Gauge theories

Rory Whybrow

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1 Gauge theories

In this section I will briefly review the purpose and construction of gauge theories, and formalise the required machinery to be used throughout.

The term 'gauge theory' refers to field theories formulated in terms of gauge fields, that transform under a particular type of transformation known as a gauge transformation; nonsense for now but all to be defined shortly. The terminology is born out of the principle of gauge invariance, stating that physical observables should not depend on the measuring system (gauge) used. Arbitrary gauge transformations (changes of measurement system) should leave any physical theory invariant. Therefore, such gauge transformations correspond to a symmetry of the theory, a gauge symmetry.

Gauge theories are formulated using an (associated vector bundle) to a principal G-bundle, over a (in most instances) Lorentzian manifold M. The gauge 'field' is realised as a principal connection on such a bundle, with the restriction of the connection to the base manifold playing the role of the 'physical' gauge field (Yang-Mills field). The transformations of such a connection under changes of local trivialisation (gauge transformations) will be central to our discussion.

1.1 Principal bundles

A principal G-bundle is the bundle (P, M, π)[1], with the additional structure of a smooth right action on P by the so called 'structure' Lie group, G. The group action is required to be free and transitive within the fibres, hence each fibre is the orbit of G acting on the base point. Therefore, each fibre can be identified as a copy of the group G, although it should be noted that whilst the fibre is diffeomorphic to the group; it does not carry a natural group structure [1]. The local trivialisations, $\psi_U : P \longrightarrow U \times G$, are required to be chosen such that they commute with the group action. Lastly, the local trivialisations can be identified as being equivalent to a local section, defined as $(x \in U \subset M)$

$$s_U := \psi_U^{-1} : U \times G \longrightarrow P \qquad s_U(x) = \psi_U^{-1}(x, e). \tag{1}$$

Such local sections/trivialisations should be interpreted as defining a local 'gauge' (measurement system) within each fibre. This is analogous to defining charts on a manifold; with a local trivialisation providing a 'coordinate' representation of the points in the fibre. For example, let $G = \mathbb{R}$, then a local trivialisation identifies the points in the fibre with $\mathbb{R}$. Therefore, a choice of trivialisation corresponds to choosing some (arbitrary) value for each (abstract) point in the fibre. Hence we should expect that changing the trivialisation should give rise to a corresponding 'gauge transformation'. The freedom in choosing an arbitrary trivialisation is therefore the realisation of 'gauge invariance'. Such a notion should be compared with the freedom to choose arbitrary charts on a manifold, related by their transition functions.

[1] Equivalently expressed as $P \xrightarrow{\pi} M$

Let (P, M, π) be a principal G-bundle, and $\rho : G \longrightarrow \mathrm{Aut}(V)$ be a representation of the structure group G (note ρ is a group homomorphism). The total space of the *associated vector bundle* is defined as

$$P \times_\rho V := (P \times V)/G, \tag{2}$$

which is a vector bundle with a right action on V by G, given by

$$(p, v) \lhd g \mapsto (p \lhd g, \rho(g^{-1}) \rhd v) \sim (p, v), \tag{3}$$

where g^{-1} was used to form a left action from the G right action, and ρ used such that the group acts in the appropriate representation of V. The equivalence relation under G is sometimes expressed with the notation $[p, v]$.

1.2 Principal Connections

An (Ehresmann) connection is an assignment such that for every point $p \in P$, the tangent space decomposes into the direct sum of vertical and horizontal parts

$$T_p P = H_p \oplus V_p. \tag{4}$$

where the vertical space is defined as the kernel of the pushforward of the projection, i.e all the vectors tangent to the fibre. Concretely, the geometric notion of a connection ω is then clear, simply that it maps vectors in $T_p P$ to their vertical part,

$$\omega : T_p P \longrightarrow V_p \qquad \Longrightarrow \; H_p = \ker(\omega). \tag{5}$$

In a principal bundle, the vertical space at each point in the fibre G can be identified with the Lie algebra (space of left invariant vector fields), such that the connection maps any vector into the Lie algebra (hence the terminology 'Lie algebra valued 1-form').

The connection must be *equivariant* under the right action of G on the fibres, stated by requiring that the horizontal subspace at p be should be pushed forward to the corresponding horizontal subspace at $p \lhd g$ by such a right action

$$(\lhd g)_* H_p = H_{p \lhd g}. \tag{6}$$

Recall the adjoint representation of a Lie group, defined as

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g}), \tag{7}$$

Since any vector in $X_p \in T_p P$ is mapped to $\mathfrak{g}$ by the connection, under the right action the resulting vector $\omega_p(X_p)$ should transform using the appropriate automorphism of $\mathfrak{g}$; hence through the adjoint representation. We find that the following diagram commutes

$$\begin{tikzcd} T_p P \arrow[r, "\omega_p"] \arrow[d, "\lhd g"'] & \mathfrak{g} \arrow[d, "\mathrm{Ad}_{g^{-1}}"] \\ T_{pg} P \arrow[r, "\omega_{pg}"] & \mathfrak{g} \end{tikzcd} \tag{8}$$

equivalently stated using the definition

$$((\lhd g)^* \omega)(X) = (\mathrm{Ad}_{g^{-1}})_*(\omega(X)) \tag{9}$$

At first glance this definition is rather terse, however, it is simply the requirement that the connection is equivariant under the right action, namely that the above diagram commutes. The significance of requiring g^{-1} can be seen from the fact that $\mathrm{Ad}_{g^{-1}}$ acts from the left for a right g action, (equally, using g would produce inconsistent results).

In addition to the equivariance condition, the connection must also satisfy the following. Given any $A \in \mathfrak{g}$ define the 'fundamental vector field', X^A, as the velocity vector field generated by the one parameter subgroup of diffeomorphisms obtained from A (using the exponential map). Any fundamental vector field when inserted into the connection must return the Lie algebra element that generated it, i.e

$$\omega(X^A) = A. \tag{10}$$

1.3 Local description of the connection

Given a connection in a principal bundle P, consider its restriction to the base manifold using a local section. The *local* data described by this restricted connection on the (physical) base manifold are what physicists would refer to as the 'Yang-Mills' or 'gauge' field. Let, $\sigma_U : U \longrightarrow P$ be a local section over some neighbourhood $U \subset M$ of the base, the Lie algebra valued 1-form $A \in \Omega^1(U, \mathfrak{g})$ is defined as[2]

$$A = \sigma_U^* \omega : T_m U \longrightarrow \mathfrak{g}\,. \tag{11}$$

Furthermore, given such a local section we can identify the corresponding local trivialisation,

$$h_U = \psi_U^{-1} : U \times G \longrightarrow P, \tag{12}$$

such that the connection appears in the trivialisation as

$$\mathcal{A} = h_U^* \omega : T_m U \times T_g G \longrightarrow \mathfrak{g}. \tag{13}$$

The full construction can be summarised with the following diagram, where the coloured arrows represent how the globally defined connection 1-form on P is pulled back respectively,

$$\begin{tikzcd}
U \times G \arrow[r, "h_U"] & P \\
U \times G \arrow[u, "\lhd g"] \arrow[r, "h_U"] & P \arrow[u, "\lhd g"'] \arrow[l, bend left, red, "h_U^*\omega"] \arrow[d, bend left, blue, "\sigma_U^*\omega"] \\
U \arrow[u, "\tilde{\sigma}_U"] \arrow[r, "id_U"] & U \arrow[u, "\sigma_U"']
\end{tikzcd} \tag{14}$$

where the map $\sigma_U = h_U \circ \tilde{\sigma}_U : U \longrightarrow U \times G$ is the projection from the base into the trivialisation. Of interest will be how the connection transforms within the fibres in a local representation, namely by examining how $\mathcal{A}$ transforms under the right action in the trivialisation.

Consider a point $m \in U$ and the corresponding trivialisation. Let $(m, g) \in U \times G$ be the point obtained by the right action of $g \in G$ on itself (i.e moving within the fibre starting at the identity of G). The connection in the trivialisation must obey the following requirement (stated very obtusely for now!), that given any tangent vector $X_m \in T_m U$ and left invariant vector field $\xi^a_g \in T_g G$

$$(\mathcal{A})_{(m,g)}(X_m, \xi^a) = \mathrm{Ad}_{g^{-1}*}(A(X_m)) + \Xi_g(\xi^a_g), \tag{15}$$

explained after elaboration of the 'Maurer-Cartan' form; defined as the map

$$\Xi_g : T_g G \longrightarrow T_e G, \tag{16}$$

[2] the subscript σ is a label not an index! It may be treated similarly to an index when considering multiple connections defined over sections σ_α, indexed by $\alpha = 1, 2, ..., n$.

that maps the vector ξ^a_g at the point g into its corresponding Lie algebra element. The Maurer-Cartan (MC) form can therefore be identified as the connection of the Lie group G, since the Lie algebra is the vertical subspace of G. The MC form is a fixed property of the group, and (for matrix groups) can be identified as the following (Lie algebra valued) 1-form

$$\Xi_g = g^{-1}\mathrm{d}g, \tag{17}$$

Eq. 15 still requires further explanation, repeated here for reference

$$(\mathcal{A})_{(m,g)}(X_m, \xi^a) = \mathrm{Ad}_{g^{-1}*}(A(X_m)) + \Xi_g(\xi^a_g), \tag{15}$$

We are considering how the connection transforms (equivariantly) under the right action of G, in a given trivialisation. This is calculated by how the connection (in the trivialisation) acts on an arbitrary tangent vector $X_m \in T_mU$, after the right action moves it through the fibre in G, i.e $(\mathcal{A})_{(m,g)}$. This is obtained by first mapping the vector $X_m \in T_mM$ into the $\mathfrak{g}$ using the Yang-Mills field A (on U only), before using the appropriate automorphism of $\mathfrak{g}$ to map the resulting vector $A(X_m)$ within the Lie algebra. The first term should therefore be read as the pushforward of the Lie algebra element obtained from the *gauge field* A acting at the point m, to the required tangent space in the trivialisation.

This is not sufficient however, as we also require knowledge of how the (Lie algebra valued) part of the connection acts on the left invariant vector field at g. This is obtained through the MC form, which pulls the vector back to its generating Lie algebra element. The result of both terms in Eq. 15 hence lie in the Lie algebra and can be added using the appropriate addition on $\mathfrak{g}$. The significance of requiring the MC form should be understood as the restriction of the connection on the group itself, namely it is the connection that arises when considering the action of G on itself, (the trivial bundle $G \times G$). Therefore any ω must coincide with the MC form when restricted to the fibre.

Concretely therefore, given a suitable matrix representation, the connection 1-form in a local trivialisation can be expressed as

$$\mathcal{A} = g^{-1}Ag + g^{-1}\mathrm{d}g \tag{18}$$

with A being the Lie algebra valued 1-form defined over the base only (gauge field).

1.4 Gauge Transformations

Since our choice of trivialisation can be arbitrarily chosen, we should expect the local representation of the connection to transform appropriately under changes of trivialisation. Recall, a local trivialisation could be viewed as assigning a local 'gauge' or measurement system within each fibre, hence, the arbitrary nature of the trivialisation is the origin of 'gauge invariance'; with changes in the choice of local trivialisation generating the corresponding 'gauge transformations'. Only with the structure of a principal G-bundle understood does it become clear why this gauge invariance is present; namely in the freedom to choose a local trivialisation. So, the rather ad-hoc transformation behaviour of the gauge fields that we observe locally, simply arises due to their restriction with the corresponding trivialisation.

To realise this, let σ_α be the local section over a neighbourhood U_α in the base, $\sigma_\alpha : U_\alpha \longrightarrow P$ ($\alpha = 1, ..., n$) with non-trivial overlaps $U_\alpha \bigcap U_\beta \neq 0$. Define the 'gauge map' (transition function), between the respective sections as the (unique) function $\mathcal{G}_{\alpha\beta} : U_\alpha \bigcap U_\beta \longrightarrow G$ such that $\forall m \in M$,

$$\sigma_\beta(m) = \sigma_\alpha(m) \lhd \mathcal{G}_{\alpha\beta}(m), \tag{19}$$

The gauge map should be viewed as the analogue of a 'transition function' between charts, where here the 'charts' are the locally defined sections (viewed as defining a local 'gauge'). Since the gauge field on the base manifold is obtained by pulling back with respect to the appropriate section, the gauge field will transform accordingly on the base under such a gauge map between sections. The principal connection will simply be pulled back by the gauge map, however the gauge fields on the base will acquire non trivial transformation behaviour as we now observe. The gauge fields are related by the 'gauge transformation'

$$A_\beta = \mathrm{Ad}_{\mathcal{G}_{\alpha\beta}{}^{-1}{}_*}(A_\alpha) + \mathcal{G}_{\alpha\beta}{}^* \Xi_m \tag{20}$$

summarised by the following diagram

$$\begin{tikzcd}
 & U_\alpha \times G \arrow[r, "\mathcal{G}_{\alpha\beta}"] \arrow[d, "h_\alpha"] & U_\beta \times G \arrow[d, "h_\beta"] & \\
 & P \arrow[r, "\mathcal{G}_{\alpha\beta}"] & P & \\
U_\alpha \arrow[ur, "\sigma_\alpha"] \arrow[r, "A_\alpha"] & \mathfrak{g} \arrow[r, "\mathrm{Ad}_{\mathcal{G}^{-1}}"] & \mathfrak{g} & U_\beta \arrow[l, "A_\beta"] \arrow[ul, "\sigma_\beta"']
\end{tikzcd} \tag{21}$$

Concretely, given an appropriate matrix representation of the Lie algebra, this can then expressed as

$$A_\beta = \mathcal{G}^{-1} A_\alpha \mathcal{G} + \mathcal{G}^{-1} \mathrm{d}\mathcal{G} \tag{22}$$

The gauge maps should be regarded as automorphisms of the principal bundle, where an automorphism is a diffeomorphism f such that, given any point in the principal bundle $p \in P$ and arbitrary group element $g \in G$

$$f : P \longrightarrow P \qquad f(p \triangleleft g) = f(p) \triangleleft g, \tag{23}$$

this maps the points in one fibre to the same fibre. This implies there is a unique diffeomorphism of the base $\bar{f} : M \longrightarrow M$ such that the following commutes

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\pi"] & P \arrow[d, "\pi"] \\
M \arrow[r, "\bar{f}"] & M
\end{tikzcd} \tag{24}$$

Since the gauge map acts entirely within the fibre (mapping points in one section to another) it can be realised as the special class of automorphisms, gauge transformations, that act as the identity on the base manifold, i.e only within the fibre such that $\bar{f} = \mathrm{id}_M$. The subset of gauge transformations is a group $\mathcal{G}(P)$.

Consider the associated bundle $P \times_{\alpha_g} G$, where α_g denotes the action of G on itself by conjugation, namely $\alpha_g h = g \triangleright h \triangleleft g^{-1}$. Given a gauge transformation $f \in \mathcal{G}$ then for any point $p \in P$ we can show that there is a unique group element $g(p) \in G$ such that $f(p) = p \triangleleft g(p)$. This leads to an isomorphism of groups (not shown how) that identifies smooth sections of the associated bundle (where G acts on itself by conjugation) with the gauge transformations of P

$$\{\mathcal{C}^\infty\text{-sections of } P \times_\alpha G\} \cong \mathcal{G}(P) \tag{25}$$

1.5 Covariant derivatives

Let $P \xrightarrow{\pi} M$ be a principal G-bundle, with connection 1-form ω, and let $E = (P \times V)/G$ be the associated vector bundle of a suitable representation $\rho : G \longrightarrow \mathrm{Aut}(V)$. Let $X_p \in T_pP$ be tangent to a curve $p(t) \subset P$, such that $p(0) = p$, $\dot{p}(0) = X_p$, and let $c(t) = \pi(p(t))$ be the projection of the curve $p(t)$ into the base.

The covariant derivative on the associated vector bundle is constructed as follows, although its geometric significance will not be clear until we discuss parallel transport in the next section. Given any pair $(p(m_0), v(m_0)) \in E$, over a base point m (recall these are elements of the *equivalence class* defining the bundle),

$$[p(m), v(m)] = [p(m) \lhd g(m), \rho(g^{-1}(m)) \rhd v(m)] \tag{26}$$

we can then construct the covariant derivative of a vector as it varies from point to point in the base. For any vector $X_{m_0} \in T_{m_0}M$, the covariant derivative of v, along the direction specified by X (as a velocity to some curve), at an arbitrary point $m \in M$ is then defined as

$$\nabla_X[p(m), v(m)] = [p(m_0), \partial_X v + \rho_*(p^*\omega(X_{m_0})) \rhd v(m_0)] \tag{27}$$

where $\rho_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is the Lie algebra representation and $\partial_X v$ denotes the vector obtained by inserting the components of v into X. Whilst the equivalence appears to complicate this definition somewhat, it can be understood simply as follows. Ignoring the equivalence for now, since this simply ensures the resulting pair remains G equivariant, consider how the vector part is affected. Under covariant differentiation, the vector becomes

$$\partial_X v + \rho_*(p^*\omega(X_{m_0})) \rhd v(m_0) \tag{28}$$

where the first term should simply be regarded as the regular derivative of the vector components along X (as a velocity specified direction). The second term can be understood by re-writing it as

$$p^*\omega(X_{m_0}) = \omega(p_*X_{m_0}) \tag{29}$$

namely that the connection acts after the vector X_{m_0} has been pushed into the total space using the curve p. The result of mapping with the connection (a Lie algebra element) is then expressed using the appropriate representation of the Lie algebra (as an endomorphism in V).

It should be noted that whilst the covariant derivative appears to depend on the chosen path p, it is in fact independent; due to the G equivariance in the total space. Hence ∇_X is determined entirely by the vector X, specifying the 'direction' in which we wish to take the derivative. The coordinate invariance ('covariance') of ∇_X is then guaranteed, given that it only depends on the vector X; a coordinate independent quantity. In the majority of cases, we consider computing the covariant derivative with respect to some particular set of coordinates (by letting $X =$ the basis vectors in question) and we are guaranteed the result should transform correctly (as a tensor) under changing coordinates due to constructing it in this way. Finally, we should note that in the case where the connection is trivial (vanishing local connection coefficients) the covariant derivative reduces to the ordinary (directional) derivative.

1.6 Parallel transport

Let (P, M, π) be a bundle, with a connection 1-form ω. For a curve within the base manifold,

$$\gamma : I \longrightarrow M : \quad \gamma(t) \subset M \qquad I = [t_0, t_1], \tag{30}$$

the *lift* $\tilde{\gamma}$, is *any* curve in the total space that projects onto the curve γ in the base. If the bundle has a connection this leads to the notion of a *horizontal* lift, the *unique* lift in which the tangent vector to the lifted curve remains horizontal along the curve. The horizontal lift is uniquely determined by the condition $p = \tilde{\gamma}(t_0)$, namely the starting point of the lift in the fibre. The uniqueness of such a lift can be realised in the following way. Define the lift using a local section, σ_α a function $\ell_\alpha : I \longrightarrow G$ as

$$\tilde{\gamma}(t) = \sigma_\alpha(\gamma(t)) \lhd h_\alpha(t), \tag{31}$$

where $h_\alpha(t_0)$ is defined by the condition $p = \tilde{\gamma}(t_0)$. The function can be obtained from the ODE

$$\dot{h}_\alpha\left(t_1\right) = -(\lhd h_\alpha\left(t_1\right))_* \left(\omega\left(\dot{s}_\alpha\left(t_1\right)\right)\right) \tag{32}$$

which for matrix groups takes the form

$$\dot{h}_\alpha = -\omega\left(\dot{s}_\alpha\right) h_\alpha \tag{33}$$

The horizontal lift then defines a notion of 'parallel transport', relating the points in one fibre to another, by requiring they both lie on the same horizontal lift to a curve connecting the base points of the respective fibres. Let $\gamma(t_0)$ and $\gamma(t_1)$ be the start and end points of a curve in the base manifold, and let $P\big|_{\gamma(t)}$ denote the fibres over the respective start and end points. The parallel transport map, $\Gamma_\gamma(p)$, is defined such that given any point in the fibre over the starting point of the curve $p \in P_{\gamma(t_0)}$,

$$\Gamma_\gamma : P\big|_{\gamma(t_0)} \longrightarrow P\big|_{\gamma(t_0)} : \quad \Gamma_\gamma(p) = \tilde{\gamma}(t_1), \tag{34}$$

simply that each point in the starting fibre is mapped to the (unique) endpoint of the corresponding horizontal lift ending at the second fibre. Therefore, the parallel transport map should be viewed as 'connecting' the points in each fibre, using the horizontal lifts defined between them.

The notion of parallel transport can be extended to vectors in an associated vector bundle, and gives a clear geometrical interpretation of the covariant derivative. Given any fixed vector $v \in V$, and a horizontal lift $\tilde{\gamma}$, define a map

$$I \longrightarrow \mathcal{V} : \quad t \mapsto [\tilde{\gamma}, v] \tag{35}$$

which assigns the vector v into the fibres at every point in the lift. This has a very clear geometric interpretation, namely that the vector should remain constant in the total space under parallel transport. This is achieved by moving the vector along a horizontal lift, which we can check by covariant differentiation with respect to the parameter t (i.e with respect to ∂_t) will leave the vector invariant as follows, using the definition in Eq. 27

$$\nabla_t[\tilde{\gamma}(t), v] = [\tilde{\gamma}(t), \partial_t v + \rho_*(\omega(\tilde{\gamma}(t)_* \partial_t)] = [\tilde{\gamma}(t), \partial_t v + \rho_*(\omega(\partial_t \tilde{\gamma}(t))] = [\tilde{\gamma}(t), 0] \tag{36}$$

this vanishes, since the vector v was fixed $\partial_t v = 0$, and given that the velocity of the horizontal lift is horizontal it will vanish when inserted into ω. Therefore, the vector v is constant along a horizontal lift in the total space, expressed by the vanishing covariant derivative with respect to the parameter t. Note that since the covariant derivative is coordinate independent, it must vanish under any arbitrary re-parametrisation of the underlying curve $\gamma(t)$.

Furthermore, we can now understand geometrically the action of the covariant derivative on a vector as the following. Given any tangent vector in the base manifold, i.e specifying a curve by its velocity, construct the appropriate lifts of the curve and vector (by pushing forward with π^{-1}).

1.7 Curvature

Given a principal bundle with connection 1-form ω, define the Lie algebra valued curvature 2-form as

$$\Omega = \mathrm{d}^\omega \omega = \mathrm{d}\omega + \frac{1}{2}[[\omega, \omega]] \tag{37}$$

where the double bracket indicates $[[\omega,\eta]](X,Y) := [\omega(X),\eta(Y)] - [\omega(Y),\eta(X)]$ and we have also introduced the *exterior covariant derivative.* This is defined as

$$\mathrm{d}^{\omega} = \mathrm{Hor}^{\omega} \circ \mathrm{d} : \Omega^{k}(P,\mathfrak{g}) \longrightarrow \Omega^{k+1}_{\mathrm{hor}}(P,\mathfrak{g}) \tag{38}$$

where the connection defines the projection

$$\mathrm{Hor}^{\omega} : \Omega^{k}(P,\mathfrak{g}) \longrightarrow \Omega^{k}_{\mathrm{hor}}(P,\mathfrak{g}) \tag{39}$$

of Lie algebra valued $k-$forms onto their horizontal part, defined such that any vertical vector inserted into the form returns zero. Hence, the exterior covariant derivative is simply the horizontal part of the exterior derivative. [2] This exterior covariant derivative then defines the covariant derivative on the associated vector bundle, given a representation $\rho : G \longrightarrow \mathrm{Aut}(V)$, one finds that C^{∞} sections of such an associated bundle are isomorphic to G equivariant functions on the principal bundle,

$$\Omega^{0}(P,V) \simeq \Gamma^{\infty}(P \times_{\rho} V). \tag{40}$$

In general, one obtains a similar isomorphism for q-forms,

$$\Omega^{q}_{\mathrm{hor}}(P,V) \simeq \Omega^{q}(P \times_{\rho} V, M). \tag{41}$$

These will come into play when when constructing covariant derivatives as follows, namely that the following diagram commutes, with the vertical arrows indicating isomorphism

$$\begin{array}{ccc} \Gamma^{\infty}(P \times_{\rho} V) & \xrightarrow{\nabla^{\omega}} & \Omega^{1}(P \times_{\rho} V) \\ \simeq\Big\uparrow & & \simeq\Big\uparrow \\ \Omega^{0}(P,V) & \xrightarrow{\mathrm{d}^{\omega}} & \Omega^{q}_{\mathrm{hor}}(P,V) \end{array} \tag{42}$$

hence the covariant derivative is obtained from the exterior differential in this way

The curvature 2-form must also be equivariant under the right action (note it is already horizontal by definition), so we obtain the following condition

$$(\lhd g)^{*}\Omega(X,Y) = \mathrm{Ad}_{g^{-1}*}\Omega(X,Y) \tag{43}$$

and equivalently for the gauge maps between the local curvature forms $\bar{\Omega}_{\alpha} = s_{\alpha}^{*}\Omega$,

$$\bar{\Omega}_{\beta}(X,Y) = \mathrm{Ad}_{\mathcal{G}^{-1}_{\alpha\beta}*}\bar{\Omega}_{\alpha}(X,Y) \tag{44}$$

1.8 Gauge Theory

Now that the geometry of associated vector bundles has been understood, the associations with gauge theories such as Yang-Mills theory can be made. Gauge theories, such a the standard model for example, are formulated by realising matter (fermionic, bosonic) fields as sections of an associated vector bundle over spacetime M, where the gauge group is identified with $SU(3) \times SU(2) \times U(1)$; giving rise to the gauge transformations of the standard model.

2 Yang-Mills and Gravity theories

2.1 Electromagnetism

Electromagnetism is the simplest example of a gauge theory, formulated as an abelian gauge theory using a principal $U(1)$-bundle. The electromagnetic potential is realised as a connection in such a bundle, and Maxwell's equations are obtained by studying the curvature and action functional defining such a connection.

Let $P \longrightarrow M$ be a $U(1)-$ principal bundle over a 4d Lorentzian manifold M, and given any connection ω let Ω be its curvature 2-form. Pullback with respect to a local section (independent of section) defines the respective forms on the base, $\mathcal{A} \in \Omega^1(M, \mathfrak{u}(1))$. Recall, the Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$, therefore we can conveniently represent Lie algebra valued forms using $\mathcal{A} = iA$, where $A \in \Omega^1(M, \mathbb{R})$ is a real form on M (similarly $\Omega = iF$). The homogeneous Maxwell equations are obtained from the Bianchi identity, namely that $\mathrm{d}F = 0$, which can be seen from

$$\mathrm{d}F = \mathrm{d}(\mathrm{d}A + \frac{1}{2}[A, A]) = 0 \tag{45}$$

which vanishes since $U(1)$ is abelian. Identifying the components of F with the respective $\vec{E}$ and $\vec{B}$ components in the canonical way

$$F = E_i \, \mathrm{d}x^i \wedge \mathrm{d}t + \epsilon_{ijk} B^i \, \mathrm{d}x^j \wedge \mathrm{d}x^k \tag{46}$$

we obtain the following from $\mathrm{d}F = 0$

$$\mathrm{d}F = \partial_i B^i \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + ... = 0 \quad \Longrightarrow \quad \partial_i B^i = 0, \tag{47}$$

namely Gauss magnetic law, and the other components (which must all independently vanish) give Faraday's law

$$\mathrm{d}F = 0 = ... + \left(\frac{\partial B}{\partial t} + \nabla \times E \right) \tag{48}$$

The remaining inhomogeneous equations are obtained by considering an appropriate action functional for the connection, obtained using the Yang-Mills Lagrangian $\mathcal{L}(\omega)$. A connection is *critical* if the corresponding Yang-Mills functional variation vanishes, determining the criticality condition is equivalent to finding the corresponding equations of motion as we will now observe.

The Yang-Mills Lagrangian is defined as the map (from $\mathcal{C}(P)$, the space of connections over P)

$$\mathcal{L} : \mathcal{C}(P) \longrightarrow \Omega^4(M, \mathfrak{u}(1)) : \quad \mathcal{L}(\omega) = \frac{1}{2} F \wedge \star F + A \wedge J \tag{49}$$

where $J \in \Omega^3(M, \mathbb{R})$ is the current density 3-form,

$$J = \rho \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - \epsilon_{ijk} j^i \, \mathrm{d}t \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \tag{50}$$

after a brief interlude to discuss the Hodge dual we will consider varying this functional.

2.2 The Hodge dual

The Hodge dual $\star$, (dependent on the orientation and choice of metric on the spacetime M) is defined as the unique linear map (where $\Lambda^k V^*$ denotes the k-th exterior power of V^*, namely the k-forms)

$$\star : \bigwedge^k V^* \longrightarrow \bigwedge^{n-k} V^* \tag{51}$$

such that $\forall \omega \in \bigwedge^k V^*$ and $\eta \in \bigwedge^{n-k} V^*$

$$\omega \wedge \eta = (-1)^{k(n-k)} \langle \star\omega, \eta \rangle \cdot \mathrm{vol} \tag{52}$$

where $\langle \cdot, \cdot \rangle$ is the inner product induced by the metric and vol the volume form of the space.

Consider now variations of the YM functional, under an infentesimal variation of the connection. The connection is defined such that on the base manifold $A(\omega_{t,\eta}, \omega_0) = A(\omega, \omega_0) + \eta t$, where ω_0 is an arbitrary reference connection and η a 1-form, $\eta \in \Omega^1(M, \mathbb{R})$, with $\mathrm{supp}(\eta) \Subset M$. The variation then yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \int_M \mathcal{L}(\omega_{t,\eta}) = 0 \qquad \Longrightarrow \qquad \int_M \eta \wedge (\mathrm{d}\star F + J) = 0 \tag{53}$$

hence the equation of motion (corresponding to the critical connection) is identified as

$$\mathrm{d}\star F + J = 0. \tag{54}$$

This can now be computed in suitable coordinates (i.e as defined previously for Minkowski space, but general spacetimes are possible), using the definition of the Hodge dual one obtains

$$\mathrm{d}(\star F) + J = -\partial_i E^i \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \left((\nabla \times B)^i - \frac{\partial E^i}{\partial t} \right) \epsilon_{ijk} \, \mathrm{d}t \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k + J = 0 \tag{55}$$

which with the definition of J as above yields the remaining two Maxwell equations

$$\partial_i E^i = 0 \tag{56}$$

namely Coulomb's law, and Ampere's law

$$\nabla \times B - \frac{\partial E}{\partial t} = j \tag{57}$$

where j is the current 3-vector. Finally, by taking the exterior derivative of the Yang-Mills equation of motion, one obtains the continuity equation

$$\mathrm{d}(\mathrm{d}\star F + J) = \mathrm{d}J = \frac{\partial \rho}{\partial t} + \partial_i J^i = 0 \tag{58}$$

2.3 General Relativity

In the previous discussion of electromagnetism, only variations with respect to the connection ω are considered. However, since the Hodge dual is explicitly constructed using the metric of the spacetime M, the YM Lagrangian should therefore depend on the chosen metric. Consider holding the connection fixed, and varying the metric using the one parameter family of metrics $g(t) : g(0) = g$. The YM Lagrangian can then be expressed as (F denotes the curvature as before)

$$\mathcal{L}_{\mathrm{YM}}(\omega, g) = \frac{1}{2} F \wedge \star F = \frac{1}{2} \langle F, F \rangle_g \cdot \mathrm{vol}_g \tag{59}$$

where $\langle \cdot, \cdot \rangle_g$ denotes the inner product induced by the metric g. Variation of such an expression w.r.t the metric $g(t)$ yields

$$\left.\frac{d}{dt}\right|_0 \mathcal{L}_{\mathrm{YM}}(\omega, g) = -\frac{1}{2} \int_M T^{\mu\nu} h_{\mu\nu} \cdot \mathrm{vol}_g = 0 \tag{60}$$

where $h_{\mu\nu} = \frac{\partial}{\partial t}\big|_0 g(t)$ is the variation of the metric and the *energy-momentum tensor* is defined as

$$T^{\mu\nu} = -F^{\alpha\mu}F^{\nu}{}_{\alpha} - \frac{1}{2}\langle F, F\rangle_g g^{\mu\nu}. \tag{61}$$

Using the definitions of F as above the energy momentum tensor can be expressed in terms of the $\vec{E}$ and $\vec{B}$ fields as

$$T^{00} = \frac{1}{2}\left(|\vec{E}|^2 + |\vec{B}|^2\right), \qquad T^{0i} = \vec{S} = \vec{E}\times\vec{B}, \qquad T^{ij} = -\sigma^{ij} = -\vec{E}\otimes\vec{E} - \vec{B}\otimes\vec{B} + T^{00}g^{\mu\nu} \tag{62}$$

In GR, we must also consider the Einstein-Hilbert action, defined as

$$\mathcal{L}_{\text{EH}}(g(t)) = -\frac{1}{2}R_g \cdot \text{vol}_g \tag{63}$$

where R_g denotes the scalar curvature of the *metric* - i.e the curvature obtained from the metric connection on the *base* manifold only; F is the principal bundle curvature. Similarly, we obtain from the metric variation

$$\frac{d}{dt}\bigg|_0 \int_M -\frac{1}{2}R_g \cdot \text{vol}_g = \frac{1}{2}\int_M \left(R_g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_g\right) h_{\mu\nu}\text{vol}_g = 0 \tag{64}$$

where $R_g^{\mu\nu}$ is the Ricci tensor for the metric g. These two action functionals are combined to yield the following critical condition for the metric

$$\frac{1}{2}\int_M \left(R_g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_g - T^{\mu\nu}\right) h_{\mu\nu}\text{vol}_g = 0 \tag{65}$$

which yields the equation of motion (Einstein field equations)

$$R_g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_g = T^{\mu\nu} \tag{66}$$

2.4 Non-abelian gauge theories

The generalisation of the abelian Yang-Mills Lagrangian to the non-abelian case of $SU(N)$ is simple, given a suitable generalisation of the Hodge dual to Lie algebra valued forms, namely that the inner product is induced by a metric on the *adjoint* bundle $P \times_{\text{Ad}} \mathfrak{g}$. The appropriate metric must therefore be a positive definite, symmetric bilinear form on the adjoint bundle, which is achieved by the following map

$$\lambda : (A, B) \in \mathfrak{su}(N) \mapsto -\text{tr}(A \cdot B) \tag{67}$$

which can easily be checked to be Ad invariant, hence λ defines a metric on the adjoint bundle, as

$$\lambda([p, A], [p, B]) = \lambda(A, B). \tag{68}$$

Therefore, for a principal $SU(N)$-bundle over a Lorentzian spacetime M, the YM Lagrangian is defined as (where $\bar{\Omega}$ denotes the local curvature 2-form on M, $\bar{\Omega} = s^*\Omega$),

$$\mathcal{L}_{\text{YM}}(\omega) = \frac{1}{2}\langle\bar{\Omega}, \bar{\Omega}\rangle \cdot \text{vol} = \frac{1}{2}\lambda(\bar{\Omega}\wedge\star\bar{\Omega}) \tag{69}$$

where the notation

$$\lambda(\bar{\Omega}\wedge\star\bar{\Omega}) = \langle\bar{\Omega}, \bar{\Omega}\rangle \cdot \text{vol} \tag{70}$$

defines the inner product $\langle\cdot,\cdot\rangle$, on the space $\Lambda^2 T^*M \otimes P \times_{\mathrm{Ad}} \mathfrak{g}$ obtained from λ and the metric on M. Therefore, the YM Lagrangian reduces to the expression obtained previously for the $U(1)$ case where the map λ is trivial.

It can be easily seen that the YM Lagrangian will be gauge invariant, since the curvature transforms using the adjoint as in Eq. 43, gauge invariance follows from the Ad invariance of the inner product defined by λ.

In general, the condition for a principal connection to be critical is obtained as the equation of motion for the Yang-Mills functional

$$\mathrm{YM}[\omega] = \frac{1}{2}\int_M \langle\bar{\Omega},\bar{\Omega}\rangle \cdot \mathrm{vol} \tag{71}$$

given a suitable inner product on the space $\Lambda^2 T^*M \otimes P \times_{\mathrm{Ad}} \mathfrak{g}$, the equation of motion is found to be

$$\mathrm{d}^\omega \star \bar{\Omega} = 0 \tag{72}$$

for the case where $\star\bar{\Omega} = \pm\bar{\Omega}$ this reduces to the regular Bianchi identity. Therefore, as we showed with electromagnetism, the YM functional fully determines the equations of motion for the connection and hence the resulting physical theory on M.

3 Einstein-Cartan-Palatini formalism

In this section we formulate general relativity (GR) in the Einstein-Cartan-Palatini (ECP) formalism, where it is naturally described in terms of the spin connection on a principal $SO(p,q)$-bundle over a Lorentzian spacetime dim $M = n = p+q$. Such a connection formulation of gravity will resemble a Yang-Mills gauge theory, and furthermore since its formulation is compatible with spinor representations of the Lorentz group (unlike Riemannian general relativity) ECP is the favourable approach when attempting quantum mechanics in curved spacetimes.

Let M be the Lorentzian manifold of dimension n, with signature (p,q). Define the vector bundle $\mathcal{V}$ over M, where $\mathcal{V}$ is isomorphic to the tangent bundle as we see shortly. The 'coframe' bundle $\mathcal{V}$, is equipped with the pseudo-Riemannian metric η of signature (p,q), with a fixed orientation. Define the 'coframe' or 'vielbien' field as the orientation preserving bundle isomorphism

$$e : TM \longrightarrow \mathcal{V}. \tag{73}$$

Such a coframe is used to pull back the Minkowsi metric on the coframe bundle to the general metric $g = e^*\eta$ on the spacetime M. If the spacetime is parallelisable,[3] namely that the tangent bundle is trivial, the coframe bundle can be regarded as the trivial bundle (note that if the spacetime is not parallelisable, this in general will only be true of local trivialisations [3]),

$$\mathcal{V} = M \times \mathbb{R}^{(p,\,q)}. \tag{74}$$

Hence, the coframe field is the 'Minkowski valued' 1-form,

$$e = e^a P_a \in \Omega(M, \mathbb{R}^{(p,\,q)}) \tag{75}$$

where P_a is the oriented pseudo-orthogonal basis of $\mathbb{R}^{(p,q)}$, which here also correspond to the translation Poincare group generators. The coframe field coefficients are 1-forms on the base manifold

$$e^a = e^a{}_\mu \mathrm{d}x^\mu \in \Omega^1(M), \tag{76}$$

[3] a globally defined smooth vector field can be used to define the tangent basis at any point

The metric on M is defined as,

$$\eta_{ab} e^a{}_\mu \otimes e^b{}_\nu = g_{\mu\nu} \tag{77}$$

namely that the general metric is pulled back from the 'internal' Minkowski metric by the coframe fields.

The coframe fields should be regarded as assigning the 'local' gauge (a local inertial frame) at every point to the general spacetime M. This is the realisation that locally, spacetime should resemble Minkowski space for any observer, here realised as the fibration of M by the local Minkowski coframes. The coframe bundle is then identified as the associated principal $SO(p, q)$ bundle to M, with group equivariance ensuring the Poincare invariance of the local frames (gauge transformations).

3.1 The spin connection

The spin connection refers to the principal connection in the principal $SO(p, q)$-bundle associated to the coframe bundle, where the terminology 'spin' arises from it's compatibility with spinor representations of the Lorentz group as we will eventually observe . Given a parallelisable manifold M, (more generaly a local trivialisation of some neighbourhood $U \subset M$), let $P \longrightarrow M$ be the principal $SO(p, q)$ bundle associated to $\mathcal{V}$, such that

$$\mathcal{V} = (M \times \mathbb{R}^{(p,q)})/SO(p, q). \tag{78}$$

The spin connection is therefore the Lie algebra valued 1-form

$$\tilde{\omega} \in \Omega^1(P, \mathfrak{so}(p, q)), \tag{79}$$

which in the case where P is trivial (or local trivialisation) is treated as a globally defined 1-form on the base $\omega = e^*\tilde{\omega}$ (equally $s^*\tilde{\omega}$ in a trivialisation)

$$\omega = \omega_\mu{}^a{}_b \mathrm{d}x^\mu \otimes L^a{}_b \in \Omega^1(M, \mathfrak{so}(p, q)), \tag{80}$$

where x^μ is the coordinate basis on M and $L^a{}_b$ the Lie algebra basis, with matrix elements given by $(L^a{}_b)_{cd} = \delta_{ac}\delta^b{}_d$. This is frequently abbreviated as

$$\omega = \omega^a{}_b L^b{}_a = \omega^{ab} L_{ba} = \omega^{ab} L_{[ba]} \tag{81}$$

where the Minkowski metric is used to raise and lower latin indices. The metric η induces the isomorphism

$$\mathfrak{so}(p, q) \simeq \bigwedge^2 \mathbb{R}^{(p,q)} \tag{82}$$

given by

$$\omega^{ab} E_{[ba]} \mapsto \omega^{[ab]} E_b \wedge E_a. \tag{83}$$

Hence, we can regard the connection as a bivector (valued 1-form) in the space $\bigwedge^2 \mathbb{R}^{(p,q)}$, with anti-symmetric upper indices $\omega^{ab} = -\omega^{ba}$.

3.2 Curvature and Torsion

The torsion and curvature forms are defined using the exterior covariant derivative induced by the spin connection

$$\mathrm{d}^\omega : \Omega^0(M, \mathcal{V}) \longrightarrow \Omega^1_{\mathrm{hor}}(M, \mathcal{V}), \tag{84}$$

as follows. Let

$$\mathrm{d}^\omega \phi = \mathrm{d}\phi + [\omega, \phi] = \mathrm{d}\phi + \omega \wedge \phi, \tag{85}$$

be the exterior covariant derivative of any $\phi \in \Omega^0(M, \mathcal{V})$ and recall that this is isomorphic to the covariant derivative on the associated bundle as in Eq. 42

$$\nabla^\omega : \Gamma^\infty \left(M, P \times_\rho \mathcal{V}\right) \longrightarrow \Omega^1 \left(M, P \times_\rho \mathcal{V}\right). \tag{86}$$

In what follows the symbol $\wedge$, denotes matrix multiplication of the Lie algebra components within the exterior product, namely

$$\omega \wedge e = (\omega^a{}_b \wedge e^b)E_a \equiv (\omega_\mu{}^a{}_b \wedge e_\nu{}^b)E_a \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{87}$$

which is a 2-form on M.

The curvature 2-form is the covariant derivative of the spin connection on M, which takes values in the associated *endomorphism* bundle of the Lie algebra $P \times_{\mathrm{ad}} \mathfrak{so}(p,q)$

$$R = \mathrm{d}^\omega \omega = \mathrm{d}\omega + \omega \wedge \omega = \mathrm{d}^\omega \omega + \frac{1}{2}[\omega, \omega] \ \in \Omega^2(M, P \times_{\mathrm{ad}} \mathfrak{so}(p,q)). \tag{88}$$

The torsion 2-form on M is the exterior covariant derivative of the coframe field,

$$T = \mathrm{d}^\omega e = \mathrm{d}e + \omega \wedge e \ \ \in \Omega^2(M, \mathcal{V}), \tag{89}$$

taking values in $\mathcal{V}$. Under the isomorphism in Eq. (82), the curvature can equally be regarded as the bivector valued 2-form on M,

$$R = R^{ab}\mathrm{E}_a \wedge \mathrm{E}_b \in \Omega^2\left(M, \wedge^2\left(\mathbb{R}^{p,q}\right)\right) \tag{90}$$

Together, they satisfy the Bianchi identities

$$\mathrm{d}^\omega T = R \wedge e \quad , \quad \mathrm{d}^\omega R = 0, \tag{91}$$

In the following we specialise to the case of $SO(1,d)$, as it will be utilised most frequently when describing Lorentzian spacetimes throughout. Here, d denotes the number of spatial dimensions $d = n - 1$.

3.3 The Cartan connection

The spin connection and coframe field can be realised more concretely in terms of a Cartan connection, that contains both the connection and coframe field information. In order to construct a Cartan connection one must first construct a reduction of the structure group of the principal bundle. Let (P, π, G) be a principal G-bundle and given a group homomorphism $\varphi : G \longrightarrow H$, let $(P \times_\varphi H)$ be the corresponding associated bundle. If the homomorphism φ consists of an embedding (of a subgroup $H \hookrightarrow G$), such that the resulting associated bundle is isomorphic to P; this constitutes a reduction of the structure group from G to H, and is thus sometimes referred to as an H- structure on M. Since $H \hookrightarrow G$ is a Lie subgroup, the coset G/H produces a homogeneous space (Klein geometry).

The coframe bundle can therefore obtained from the tangent bundle via a reduction of the $GL(n, \mathbb{R})$ structure group on TM, to the Lorentz group on $\mathbb{R}^{1,d}$, $(SO(1,d) \hookrightarrow GL(n, \mathbb{R}))$. The coframe field is then understood as the isomorphism between the tangent bundle and the associated $SO(1,d)$ bundle corresponding to the coframe bundle, namely $e : TM \longrightarrow \mathcal{V} = M \times_{SO(1,d)} \mathbb{R}^{1,d}$.

The Cartan $(H \hookrightarrow G)$-connection, Ω, is the G-principal connection on M such that, given a suitable reduction of the structure group G to H, the connection satisfies

$$TM \xrightarrow{\Omega} \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}/\mathfrak{h} \tag{92}$$

where the last arrow indicates isomorphism. Therefore, the Cartan connection identifies each tangent space $T_xM \in TM$ with the tangent space

$$T_xM \longrightarrow \mathfrak{g}/\mathfrak{h} \cong T_e(G/H) \tag{93}$$

of the corresponding (homogeneous) coset space.

In what follows the particular case will be relevant, as before let $H \hookrightarrow G$. Define a suitable reduction of the structure group ($H \hookrightarrow GL(n, \mathbb{R})$ such that this provides a H-structure and coframe on M. If the group G is of the form $G = \mathbb{R}^{1,\,d} \rtimes H$, the $H \hookrightarrow (G = \mathbb{R}^{1,\,d} \rtimes H)$ Cartan connection is then equivalent to a H-structure on M (defining the coframe field) and a H-principal connection. The Cartan connection then consists of two components, namely

$$\Omega \in \Omega^1(M, \mathbb{R}^{1,\,d} \rtimes \mathfrak{h}) \cong (e, \omega) \in \Omega^1(M, \mathbb{R}^{1,\,d}) \rtimes \Omega^1(M, \mathfrak{h}) \tag{94}$$

where the coframe is then realised as the $\mathbb{R}^{1,\,d}$ valued part of the Cartan connection, with the H-principal connection being the Lie algebra valued component.

In the ECP formalism, the coframe and spin connection are obtained through an ($SO(1, d) \hookrightarrow ISO(1, d)$) Cartan connection on M, $\Omega \in \Omega^1(M, \mathfrak{iso}(1, d))$. Hence, the Cartan connection identifies each tangent space with

$$T_xM \cong \mathfrak{iso}(1, d)/\mathfrak{so}(1, d) = \mathbb{R}^{1,\,d} \tag{95}$$

hence exhibiting the required isomorphism producing the bundle of Minkowski coframes (every tangent space is Minkowski due to the homogeneity). Since the Poincare group is of the form $ISO(1, d) = \mathbb{R}^{1,\,d} \rtimes SO(1, d)$, the Cartan connection generates an $SO(1, d)$-structure on M with the corresponding coframe field. Therefore, one can regard the Cartan connection as having two components, $\Omega = (e, \omega)$, where $e \in \Omega^1(M, \mathbb{R}^{1,\,d})$ corresponds to the coframe and $\omega \in \Omega^1(M, \mathfrak{so}(1, d))$ the spin connection as defined previously. A manifold with a Cartan connection is usually referred to as a Cartan geometry, where the tangent spaces are realised as homogeneous spaces.

The curvature and torsion are then both obtained similarly from the Cartan connection using the exterior covariant derivative (defined by the principal connection ω),

$$\mathrm{d}^\omega\Omega = \mathrm{d}\Omega + \omega \wedge \Omega = (\mathrm{d}e, \mathrm{d}\omega) + (\omega \wedge e, \omega \wedge \omega) = (\mathrm{d}^\omega e, \mathrm{d}^\omega\omega) = (T, R) \tag{96}$$

3.4 Principal bundle automorphisms

As mentioned previously, the $SO(1, d)$ equivariance of the coframe bundle provides the local Lorentz invariance. This is realised as the gauge transformations of the principal $SO(1, d)$ bundle associated to the coframe bundle. In addition to the Lorentz (gauge) transformations within each coframe, we must also respect the diffeomorphism invariance of the base manifold M as required by general relativity.

The transformation behaviour of the coframe and spin connection can be understood by considering automorphisms of the principal bundle, which correspond to gauge transformations and diffeomorphisms of the base manifold as we now show. The action of the automorphism group can be decomposed into the semi-direct product group action of the gauge and diffeomorphism groups.

The group structure of the semi-direct product is most easily understood in terms of principal bundle automorphisms, defined previously as the diffeomorphism $f : P \longrightarrow P$ such that

$$\forall p \in P, g \in G, \quad f(p \lhd g) = f(p) \lhd g \tag{97}$$

i.e it is a G-equivariant diffeomorphism of the principal bundle into itself. We will now explore how such a principle bundle automorphism can be realised as a gauge transformation along with a diffeomorphism of M. Let $P = M \times G$ be a trivial bundle and $f : M \times G \longrightarrow M \times G$ be an automorphism. The action of the automorphism can then be expressed, given $p = (x, g) \in M \times G$ as

$$f : (x, g) \mapsto (f_1(x, g), f_2(x, g)) \in M \times G, \tag{98}$$

which can then be used to verify the G-equivariance property, that

$$f(x, g) \lhd h \stackrel{!}{=} f((x, g) \lhd h) \tag{99}$$

for the LHS

$$f(x, g) \lhd h = (f_1(x, g), f_2(x, g)) \lhd h = (f_1(x, g), f_2(x, g) \lhd h) \tag{100}$$

and similarly the RHS

$$f(x, g \lhd h) = (f_1(x, g \lhd h), f_2(x, g \lhd h)). \tag{101}$$

Hence, the following two conditions must hold

$$f_1(x, g) = f_1(x, g \lhd h) \tag{102}$$
$$f_2(x, g) \lhd h = f_2(x, g \lhd h) \tag{103}$$

the first is simply that the map $f_1 : M \times G \longrightarrow M$ is constant under the group action, since,

$$f_1(x, g) = f(x, e \lhd g) = f_1(x, e). \tag{104}$$

Therefore, the map f_1 is equivalent to a diffeomorphism $\underline{f_1}$ of M, by letting

$$\underline{f_1} : M \longrightarrow M, \quad \underline{f_1}(x) = f_1(x, e) = f_1(x, g) \tag{105}$$

The second condition can be re-written as, letting $g = e$

$$f_2(x, e) \lhd h = f_2(x, e \lhd h) = f_2(x, h) \tag{106}$$

from which we see that knowledge of how f_2 maps the identity in G is enough to construct the entire map. Therefore, since $f_2 : M \times G \longrightarrow G$ is independent of the group element chosen, it corresponds to the map $\underline{f_2} : M \longrightarrow G$, producing a gauge transformation. Therefore, the combined action of the gauge transformation along with a diffeomorphism is described by a principal bundle automorphism in this way. This can be summarised with the following diagram

$$\begin{tikzcd} M \times G \arrow[r, "f"] \arrow[d, "\pi_1"] & M \times G \arrow[d, "\pi_1"] \\ M \arrow[r, "\exists! \underline{f_1}"] & M \end{tikzcd} \tag{107}$$

where $\underline{f_1}$ is the *unique* diffeomorphism of M such that the diagram commutes. This can be verified by considering arbitrary points $p, p' \in \pi_1^{-1}(x)$ in the fibre over $x \in M$, and noting that $\exists! g \in G :\ p' = p \lhd g$. Since f is an automorphism, it satisfies the following

$$\pi_1(f\,(p')) = \pi_1(f(p \lhd g)) = \pi_1(f(p) \lhd g) = \pi_1(f(p)) \tag{108}$$

namely that every point (p, p') in the fibre over x is mapped into the same target fibre over a new (potentially the same) base point, since the projections of the points after the automorphism are equivalent.

Hence, there must exist a unique diffeomorphism of M, such that the base point x is mapped to the corresponding base point

$$x' = \underline{f_1}(x)\pi(f(p)) = \pi(f(p')) \tag{109}$$

summarised with the following diagram

$$\begin{tikzcd} (p,p') \arrow[r, "f"] \arrow[d, "\pi_1"] & (f(p), f(p')) \arrow[d, "\pi_1"] \\ x \arrow[r, "\underline{f_1}"] & x' \end{tikzcd} \tag{110}$$

Hence, gauge transformations and diffeomorphisms of M can both be understood as principle bundle automorphisms in this way, with gauge transformations corresponding to automorphisms that act as the identity on the base $\underline{f_1} = \mathrm{id}_M$.

Therefore, the action of an arbitrary automorphism is then equivalent to a diffeomorphism of the base along with a gauge transformation. This can be expressed, given $(\lambda, \phi) \in \Gamma^\infty(P \times_{\mathrm{Ad}} G) \times \mathrm{Diff}(M)$, denote the corresponding automorphism by $f_{(\lambda,\phi)} \in \mathrm{Aut}(P)$, then the action on an arbitrary point $p = (x, g) \in P$ is given by,

$$f_{(\lambda,\phi)}(p) = f_{(\lambda,\phi)}(x,g) = (\phi(x), g \lhd \lambda(x)) \tag{111}$$

The automorphism group of P can then be identified with the relevant subgroups, the gauge group of principal bundle and the diffeomorphism group of the base. The automorphism group then fits into the sequence of inclusions

$$0 \hookrightarrow \mathcal{G}(P) \hookrightarrow \mathrm{Aut}(P) \longrightarrow \mathrm{Diff}(M) \tag{112}$$

with the last map being the identification of the diffeomorphism $\underline{f_1}$ of the base corresponding to f. This map is in only surjective, since there are automorphisms that correspond to the same diffeomorphism (the gauge transformations). In the trivial case (henceforth assume trivial) the automorphism group is given by

$$\mathrm{Aut}(P) \cong \Gamma^\infty(P \times_{\mathrm{Ad}} G) \rtimes \mathrm{Diff}(M). \tag{113}$$

where the gauge group is identified with smooth sections of the adjoint bundle. The semidirect product group structure can be inferred from the automorphism group as follows, given arbitrary elements

$$(\lambda, \phi), (\lambda', \phi') \in \Gamma^\infty(P \times_{\mathrm{Ad}} G) \times \mathrm{Diff}(M) \tag{114}$$

denote the corresponding automorphism as $f_{(\lambda,\phi)}$. Then using the composition on $\mathrm{Aut}(P)$ we can express the composition of automorphisms acting on an arbitrary element $p = (x, g) \in P$

$$f_{(\lambda',\phi')} \circ f_{(\lambda,\phi)}(x,g) = f_{(\lambda',\phi')}(\phi(x), g \lhd \lambda(x)) = (\phi' \circ \phi(x), (g \lhd \lambda(x)) \lhd \lambda'(\phi(x))), \tag{115}$$

which can be rewritten using the pullback as

$$(\phi' \circ \phi(x),\ (g \lhd \lambda(x)) \lhd \phi^*\lambda'(x)), \tag{116}$$

this then corresponds to the automorphism generated by the element

$$(\lambda, \phi) \cdot (\lambda', \phi') = (\phi^*\lambda' \cdot \lambda, \phi' \circ \phi) \in \Gamma^\infty(P \times_{\mathrm{Ad}} G) \rtimes \mathrm{Diff}(M), \tag{117}$$

(where $\cdot$ denotes the multiplication on the semidirect product group) Therefore, the action of the second gauge transformation is affected by the preceding diffeomorphism. Hence, since the diffeomorphism group additionally acts on the gauge transformations this must be taken into account in the product structure in this way.

From the construction of the semi-direct product the following situation emerges. Consider first acting with a gauge transformation, followed by a diffeomorphism, this can be expressed using the semi-direct product multiplication as

$$f_{(\mathrm{id},\phi)} \circ f_{(\lambda,\mathrm{id})} : \qquad (\lambda, \mathrm{id}) \cdot (\mathrm{id}, \phi) = (\lambda \triangleleft \phi^* \mathrm{id}, \mathrm{id} \cdot \phi) = (\lambda, \phi), \tag{118}$$

however, when first acting with the diffeomorphism

$$f_{(\lambda,\mathrm{id})} \circ f_{(\mathrm{id},\phi)} : \qquad (\mathrm{id}, \phi) \cdot (\lambda, \mathrm{id}) = (\mathrm{id} \triangleleft \phi^* \lambda, \phi) = (\phi^* \lambda, \phi), \tag{119}$$

since after first acting with a diffeomorphism, the proceeding gauge transformation also transforms, the action of the two transformations does not necessarily commute. This was already implied from the semi-direct product structure, and here we see that the combined transformations correspond to different elements in the semi-direct product. Therefore, pairs in the semi-direct product will correspond to application of the appropriate transformations in the particular order as defined above, where in general $(\lambda, \phi) \neq (\phi^* \lambda, \phi)$ do not correspond to the same automorphism unless $\lambda = \phi^* \lambda$. Henceforth, we will assume that an arbitrary gauge + diffeomorphism transformation correspond to the same transformation, i.e that $\lambda = \phi^* \lambda$, such that

$$f_{\lambda,\phi}(x, g) = (\phi(x), g \triangleleft \lambda_\phi(x)) \tag{120}$$

3.5 Coframe bundle automorphisms

The results of the previous section can now be used to determine the automorphisms of the $SO(1, d)$ principal bundle associated to the (trivial) coframe bundle

$$\mathcal{V} = (M \times \mathbb{R}^{1,\,d})/SO(1, d). \tag{121}$$

obtained from the $(SO(1, d) \hookrightarrow ISO(1, d))$ Cartan connection on the principal $ISO(1, d)$ bundle over M (recall this identifies a coframe bundle with $SO(1, d)$-structure over M). The automorphisms of such a principal bundle are then equivalently given by the semi-direct product

$$(\lambda, \phi) \in \Gamma^\infty(M \times_{\mathrm{Ad}} SO(1, d)) \rtimes \mathrm{Diff}(M) \cong \mathrm{Aut}(P) \tag{122}$$

with the group structure as defined previously. This corresponds to the local Lorentz invariance of the (Minkowski) coframes, along with the diffeomorphism invariance of the base manifold.

The coframe and spin connection will transform under the semi-direct product action as follows. Consider the action independently of the semi-direct product for now. The coframe (components), realised as a section $\Omega^1(M, \mathbb{R}^{1,\,d})$, transform in the fundamental vector representation of $SO(1, d)$, hence by the right action of the group under finite gauge transformations

$$e \mapsto \lambda^{-1} e \tag{123}$$

The spin connection, realised as a Lie algebra valued 1-form on the base will transform under the gauge transformation as

$$\omega \mapsto \lambda^{-1} \omega \lambda + \lambda^{-1} \mathrm{d}\lambda. \tag{124}$$

Furthermore, both the coframe and spin connection will transform by the pullback of a finite diffeomorphism.

$$e \mapsto \phi^* e, \quad \omega \mapsto \phi^* \omega \tag{125}$$

Given an arbitrary gauge + diffeomorphism transformation, in terms of a principal bundle automorphism (λ, ϕ), the coframe and spin connection will be pulled back by such an automorphism; using the action of the semidirect product group as defined previously

$$f^*_{(\lambda,\phi)}e \mapsto \lambda_\phi^{-1}(\phi^* e), \tag{126}$$

Likewise, for the spin connection,

$$f^*_{(\lambda,\phi)}\omega \mapsto \lambda_\phi^{-1}\phi^*\omega\lambda_\phi + \lambda_\phi^{-1}\mathrm{d}\lambda_\phi \tag{127}$$

The automorphism can then be described in terms of the $(SO(1,d) \hookrightarrow ISO(1,d))$ Cartan connection, which simply acts by pullback

$$f^*_{(\lambda,\phi)}\Omega = (f^*_{(\lambda,\phi)}e, f^*_{(\lambda,\phi)}\omega) = (\lambda_\phi^{-1}(\phi^* e), \lambda_\phi^{-1}\phi^*\omega\lambda_\phi + \lambda_\phi^{-1}\mathrm{d}\lambda_\phi) \tag{128}$$

4 Einstein-Palatini-Cartan Action

The ECP action functional (in dimension $n \geq 3$) is given by,

$$S_{\mathrm{ECP}}[e,\omega] = \frac{1}{2\kappa^2}\int_M \mathrm{Tr}\left(\frac{1}{n-2} e \overset{n-2}{\wedge} e \wedge R + \frac{\Lambda}{n} e \overset{n}{\wedge} e\right), \tag{129}$$

where $\Lambda \in \mathbb{R}$ is the cosmological constant and Tr denotes the Hodge dual in this context, given by

$$\mathrm{Tr} : \Omega^n\left(M, \bigwedge^n \mathbb{R}^{(n-1,\,1)}\right) \to \Omega^n(M). \tag{130}$$

Throughout we can normalise the gravitational constants to be $2\kappa^2 = 1$ without loss of generality. When acting on the coordinate basis for $\mathbb{R}^{(n-1,\,1)}$ (in a trivialisation), this has the following action

$$\mathrm{Tr}(E_{a_i} \overset{n}{\wedge} E_{a_n}) = \varepsilon_{a_i \ldots a_n} \tag{131}$$

namely producing the $n-$dimensional Levi-Civita symbol which can then be integrated. This action functional is the analogue of the YM functional, and we consider variation of both the coframe field and spin connection. The resulting equations of motion are then found to be

$$\mathcal{F}_e(e,\omega) = e \overset{n-3}{\wedge} e \wedge R \, + \, \Lambda\, e \overset{n-1}{\wedge} e = 0, \tag{132}$$

for the coframe variation and

$$\mathcal{F}_\omega(e,\omega) = e \overset{n-3}{\wedge} e \wedge T = 0, \tag{133}$$

for the spin connection variation. Of interest will be the $n = 3, 4$ cases, presented here for clarity. For the $n = 3$ case we find the remarkably simple equations

$$\mathcal{F}_e = R + \Lambda\, e \wedge e = 0, \tag{134}$$

$$\mathcal{F}_\omega = T = \mathrm{d}^\omega e = 0. \tag{135}$$

The torsionless condition $\mathrm{d}^\omega e = 0$, can be solved to determine the connection (up to gauge transformations) in terms of the coframe $\omega = \omega(e)$. The unique solution corresponds to the Levi-Civita connection when pulled back to the metric $g = e^*\eta$. Eq. (134) then corresponds to the vacuum Einstein equation with a cosmological constant. The metric compatibility condition also follows from the torsionless condition, namely that

$$\mathrm{d}^{e^*\omega} g = \mathrm{d}g + [e^*\omega, g] = \mathrm{d}e^*\eta + e^*[\omega,\eta] = e^*\mathrm{d}\eta + e^*[\omega,\eta] = e^*\mathrm{d}^\omega\eta = 0 \tag{136}$$

which vanishes due to the fact that ω is also required to be η metric compatible (i.e $\mathrm{d}^\omega \eta = 0$) [4]. Similarly, for the $n = 4$ case, we obtain

$$\mathcal{F}_e = e \wedge R + \Lambda\, e \wedge e \wedge e = e \wedge (R + \Lambda e \wedge e) = 0, \tag{137}$$

$$\mathcal{F}_\omega = e \wedge T = e \wedge \mathrm{d}^\omega e = 0. \tag{138}$$

Again, the last equation corresponds to the torsionless condition $\mathrm{d}^\omega e = 0$ which determines the Levi-Civita connection $\omega = \omega(e)$ for the metric $g = e^*\eta$ as previously.

4.1 Euler-Lagrange equations

Consider variation under an arbitrary infinitesimal variation of the parameters, $(e + \delta e, \omega + \delta\omega)$, where the variation in the action (to first order) is then given as (first in e),

$$\delta_e S = \int_M \mathrm{Tr}\left(\delta e \wedge R \;+\; \Lambda\, \delta e \wedge e \wedge e\right) = 0 \tag{139}$$

which under arbitrary variation returns the Euler Lagrange equation as stated previously,

$$\mathcal{F}_e = R + \Lambda\, e \wedge e = 0. \tag{140}$$

Note the cyclicity of the trace was used when evaluating the first order terms obtained from

$$\mathrm{Tr}\left(\frac{1}{3}(e + \delta e)^3\right) \longrightarrow \mathrm{Tr}\left(\delta e \wedge e \wedge e\right) \tag{141}$$

Similarly, the first order variation in ω obtains

$$\delta_\omega S = \int_M \mathrm{Tr}\left(e \wedge \mathrm{d}(\delta\omega) + e \wedge \omega \wedge \delta\omega\right) = 0 \tag{142}$$

which after using the cyclicity of the trace can be expressed as

$$\delta_\omega S = \int_M \mathrm{Tr}\left(e \wedge \mathrm{d}(\delta\omega) + (\omega \wedge e) \wedge \delta\omega\right) = 0. \tag{143}$$

The first term can then be integrated by parts in order to isolate the $\delta\omega$ as

$$\delta_\omega S = \int_{\partial M} \mathrm{Tr}\left(e \wedge \delta\omega\right) + \int_M \mathrm{Tr}\left(\mathrm{d}e \wedge \delta\omega + (\omega \wedge e) \wedge \delta\omega\right) = 0, \tag{144}$$

which under arbitrary variations leads to the following condition in the bulk

$$\mathcal{F}_\omega = \mathrm{d}e + \omega \wedge e = \mathrm{d}^\omega e = T = 0 \tag{145}$$

and if the boundary is present, e must vanish on the boundary due to the first term in Eq. (144).

4.2 Gauge + Diffeomorphism invariance

This section will elaborate on the gauge and diffeomorphism invariance that is required of the ECP action, and focus on developing the situation where the spacetime in question has a boundary. In testing the action for invariance, and indeed in deriving the resulting equations of motion, one often integrates by parts and in most cases assumes a vanishing contribution to the variation at the boundary. The proceeding work will be interested in examining what additional effects arise when including the boundary in such calculations.

For concreteness, integration by parts can be understood using Stoke's theorem as follows. Begin with a simple application of the Leibniz rule

$$\int_M \mathrm{d}(\omega \wedge \eta) = \int_M \mathrm{d}\omega \wedge \eta + (-1)^{|\omega|}\, \omega \wedge \mathrm{d}\eta. \tag{146}$$

Stoke's theorem then applies to the left hand side, to obtain

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}\omega \wedge \eta + (-1)^{|\omega|}\, \omega \wedge \mathrm{d}\eta, \tag{147}$$

which is frequently used during 'integration by parts'

$$\int_M \mathrm{d}\omega \wedge \eta = \int_{\partial M} \omega \wedge \eta - (-1)^{|\omega|} \int_M \omega \wedge \mathrm{d}\eta \tag{148}$$

For simplicity, we work with a 3 dimensional base spacetime M, of Lorentzian signature $(1,2)$. The ECP action is most concretely expressed in components as

$$S = \int_M \left(R^{ab} \wedge e^c + \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \varepsilon_{abc} = \int_M \left(\left(\mathrm{d}\omega^{ab} + {\omega^a}_k \wedge \omega^{kb} \right) \wedge e^c + \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \varepsilon_{abc} \tag{149}$$

such that wedges are understood explicitly in terms of the form components. We now want to consider the invariance of this action, first under variation of the parameters (to obtain Euler-Lagrange equations) and then under finite/infinitesimal gauge transformations and diffeomorphisms.

4.3 Gauge + diffeomorphism invariance

The ECP action functional should be invariant under infinitesimal gauge transformations and diffeomorphisms (regardless of the order applied). The infinitesimal action of a pair $(\lambda, \xi) \in \Omega^0(M, \mathfrak{so}(p,q)) \rtimes \Gamma(TM)$, will be expressed as $\delta_{\lambda\xi}(e, \omega) = (\delta_\lambda e + \delta_\phi e, \delta_\lambda \omega + \delta_\phi \omega)$. Under such a variation, the change in the action can be expressed using the analogous variational equations as in Eqs. (139), (142) as

$$\delta_{\lambda\xi} S = \int_M \mathrm{Tr}\left(\delta_{\lambda\xi}(e) \wedge (R + \Lambda e \wedge e) + (e \wedge \mathrm{d}(\delta_{\lambda\xi}\omega) + (\omega \wedge e) \wedge \delta_{\lambda\xi}\omega) \right) \tag{150}$$

where the $\mathrm{d}(\delta_{\lambda\xi}\omega)$ is integrated by parts to yield

$$\delta_{\lambda\xi} S = \int_M \mathrm{Tr}\left(\delta_{\lambda\xi}(e) \wedge (R + \Lambda e \wedge e) \right) + \int_{\partial M} \mathrm{Tr}\left(e \wedge \delta_{\lambda\xi}\omega \right) + \int_M \mathrm{Tr}\left(\mathrm{d}e \wedge \delta_{\lambda\xi}\omega \right) + \int_M \mathrm{Tr}((\omega \wedge e) \wedge \delta_{\lambda\xi}\omega) \tag{151}$$

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