may take the edge mode to consist of a boost only. We find the constraints are consistent with a bounded (half) Minkowski spacetime and find an arbitrary constant edge mode h and boundary diffeomorphism φ in this case (the diffeomorphism φ is always arbitrary). When evaluating the constraints in the half de Sitter spacetime, we found inconsistent results when including a single dressed boundary term. This is resolved by introducing a dressed boundary cosmological constant term, corresponding to the cosmological constant on the boundary of two dimensional hypersurfaces in M. In principal, one must introduce three (in the (2+1) dimensional case) boundary cosmological constant terms, corresponding to the boundaries of the spacelike and two timelike hypersurfaces in M. However, we find the solution only yields consistent results when letting the timelike boundary constants be zero, and only identifying a cosmological constant on the boundary of *spacelike* hypersurfaces in M. After such an identification, we find the half de Sitter spacetime is consistent with the dressed boundary constraints and furthermore obtain the edge mode $h = \mathbb{I}$ as a unique constant solution. Finally, we verified the ansatz of a spatially constant edge mode is not consistent due to requiring the boundary constant be dynamical.

2 Gauge theories and Edge modes

This section overviews the machinery used in constructing (non-abelian Yang-Mills) gauge theories, before introducing the phenomena of 'edge modes' in the form of a topological boundary condition.

2.1 Gauge theories

The term 'gauge theory' refers to field theories formulated using gauge fields, fields that transform under a particular type of transformation known as a gauge transformation; meaningless for now but all to be defined shortly. The terminology is born out of the principle of gauge invariance, stating that physical observables should not depend on the measuring system (gauge) used. Arbitrary gauge transformations (changes of measurement system) should leave any physical theory invariant. Therefore, such gauge transformations correspond to a symmetry of the theory; a gauge symmetry.

Gauge theories are mathematically formulated using (an associated vector bundle to) a principal G-bundle, over a Lorentzian manifold M (for generality assume G is non-abelian). The gauge 'field' is identified as a principal connection on such a bundle, with the restriction of the connection to the base manifold corresponding to the 'physical' gauge field ('Yang-Mills' field). The transformations of such a connection when expressed in a local trivialisation (gauge transformations) will be central to our discussion.

2.1.1 Gauge fields and connections

Gauge fields are formalised using a principal (Ehresmann) connection $\tilde{\omega}$ in a principal G-bundle $P \xrightarrow{\pi} M$. The bundle in question is usually an associated vector[1] bundle describing a collection of (non gauge) fields, that transform under a representation of G. Geometrically, the connection can be understood to partition TP into *horizontal* and *vertical* subspaces, where the vertical space is identified as the Lie algebra (tangent to the fibre). Whilst the connection may be simply expressed as a G-equivariant Lie algebra valued 1-form on the principal bundle, it obtains non-trivial transformation behaviour when expressed in a trivialisation as we explore shortly.

Let P be a principal G-bundle over M, the principal connection is the G-equivariant, Lie algebra valued[2] 1-form $\tilde{\omega} \in \Omega^1(P, \mathfrak{g})$, hence for any vector field $X \in TP$

$$R_g^* \tilde{\omega}(X) = \mathrm{Ad}_{g^{-1}*} \tilde{\omega}(X) \tag{1}$$

furthermore, any fundamental vector field $\xi^g = \exp(tg)$ inserted into the connection simply returns the generating vector field $g \in \mathfrak{g}$

$$\tilde{\omega}(X^\xi) = \xi \tag{2}$$

which is thus realised as the requirement that $\tilde{\omega}$ restricts to the Maurer-Cartan form on vertical vector fields.

In a trivialisation, the connection may be pulled back to the base manifold using a local section as $\omega = \sigma^* \tilde{\omega} \in \Omega^1(M, \mathfrak{g})$, where the (principal) connection must satisfy the following. Given any

[1] or spinor, depending on the context

[2] see appendix for details

vector field $X \in TP$ which is pushed forward to M as $X_M = \pi_* X \in TM$, the principal connection satisfies

$$R_g^* \tilde{\omega}(X) = \mathrm{Ad}_{g^{-1}*}\, \omega(X_M) + \Xi_g(\xi_g) \tag{3}$$

namely the G-equivariance property and that $\tilde{\omega}$ restricts to the Maurer-Cartan form on the fibre.

In the case where the bundle is trivial, the connection may simply be regarded as a globally defined 1-form on the base manifold. Concretely, given a suitable matrix representation of the group G (henceforth always assumed), the connection for a trivial bundle may be expressed as the 1-form that satisfies

$$R_g^* \omega = g^{-1} \omega g + g^{-1} \mathrm{d} g \tag{4}$$

where $\Xi_g = g^{-1}\mathrm{d}g$ is the Maurer-Cartan form in the matrix representation of G.

2.1.2 Gauge Transformations

Local sections/trivialisations should be interpreted as defining a local 'gauge' (measurement system) within each fibre, analogous to defining charts on a manifold, with a local trivialisation providing a 'coordinate' representation of the points in the fibre. For example, let $G = \mathbb{R}$, then a local trivialisation identifies the points in the fibre with $\mathbb{R}$. Therefore, a choice of trivialisation corresponds to choosing some (arbitrary) value for each (abstract) point in the fibre. The freedom in choosing an arbitrary trivialisation is the realisation of 'gauge invariance', just as one has the freedom to choose arbitrary charts on a manifold (diffeomorphism invariance). Hence, one should expect that changing the trivialisation should give rise to a corresponding 'gauge transformation'; analogous to the chart transition functions.

Given local sections/trivialisations $\sigma_\alpha, \sigma_\beta$ over an intersecting region on the base $U_\alpha \bigcap U_\beta \neq \varnothing$, one may construct a 'gauge transformation' between gauge fields $\omega_\alpha, \omega_\beta = \sigma_\alpha \tilde{\omega}, \sigma_\beta \tilde{\omega}$ such that gauge fields defined on overlapping sections are in correspondence. This is achieved by defining the group element $g_{\alpha\beta} \in C^\infty(U_\alpha \bigcap U_\beta, G)$ that acts on the connection through a gauge transformation as

$$\omega_\beta = g_{\alpha\beta}^{-1}\, \omega_\alpha\, g_{\alpha\beta} + g_{\alpha\beta}^{-1}\, \mathrm{d} g_{\alpha\beta} \tag{5}$$

where the subscripts are labelling sections and do not represent indices. Hence, in order to conclude the two gauge fields are equivalent, one requires the corresponding gauge transformation between them.

2.1.3 Curvature

Given a principal connection one may also compute the curvature 2-form $\tilde{\Omega} \in \Omega^2(P, \mathfrak{g})$, defined by the Cartan structural equation as

$$\tilde{\Omega} = \mathrm{Hor}^{\tilde{\omega}} \circ \mathrm{d}\tilde{\omega} = \mathrm{d}\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] \tag{6}$$

where the connection defines the horizontal projection

$$\mathrm{Hor}^{\tilde{\omega}} : \Omega^k(P, \mathfrak{g}) \longrightarrow \Omega^k_{\mathrm{hor}}(P, \mathfrak{g}) \tag{7}$$

of Lie algebra valued k-forms onto their horizontal part. The curvature 2-form must be equivariant under the right action, and since it is horizontal it simply transforms with the adjoint action, hence $\forall X, Y \in TP$

$$R_g^* \tilde{\Omega}(X,Y) = \mathrm{Ad}_{g^{-1}*} \tilde{\Omega}(X,Y), \quad R_g^* \Omega(X,Y) = \mathrm{Ad}_{g^{-1}*} \Omega(X,Y) \tag{8}$$

where the local curvature form $\Omega \in \Omega^2(M, \mathfrak{g})$ is also horizontal and G-equivariant. Hence, the curvature can easily be used to construct gauge invariant actions due to it's simple transformation behaviour.

2.1.4 Principal bundle automorphisms

Here we review the semidirect product structure of trivial principal bundle automorphisms to be used throughout. Arbitrary transformations involving both diffeomorphisms and gauge transformations will be suitably described by automorphisms f of the appropriate principal bundle, which can be seen to decompose into a gauge transformation and a diffeomorphism of M. The automorphism group fits into the sequence of group homomorphisms

$$0 \longrightarrow \mathcal{G}(P) \longrightarrow \mathrm{Aut}(P) \longrightarrow \mathrm{Diff}(M) \tag{9}$$

here $\mathcal{G}(P)$ is the gauge group (of all gauge transformations of P) and the last map is only surjective, since there are many automorphisms of P that may correspond to the same diffeomorphism (e.g all pure gauge transformations, which correspond $\mathrm{Diff}(M) = \mathrm{id}_M$). In the trivial case (henceforth assumed) the automorphism group is given by the semi-direct product

$$\mathrm{Aut}(P) \cong \mathrm{Diff}(M) \ltimes C^\infty(M, G) \tag{10}$$

since the gauge group in the trivial case is simply given by $C^\infty(P \times_{\mathrm{Ad}} G) = C^\infty(M, G)$. Given $(\phi, \lambda) \in \mathrm{Diff}(M) \times C^\infty(M, G)$, denote the corresponding automorphism by $f_{(\phi,\lambda)} \in \mathrm{Aut}(P)$, the action on an arbitrary point $p = (x, g) \in P$ is given by,

$$f_{(\phi,\lambda)}(p) = f_{(\phi,\lambda)}(x, g) = (\phi(x), \lambda(x) \cdot g) \tag{11}$$

where $\cdot$ denotes the group multiplication. The semidirect product group structure can be inferred from the automorphism group as follows, given arbitrary elements

$$(\phi, \lambda), (\phi', \lambda') \in \mathrm{Diff}(M) \times C^\infty(M, G) \tag{12}$$

denote the corresponding automorphism as $f_{(\phi,\lambda)}$. Then using the composition on $\mathrm{Aut}(P)$ we can express the composition of automorphisms acting on an arbitrary element $p = (x, g) \in P$

$$f_{(\phi',\lambda')} \circ f_{(\phi,\lambda)}(x, g) = f_{(\phi',\lambda')}(\phi(x), \lambda(x) \cdot g) = \big(\phi' \circ \phi(x),\ \lambda'(\phi(x)) \cdot \lambda(x) \cdot g\big), \tag{13}$$

which can be rewritten using the pullback as

$$f_{(\phi',\lambda')} \circ f_{(\phi,\lambda)}(x, g) = (\phi' \circ \phi(x),\ \phi^* \lambda'(x) \cdot \lambda(x) \cdot g) \tag{14}$$

this then corresponds to the automorphism generated by the element

$$(\phi', \lambda') \overset{\ltimes}{\circ} (\phi, \lambda) = (\phi' \circ \phi,\ \phi^* \lambda' \cdot \lambda) \tag{15}$$

where $\overset{\ltimes}{\circ}$ denotes the composition on the semidirect product group. Therefore, the action of the second gauge transformation is affected by the preceding diffeomorphism. Hence, since the diffeomorphism group additionally acts on the gauge transformations this must be taken into account in the product structure in this way.

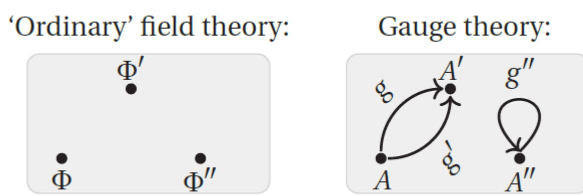
2.2 Gauge Groupoids

This section will briefly[3] discuss how gauge fields should be realised as elements of a groupoid (as opposed to a set), with the relevant example being the groupoid of (trivial) principal G-bundles on M. Where a group can be thought of as the set of transformations (morphisms) acting on a single object, a groupoid is then realised as the set of morphisms between a collection of objects, with the group then being the special case of a groupoid with a single object. Without delving too far into the categorical nature of groupoids, elements of the groupoid are objects $A \in \mathfrak{G}$, and between every pair of objects (including the object with itself) there exists a set of morphisms g, between them, denoted in the following way

$$\mathfrak{G} = \begin{cases} \text{Obj}: \ A \\ \text{Mor}: \ A \xrightarrow{g} A' \end{cases} \tag{16}$$

where the objects A need not necessarily be of the same 'type'[4], one must simply be able to define morphisms between them. Therefore, a groupoid consists of a collection (set) of objects with *extra* structure defined between them, namely the morphisms (gauge transformations).

When considering gauge theories, the objects of the groupoid are the gauge fields and the morphisms the gauge transformations between them. The groupoid is therefore able to capture the information specified by the gauge fields along with their gauge transformations, rather than simply being a set of gauge fields. Furthermore, the groupoid structure redefines the notion of two gauge fields being 'the same', since extra information is required to conclude that two gauge fields are 'the same'; namely a gauge transformation between them. Therefore, gauge transformations are understood to be 'witnesses' to the property that two objects in the groupoid are 'the same'; in this sense meaning isomorphic up to gauge transformations. This is visualised with the following figure given in Ref. [2]



The space of all gauge fields is usually taken to be the gauge equivalence class of gauge fields under gauge transformations, namely the 'gauge orbit space' of the collection of gauge fields (as a simple set). However, this neglects information contained within higher order structures, such as the first homotopy group (fundamental group), that contains information about non-trivial 'loops' within the space of fields; i.e the gauge transformations of a single gauge field to itself (see figure). The groupoid perspective therefore naturally captures such higher order structures, with the gauge orbit space realised as the 0-th homotopy group of the groupoid. One may consider even higher order structures, in the form of gauge transformations of the gauge transformations themselves; the structure of which would be described by a 2-groupoid, up to potentially ∞-groupoids [3]. However, for simplicity we will neglect such higher structures and focus on the case of 1-groupoids (standard gauge theory).

[3]for more details see section 3 of Ref. [2] for example

[4]we will consider groupoids with both gauge fields and transformations as objects, with (other) gauge transformations as morphisms

2.3 Boundary conditions (Edge modes)

When constructing gauge theories one implicitly works with a base (usually Minkowski) spacetime that has no boundary. This allows for simplifications when deriving equations of motion, where boundary contributions arising after integration by parts can usually be dropped. However, if the boundary is non-trivial, additional boundary structure (edge modes) can potentially ensure such terms do not cause issues on the boundary as we explore later in the Einstein-Cartan theory.

Here we briefly motivate the introduction and history of edge modes, that were first introduced in condensed matter systems in order to preserve current conservation that is broken by the presence of a boundary (most notably arising in the quantum hall effect [4]). Edge modes in this context are realised as edge currents that lead to total current conservation in the bulk and boundary. Furthermore, edge modes have been seen to be required in both gauge and gravity theories with non trivial boundary, where they were first introduced to preserve gauge invariance that was otherwise broken by the presence of a boundary. The edge modes are gauge transformations (Lorentz transformations and diffeomorphisms in the gravity case) on the boundary, that ensure the gauge fields themselves need not be constrained on the boundary (thus preserving gauge invariance that would otherwise be broken). More generally however, edge modes are seen to be required in theories that remain gauge invariant even in the presence of a boundary (such as Yang-Mills theories), where one may find the pre-symplectic form is not gauge invariant unless one introduces edge modes [5]. Furthermore, as was noted in Ref. [6], it is always possible to introduce a 'fiducial' boundary that disappears upon gluing two sub-regions. Hence, one requires edge modes regardless of whether the theory contains a 'physical' boundary that does not disappear upon gluing [6], however, only physical boundaries can carry 'charged' edge modes that influence the bulk field content in some way [1].

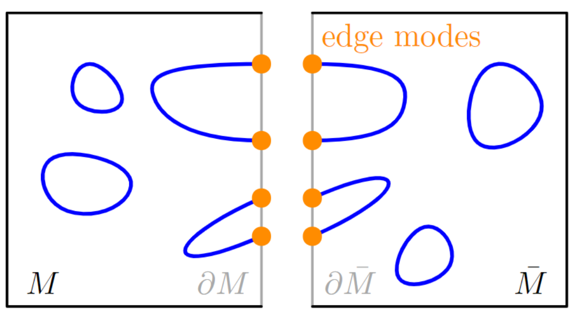


Figure 1: From Ref. [6] visualising edge modes on the boundary

Figure 2.3 visualises edge modes on the boundary. The loops (Wilson loops) represent gauge invariant quantities (the holonomy of the connection around the loop - related to the curvature). The edge modes are gauge transformations on the boundary that ensure the *gauge fields themselves* restricted to the boundary are *not* constrained, by providing the boundary condition that witnesses the restricted gauge field is gauge equivalent to a fixed[5] gauge field on the boundary. Such a condition ensures the gauge fields themselves are not constrained on the boundary, with the gauge equivalence on the boundary ensured by the edge mode. It is important to stress that such a boundary condition must be specified a priori along with the bulk field content; implemented by forming a homotopy pullback of bulk and boundary field groupoids as we now discuss.

[5]to be specified when constructing the theory

2.3.1 Homotopy pullback

In non-abelian Yang-Mills theories, the collection of gauge fields on M may be realised as the groupoid of (trivial) principal G-connections on M, specified by the objects (gauge fields/connections) and their morphisms (gauge transformations) as

$$\mathfrak{F}(M) = \mathbf{B}G_{\text{con}}(M) = \begin{cases} \text{Obj}: \ A \in \Omega^1(M, \mathfrak{g}) \\ \text{Mor}: \ A \xrightarrow{g} A^g = g^{-1}Ag + g^{-1}\mathrm{d}g \end{cases} \tag{17}$$

where $\mathbf{B}G_{\text{con}}(M)$ is notation[6] for the groupoid of principal G-connections on M and A^g the gauge transformation of A.

In this work we implement the boundary condition (edge modes) in a homological fashion by constructing the homotopy pullback of the bulk and boundary field groupoids, where the pullback groupoid naturally contains the information specified by the boundary condition (the edge mode). The boundary condition is understood as the (topological) condition that ensures the bulk principal groupoid of fields is (gauge) equivalent to a fixed boundary groupoid, as witnessed by the edge mode (gauge transformation). This condition may be regarded as 'topological' in the sense it only affects the structure of the underlying principal bundles; whilst the gauge fields themselves are not constrained on the boundary. The freedom to allow the gauge fields to be unconstrained on the boundary is then hence ensured by the edge mode.

The boundary condition requires that the bulk principal bundle $P \longrightarrow M$ restricted to the boundary $P|_{\partial M} \longrightarrow \partial M$, must be (gauge) equivalent to a fixed principal bundle on the boundary $\bar{P} \longrightarrow \partial M$. This is constructed by forming a homotopy pullback of the corresponding groupoids,

$$\begin{tikzcd} \hat{\mathfrak{F}}(M) \arrow[r, dashed] \arrow[d, dashed] & \mathbf{B}G_{\text{con}}(M) \arrow[d, "j^*"] \\ \{*\} \arrow[r, "p"] & \mathbf{B}G(\partial M) \end{tikzcd} \qquad \begin{tikzcd} \{*\} \arrow[d, "p"] & \mathbf{B}G_{\text{con}}(M) \arrow[d, "j^*"] \\ \bar{P} \arrow[r, leftrightarrow, "h"] & P|_{\partial M} \end{tikzcd} \tag{18}$$

here $\{*\}$ denotes an arbitrary singleton set (separate from M) such that $p(*) \longrightarrow \bar{P}$ defines the fixed principal bundle $\bar{P}$. The homotopy pullback is then identified with the groupoid of fields

$$\hat{\mathfrak{F}}(M) = \begin{cases} \text{Obj}: \ (*, A, h) \in \{*\} \times \Omega^1(M, \mathfrak{g}) \times \Omega^0(\partial M, G) \\ \text{Mor}: \ (*, A, h) \xrightarrow{g} (*, A^g, h\, g|_{\partial M}) \end{cases} \tag{19}$$

where the arbitrary point $*$ is usually supressed, such that that the field content consists of the bulk fields A, and the additional gauge transformation h (edge mode) on the boundary, witnesses to the condition that the restriction of the bulk principal bundle is isomorphic to the (fixed) boundary bundle. Therefore, the boundary condition may be regarded as topological, in the sense the condition only applies to the underlying principal bundle; whilst the boundary fields are not constrained by such a condition.

[6] specifically referring to the 'classifying space', the details of which won't be needed here

3 Einstein-Cartan Gravity

This section will construct the Einstein-Cartan (EC) formalism for general relativity (GR) using the appropriate tools from Cartan geometry. Cartan geometry can be understood as the extension of homogeneous (Klein) geometries, to more general spaces that allow curvature, but that locally are represented by homogeneous (model) spaces. This is realised by identifying a (reductive) Cartan connection, consisting of an Ehresman connection and a solder form (coframe); which identifies each tangent space with the corresponding homogeneous space.

3.1 Cartan geometry

A homogeneous space (X, G) is the smooth manifold[7] X along with a transitive G action that acts by diffeomorphisms. One may then realise geometric features of such a space that are stabilised by a certain subgroup $H \subset G$, where the space of such features may be identified with the coset space G/H. One often simply refers to the coset space of geometric features G/H as the 'homogeneous space', although this is a slight abuse of terminology it will be used throughout. For example, let $X = \mathbb{R}^n$ with G being the Euclidean symmetry group $ISO(n)$ (space of all symmetries of $\mathbb{R}^n$), one may consider the subgroup $SO(n)$ which preserves the geometric feature of 'length' (preserving the quadratic form that may be used to define distances), one thus identifies the homogeneous space as $ISO(n)/SO(n) = \mathbb{R}^n$. Throughout, we will focus on homogeneous spaces preserving the geometric feature of *spacetime* distance, namely spaces stabilised by the Lorentz group $SO(1, d)$.

Given Lie groups $H \hookrightarrow G$, one may form the infinitesimal homogeneous space $\mathfrak{g}/\mathfrak{h}$, which forms the 'model' space to be used in defining the Cartan geometry on M, where the coframe defines an isomorphism between the tangent spaces and such a model space. Given a principal H-bundle $P \xrightarrow{\pi} M$, and Lie groups $H \subseteq G$, we specialise to the *reductive* case, in which both $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{h}$ are $\mathrm{Ad}(H)$ invariant subspaces of $\mathfrak{g}$, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} \tag{20}$$

is the direct sum of $\mathrm{Ad}(H)$ representations of $\mathfrak{g}$, which will be used shortly to decompose any $\mathfrak{g}$ valued form into $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ valued components.

3.1.1 The Cartan connection

The Cartan geometry over M, modelled on the infinitesimal space $\mathfrak{g}/\mathfrak{h}$, is specified by defining a Cartan connection

$$\tilde{A} \in \Omega^1(P, \mathfrak{g}), \quad \tilde{A} : TP \longrightarrow \mathfrak{g} \tag{21}$$

required to a H-equivariant, $\mathfrak{g}$ valued 1-form on P that restricts to the Maurer-Cartan form on the fibres of P. Note that whilst $\tilde{A}$ is defined over the H-principal bundle P, it takes values in the larger Lie algebra $\mathfrak{g}$. Furthermore, the Cartan connection is required to be a vector space isomorphism

$$\forall p \in P, \quad \tilde{A} : T_pP \xrightarrow{\cong} \mathfrak{g} \tag{22}$$

[7]The space X may in general be more abstract, however, we focus here on the case of smooth manifolds

which can then be composed with the canonical *surjection* onto the coset $\mathfrak{g} \xrightarrow{/} \mathfrak{g}/\mathfrak{h}$ to obtain the (Cartan) coframe field $\tilde{e}$,

$$\begin{tikzcd} TP \arrow[r, "\tilde{A}"] \arrow[rr, bend left, "\tilde{e}"] & \mathfrak{g} \arrow[r, "/"] & \mathfrak{g}/\mathfrak{h} \end{tikzcd} \tag{23}$$

which should be noted takes values in the *vector space* $\mathfrak{g}/\mathfrak{h}$ and furthermore is *not* an isomorphism (not bijective). Therefore, it would appear to be a different object to what is usually called the 'coframe' in the EC formalism, which is taken to map TM into a 'fake tangent bundle', not just a vector space. However, since there is a bijective correspondence between H-equivariant, horizontal, V (vector space) valued forms and sections of the associated bundle, [7]

$$\Omega^k_{\text{hor}}(P, V) \leftrightarrow \Omega^k(M, P \times_\rho V) \tag{24}$$

we may regard the Cartan coframe as being equivalent to a section of the associated H-bundle as

$$\tilde{e} \in \Omega^1_{\text{hor}}(P, \mathfrak{g}/\mathfrak{h}) \leftrightarrow \Omega^1(M, P \times_H \mathfrak{g}/\mathfrak{h}) \ni e \tag{25}$$

since $\tilde{e}$ is H-equivariant and horizontal by definition ($\mathfrak{g}/\mathfrak{h}$ is in the kernel of $\tilde{\omega}(X) \subseteq \mathfrak{h}$ and hence $\tilde{e}$ is horizontal).

In the case of a reductive Cartan geometry (henceforth assumed), $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ is a direct sum of H-equivariant subspaces of $\mathfrak{g}$, hence the Cartan connection (or any other H-equivariant $\mathfrak{g}$ valued form) may be regarded as having two components

$$\tilde{A} = \tilde{e} \oplus \tilde{\omega}, \quad \tilde{e} \in \Omega^1(P, \mathfrak{g}/\mathfrak{h}), \quad \tilde{\omega} \in \Omega^1(P, \mathfrak{h}) \tag{26}$$

which correspond to the coframe $\tilde{e}$ as defined previously, and an Ehresmann connection $\tilde{\omega}$ on P, summarised with the following diagram [8]

$$\begin{tikzcd} & \mathfrak{h} \arrow[d, "\subset"] \\ TP \arrow[ur, bend left, "\tilde{\omega}"] \arrow[r, "\tilde{A}"] \arrow[dr, bend right, "\tilde{e}"'] & \mathfrak{g} \arrow[d, "/"] \\ & \mathfrak{g}/\mathfrak{h} \end{tikzcd} \tag{27}$$

3.1.2 Cartan Curvature

Given a Cartan connection one may define compute the curvature 2-form, defined by the structural equation

$$\tilde{F} = \mathrm{d}\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}], \qquad \tilde{F} \in \Omega^2(P, \mathfrak{g}) \tag{28}$$

which again in the reductive case decomposes into components

$$\tilde{F} = \tilde{T} \oplus \tilde{\mathcal{R}}, \qquad \tilde{T} \in \Omega^2(P, \mathfrak{g}/\mathfrak{h}),\ \tilde{\mathcal{R}} \in \Omega^2(P, \mathfrak{h}) \tag{29}$$

where the $\mathfrak{g}/\mathfrak{h}$ valued part is the torsion $\tilde{T}$, obtained by composing the curvature with the canonical projection on to the space $\mathfrak{g}/\mathfrak{h}$, and similarly the *corrected* curvature $\tilde{\mathcal{R}}$ is the $\mathfrak{h}$ valued part.

The corrected curvature is related to the curvature of the Ehresmann connection $\tilde{\omega}$, however, it is 'corrected' in the sense that $\tilde{\mathcal{R}}$ only vanishes for the model geometry itself, therefore, the model geometry defines what it means to be 'flat' in the usual sense; given it's curvature vanishes. If the Cartan curvature vanishes, both the torsion and corrected curvature parts must independently vanish.

3.2 Spacetime and Cartan geometry

Pseudo-Riemannian geometry may be re-formulated using the Cartan formalism by identifying the tangent spaces of a (pseudo-Riemannian) manifold with homogeneous Minkowski spacetimes. This is usually stated by defining a global coframe bundle, providing an isomorphism between the TM and a trivial vector bundle with typical fibre $\mathbb{R}^{1,\,d}$ (usually obtained through a reduction of the frame bundle structure group to $SO(1,d) \hookrightarrow GL(n,\mathbb{R})$). One then defines an Ehresmann (spin) connection on the $SO(1,d)$ bundle associated to the coframe bundle, and hence its corresponding curvature form. Here we begin with a slightly more general approach, by realising both the coframe and spin connection as parts of a (reductive) Cartan connection, taking values in the Poincaré Lie algebra, $\mathfrak{iso}(1,d) := \mathfrak{so}(1,d) \ltimes \mathbb{R}^{1,\,d}$.

Given a smooth manifold, M, of dimension $\dim(M) = n = 1 + d$, which for simplicity we assume is parallelisable,[8] we consider model geometries that are stabilised by the Lorentz group, and find three possible cases in $n = 1 + d$ dimensions (Table 1) where H refers to the Lorentz group and G the larger group in each case.

	Anti de Sitter ($\Lambda < 0$)	Minkowski ($\Lambda = 0$)	de Sitter ($\Lambda > 0$)
G/H	$SO(2,d)/SO(1,d) \cong \mathrm{AdS}_n$	$ISO(1,d)/SO(1,d) \cong \mathbb{R}^{1,\,d}$	$SO(1,n)/SO(1,d) \cong \mathrm{dS}_n$
$\mathfrak{g}/\mathfrak{h}$	$\mathfrak{so}(2,d)/\mathfrak{so}(1,d) \cong \mathbb{R}^{1,\,d}$	$\mathfrak{iso}(1,d)/\mathfrak{so}(1,d) \cong \mathbb{R}^{1,\,d}$	$\mathfrak{so}(1,n)/\mathfrak{so}(1,d) \cong \mathbb{R}^{1,\,d}$

Table 1: Homogeneous model spacetimes, AdS included for completeness

The Lie algebra of the larger group $\mathfrak{g}$ is seen to be *reductive* in each case, since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ may be expressed as the direct sum of $SO(1,d)$ invariant subspaces. Furthermore, the Lie algebras of the respective groups $\mathfrak{g}$ can conveniently be expressed using the Poincaré algebra, where the P_c are additionally taken to depend on the sign of the cosmological constant as [9]

$$[P_i, P_j] = -\epsilon\,\varepsilon_{ijk} J^k \quad [P_0, P_i] = -\epsilon B_i \tag{30}$$

$$\epsilon = \mathrm{sgn}(\Lambda) \begin{cases} 1 & \mathfrak{g} = \mathfrak{so}(1,n) \\ 0 & \mathfrak{g} = \mathfrak{iso}(1,d) \\ -1 & \mathfrak{g} = \mathfrak{so}(2,d) \end{cases} \tag{31}$$

where J, B are the rotation and boost generators respectively. The length scale ℓ may also be introduced, such that the translation parameters p^a/ℓ correspond to the components of a normalised translation vector in the model spacetime. This length scale will appear in terms with the basis elements P_a, e.g. the coframe, however it will not be made explicit unless required to elaborate the cosmological constant, which is then taken to be proportional to this length scale, given by [9]

$$\Lambda = \frac{3\epsilon}{\ell^2} \tag{32}$$

[8] later we specialise to the case n = 3 which is always parallelisable

We refer to the larger group $\mathfrak{g}$ as $\mathfrak{iso}(1,d)$ throughout, where one may substitute $\mathfrak{g} = \mathfrak{so}(1,n)$ or $\mathfrak{so}(2,d)$ as appropriate for the sign of Λ. Throughout we present the full EC action for the $\Lambda \neq 0$ spacetimes, where one obtains the Minkowski spacetime in the limit $\Lambda \longrightarrow 0$.

3.2.1 The Cartan connection

Since the geometry is reductive, the Cartan connection $\tilde{A} \in \Omega^1(P, \mathfrak{iso})$ decomposes into the direct sum of $SO(1,d)$-equivariant subspaces of $\mathfrak{iso}(1,d)$, where the components are the (Cartan) coframe $\tilde{e}$ and spin connection $\tilde{\omega}$

$$\tilde{A} = \tilde{\omega} \oplus \tilde{e}, \quad \tilde{A} \in \Omega^1(P, \mathfrak{iso}(1,d)) \tag{33}$$

$$\tilde{\omega} \in \Omega^1(P, \mathfrak{so}(1,d)), \quad \tilde{e} \in \Omega^1(P, \mathbb{R}^{1,\,d}) \tag{34}$$

where P is the trivial $SO(1,d)$-bundle over M. Since P (and TM) are trivial, we may simply regard these as globally defined 1-forms on the base manifold as

$$A = \omega \oplus e, \quad A \in \Omega^1(M, \mathfrak{iso}(1,d)) \tag{35}$$

$$\omega \in \Omega^1(M, \mathfrak{so}(1,d)), \quad e \in \Omega^1(M, \mathbb{R}^{1,\,d}) \tag{36}$$

which has the advantage in unifying the algebra of many calculations, since expressions such as $[\omega, e]$ or $[\omega, \omega]$ are then easily evaluated using the Poincaré algebra.

We may express the Cartan connection components in terms of the spin connection and coframe components (defined shortly), given that $A = \omega \oplus e$.

$$A = A^I{}_J \in \Omega^1(M, \mathfrak{iso}(1,\,d)), \qquad A^a{}_b = \omega^a{}_b \in \Omega^1(M, \mathfrak{so}(1,\,d)), \qquad A^a{}_n = e^a \in \Omega^1(M, \mathbb{R}^{1,\,d}) \tag{37}$$

where the indices $(I, J) = 0, 1, .., n$ and $(a, b) = 0, 1, ..., d$ such that the components $A^a{}_n = -A^n{}_a$ define a vector in $\mathbb{R}^{1,\,d}$ (the coframe components). Given that each element in the direct sum consists of an $SO(1,d)$-equivariant representation, gauge transformations of the principal bundle will affect the coframe and spin connection components independently.

3.2.2 Coframe

The Cartan connection components can now be identified with the coframe and spin connection used in the EC formalism. As discussed previously, we may regard the Cartan coframe as being equivalent to a section of the associated $SO(1,d)$-bundle as

$$\tilde{e} \in \Omega^1_{\text{hor}}(P, \mathbb{R}^{1,\,d}) \;\leftrightarrow\; \Omega^1(M, P \times_{SO(1,d)} \mathbb{R}^{1,\,d}) \ni e \tag{38}$$

such that one may therefore identify the coframe bundle as

$$\mathcal{T} = P \times_{SO(1,d)} \mathbb{R}^{1,\,d} = M \times \mathbb{R}^{1,\,d} \tag{39}$$

given that P is trivial. The $\mathbb{R}^{1,\,d}$ valued 1-form e is usually expressed in components, given the coordinate basis for M and the basis P_a of $\mathbb{R}^{1,\,d}$ (the translation generators of the Poincaré algebra) as

$$e \in \Omega^1(M, \mathcal{T}), \quad e = e^a{}_\mu \mathrm{d}x^\mu \otimes P_a \tag{40}$$

suppressing the length scale. This coframe bundle is isomorphic to the tangent bundle, however, it is equipped with the (non-degenerate) Minkowski metric η of the model geometry[9] $\mathfrak{iso}/\mathfrak{so} \cong \mathbb{R}^{1,\,d}$. Such a model metric is then pulled back to the general metric on TM using the coframe as $g = e^*\eta$, such that in components the metric on M can be expressed as

$$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu \otimes e^b{}_\nu \tag{41}$$

3.2.3 Spin connection

The spin connection refers to the $\mathfrak{so}(1,d)$ valued part of the Cartan connection, $\omega \in \Omega^1(M, \mathfrak{so}(1,d))$ realised as the principal connection on the principal $SO(1,d)$-bundle associated to the coframe bundle. Therefore, expressed using the matrix representation of $\mathfrak{so}(1,d)$, with basis elements $L^a{}_b$, the spin connection components can be expressed as

$$\omega = \frac{1}{2}\omega_\mu{}^a{}_b \mathrm{d}x^\mu \otimes L^b{}_a \tag{42}$$

Furthermore, given that the Lie algebra $\mathfrak{so}(1,d)$ is isomorphic to the space

$$\mathfrak{so}(1,d) \cong \bigwedge^2 \mathbb{R}^{(1,d)} \tag{43}$$

one may also regard the spin connection (supressing spacetime indices) as the bivector

$$\omega = \frac{1}{2}\omega^{ab} \otimes L_{ab} \cong \frac{1}{2}\omega^{ab} \otimes P_a \wedge P_b \tag{44}$$

where the components and basis elements are understood to be anti-symmetric.

3.2.4 Curvature

The Cartan curvature can now be computed using the structural equation, which using the wedge notation is given by

$$F^I{}_J = \mathrm{d}A^I{}_J + A^I{}_K \wedge A^K{}_J \tag{45}$$

where the $\mathfrak{iso}(1,d)$ valued form again splits into $\mathfrak{so}(1,d) \oplus \mathbb{R}^{1,\,d}$ components in the reductive case. For the $\mathfrak{so}(1,d)$ valued component (the corrected curvature) one obtains (n is an integer not an index)

$$\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_k \wedge \omega^k{}_b + A^a{}_n \wedge A^n{}_b = \mathrm{d}^\omega \omega^a{}_b - \frac{1}{\ell^2} e^a \wedge e_b = R^a{}_b - \frac{\Lambda}{3} e^a \wedge e_b \tag{46}$$

using the definition of the exterior covariant derivative w.r.t the connection ω, and recognising the curvature of the principal connection $\mathrm{d}^\omega \omega^a{}_b = R^a{}_b$. The last term is the corrective term, arising due to the fact that $K = 1, ..., n$ is fully summed in the Cartan curvature, hence the n-th term must be present when projecting $F^I{}_J$ into the respective subspaces. Here the length scale has been made explicit, such that the cosmological constant term is present .

Similarly, from the $\mathbb{R}^{1,\,d}$ valued component one obtains the torsion 2-form $T^a \in \Omega^2(M, \mathbb{R}^{1,\,d})$

$$T^a = F^a{}_n = \mathrm{d}A^a{}_n + A^a{}_K \wedge A^K{}_n = \mathrm{d}e^a + A^a{}_b \wedge A^b{}_n = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = \mathrm{d}^\omega e^a \tag{47}$$

[9]assumed to be a metric Klein geometry

Hence, we can identify the condition that this Cartan geometry be flat, namely $F = 0$, which requires both the torsion and corrected curvature vanish, $T = \mathcal{R} = 0$, giving

$$R^a{}_b = \frac{\Lambda}{3} e^a \wedge e_b, \quad \text{and} \quad T = 0 \tag{48}$$

which we see corresponds to the vanishing torsion condition (as will be obtained shortly) and the definition of the corrected curvature used in the EC action. Furthermore, one realises that such a condition allows for the spin connection to be determined explicitly in terms of e, since

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_k \wedge \omega^k{}_b = \frac{\Lambda}{3} e^a \wedge e_b \tag{49}$$

such a solution then corresponds to the (unique) Levi-Civita connection with vanishing torsion.

3.3 (2+1) Einstein-Cartan action

The EC action functional in $(2+1)$ dimensions may be expressed using the corrected curvature as

$$S_{EC} = \frac{1}{2\kappa^2} \int_M \left(\mathcal{R}^{ab} \wedge e^c\right) \varepsilon_{abc} = \frac{1}{2\kappa^2} \int_M \left(R^{ab} \wedge e^c - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c\right) \varepsilon_{abc} \tag{50}$$

where Λ is the cosmological constant and we take $2\kappa^2 = 1$ throughout.

3.3.1 Action variation

Consider variation under an arbitrary infinitesimal variation of the parameters, $(e + \delta e, \omega + \delta\omega)$, where the variation in the action (to first order) is then given as (first in e),

$$\delta_e S = \int_M \left(R^{ab} \wedge \delta e^c \; - \; \Lambda \, e^a \wedge e^b \wedge \delta e^c\right) \varepsilon_{abc} = 0 \tag{51}$$

which under arbitrary variations yields the Einstein field equation,

$$\mathcal{F}_e = R^{ab} - \Lambda \, e^a \wedge e^b = 0 \tag{52}$$

Likewise, the first order variation in ω obtains

$$\int_M \left(\mathrm{d}(\delta\omega^{ab}) \wedge e + (\delta\omega^a{}_k \wedge \omega^{kb} + \omega^a{}_k \wedge \delta\omega^{kb}) \wedge e^c\right) \varepsilon_{abc} = 0, \tag{53}$$

where the first term is integrated by parts and the quadratic terms simplify to give

$$\delta_\omega S = \int_{\partial M} j^* \left(\delta\omega^{ab} \wedge e^c\right) \varepsilon_{abc} + \int_M \delta\omega^{ab} \wedge \left(\mathrm{d}e^c + \omega^c{}_k \wedge e^k\right) \varepsilon_{abc} = 0 \tag{54}$$

where $j : \partial M \hookrightarrow M$ denotes the boundary inclusion. Hence, under arbitrary variations in ω this leads to the following condition in the bulk

$$\mathcal{F}_\omega = \mathrm{d}e^a + \omega^a{}_k \wedge e^k = \mathrm{d}^\omega e = T^a = 0 \tag{55}$$

namely that the torsion 2-form $T^a \in \Omega^2(M,)$ vanishes in M. If the boundary is present, e appears to vanish on the boundary due to the first term in Eq. (54). However, since the coframe field (and hence the metric) are required to be non-degenerate, this would appear to cause issues for the boundary geometry which subsequently must be compensated for. This is acheived through the introduction of edge modes, as will be formulated in proceeding sections.

3.4 Gauge + Diffeomorphism invariance

One now wishes to examine how the action behaves under principal bundle automorphisms, which we have seen correspond to a diffeomorphism of the base spacetime along with a gauge transformation in the bundle. Hence, we wish to determine the automorphisms of the $SO(1,d)$ principal bundle to which the coframe bundle is associated. The automorphisms of such a principal bundle are then equivalently given by the semi-direct product

$$(\lambda, \phi) \in \mathrm{Diff}(M) \ltimes C^\infty(M, SO(1,d)) \cong \mathrm{Aut}(P) \tag{56}$$

with the group structure as defined previously. This corresponds to the local Lorentz invariance of the coframes, along with the diffeomorphism invariance of the base manifold.

The coframe and spin connection will transform under the semi-direct product action as follows, where we consider the action independently of the semi-direct product for now. The coframe *components*, realised as a section $\Omega^1(\mathbb{R}^{1,d})$, transform in the fundamental vector representation of $SO(1,d)$, hence by the right action of the group under finite gauge transformations

$$e \mapsto \lambda^{-1} e \tag{57}$$

The spin connection will transform under the gauge transformation as

$$\omega \mapsto \omega^\lambda = \lambda^{-1}\omega\lambda + \lambda^{-1}\mathrm{d}\lambda. \tag{58}$$

and the curvature transforms simply with the adjoint action as

$$R \mapsto \lambda^{-1} R \, \lambda \tag{59}$$

Furthermore, since the pullback commutes with the exterior derivative and wedge product, all objects in question will simply transform by the pullback of a finite diffeomorphism.

$$e \mapsto \phi^* e, \quad \omega \mapsto \phi^*\omega, \quad R \mapsto \phi^* R \tag{60}$$

3.4.1 Finite diffeomorphisms

First consider the action of finite diffeomorphisms on the ECP action, $\phi : M \longrightarrow M$, where the curvature and coframe simply transform via the pullback of the diffeomorphism

$$S' = \int_{\phi(M)} \left(\phi^* R^{ab} \wedge \phi^* e^c - \frac{\Lambda}{3} (\phi^* e^a \wedge \phi^* e^b \wedge \phi^* e^c \right) \varepsilon_{abc} = \int_M \phi^* \left(R^{ab} \wedge e^c - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \varepsilon_{abc} \tag{61}$$

since $\phi(M) = M$ and the pullback is associative. Furthermore, since the pullback also commutes with integration, in general

$$\int_{\phi(M)} \omega = \int_M \phi^* \omega \tag{62}$$

such that the last term may be written as

$$S' = \int_M \phi^* \left(R^{ab} \wedge e^c - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \varepsilon_{abc} = \int_{\phi(M)} \left(R^{ab} \wedge e^c - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \varepsilon_{abc} \tag{63}$$

which again using $\phi(M) = M$ shows that $S' = S$; namely the action is invariant under the action of finite diffeomorphisms (of M into M).

3.4.2 Finite gauge transformations

Now consider how the action changes under the action of local finite gauge transformations (local Lorentz transformations), $\lambda(x) \in C^\infty(M, SO(1,d))$, which we express in components using the matrix representation of $SO(1,d)$ and usually omit the x variable. Hence, in components, the relevant objects transform as defined previously as

$$e^a \mapsto \lambda^{-1\,a}{}_{\tilde{a}}\, e^{\tilde{a}} \tag{64}$$

and for the spin connection

$$\omega^{ab} \mapsto \lambda^{-1\,a}{}_{\tilde{a}}\, \lambda^{-1\,b}{}_{\tilde{b}}\, \omega^{\tilde{a}\tilde{b}} + \lambda^{-1\,a}{}_{\tilde{c}}\, \mathrm{d}\lambda^{\tilde{c}}{}_{d}\eta^{db}. \tag{65}$$

Finally, the curvature

$$R^{ab} \mapsto \lambda^{-1\,a}{}_{\tilde{a}}\, \lambda^{-1\,b}{}_{\tilde{a}}\, R^{\tilde{a}\tilde{b}} \tag{66}$$

Hence, the action transforms as

$$S' = \int_M \left((\lambda^{-1\,a}{}_{\tilde{a}}\, \lambda^{-1\,b}{}_{\tilde{a}}\, R^{\tilde{a}\tilde{b}}) \wedge (\lambda^{-1\,c}{}_{\tilde{c}}\, e^{\tilde{c}}) - \frac{\Lambda}{3} (\lambda^{-1\,a}{}_{\tilde{a}}\, e^{\tilde{a}}) \wedge (\lambda^{-1\,b}{}_{\tilde{b}}\, e^{\tilde{b}}) \wedge (\lambda^{-1\,c}{}_{\tilde{c}}\, e^{\tilde{c}}) \right) \varepsilon_{abc} \tag{67}$$

which, since expressed entirely in components can be simply re written as

$$S' = \int_M \lambda^{-1\,a}{}_{\tilde{a}}\, \lambda^{-1\,b}{}_{\tilde{a}}\, \lambda^{-1\,c}{}_{\tilde{c}} \left(R^{\tilde{a}\tilde{b}} \wedge e^{\tilde{c}} - \frac{\Lambda}{3} e^{\tilde{a}} \wedge e^{\tilde{b}} \wedge e^{\tilde{c}} \right) \varepsilon_{abc} \tag{68}$$

and using that the volume form is (proper orthocronous) Lorentz invariant, such that we can write

$$\lambda^{-1\,a}{}_{\tilde{a}}\, \lambda^{-1\,b}{}_{\tilde{a}}\, \lambda^{-1\,c}{}_{\tilde{c}}\, \varepsilon_{abc} = \varepsilon_{\tilde{a}\tilde{b}\tilde{c}} \tag{69}$$

the Lorentz invariance of the action is then clear

$$S' = \int_M \left(R^{\tilde{a}\tilde{b}} \wedge e^{\tilde{c}} - \frac{\Lambda}{3} e^{\tilde{a}} \wedge e^{\tilde{b}} \wedge e^{\tilde{c}} \right) \varepsilon_{\tilde{a}\tilde{b}\tilde{c}} \tag{70}$$

3.4.3 Coframe bundle automorphisms

Given an arbitrary gauge + diffeomorphism transformation in terms of a principal bundle automorphism $f_{(\phi,\lambda)}$, this simply acts by pullback of the Cartan connection,

$$A \mapsto f^*_{(\phi,\lambda)} A = (f^*_{(\phi,\lambda)} e, f^*_{(\phi,\lambda)} \omega) \tag{71}$$

which can equivalently be realised using the semidirect product group action, and the transformation behaviour of (e, ω) under arbitrary gauge and diffeomorphisms as defined previously,

$$f^*_{(\phi,\lambda)} A = (f^*_{(\phi,\lambda)} e, f^*_{(\phi,\lambda)} \omega) = \Big(\phi^*\lambda^{-1}(\phi^* e),\ \ \phi^*\lambda^{-1}(\phi^*\omega)\phi^*\lambda + \phi^*\lambda^{-1}\mathrm{d}\phi^*\lambda \Big) = (\phi^*(\lambda^{-1} e),\ \phi^*\omega^\lambda) \tag{72}$$

since the pullback distributes over the wedge product. Therefore, since we have shown previously that the EC action is both (finite) gauge and diffeomorphism invariant, the action will subsequently be invariant under arbitrary coframe bundle automorphisms.

3.4.4 Einstein-Cartan groupoid

We now present the field content of the EC theory using the groupoid language, where the fields are taken to be the Cartan connection components with morphisms being the principal bundle automorphisms described previously. One therefore deduces the field groupoid as

$$\mathfrak{F}_{EC}(M)\begin{cases}\text{Obj}: \ (\omega, e) \in \Omega^1(M, \mathfrak{so}(1,d)) \times \Omega^1(M, \mathbb{R}^{1,\,d}) \\ \text{Mor}: \ (\omega, e) \xrightarrow{f_{(\phi,\lambda)}} \left(\phi^*\omega^\lambda, \ \phi^*(\lambda^{-1}e)\right)\end{cases} \tag{73}$$

where $f_{(\phi,\lambda)}$ is a principal bundle automorphism as defined previously.

4 Einstein-Cartan gravity with boundary

To simplify the analysis, assume the manifold is diffeomorphic to the manifold $M = \Sigma \times \mathbb{H}$, where the half space $\mathbb{H} = [0, \infty)$ and the boundary $\partial M = \Sigma$ for some $(1 + d - 1)$-dimensional (in this case (1+1) dimensional) sub-manifold Σ. Assume M has coordinates $x^\mu = (x^0, x^1, x^2) \in \mathbb{H} \times \mathbb{R}^2$ in the bulk and $j^*x^\mu = (x^0, x^1, 0) \implies \hat{x}^i = (\hat{x}^0, \hat{x}^1) \in \mathbb{R}^{1,\,1}$ on the boundary where a single component vanishes.

4.1 Boundary transformations

Here we show the EC action is gauge and diffeomorphism invariant in the presence of a boundary. Trivially, the action will be gauge invariant on the boundary since any Lorentz transformation simply restricts to a boundary Lorentz transformation; such that the EC action is gauge invariant as before. However, for diffeomorphisms one must consider the following situation that arises when considering an infinitesimal diffeomorphism.

4.1.1 Infinitesimal diffeomorphisms

Infinitesimal diffeomorphisms are generated by vector fields $\xi \in \Gamma(TM)$, with the fields (coframe, spin connection) transforming using the Lie derivative, computed using Cartan's magic formula, for example

$$\delta_\xi e = \mathcal{L}_\xi e = \mathrm{d} \circ \iota_\xi e + \iota_\xi \circ \mathrm{d}e, \tag{74}$$

which is now used to obtain the transformation behaviour of the EC action under (arbitrary) infinitesimal diffeomorphisms.

Let L be the EC lagrangian as defined previously, the variation can be expressed simply as a total Lie derivative, by repeated application of the Leibniz rule (suppressing indices)

$$\delta_\xi S = \int_M \left(\delta_\xi R \wedge e + R \wedge \delta_\xi e - \frac{\Lambda}{3}\delta_\xi e \wedge e \wedge e + \ldots\right) = \int_M \delta_\xi (R\wedge e) - \delta_\xi\left(\frac{\Lambda}{3} e \wedge e \wedge e\right) = \int_M \delta_\xi L, \tag{75}$$

hence, since L is a top form, this is evaluated using Cartan's formula simply as

$$\delta_\xi S = \int_M \mathcal{L}_\xi L = \int_M \mathrm{d}(\iota_\xi L). \tag{76}$$

where typically, if the manifold has no boundary, one uses Stokes' theorem to conclude that such a variation automatically vanishes. However, if the boundary is non trivial this will not necessarily be true

$$\partial M \neq \varnothing, \qquad \Longrightarrow \qquad \delta_\xi S = \int_{\partial M} j^* \left(\iota_\xi L\right) \overset{!}{\neq} 0, \tag{77}$$

therefore, diffeomorphism invariance in the presence of a boundary requires that $\iota_\xi L$ vanish on the boundary; as we now verify.

4.1.2 Boundary diffeomorphisms

In general, a diffeomorphism $\phi : M \longrightarrow M$ will preserve the boundary of M [10], i.e no diffeomorphism exists that can map boundary points to interior points and vice versa; verified by the following consideration. The neighbourhood of a boundary point is homeomorphic to the half space $\mathbb{H}^n$, therefore, no diffeomorphism interchanging boundary/interior points can be constructed, since $\mathbb{H}^n$ is not homeomorphic[10] to $\mathbb{R}^n$. Therefore, an arbitrary diffeomorphism of M must restrict to a diffeomorphism of the boundary under the inclusion, $j : \partial M \hookrightarrow M$, namely

$$j^*\phi : \partial M \longrightarrow \partial M, \tag{78}$$

such that the infinitesimal diffeomorphism will be tangent to the boundary. This is in fact required to define the Lie derivative; understood as flowing along an integral curve *on the boundary* [10].

Consider the infinitesimal diffeomorphism generated by the vector field $\xi^\sigma \partial_\sigma$, such that the restriction $\xi|_{\partial M}$ is tangent to ∂M (i.e the normal component vanishes which in this case is $\xi^2|_{\partial M} = 0$). Computing the interior product with the Lagrangian top form $\frac{1}{3!} L_{\mu\nu\rho} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho$, and evaluating this at the boundary as in Eq. 77 one finds that

$$j^*(\iota_\xi L) = j^* \left(\frac{1}{2!} \xi^\sigma L_{\sigma\nu\rho} \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \right) = \frac{1}{2!} (\xi^i L_{ijk}) \Big|_{\partial M} \mathrm{d}x^j \wedge \mathrm{d}x^k = 0 \tag{79}$$

where under the boundary inclusion the ξ^2 component vanishes and the un-contracted indices on L restrict to $\nu, \rho \longrightarrow j, k = 0, 1$, such that the remaining terms vanish due to the anti-symmetry[11] of L and the boundary term in Eq.(77).

Therefore, the EC action is diffeomorphism invariant in the presence of a boundary, simply due to the topological requirement that the boundary/bulk character of points be preserved under the diffeomorphism.

4.2 Boundary variation

Here we examine the problematic boundary term that arises in the variation of the EC action with respect to the spin connection, that seems to imply the coframe is degenerate on the boundary. We then introduce edge modes and construct a 'dressed' boundary action in terms of the edge modes in an attempt to compensate for such a degeneracy.

[10] $\mathbb{R}^n$ can be disconnected by removing the origin (hyperplane), whilst $\mathbb{H}^n$ remains connected

[11] this argument holds in arbitrary dimensions, where Greek indices run from $1, ..., n$ and Latin indices $1, ..., d$

4.2.1 Spin connection variation

Recall the following expression for the variation of the EC action with respect to the spin connection Eq. (54), obtained after integrating by parts

$$\delta_\omega S = \int_{\partial M} j^* \left(\delta\omega^{ab} \wedge e^c\right) \varepsilon_{abc} + \int_M \delta\omega^{ab} \wedge \left(\mathrm{d}e^c + \omega^c{}_k \wedge e^k\right) \varepsilon_{abc} = 0, \tag{80}$$

which obtains the vanishing torsion constraint in M and if the boundary is non-trivial, implies $j^*e = 0$, namely the coframe is degenerate on the boundary. Since the metric (and hence the coframe) are required to be non-degenerate, this causes issues with the boundary geometry which subsequently must be compensated for. This is attempted in this work by the introduction of edge modes.

4.3 Edge modes in the Einstein-Cartan theory

We now introduce edge modes in the EC theory, which we find can potentially compensate for the degeneracy of the coframe on the boundary. The field content can be understood to contain the coframe and spin connection as previously, in addition to *two* edge modes, corresponding to a boundary $SO(1,d)$ gauge transformation h and boundary diffeomorphism φ, computed through the homotopy pullback

$$\begin{tikzcd} \hat{\mathfrak{F}}(M) \arrow[r, dashed] \arrow[d, dashed] & \mathbf{B}SO_{\mathrm{Cart}}(M) \arrow[d, "j^*"] \\ \{*\} \arrow[r, "p"] & \mathbf{B}SO(\partial M) \end{tikzcd} \qquad \begin{tikzcd} \{*\} \arrow[d, "p"] & \mathbf{B}SO_{\mathrm{Cart}}(M) \arrow[d, "j^*"] \\ \bar{P} \arrow[r, leftrightarrow, "(h,\varphi)"] & P|_{\partial M} \end{tikzcd} \tag{81}$$

where $\mathbf{B}SO_{\mathrm{Cart}}(M)$ indicates we are considering the Cartan connection on M (in terms of the coframe and spin connection) and the diffeomorphism invariance of $\mathbf{B}SO(M)$ in each case is assumed. Therefore, the field content is identified as the groupoid

$$\hat{\mathfrak{F}}_{\mathrm{ECP}}(M) = \begin{cases} \mathrm{Obj}: \ (\omega, e, \varphi, h) \in \Omega^1(M, \mathfrak{so}) \times \Omega^1(M, \mathbb{R}^{1,\,d}) \times \mathrm{Diff}(\partial M) \times \Omega^0(\partial M, SO(1,d)) \\ \mathrm{Mor}: \ (\omega, e, \varphi, h) \overset{f_{(\phi,\lambda)}}{\longrightarrow} \left(\phi^*\omega^\lambda,\ \phi^*(\lambda^{-1}e),\ (\varphi, h) \overset{\rtimes}{\circ} (\phi,\lambda)|_{\partial M}\right) \end{cases} \tag{82}$$

where the morphisms are the principal bundle automorphisms acting through the semi-direct product as defined previously. Here $\overset{\rtimes}{\circ}$ denotes the composition on the semi-direct product as defined in Eq. (15)) such that the automorphism of the edge modes may be expressed as

$$(\varphi, h) \overset{f_{(\phi,\lambda)|}}{\longrightarrow} (\phi_|^*\varphi, \phi_|^* h \circ \lambda_|) \tag{83}$$

where we have abbreviated the boundary restriction as $\phi|_{\partial M} \equiv \phi_|$, which recall restricts to a diffeomorphism of the boundary $\phi_| : \partial M \longrightarrow \partial M$; hence the composition with φ is well defined. Likewise, a Lorentz transformation $\lambda(x)$ (trivially) restricts to a Lorentz transformation on the boundary $\lambda_|(\hat{x})$ where $\hat{x}$ are the boundary coordinates.

4.3.1 The Dressed boundary action

Now that the edge modes have been identified, one considers attempting to modify the ECP action by introducing extra boundary terms to compensate for the degeneracy of the coframe on the boundary; caused by the first term in the ω variation (Eq. (54)) repeated here

$$\delta_\omega S = \int_{\partial M} \left(\delta\omega^{ab} \wedge e^c\right)_| \varepsilon_{abc} + ... = 0 \tag{84}$$

where $(...)_|$ abbreviates the boundary inclusion.

To compensate, one might simply consider adding the following term to the action

$$S_{\partial M} = -\int_{\partial M} (\omega^{ab} \wedge e^c)_| \varepsilon_{abc} \quad \overset{\delta\omega}{\Longrightarrow} \quad \delta S_{\partial M} = -\int_{\partial M} (\delta\omega^{ab} \wedge e^c)_| \tag{85}$$

however, since ω is a connection this term is clearly not gauge invariant. Instead, one considers the following 'dressed' boundary action, that contains a term with the following 'dressed' boundary fields; defined using the edge modes as the following

$$\hat{\omega} = f^*_{(\varphi,\, h)^{-1}} \omega_| \in \Omega^1(\partial M, \mathfrak{so}(1, d)), \qquad \hat{e} = f^*_{(\varphi,\, h)^{-1}} e_| \in \Omega^1(\partial M, \mathbb{R}^{1,\, d}) \tag{86}$$

$$\hat{\omega} = {\varphi^{-1}}^*(h\omega_| h^{-1} + h\, \mathrm{d}h^{-1}), \qquad \hat{e} = {\varphi^{-1}}^*(he_|) \tag{87}$$

which simply corresponds to pre-composing the fields on the boundary with an 'inverse' pair of edge modes. Adding a boundary term of the form given in Eq. (85) replaced with the dressed fields will produce a gauge/diffeomorphism invariant action that is sufficient to compensate for the degenerate boundary coframe as we now verify.

Using the dressed fields in terms of the edge modes, add the following boundary term to the ECP action

$$S_{\partial M} = -\int_{\partial M} (\hat{\omega}^{ab} \wedge \hat{e}^c)\, \varepsilon_{abc} \tag{88}$$

which must be invariant under gauge transformations and diffeomorphisms of the boundary, namely the automorphism $f_{(\phi,\lambda)_|}$. The naked fields are transformed (gauge/diffeomorphism) before the dressing is applied, furthermore, the edge modes also transform such that the dressed transformed field becomes

$$\hat{\omega} \longrightarrow f^*_{(\varphi',\, h')^{-1}}(\omega'_|) = f^*_{\left((\varphi,h) \overset{\rtimes}{\circ} (\phi,\lambda)_|\right)^{-1}} (\phi^* \omega^\lambda)_| = f^*_{(\phi,\lambda)_|^{-1} \overset{\rtimes}{\circ} (\varphi,h)^{-1}} (\phi^* \omega^\lambda)_| \tag{89}$$

then acting with the (inverse) transformation from the right yields

$$f^*_{(\varphi,\, h)^{-1}}((\phi^*\omega^\lambda)_| \lhd (\phi, \lambda)_|^{-1}) = f^*_{(\varphi,\, h)^{-1}} \omega_| = \hat{\omega} \tag{90}$$

such that the inverse transformation acting on $\omega'_|$ returns $\omega_|$. Hence, the dressed field is (boundary) gauge/diffeomorphism invariant, with an identical calculation for the dressed coframe $\hat{e}$.

4.4 Boundary dynamics

Here we obtain constraints that the dressed coframe and spin connection must satisfy by considering how the dressed action affects the variation.

4.4.1 Dressed action variation

Now that a sufficiently gauge and diffeomorphism invariant term has been added to the EC action, one may now compute the equations of motion and verify that the problematic term in Eq. (54) is cancelled by the additional boundary term. The EC action with boundary term is expressed as

$$S = S_{\text{EC}} + S_{\partial M} = \int_M \left(R^{ab} \wedge e^c - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \varepsilon_{abc} - \int_{\partial M} \left(\hat{\omega}^{ab} \wedge \hat{e}^c \right) \varepsilon_{abc} \tag{91}$$

where we focus on the additional terms arising due to the variation on the boundary. Using the dressing, we may re-write the boundary term in terms of the edge modes as (suppressing edge mode indices)

$$S_{\partial M} = - \int_{\partial M} {\varphi^{-1}}^* \left((h\omega_| h^{-1} + h\,\mathrm{d}h^{-1})^{ab} \wedge (he_|)^c \right) \varepsilon_{abc} \tag{92}$$

where we may additionally suppress the edge diffeomorphism since the action is always diffeomorphism invariant (regardless of boundary). Therefore, under an arbitrary variation in ω, the boundary term yields

$$\delta_\omega S_{\partial M} = - \int_{\partial M} \left(h\,\delta\omega_|\, h^{-1} \right)^{ab} \wedge (h\,e_|)^c\, \varepsilon_{abc} \overset{h}{\cong} - \int_{\partial M} \delta\omega_|{}^{ab} \wedge e_|^c\, \varepsilon_{abc} \tag{93}$$

where an identical calculation as in section 3.4.2 using the $SO(1,d)$ invariance of the ε symbol was used. Therefore, given a suitable edge mode witnessing the equivalence, this term cancels the problematic boundary term in Eq. (54), such that the equation of motion for the ω variation is unchanged; found previously as

$$\mathcal{F}_\omega = \mathrm{d}e^a + \omega^a{}_k \wedge e^k = \mathrm{d}^\omega e^a = T^a = 0, \tag{94}$$

requiring the torsion vanish in the bulk (and hence the boundary by continuity).

4.4.2 Dressed spin connection constraint

Now consider how the coframe variation will affect the boundary term, which may be expanded as

$$S_{\partial M} = - \int_{\partial M} \left(h\,\omega_|\, h^{-1} \right)^{ab} \wedge (h\,e_|)^c\, \varepsilon_{abc} - \int_{\partial M} (h\,\mathrm{d}h^{-1})^{ab} \wedge (he_|)^c\, \varepsilon_{abc} \tag{95}$$

in order to simplify the calculation. Under the variation, this becomes

$$\delta_e S_{\partial M} = - \int_{\partial M} \left(h\,\omega_|\, h^{-1} \right)^{ab} \wedge (h\,\delta e_|)^c\, \varepsilon_{abc} - \int_{\partial M} (h\,\mathrm{d}h^{-1} h\, h^{-1})^{ab} \wedge (h\delta e_|)^c\, \varepsilon_{abc} \tag{96}$$

where $\mathbb{I} = hh^{-1}$ has been inserted into the second term in order to re-write this as

$$\delta_e S_{\partial M} = - \int_{\partial M} \left(h(\omega_| + dh^{-1}h)h^{-1} \right)^{ab} \wedge \left(h\,\delta e_| \right)^c\, \varepsilon_{abc} \overset{h}{\cong} - \int_{\partial M} \left(\omega_| + dh^{-1}h \right)^{ab} \wedge \left(\delta e_| \right)^c\, \varepsilon_{abc} \tag{97}$$

which under an arbitrary variation yields the following condition

$$\hat{\mathcal{F}}_{\hat{\omega}} = (\omega_| + \mathrm{d}h^{-1}h) = 0 \qquad \Longrightarrow \qquad \hat{\omega} = h\omega_| h^{-1} + h\mathrm{d}h^{-1} = 0 \tag{98}$$

which we realise as the requirement that the dressed spin connection $\hat{\omega}$ be gauge equivalent to zero on the boundary; witnessed by the edge modes (recall the diffeomorphism was suppressed).

Hence, the bulk equations of motion for both the coframe and spin connection variation are unchanged and one finds the additional condition that the spin connection be gauge equivalent to zero on the boundary. Furthermore, the degenerate boundary coframe has been compensated by introducing such a boundary term into the action.

4.4.3 Edge mode variation

One may also examine the dynamics of the edge modes in the boundary term,

$$S_{\partial M} = -\int_{\partial M} \left(h\omega_| h^{-1} + h\,\mathrm{d}h^{-1}\right)^{ab} \wedge (he_|)^c\,\varepsilon_{abc} \tag{99}$$

where only variations in h are considered (since variations in φ will have no effect on the action it is again suppressed). Again, it is convenient to split this term into two parts and insert $\mathbb{I} = hh^{-1}$,

$$S_{\partial M} = -\int_{\partial M} \left(h\,\omega_|\,h^{-1}\right)^{ab} \wedge (h\,e_|)^c\,\varepsilon_{abc} - \int_{\partial M} (h\,\mathrm{d}h^{-1}\,h\,h^{-1})^{ab} \wedge (he_|)^c\,\varepsilon_{abc} \tag{100}$$

such that the Lorentz invariance (of ε) can be used to re-write this as

$$S_{\partial M} = -\int_{\partial M} \omega_|^{ab} \wedge e_|^c\,\varepsilon_{abc} - \int_{\partial M} (\mathrm{d}h^{-1}\,h)^{ab} \wedge e_|^c\,\varepsilon_{abc} \tag{101}$$

such that the variation in h only affects the final term.

Under an infinitesimal variation $\xi \in C^\infty(\partial M, \mathfrak{so}(1,d))$, the edge mode will transform as

$$h \longrightarrow e^{\varepsilon\cdot\xi}h \quad \Longrightarrow \quad \mathrm{d}h^{-1}\,h \longrightarrow \mathrm{d}(h^{-1}\,e^{-\varepsilon\cdot\xi})e^{\varepsilon\cdot\xi}h = \mathrm{d}h^{-1}\,h - \varepsilon h^{-1}(\mathrm{d}\xi)h \tag{102}$$

such that

$$\delta_h(\mathrm{d}h^{-1}\,h) = -h^{-1}(\mathrm{d}\xi)h\,. \tag{103}$$

Hence the variation may be expressed

$$\delta_h S_{\partial M} = \int_{\partial M} (h^{-1}\,(\mathrm{d}\xi)\,h)^{ab} \wedge (e_|)^c\,\varepsilon_{abc} = 0 \tag{104}$$

where one may now insert again $\mathbb{I} = h^{-1}h$ such that the Lorentz invariance can be used to re-write this as

$$\delta_h S_{\partial M} = \int_{\partial M} (h^{-1}\,(\mathrm{d}\xi)\,h)^{ab} \wedge (h^{-1}h\,e_|)^c\,\varepsilon_{abc} = \int_{\partial M} (\mathrm{d}\xi)^{ab} \wedge (h\,e_|)^c\,\varepsilon_{abc} = 0 \tag{105}$$

which finally may be integrated by parts to yield ($\partial(\partial M) = \varnothing$)

$$\delta_h S_{\partial M} = -\int_{\partial M} \xi^{ab} \wedge \mathrm{d}\big(h\,e_|\big)^c\,\varepsilon_{abc} = 0 \tag{106}$$

hence, under an arbitrary variation ξ, the following must be satisfied

$$\mathrm{d}(h\,e_|)^c = 0 \qquad \equiv \qquad \mathrm{d}\hat{e}^c = 0 \tag{107}$$

namely that the dressed coframe be a closed form (recall the diffeomorphism was suppressed).

4.4.4 Constraint summary

We have verified that the dressed action reproduces the correct equations of motion and further obtained conditions the fields must satisfy when restricted to the boundary. In summary, the full set of equations of motion and boundary conditions for the EC theory with dressed boundary action are

$$M\begin{cases}\mathcal{F}_\omega: & \mathrm{d}e^a + \omega^a{}_k \wedge e^k = 0\\ \mathcal{F}_e: & R^{ab} + \Lambda\, e^a \wedge e^b = 0\end{cases} \qquad \partial M\begin{cases}\hat{\mathcal{F}}_{\hat\omega}: & (\omega_| + \mathrm{d}h^{-1}h)^{ab} = 0\\ \hat{\mathcal{F}}_{\hat e}: & \mathrm{d}(h\, e_|)^c = 0\end{cases} \tag{108}$$

5 (2+1) Cosmological boundary solutions

Here we attempt to verify that the boundary constraints are compatible with (2+1) dimensional cosmological (Minkowski and de Sitter) half space vacuum solutions. We take M to be diffeomorphic to the half space $\Sigma \times \mathbb{H}$ with coordinates (t, x, y), where the *timelike* boundary hypersurface $\partial M = \Sigma \cong \mathbb{R}^{1,1}$ is located at $y = 0$.

Since the boundary spacetime Σ is (1+1) dimensional, we consider the possible (Lorentz) edge modes to consist of boosts along the boundary only, taking the ansatz

$$h = \begin{pmatrix} \cosh\psi & -\sinh\psi & 0\\ -\sinh\psi & \cosh\psi & 0\\ 0 & 0 & 1\end{pmatrix} \equiv \begin{pmatrix} c_\psi & -s_\psi\\ -s_\psi & c_\psi\end{pmatrix} \tag{109}$$

where we have abbreviated $s_\psi = \sinh\psi$ and since h only acts on the boundary the x^2 coordinate may be suppressed.

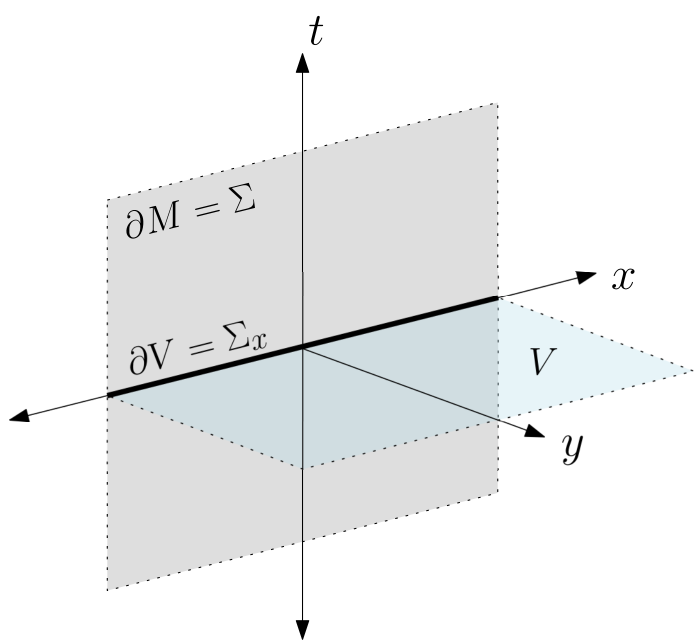


Figure 2: Schematic of the (2+1) half spacetime boundary geometry, where Σ, V are timelike/spacelike hypersurfaces in M

5.1 Half Minkowski spacetime

For the Minkowski spacetime ($\Lambda = 0$) the coframe is simply $e^a = \mathrm{d}x^a$ and both ω and R vanish. Hence, from the dressed spin connection constraint one obtains $\mathrm{d}h^{-1}h = 0$, and furthermore from the coframe constraint $\mathrm{d}h = 0$, such that both constraints are satisfied.

Given the edge mode h must now consist of a boost only, one may obtain the PDE for the rapidity obtained by evaluating $\mathrm{d}h = 0$, where

$$\mathrm{d}h = \mathrm{d}\psi \begin{pmatrix} s_\psi & -c_\psi \\ -c_\psi & s_\psi \end{pmatrix} = 0, \qquad \Longrightarrow \qquad \mathrm{d}\psi = \partial_t\psi\,\mathrm{d}t + \partial_x\psi\,\mathrm{d}x = 0 \tag{110}$$

which implies both $\partial_t\psi, \partial_x\psi = 0$ namely the boost (edge mode) is constant. Therefore, the boundary constraints are suitably satisfied given an arbitrary constant edge mode h and such the half Minkowski spacetime is a consistent solution.

5.2 Half de Sitter spacetime

Now we consider evaluating the constraints in the half de Sitter spacetime ($\Lambda > 0$). We present the relevant geometric objects in (2+1) dimensions, refer to the appendix for more details.

5.2.1 Geometry

Let M be the (2+1) dimensional de Sitter half spacetime, which in coordinates (t, x, y) has a timelike boundary hypersurface $\partial M = \Sigma$ located at $y = 0$. The coframe components for the de Sitter spacetime are

$$e^0 = \mathrm{d}t, \;\; e^i = a(t)\,\mathrm{d}x^i \tag{111}$$

where $a(t) = e^{Ht}$ and the boundary restriction simply sets $e^2_| = 0$. The Hubble parameter H can be obtained from Einstein's equation (see appendix) to be $H = \sqrt{\Lambda}$.

In (2+1) dimensions the spin connection will have three independent components and after using the ansatz ${\omega^1}_2 = {\omega^2}_1 = 0$ one finds two independent components, obtained as ($i = 1, 2$)

$$\omega^i{}_0 = \dot{a}\,\mathrm{d}x^i \tag{112}$$

The curvature is not explicitly required although it is calculated in the appendix in order to determine the relation $H = \sqrt{\Lambda}$ from the Einstein equation.

5.2.2 Dressed coframe constraint

Consider evaluating the condition $\mathrm{d}(h\,e_|) = 0$ obtained from the variation of the edge mode. Using h as defined in Eq. (109)

$$\hat{e}^c = (he_|)^c = \begin{pmatrix} c_\psi \mathrm{d}t - a\,s_\psi \mathrm{d}x \\ -s_\psi \mathrm{d}t + a\,c_\psi \mathrm{d}x \end{pmatrix} \tag{113}$$

where since the $c = 2$ component vanishes it is suppressed. The exterior derivative may now be evaluated, giving the 2-form

$$\mathrm{d}\hat{e}^c = \begin{pmatrix} -\dot{a}s_\psi - s_\psi\partial_x\psi - ac_\psi\partial_t\psi \\ \dot{a}c_\psi + c_\psi\partial_x\psi + s_\psi a\partial_t\psi \end{pmatrix} \mathrm{d}t \wedge \mathrm{d}x = 0 \tag{114}$$

such that each component $\mathrm{d}\hat{e}^0 = \mathrm{d}\hat{e}^1 = 0$ must independently vanish. One thus obtains the following system of PDEs

$$-s_\psi \dot{a} - s_\psi \partial_x \psi - a c_\psi \partial_t \psi = 0 \tag{115}$$

$$c_\psi \dot{a} + c_\psi \partial_x \psi + a s_\psi \partial_t \psi = 0 \tag{116}$$

which combine to give

$$(s_\psi + c_\psi)\dot{a} + (s_\psi + c_\psi)\partial_x \psi + (c_\psi + s_\psi) a \partial_t \psi = 0 \tag{117}$$

which simplifies to

$$\partial_t \psi = -\frac{\dot{a}}{a} - \frac{1}{a}\partial_x \psi \tag{118}$$

which we consider evaluating once the spin connection constraint has been obtained.

5.2.3 Dressed spin connection constraint

In addition to the closed dressed coframe requirement one must also verify that the spin connection be gauge equivalent to zero on the boundary, namely $\omega_| + \mathrm{d}h^{-1}h = 0$. Since the spin connection has two independent components, $\omega^i{}_0$, the boundary restriction simply has one independent component

$$\omega^1{}_0 = \dot{a}\,\mathrm{d}x \tag{119}$$

therefore, one immediately sees the edge mode must be independent of t (since $\omega_|$ only contains a $\mathrm{d}x$ component) as we now verify.

Consider evaluating the spin connection constraint for ω as found previously

$$\omega_|{}^i{}_0 = -(\mathrm{d}h^{-1})^i{}_c h^c{}_0 \qquad \Longrightarrow \qquad \omega^1{}_0 = -(\mathrm{d}h^{-1})^1{}_c h^c{}_0 \tag{120}$$

and using

$$\mathrm{d}h^{-1} = \begin{pmatrix} s_\psi & c_\psi \\ c_\psi & s_\psi \end{pmatrix} (\partial_t \psi\,\mathrm{d}t + \partial_x \psi\,\mathrm{d}x)\,, \qquad \Longrightarrow \qquad h\mathrm{d}h^{-1} = \begin{pmatrix} 0 & \mathrm{d}\psi \\ \mathrm{d}\psi & 0 \end{pmatrix} \tag{121}$$

one obtains

$$\omega^1{}_0 = \dot{a}\mathrm{d}x = -(c_\psi^2 - s_\psi^2)\,(\partial_t \psi\,\mathrm{d}t + \partial_x \psi\,\mathrm{d}x) \tag{122}$$

such that one requires

$$\partial_t \psi = 0, \qquad \partial_x \psi = -\dot{a} \tag{123}$$

which one now easily verifies is a solution to Eq. (118). Therefore, whilst the edge mode appears to be independent of time, there is non-trivial time dependence contained within the $\dot{a}$.

5.2.4 Vanishing acceleration

In summary, we have identified the following conditions that must be satisfied for a (1+1) dimensional, timelike boundary hypersurface in (2+1) de Sitter half space. From the dressed coframe condition one obtains the following PDE for the edge mode rapidity

$$\partial_t \psi = -\frac{\dot{a}}{a} - \frac{1}{a}\partial_x \psi \tag{124}$$

and from the spin connection constraint one further obtains that

$$\partial_t \psi = 0, \quad \partial_x \psi = -\dot{a} \tag{125}$$

which is compatible with the dressed coframe constraint. The solution is therefore

$$\psi = -\dot{a}x + c_0 \tag{126}$$

where c_0 can be taken to be 0 without loss of generality. However, if one considers the condition that $\partial_t \psi = 0$ from the above solution one finds

$$\ddot{a} = 0 \tag{127}$$

which is not what is expected for the de Sitter universe, expected to have accelerated expansion!

6 Hypersurface boundary cosmological constant

In order to resolve the issue of vanishing acceleration consider adding the following three boundary cosmological constant terms (indexed by a) into the action

$$-\int_{\partial M} \frac{\lambda^a}{2} \left(\hat{e}^b \wedge \hat{e}^c\right) \varepsilon_{abc} \tag{128}$$

where the components λ^a are the cosmological constants on the boundary of two dimensional hypersurfaces in M, spanned by the $e^b, b \neq a$ (the relation to Λ will be determined shortly). Hence, the full EC boundary action may be expressed as

$$S_{\partial M} = -\int_{\partial M} \left(\hat{\omega}^{ab} \wedge \hat{e}^c\right) \varepsilon_{abc} + \frac{\lambda^a}{2} \left(\hat{e}^b \wedge \hat{e}^c\right) \varepsilon_{abc} \tag{129}$$

6.1 Augmented boundary constraints

Here we derive the augmented boundary constraints obtained after including such a dressed boundary action with boundary cosmological constant term. We then verify the augmented coframe and spin connection constraints are consistent with the de Sitter solution.

6.1.1 Augmented spin connection constraint

The first order variation of the boundary action with respect to the coframe may be expressed (suppressing edge mode Lorentz indices) as

$$\delta_e S_{\partial M} = -\int_{\partial M} \left(\hat{\omega}^{ab} \wedge (h\delta e)^c\right) \varepsilon_{abc} + \lambda^a \left(\hat{e}^b \wedge (h\delta e)^c\right) \varepsilon_{abc} \tag{130}$$

such that under arbitrary variations δe (and after anti-symmetrising the indices ab on the second term) one obtains

$$\mathcal{F}_{\hat{\omega}} = \hat{\omega}^{ab} + \lambda^{[a} \hat{e}^{b]} = 0 \tag{131}$$

hence obtaining the 'augmented spin connection constraint', which may further be expressed by left/right multiplying with h^{-1} and h respectively as

$$(\omega_| + \mathrm{d}h^{-1}h)^{ab} + (h^{-1}\lambda)^{[a} e_|^{b]} = 0 \tag{132}$$

which is used in calculations that follow.

6.1.2 Augmented coframe constraint

Introducing such a boundary cosmological constant term will also influence the edge mode variation, where previously one found

$$\delta_h S_{\partial M} = -\int_{\partial M} \xi^{ab} \wedge \mathrm{d}\big(h\, e_|\big)^c \varepsilon_{abc} = 0 \tag{133}$$

which is now augmented by adding the variation of the boundary cosmological constant term. The augmented boundary term may be re-written using the Lorentz invariance (and permuting indices $ab \leftrightarrow c$) as

$$-\int_{\partial M} (he_|)^a \wedge (he_|)^b \wedge \frac{\lambda^c}{2} \varepsilon_{abc} = -\frac{1}{2}\int_{\partial M} e_|^a \wedge e_|^b \wedge (h^{-1}\lambda)^c \varepsilon_{abc} \tag{134}$$

such that the first order variation may be expressed

$$\delta_h S_{\partial M} = +\frac{1}{2}\int_{\partial M} e_|^a \wedge e_|^b \wedge (h^{-1}\xi\lambda)^c \varepsilon_{abc} = +\frac{1}{2}\int_{\partial M} (he_|)^a \wedge (he_|)^b \wedge \xi^c{}_d \lambda^d \varepsilon_{abc} \tag{135}$$

after using the Lorentz invariance once more. One may now use the following identity to write

$$\varepsilon_{abc}\varepsilon^{a'b'c} = \delta_a^{a'}\delta_b^{b'} - \delta_a^{b'}\delta_b^{a'} \tag{136}$$

$$\frac{1}{2}\varepsilon_{cde}\varepsilon^{c'd'e}\varepsilon_{abc'}\delta^d_{d'} = \frac{1}{2}(\varepsilon_{cdb}\delta_a^{d'} - \varepsilon_{cda}\delta_b^{d'})\delta^d_{d'} = \varepsilon_{abc} \tag{137}$$

such that (after raising/lowering indices on the boundary term) one finds

$$+\frac{1}{2}\int_{\partial M} \xi^{cd}\lambda_d (he_|)^a \wedge (he_|)^b \varepsilon_{abc} = \frac{1}{4}\int_{\partial M} \xi^{cd}(he_|)^a \wedge (he_|)^b \lambda_{d'}(\varepsilon_{cdb}\delta_a^{d'} - \varepsilon_{cda}\delta_b^{d'}) \tag{138}$$

which may be written as (after relabelling $c \leftrightarrow a, d \leftrightarrow b$)

$$\frac{1}{4}\int_{\partial M} \xi^{ab}(he_|)^c \wedge (he_|)^d (\lambda_c\varepsilon_{abd} - \lambda_d\varepsilon_{abc}) \tag{139}$$

which after relabelling $c \leftrightarrow d$ on the first term this simplifies to

$$-\frac{1}{2}\int_{\partial M} \xi^{ab}(he_|)^c \wedge (he_|)^d \lambda_d \varepsilon_{abc} \tag{140}$$

Therefore, the full boundary variation may be expressed

$$\delta_h S_{\partial M} = -\int_{\partial M} \xi^{ab} \wedge \mathrm{d}\big(h\, e_|\big)^c \varepsilon_{abc} - \frac{1}{2}\int_{\partial M} \xi^{ab}(he_|)^c \wedge (he_|)^d \lambda_d \varepsilon_{abc} \tag{141}$$

such that one finds the following augmented coframe constraint

$$\hat{\mathcal{F}}_{\hat{e}} = \mathrm{d}\big(h\, e_|\big)^c + \frac{1}{2}(he_|)^c \wedge (he_|)^a \lambda_a = \mathrm{d}\hat{e}^c + \frac{1}{2}\hat{e}^c \wedge \hat{e}^a \lambda_a = 0 \tag{142}$$

6.1.3 Augmented constraint summary

In summary, the full EC action with boundary term may now be expressed as

$$S_{\partial M} = \int_M \left(R^{ab} \wedge e^c - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \varepsilon_{abc} - \int_{\partial M} \left(\hat{\omega}^{ab} \wedge \hat{e}^c\right) \varepsilon_{abc} + \frac{\lambda^a}{2} \left(\hat{e}^b \wedge \hat{e}^c\right) \varepsilon_{abc} \tag{143}$$

where the full set of equations of motion and boundary constraints are

$$M \begin{cases} \mathcal{F}_\omega : & \mathrm{d}e^a + \omega^a{}_k \wedge e^k = 0 \\ \mathcal{F}_e : & R^{ab} + \Lambda\, e^a \wedge e^b = 0 \end{cases} \qquad \partial M \begin{cases} \hat{\mathcal{F}}_{\hat{\omega}} : & \hat{\omega}^{ab} + \lambda^{[a} \hat{e}^{b]} = 0 \\ \hat{\mathcal{F}}_{\hat{e}} : & \mathrm{d}\hat{e}^c + \frac{1}{2} \hat{e}^c \wedge \hat{e}^a \lambda_a = 0 \end{cases} \tag{144}$$

6.2 Spatial hypersurface boundary cosmological constant

In general, one must consider adding all three terms indexed by λ^a, where each λ^a component carries a potentially different value for the boundary cosmological constant of the respective hypersurfaces. The $a = 2$ component automatically vanishes when added to the action (due to the ansatz $e^2_| = 0$) and furthermore if one takes $\lambda^0 = \lambda^1$ and includes both terms in the action they will cancel to again yield vanishing acceleration as we show shortly. Therefore, one may consider tuning the parameters $\lambda^0 \neq \lambda^1$ in order to produce positive acceleration, however, one may equally conclude the $a = 0$ term is distinguished and let $a = 1, 2$ vanish as we now consider.

6.2.1 Consistent de Sitter solution

Letting $\lambda^0 = \lambda$ and $\lambda^i = 0$ corresponds to identifying a non-zero cosmological constant on the boundary of *spacelike* hypersurfaces in M, in this case given by $V = \Sigma_x \times \mathbb{H}$ where the boundary $\partial V = \Sigma_x \subset \Sigma$ corresponds to the one dimensional subset of Σ given by $\Sigma_x = (0, x) \subset \Sigma$ (see Fig. 5).

Hence, the boundary constraints simplify to ($\lambda^0 = -\lambda_0$)

$$\begin{cases} \hat{\mathcal{F}}_{\hat{\omega}} : & \hat{\omega}^{ab} + \lambda^{[0} \hat{e}^{b]} = \hat{\omega}^{ab} + \frac{\lambda}{2} \hat{e}^b = 0 \\ \hat{\mathcal{F}}_{\hat{e}} : & \mathrm{d}\hat{e}^c - \frac{\lambda}{2} \hat{e}^c \wedge \hat{e}^0 = 0 \end{cases} \tag{145}$$

One may now verify such augmented constraints are compatible with the de Sitter solution, by taking the ansatz $\lambda^0 = \lambda, \lambda^i = 0$, which as shown previously produces positive acceleration. Hence, one obtains from the augmented spin connection constraint using $\hat{\omega}^{ab} = \frac{1}{2}(\omega^{01}_| + \mathrm{d}\psi)$, the following

$$\partial_x \psi = - \dot{a} - \lambda a c_\psi \tag{146}$$

$$\partial_t \psi = \lambda s_\psi \tag{147}$$

From the augmented coframe constraint, after using the expression for $\hat{e}$ in Eq. (113) one finds

$$\mathrm{d}\hat{e}^c - \frac{\lambda}{2} \hat{e}^c \wedge \hat{e}^0 = \mathrm{d}\hat{e}^c - \frac{\lambda}{2} \begin{pmatrix} 0 \\ -a \end{pmatrix} \mathrm{d}t \wedge \mathrm{d}x = 0 \tag{148}$$

Therefore, after using the expression for $\mathrm{d}\hat{e}^c$ in Eq. (114) and requiring each component vanish obtains the PDE (again cancelling the implicit $\frac{1}{2}$ from the 2-form components $\mathrm{d}\hat{e}^c$)

$$(s_\psi + c_\psi)\dot{a} + (s_\psi + c_\psi)\partial_x \psi + (c_\psi + s_\psi) a \partial_t \psi + \lambda a = 0 \tag{149}$$

which them simplifies to give

$$\partial_t \psi = -\frac{\dot{a}}{a} - \frac{1}{a}\partial_x \psi - \lambda \frac{1}{s_\psi + c_\psi} \tag{150}$$

which one now verifies is compatible with the solution obtained from the augmented spin connection constraint, since after substituting Eqs. (155) and (162) into Eq. (150) one finds

$$\lambda s_\psi = \lambda c_\psi - \lambda \frac{1}{s_\psi + c_\psi} \quad \implies \quad \frac{1}{s_\psi + c_\psi} = c_\psi - s_\psi \quad \implies \quad c_\psi^2 - s_\psi^2 = 1 \tag{151}$$

which is hence satisfied $\forall \psi$ and thus the constraints are compatible.

Furthermore, one may identify by taking mixed partial derivatives of the augmented spin connection constraint equations that

$$\partial_t \partial_x \psi = -\ddot{a} - \lambda \dot{a} c_\psi - \lambda a s_\psi \partial_t \psi = \lambda c_\psi \partial_x \psi = \partial_x \partial_t \psi \tag{152}$$

which after substituting $\partial_t \psi$ and $\partial_x \psi$ one obtains

$$-\ddot{a} - \lambda \dot{a} c_\psi - \lambda^2 a s_\psi^2 = -\lambda \dot{a} c_\psi - \lambda^2 a c_\psi^2 \tag{153}$$

such that one finds the acceleration

$$\frac{\ddot{a}}{a} = \lambda^2 = \Lambda \tag{154}$$

where we have also used the Einstein field equation (see appendix) $\frac{\ddot{a}}{a} = \Lambda$. If one had allowed for $\lambda^1 \neq 0$ term (where the resulting acceleration can be obtained simply[12] by swapping $s_\psi \leftrightarrow c_\psi$ in the above expression) one would find such a term produces an equal but negative acceleration; hence it is desirable to only include the $a = 0$ term.

6.2.2 Identity edge mode solution

Since these constraints are satisfied $\forall \psi$ we may take the ansatz $\psi = const$, namely $\partial_t \psi, \partial_x \psi = 0$ and study the resultant solutions. From the augmented spin connection constraint (Eqs. (155), (162)) one finds

$$0 = -\dot{a} - \lambda a c_\psi \tag{155}$$

$$0 = \lambda s_\psi \tag{156}$$

such that these are only satisfied for $\psi = 0$ and $\lambda = -H$ which is hence the unique constant solution. Therefore, we may take $\psi = 0, \implies h = \mathbb{I}$, where in this simple case the dressed fields are simply equivalent to the restriction, i.e $\hat{e} = e_|$, $\hat{\omega} = \omega_| = \omega^{01}$. Therefore, from the augmented spin connection constraint one obtains

$$\hat{\omega}^{ab} + \lambda^{[a} \hat{e}^{b]} = \omega^{01} + \lambda e^1_| = \dot{a} \mathrm{d}x + \lambda a \mathrm{d}x = 0 \tag{157}$$

which therefore requires

$$\lambda = -\frac{\dot{a}}{a} = -H \tag{158}$$

[12] to see this explicitly note since $\lambda^1 = \lambda_1$ the terms will differ by a sign, effectively obtained here by swapping $c_\psi \leftrightarrow s_\psi$

as implied previously. Furthermore, from the augmented coframe constraint one verifies

$$\mathrm{d}e^0_| - \lambda e^0_| \wedge e^0_| = 0 \tag{159}$$

$$\mathrm{d}e^1_| - \lambda e^1_| \wedge e^0_| = \dot{a}\mathrm{d}t \wedge \mathrm{d}x + \lambda a\,\mathrm{d}t \wedge \mathrm{d}x = 0 \tag{160}$$

which is hence satisfied for $\lambda = -\frac{\dot{a}}{a}$, compatible with the result obtained previously from the augmented spin connection constraint.

Hence, we have successfully shown that the de Sitter solution is consistent with the edge mode boundary conditions, providing one identifies a suitable cosmological constant term on the boundary of spacelike hypersurfaces in M. In order to satisfy the constraints one requires a constant identity edge mode h, along with an arbitrary diffeomorphism φ that was suppressed throughout.

6.2.3 Spatially constant edge mode ansatz

Here we consider the ansatz $\partial_x\psi = 0$, namely for a spatially constant edge mode. From the augmented spin connection constraint one finds

$$\frac{\dot{a}}{a} = -\lambda c_\psi \tag{161}$$

$$\partial_t\psi = \lambda s_\psi \tag{162}$$

such that after substituting $\lambda = -\frac{\dot{a}}{a}\frac{1}{c_\psi}$ one obtains

$$\partial_t\psi = -\frac{\dot{a}}{a}\tanh\psi = -H\tanh\psi \tag{163}$$

which is separable and may be solved using

$$\int \coth\psi\,\mathrm{d}\psi = -\int H\,\mathrm{d}t \qquad \Longrightarrow \qquad \ln(\sinh\psi) = -Ht + c_0 \tag{164}$$

which hence gives (letting c_0 =0)

$$\psi = \sinh^{-1}(e^{-Ht}) = \sinh^{-1}(a^{-1}) \tag{165}$$

which one easily verifies after substituting solves the augmented spin connection constraint equations (using $\cosh(\sinh^{-1}(x)) = \sqrt{x^2+1}$ and $\tanh(\sinh^{-1}(x)) = \frac{x}{\sqrt{x^2+1}}$ with $x = a^{-1}$)

$$\frac{\dot{a}}{a} = -\lambda\sqrt{a^{-2}+1} \tag{166}$$

$$-\frac{\dot{a}}{a}\frac{a^{-1}}{\sqrt{a^{-2}+1}} = \lambda a^{-1} \tag{167}$$

which are equivalent, and such one realises

$$\lambda = -\frac{\dot{a}}{a}\frac{1}{\sqrt{a^{-2}+1}} = -\frac{\dot{a}}{\sqrt{1+a^2}} \tag{168}$$

which implies non-trivial time dependence for λ that we consider shortly, after verifying the ansatz is compatible with the augmented coframe constraint.

After substituting into the PDE obtained from the augmented coframe constraint Eq. (150) (repeated here)

$$\partial_t \psi = -\frac{\dot{a}}{a} - \frac{1}{a}\partial_x \psi - \lambda \frac{1}{s_\psi + c_\psi} \tag{169}$$

one finds

$$-\frac{\dot{a}}{a}\frac{a^{-1}}{\sqrt{a^{-2}+1}} = -\frac{\dot{a}}{a} - \lambda \frac{1}{a^{-1} + \sqrt{1+a^{-2}}} \tag{170}$$

which may be re-arranged

$$\frac{\dot{a}}{a}\left(1 - \frac{1}{\sqrt{a^2+1}}\right) = -\lambda \frac{a}{1+\sqrt{1+a^2}} \tag{171}$$

which further simplifies to

$$\frac{\dot{a}}{a^2}\left(\frac{a^2}{\sqrt{a^2+1}}\right) = -\lambda \qquad \Longrightarrow \qquad \lambda = -\frac{\dot{a}}{\sqrt{1+a^2}} \tag{172}$$

as obtained from the augmented spin connection constraint, hence the result is compatible.

Hence, this result implies λ is *not* constant, however, in the limit of large a

$$\lim_{a\to\infty} : \lambda \longrightarrow -\frac{\dot{a}}{a} \tag{173}$$

as was obtained previously for the identity edge mode case. However, since λ is required to be constant in order to maintain the diffeomorphism invariance of the EC action, this implies the spatially constant edge mode solution must be inconsistent since it requires λ be dynamical. Therefore, as both constraints imply the same non-trivial time dependence for λ one thus concludes the solution $\partial_x \psi = 0$ is inconsistent, since in order to maintain the diffeomorphism invariance of the EC action λ must be constant.

6.3 Half spacetime solution summary

Hence, we have verified the constraints derived from the dressed boundary action are consistent with (2+1) dimensional half Minkowski and de Sitter spacetimes with timelike boundary and obtained the resulting edge mode in each case. A constant edge mode h is required in the Minkowski spacetime, whilst for the de Sitter spacetime requires an identity edge mode $h = \mathbb{I}$ (both cases also implicitly include the arbitrary diffeomorphism).

For de Sitter spacetime, one requires a dressed cosmological constant term in the action that corresponds to the cosmological constant on the boundary of 2-dimensional hypersurfaces in M. It was found that allowing for a non-zero value of such a boundary cosmological constant on *spacelike* hypersurfaces in M produces consistent results, where in the de Sitter case for the identity edge mode, one finds the spacelike boundary cosmological constant to be given by $\lambda = -H$, $\lambda^2 = \Lambda$. It was also shown that a spatially constant edge mode ansatz $\partial_x \psi = 0$ was inconsistent as it required a dynamical boundary cosmological constant, thus breaking the diffeomorphism invariance of the EC action. It is possible that allowing for perturbations in both $\partial_t \psi, \partial_x \psi \neq 0$ can yield a consistent solution, however, full analysis of the in-homogeneous system of PDEs was beyond the timeframe of the present work.

7 Summary

We have studied the identification of edge modes in the Einstein-Cartan theory, topological boundary conditions ensuring that the restriction of the bulk Cartan connection (coframe, spin connection) is gauge equivalent to a fixed Cartan connection on the boundary. The edge modes were identified using the homotopy pullback, where the resulting groupoid is found to contain edge modes consisting of a boundary diffeomorphism and Lorentz transformation, with morphisms given by the semidirect product group (the automorphism group of the SO-principal bundle).

After introducing edge modes, we constructed additional boundary action terms, using 'dressed' fields obtained by pre-composing the boundary fields with an inverse pair of edge modes. After introducing the dressed boundary term one finds that under the ω variation such a term cancels the problematic term arising on the boundary previously, such that one no longer requires the coframe be degenerate on the boundary. Furthermore, by considering how the edge modes affect the equations of motion we obtain constraints that must be satisfied by the dressed fields on the boundary and attempt to verify them using a bounded Minkowski and de Sitter solution. The Minkowski case works trivially, where one finds the edge mode is simply constant, however, a vanishing acceleration was found in the de Sitter case.

The dressed boundary action initially considered did not contain a cosmological constant term, however, it was found an additional boundary cosmological constant term was required in order to produce non-zero acceleration in the de Sitter case. After adding a suitable two dimensional hypersurface boundary cosmological constant term, it was found the resulting augmented constraints are consistent with the de Sitter solution and produce non-zero acceleration. In this case the unique solution is given by $h = \mathbb{I}$, along with the arbitrary diffeomorphism. Here, one finds the hypersurface boundary cosmological constant $\lambda = -H, \lambda^2 = \Lambda$. However, for the ansatz of a spatially constant edge mode, whilst this is consistent with the constraints it requires the boundary 'constant' λ to be time dependent; implying such an ansatz cannot be consistent as this would break the diffeomorphism invariance of the action.

8 Conclusion

In an attempt to resolve the degeneracy in the standard formalism implied for the boundary coframe we have introduced edge modes in the EC theory, and focused on the (2+1) dimensional case for simplicity. After using the homotopy pullback to identify the required edge modes (Lorentz transformation + boundary diffeomorphism) we construct 'dressed' fields in terms of the edge modes, allowing additional boundary terms to be added to the EC action. Under the variation, such boundary terms are found to eliminate the degeneracy previously implied for the coframe. Furthermore, we have derived constraints that must be satisfied by the dressed boundary fields and verified that such constraints are compatible for (2+1) half Minkowski and de Sitter vacuum solutions. In order to maintain consistency in the de Sitter case, one is required to identify a cosmological constant on the boundary of spacelike hypersurfaces in M. Finally, we showed a spatially constant edge mode ansatz is not consistent, such that one requires a constant identity edge mode in the de Sitter case (constant but arbitrary in the Minkowski case).

Appendices

A Lie algebra valued forms

This section formalises the conventions to be used when dealing with Lie algebra valued, or more generally linear representation space valued, differential forms. For example, given a principal G-bundle with a Lie algebra, $\mathfrak{g}$ and a Lie algebra representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$, and corresponding Lie algebra and representation valued forms

$$\omega \in \Omega^r(P, \mathfrak{g}), \quad \phi \in \Omega^s(P, V) \tag{174}$$

the wedge product can be defined using the representation as

$$\rho_*(\omega) \wedge \phi \in \Omega^{r+s}(P, V) \tag{175}$$

which is an $r+s$ form taking values in the *representation* V. The pushforward using the representation is usually omitted for brevity and understood from context based on the forms in question. Given suitable bases t_a, e_i for $\mathfrak{g}$ and V respectively, this can be expressed in components as

$$(\omega^a \otimes t_a) \wedge (\phi^i \otimes e_i) = \omega^a \wedge \phi^i \otimes \rho_*(t_a)(e_i) \tag{176}$$

A special case that is frequently encountered is the case where the space V is the space $\mathfrak{g}$ under the adjoint representation (given by the commutator of the matrix group), such that given $\omega, \phi \in \Omega^\bullet(P, \mathfrak{g})$, the wedge product can be expressed using the commutator as (ommiting ρ_*)

$$\omega \wedge \phi = \omega^a \wedge \phi^b \otimes \mathrm{ad}_* (t_a) (t_b) = \omega^a \wedge \phi^b \otimes [t_a, t_b] \tag{177}$$

which is hence frequently abbreviated as $[\omega, \phi]$.

B Homotopy Pullback

Here we briefly formulate the homotopy pullback used in constructing the boundary condition that we will see corresponds to the 'edge modes'. Given groupoids $\mathcal{G}, \mathcal{H}, \mathcal{K}$ with functors $f : \mathcal{G} \longrightarrow \mathcal{K}$, $g : \mathcal{H} \longrightarrow \mathcal{K}$ between the respective groupoids, one may form the homotopy pullback (the 2-categorical pullback) summarised with the following diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{H} \arrow[d, "g"] \\ \mathcal{G} \arrow[r, "f"] & \mathcal{K} \end{tikzcd} \tag{178}$$

where the dashed arrows indicate forming the homotopy pullback $\mathcal{P}$, the groupoid who's elements are triples (x, y, k), where $(x, y) \in \mathcal{G} \times \mathcal{H}$ and k is an isomorphism in $\mathcal{K}$ such that

$$k : f(x) \longrightarrow g(y) \tag{179}$$

where k must also commmute with the morphisms on $\mathcal{G}$ and $\mathcal{H}$ (not shown explicitly). We regard k as the 'witness' to the property that the images of $f(x)$ and $g(y)$ are equivalent in $\mathcal{K}$, in the sense that k 'witnesses' this property by providing the isomorphism between them.

C Cartan de Sitter geometry

Let M be the (2+1) dimensional de Sitter half spacetime, which in coordinates (t, x, y) has a timelike boundary hypersurface $\partial M = \Sigma$ located at $y = 0$. The coframe components for the de Sitter spacetime are (natural units)

$$e^0 = \mathrm{d}t, \;\; e^i = a(t)\, \mathrm{d}x^i \tag{180}$$

where $a(t) = e^{Ht}$ and the boundary restriction simply sets $e^2_| = 0$.

In (2+1) dimensions the spin connection will have three independent components which are easily obtained through evaluating the torsion constraint,

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0, \tag{181}$$

where

$$\mathrm{d}e^0 = 0, \quad\; \mathrm{d}e^i = \dot{a}\, \mathrm{d}t \wedge \mathrm{d}x^i, \tag{182}$$

such that one obtains

$$T^0 : \quad \mathrm{d}e^0 + \omega^0{}_b \wedge e^b = a\,\omega^0{}_i \wedge \mathrm{d}x^i = 0 \tag{183}$$

$$T^i : \quad \mathrm{d}e^i + \omega^i{}_b \wedge e^b = \dot{a}\, \mathrm{d}t \wedge \mathrm{d}x^i + \omega^i{}_0 \wedge \mathrm{d}t + a\,\omega^i{}_j \wedge \mathrm{d}x^j = 0 \tag{184}$$

We may now consider the ansatz $\omega^1{}_2 = -\omega^2{}_1 = 0$ and check the above are satisfied (any solution is guaranteed to be unique). Hence, one obtains

$$a\,\omega^0{}_i \wedge \mathrm{d}x^i = 0 \tag{185}$$

$$\dot{a}\, \mathrm{d}t \wedge \mathrm{d}x^i + \omega^i{}_0 \wedge \mathrm{d}t = 0, \tag{186}$$

which is hence satisfied for

$$\omega^i{}_0 = \dot{a}\, \mathrm{d}x^i. \tag{187}$$

Furthermore, one verifies using the definition

$$\Gamma^\nu{}_{\mu\lambda} = e^\nu{}_a \partial_\mu e_\lambda{}^a + e^\nu{}_a e_\lambda{}^b \omega_\mu{}^a{}_b \tag{188}$$

that such a spin connection produces the correct Levi-Civita connection coefficients (here $e^\nu{}_a$ is the inverse coframe defined by $e^\nu{}_a e_\mu{}^a = \delta^\nu{}_\mu$ i.e $e_0 = \partial_0, e_i = \frac{1}{a}\partial_i$ One thus verifies

$$\Gamma^i{}_{0j} = e^i{}_a \partial_0 e_j{}^a + e^i{}_a e_j{}^b \omega_0{}^a{}_b = \frac{\dot{a}}{a}\, \delta^i{}_j \tag{189}$$

$$\Gamma^0{}_{ij} = e^0{}_a \partial_i e_j{}^a + e^0{}_a e_j{}^b \omega_i{}^a{}_b = e^0{}_0 e_j{}^k \omega_i{}^0{}_k = e_j{}^k \dot{a} \delta_{ik} = a\dot{a}\, \delta_{ij} \tag{190}$$

as required. Furthermore, this spin connection will work in any (1+d) dimensional (isotropic) spacetime with coframes of the form $e^i = a(t)\mathrm{d}x^i$ and taking $\omega^i{}_j$ to vanish as the ansatz.

Finally, one may calculate the curvature 2-form as

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{191}$$

from which one obtains the nonvanishing components as

$$R^i{}_0 = \mathrm{d}\omega^i{}_0 + \omega^i{}_c \wedge \omega^c{}_0 = \mathrm{d}\omega^i{}_0 = \ddot{a}\,\mathrm{d}t \wedge \mathrm{d}x^i \tag{192}$$

$$R^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_c \wedge \omega^c{}_j = \omega^i{}_0 \wedge \omega^0{}_j = \dot{a}^2 \mathrm{d}x^i \wedge \mathrm{d}x_j \tag{193}$$

which can equivalently be expressed using the coframe as

$$R^i{}_0 = -R^{i0} = R^{0i} = \frac{\ddot{a}}{a} e^0 \wedge e^i \tag{194}$$

$$R^{ij} = \eta^{jk} R^i{}_k = \left(\frac{\dot{a}}{a}\right)^2 e^i \wedge e^j \tag{195}$$

Hence the Einstein field equation has two components, given by

$$R^{0i} - \Lambda e^0 \wedge e^i = \left(\frac{\ddot{a}}{a} - \Lambda\right) e^0 \wedge e^i = 0 \quad \Longrightarrow \quad \frac{\ddot{a}}{a} = \Lambda \tag{196}$$

$$R^{ij} - \Lambda e^i \wedge e^j = \left(\left(\frac{\dot{a}}{a}\right)^2 - \Lambda\right) e^i \wedge e^j = 0 \quad \Longrightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \Lambda \tag{197}$$

such that one finds

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \Lambda \tag{198}$$

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