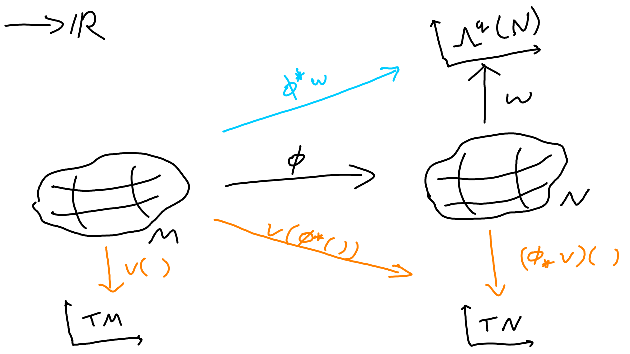
Pullback: Given a map $\phi: M \longrightarrow N$: pullback of function $f: N \longrightarrow \mathbb{R}$. $\phi^* f = f \circ \phi : M \longrightarrow \mathbb{R}$

of general p-form: (note $\phi^* d\omega = d(\phi^* \omega)$) $\quad \phi^* \omega = \phi^* \omega_a dy^a = \underbrace{(\phi^* \omega_a)}_{\omega(x)} \underbrace{d(\phi^* y^a)}_{dx^a}$

$\omega \in \Lambda^q(N)$ $\quad \downarrow \omega(y) : \phi : x \mapsto y$



Pushforward: Given a vector $v \in TM$ can define $\tilde{v} \in TN$ by $(\phi_* v)(f) = v(\phi^* f)$ for $f \in C^\infty(N)$

pushforward vector given by vector acting on pullback.

Lie Derivative On general tensors $\boxed{\mathcal{L}_v T = \left.\frac{\partial}{\partial t}\right|_{t=0} \phi_t^* T}$ where v is the vector generated by the diffeomorphism $\phi_t : \mathbb{R} \times M \longrightarrow M$, $(t, x) \longmapsto y$

to derive consider $\partial_t|_0$ of $\phi^* = v(T(T'))$

for the T acting on arbitrary T'. $y \mapsto T(y)$ eg $\mathcal{L}_v \theta(u) = \left.\frac{\partial}{\partial t}\right|_{t=0} \phi_t^* \theta(u) = v(\theta(u)) = \dot{\theta}(u) + \theta(\dot{u}) = \dot{\theta}(u) + \theta_b(v^a \partial_a u^b - u^a \partial_a v^b) = v^a \partial_a(\theta_b u^b) - u^b \partial_b(\theta_a v^a) + v^a u^b(\partial_b \theta_a - \partial_a \theta_b)$

$\dot{u} \to [v,u]$

velocity vector $\quad$ Leibniz rule on ∂_t $\quad$ re write to put θ inside $v^a \partial_a(\theta u)$ $\Rightarrow$ then write $v(\theta(u))$.. $\quad = v(\theta(u)) - \underbrace{u(\theta(v))}_{i_u d i_v \theta} + i_v i_u d\theta$

cancel all i_u

$\Rightarrow$ so $\dot{\theta}(u) = u(\theta(v)) + i_u i_v d\theta \Rightarrow \boxed{\mathcal{L}_v \theta = \dot{\theta} = d i_v \theta + i_v d\theta}$

Vectors: $\mathcal{L}_v$ is given by Lie bracket $\boxed{\mathcal{L}_v u = \dot{u} = [v, u]}$: in components $\dot{u}^b = \underbrace{v^a \partial_a u^b - u^a \partial_a v^b}$ $\to$ ∂^2 terms always cancel.

Practical way: $\mathcal{L}_v u = \mathcal{L}_v u^a \partial_a = \underbrace{(\mathcal{L}_v u^a)}_{(v^b \partial_b u^a)\partial_a} \partial_a + u^a \underbrace{\mathcal{L}_v \partial_a}_{[v, \partial_a] = v^b \partial_b \partial_a - \partial_a(v^b \partial_b) = -(\partial_a v^b)\partial_b} = (v^b \partial_b u^a - u^b \partial_b v^a)\partial_a \Rightarrow \dot{u} = [v,u]^b \partial_b = (\underbrace{u(v) - v(u)}_{\dot{u}^b})^b \partial_b = \dot{u}^b \partial_b$

$\mathcal{L}_v u = \dot{u} = [v,u]$ is a vector (linear) (ie obeys Leibniz/is a derivation)

1-Forms: $\mathcal{L}_v \theta = \mathcal{L}_v(\theta_a dx^a) = (\mathcal{L}_v \theta_a) dx^a + \theta_a \mathcal{L}_v(dx^a) = (\mathcal{L}_v \theta_a) dx^a + \theta_a d(\mathcal{L}_v x^a) = v^b(\partial_b \theta_a)\ {}^a + \theta_a d(v^b \partial_b x^a)$

use Leibniz rule $\quad$ $\mathcal{L}_v$ commutes with d $\quad$ $\mathcal{L}_v$ on fns is just $\frac{\partial}{\partial x}$

Cartan's Magic Formula: $\boxed{\mathcal{L}_v \theta = d(i_v \theta) + i_v d\theta = d\langle \theta, v \rangle + \langle d\theta, v \rangle}$

in components, use product rule: $\mathcal{L}\theta = \dot{\theta}_a = \partial_a(\theta_b v^b) + v^b(\partial_b \theta_a - \partial_a \theta_b) = v^b \partial_b \theta_a + \theta_b \partial_a v^b = \dot{\theta}_a\ (dx^a)$

Killing vectors: ξ is a K.V.F if $\mathcal{L}_\xi g = 0 \to \mathcal{L}_\xi g = \left.\frac{\partial}{\partial t}\right|_{t=0} \underbrace{\phi^* g}_{=g \text{ if } \phi^* g \text{ is isometry}} = 0$ $\to$ ie a group of diffeomorphisms ϕ, generated by the K.V.F ξ, ϕ is then the isometry group of the metric which leads to $\mathcal{L}_v g = 0$ (g constant tensor).

$\mathcal{L}_v$ of metric is derived: $\mathcal{L}_v g = \mathcal{L}_v(g_{ab} dx^a \otimes dx^b) = \mathcal{L}_v(g_{ab}) dx^a \otimes dx^b + g_{ab}(d(\mathcal{L}_v x^a) \otimes dx^b + dx^a \otimes d(\mathcal{L}_v x^b))$

Lie groups: Lie group G is manifold with additional operation multiplication : $\mu(g, g') \in G$ (satisfy group axioms) $\to \exists$ inverse $\mu(g, g^{-1}) = e$. (mult + inverse must be smooth).

All LG can be realized as matrix subgroups of $M(n, \mathbb{R})$. $GL(n,\mathbb{R})$ - subgroup of invertible matrices ie $\boxed{m \in GL(n,\mathbb{R}) : \det m \neq 0.}$

Classical LG:

$$\boxed{\underset{\det m \neq 0}{GL(n,\mathbb{R})} \supset \underset{\det m = 1}{SL(n,\mathbb{R})} \supset \underset{mm^T = \mathbb{1}}{O(n,\mathbb{R})} \supset \underset{\det m = 1}{SO(n,\mathbb{R})} \ / \ \underset{\text{(Minkowski norm)}}{O(n,s,\mathbb{R}) \supset SO(n,s,\mathbb{R})}}$$

$\to$ isometry group of pseudo lorentz $\mathbb{R}^{n,s}$. $SO(n,s)$

Orthogonal group preserves quadratic form : $x^T x = (x^1)^2 + \dots (x^n)^2 \Rightarrow$ pseudo-orth. preserves $x^T \eta x$. $(x^1)^2 \dots + (x^n)^2 - \dots (x^s)^2$ ie pseudo-ortho quadratic form with $\eta = (\underbrace{+1 \dots, +1}_{n}, \underbrace{-1 \dots, -1}_{s})$

Complex LG:

$$\boxed{GL(n,\mathbb{C}) \supset \underset{\text{"}SL(2n,\mathbb{R})\text{"}}{SL(n,\mathbb{C})} \supset \underset{m^\dagger m = \mathbb{1}}{U(n)} \supset \underset{\det m = 1.}{SU(n)}}$$

$\to$ can also have $U(n,s)$ preserving $x^\dagger \eta x = |x^1|^2 \underset{n}{\dots} + |x^n|^2 \underset{s}{-} |x^s|^2$

Group Actions: Left action of G on M is a map $\lambda : G \times M \longrightarrow M$ that satisfies $\lambda(\mu(g,h), x) = \lambda(g, \lambda(h,x))$

$gh = \mu(g,h)$ $\quad$ $(\lambda : g \longmapsto \lambda_g)$

equivalent: $\lambda_g : M \longrightarrow M$. s.t $\lambda_{gh} = \lambda_g \circ \lambda_h$: Left action λ is map from G to λ_g . group of diffeomorphisms on M : $\lambda_g : M \longrightarrow M$

Effective: Every nontrivial λ_g acts nontrivially $\Rightarrow$ ie can only have $\lambda_g = \mathbb{1}$ if $g = e$ ($g = e$ trivial element)

Free: No fixed points : ie if $\lambda_g(x) = x$ must have $g = e$

Transitive: Any two points can be connected by $\lambda_g \Rightarrow$ (action has single orbit in M) ie $\forall x, y \in M : \exists g \in G . \lambda_g(x) = y$

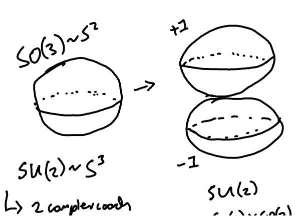
Group homomorphism: $\psi : G \longrightarrow G'$: compatible with product $\psi(gh) = \psi(g)\psi(h)$.

$\boxed{\text{(Left) coset} : G/H = \{g \in G : gh \sim g\} \text{ ie those } g \in G \text{ s.t } gh \sim g \ \forall h \in H}$

Normal subgroup: $N \subset G$: $\forall g \in G$. $g N g^{-1} \subset N$.

Theorem: if H normal subgroup of G, then coset is a group.

ie for $h, h' \in G/H$: $gh \circ g'h' = g \circ g'$ since $gh \sim g$ etc.

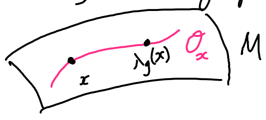


eg $SU(2)/\mathbb{Z}_2 = SO(3)$

spinor rotation $\sim$ vector rot modulo $\mathbb{Z}_2$.

Lie Algebras The orbit of a group action $\mathcal{O}_x : \{y \in M : \lambda_g(x) = y\}$ ie the points y that can be reached from x by the left group action.

Stabiliser of x: $H_x : \{h \in H_x : \lambda_h(x) = x\}$ ie group elements s.t. x is stable point (note group action then cannot be free).



Theorem. Let $H_x \subset G$ be the stabiliser of a point x, with orbit $\mathcal{O}_x$. Then $G/H_x \cong \mathcal{O}_x$ (canonically isomorphic)
$\to$ since H_x normal subgroups of G: G/H_x is a group: $\mathcal{O}_x$ is then a subgroup of $G \to$ the equivalence class of the stabiliser group.

Orbits are therefore manifolds sitting inside the lie group.
$\to$ Symmetric manifolds are orbits inside Lie groups. (coset spaces) eg $S^2 = SO(3)/SO(2)$, $\mathbb{H}^2 = SL(2,\mathbb{R})/SO(2)$

Kernel: For a group homomorphism: $\psi : G \to G'$: $\ker_\psi = \mathrm{preim}_\psi(e')$ (e': G' identity). $\ker_\psi = \{g \in G : \psi(g) = e'\}$ ie preimage of identity on G'

Theorem: $\ker_\psi$ is normal subgroup of G : Target group $G' \cong G/\ker_\psi$. Ie G' is coset space of group elements equivalent under $\ker_\psi$: ψ is isomorphism.
$\Rightarrow$ eg: Consider $\psi : SU(2) \to SO(3)$. $\ker_\psi = \{\mathbb{1}, -\mathbb{1}\} = \mathbb{Z}_2$. Hence $SO(3) \simeq SU(2)/\mathbb{Z}_2$. ψ gives isomorphism $so(3) \sim su(2)$

Kernel of $SU(2) \to$ ie elements in $SU(2)$ that map to $\mathbb{1}$ in $SO(3)$

Lie Algebras: $\mathcal{L}(G)$ Lie algebra of G is a vector space equipped with bilinear map (Lie bracket) $\mathcal{L} \times \mathcal{L} \to \mathcal{L} : (A,B) \mapsto [A,B]$. $\to$ Lie bracket usually matrix commutator (matrix lie algebra).

- Lie bracket must be antisymmetric $[A,B] = -[B,A]$ • + satisfy Jacobi $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$
  - Left invariant V.F on manifold form lie algebra (of corresponding Lie group).
  - $M(n,\mathbb{R})$ has lie algebra given by matrix commutator $\to$ all algebras can be realised as matrix algebras.

Homomorphisms: Lie Homo must preserve $[\,,\,]$ ie $\boxed{\phi([A,B]) = [\phi(A), \phi(B)]}$ for $\phi : \mathcal{L} \to \mathcal{L}'$

Def: Left invariant vector field : vector invariant under pushforward ie $\boxed{\lambda_{g*} V = V}$ $\to$ Lie algebra of G is space of all left invariant V.F equipped with Lie bracket ($[\,,\,]$ is also LIVF)

3 equivalent def: ① $\mathcal{L}$ is space of L.I.V.F on G: Space of L.I.V.F is vector space + has lie bracket

② $\mathcal{L}(G)$ is tangent space @ identity : $\mathcal{L} = T_eG$: Any vector field on G can be restricted to T_eG and vice versa : Calculate $[\,,\,]$ and then restrict/extend resulting V.F

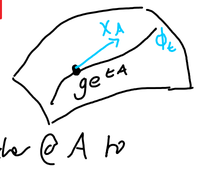
③ $\mathcal{L}(G)$ is vectors generated by OPSG (1 p subgroup) $g_t : G \to G$. : LIVF are then velocity vectors to this OPSD

NOTE: in ③ – have one parameter subgroup $g : \mathbb{R} \times G \to G$ of group homomorphisms. The left action λ_{g_t} then generates OPSG of diffeomorphisms
$\hookrightarrow$ of G $\hookrightarrow$ LIVF as velocity.

• When a group G acts on a manifold the orbits of the group action correspond to the integral curve that is generated by the 1p subgroup. The LIVF represent infinitesimal transformations $\to$ where the Lie group acts on M by symmetries.

Matrix groups: Compare these defns for matrix groups discussed. Proposition. All 1PSG of $GL(n,\mathbb{R})$ are of form $g_t = e^{tA}$ (matrix exponent) $A \in M_n(\mathbb{R})$

Proof: Let ϕ_t be 1PSD : Let $A = \frac{d}{dt}\phi_t\big|_{t=0}$ now use $\frac{d}{dt} g_t\big|_{t=0} = \frac{d}{ds} g_{t+s}\big|_{t=0} = \underbrace{\left(\frac{d}{ds} g_s\right)}_{A} g_t\big|_{t=0} \to \dot{g}_t\big|_{t=0} = A g_t\big|_{t=0}$ soln $\boxed{g_t = e^{tA}}$



Compare defns. ③ $\leftrightarrow$ ①: Check velocity vector of ge^{tA} is also LIVF. – use left action of $\phi_t = e^{tA}$ to produce VF on G: ge^{tA} translation of vector @ A to VF using left action of G on G.

velocity $X_A f = \frac{d}{dt}\Big|_{t=0} \phi^* f = \frac{d}{dt}\Big|_{t=0} f(\underbrace{ge^{tA}}_{\tilde{g}_{ij}})$ write $ge^{tA_{ij}} = \tilde{g}_{ij} = g_{ij}$ (@ $t=0$) $\Rightarrow \frac{d}{dt}\Big|_{t=0} ge^{tA_i} = (gA)_{ij}$

so have $X_A f = \left(\frac{\partial f}{\partial \tilde{g}_{ij}} \frac{\partial \tilde{g}_{ij}}{\partial t}\right)\Big|_{t=0} \Rightarrow (gA)_{ij} \frac{\partial}{\partial g_{ij}} f = X_A f \Rightarrow \boxed{X_A = (gA)_{ij} \frac{\partial}{\partial g_{ij}}}$ $\to$ claim this is LIVF. $\to$ also satisfies $[\,,\,]$ reqs. $\Rightarrow$ $\boxed{\text{Show LI by pushforward: let } h \in G \quad g' = hg : \lambda_{g'*} X_A \stackrel{?}{=} X_A}$

• pushforward of coordinates: $\frac{\partial}{\partial g_{ij}} = \frac{\partial g'_{ke}}{\partial g_{ij}} \frac{\partial}{\partial g'_{ke}} = \underbrace{\frac{\partial (h_{km} g_{me})}{\partial g_{ij}}}_{h_{ki}\delta_{ej}} \frac{\partial}{\partial g'_{ke}} = h_{ki} \frac{\partial}{\partial g'_{kj}}$

so $\lambda_{g'*} X_A = \underbrace{(gA)_{ij} h_{ki}}_{g'_{kj}} \frac{\partial}{\partial g'_{kj}} = \underbrace{(hgA)_{kj}}_{g'A} \frac{\partial}{\partial g'_{kj}} \simeq (gA)_{ij} \frac{\partial}{\partial g_{ij}}$ ie invariant under λ_{g*} pushforward $\to$ LIVF.

• Can easily compute $\boxed{[X_A, X_B] = X_{[A,B]}}$ ie the lie bracket of LIVF reduces to a lie bracket on the corresponding matrix lie algebra elements $A, B \in M_n(\mathbb{R})$.
$\hookrightarrow$ have a homomorphism from the algebra of $GL(n,\mathbb{R})$ (the LIVF X_A) to the algebra $\mathcal{L}(M_n(\mathbb{R}))$ the matrix lie algebra.

• **Essentially:** the LIVF restricted to T_eG correspond to matrices $A_{ij} \in M(n,\mathbb{R})$, by the fact that all 1PSG are of form $\phi_t = e^{tA_{ij}}$: The LIVF are corresponding VF to this ϕ_t.

Stokes Theorem: $\boxed{\int_M dw = \int_{\partial M} w}$ • Integral of $w \in \Lambda^{p-1}(M)$ ($dw \in \Lambda^p(M)$) $(p-1)$ form on compact region of M.

$\to$ integral of dw given by integral of w on ∂M: boundary of region M.

• $\dim M = p$. $\dim(\partial M) = p-1$; $w : (p-1)$ form

Orientation: If N has orientation $(x^1, \ldots, x^p)$; swapping any two coords gives (-1)
∂N has orientation $(-1)^p (x^2, \ldots, x^{p-1})$ ie $\mathrm{sgn}(\sigma)$ perm