

1 Functions as Sets

Given a function $f : A \rightarrow B$ we can derive some related sets.

Definition 1.1. The *relation*

$$\hat{f} := \{\langle a, f(a) \rangle \mid a \in A\}.$$

Definition 1.2. The *image set*

$$f(A) := \{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

Equivalently

$$f(A) := \left\{ b \mid \langle a, b \rangle \in \hat{f} \text{ for some } a \in A \right\}.$$

2 Composition

The power of composition is that it can't be resisted. Say I hand you an $f : A \rightarrow B$ and a $G : B \rightarrow C$. Then there is a clear procedure for getting from A to C .

- (1) Take your $a \in A$.
- (2) Apply f to a , yielding $f(a)$ which is some $b \in B$.
- (3) Apply g to b , yielding $g(b)$, which is some $c \in C$

This procedure yields the

Definition 2.1. *Function composition*

$$g \circ f.$$

Fact 2.2. *Functions are associative, or*

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Exercise 2.1. Convince yourself that functions are associative.

2.1 Identities

Given any set B there is an important function that comes for free. We call it $1_B : B \rightarrow B$ and it's given by

$$1_B(b) = b$$

for any $b \in B$.

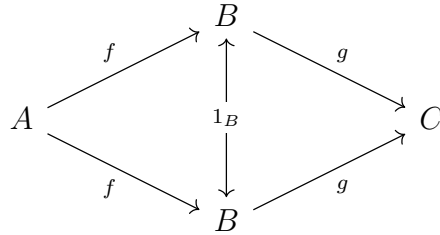
Fact 2.3. *Identities are absorbed left and right. In other words, if we have $f : A \rightarrow B$ and a $G : B \rightarrow C$, then*

$$1_B \circ f = f$$

and

$$g \circ 1_B = g.$$

That gives us our first commutative diagram, for which any path is equivalent:



3 Category Axioms

We are ready for the abstract view of the above.

Definition 3.1. A *category* has

1. a collection of objects:

$$a, b, c.$$

2. a collection of arrows, each with specific domain and codomain:

$$f : a \rightarrow b,$$

$$g : b \rightarrow c.$$

3. an associative composition operation that yields a unique arrow ‘skipping’ aligned domains and codomains:

$$g \circ f : a \rightarrow c.$$

4. an identity arrow for each object:

$$f \circ 1_A = f$$

$$1_B \circ f = f$$

$$g \circ 1_B = g$$

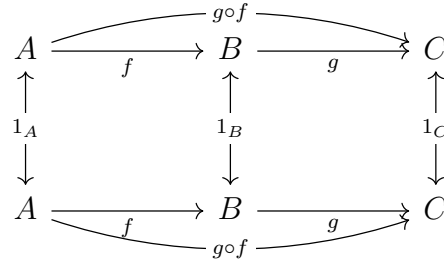
$$1_C \circ g = g$$

Fact 3.2. *Diagrams are a convenient way to present categories:*

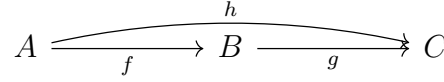
$$A \xrightarrow{f} B \xrightarrow{g} C$$

But you must keep in mind there are some implicit things not being shown here. The above is a compact

version of the following:



Fact 3.3. Saying a diagram ‘commutes’ is a convenient way to present equivalent compositions of arrows. For example, if



commutes then we are saying that

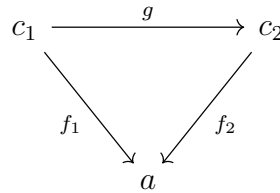
$$g \circ f = h.$$

4 Comma Categories

Definition 4.1. The comma-category $C \downarrow a$ is formed from any category C and any C -object a . Its objects are all the C -arrows with codomain a . (IE, $f_1 : c_1 \rightarrow a$ and $f_2 : c_2 \rightarrow a$). Its arrows are all C -arrows between the objects’ domains, that commute with the ‘object arrows’. (IE, $g : c_1 \rightarrow c_2$ so that $f_1 = f_2 \circ g$). So $C \downarrow a$ looks like this:

$$f_1 \xrightarrow{g} f_2$$

and indicates that this diagram commutes in the original category:



(TODO: Verify category axioms)

Example. Take C to be the preorder on natural numbers, and a to be a given number. For example, let $a = 3$.

$$1 \longrightarrow 2 \longrightarrow \mathbf{3} \longrightarrow 4 \longrightarrow 5 \longrightarrow \dots$$

Objects in $C \downarrow 3$ are statements of ‘ $n \leq 4$ ’ relationships. Arrows in $C \downarrow 3$ from ‘ $m \leq 3$ ’ to ‘ $n \leq 3$ ’ are ‘ $m \leq n$ ’.

$$(1 \leq 3) \xrightarrow{1 \leq 2} (2 \leq 3) \xrightarrow{2 \leq 3} (3 \leq 3)$$

Example. Recall $\mathbf{Matr}(\mathbf{k})$ has the natural numbers \mathbb{N} as objects, and $(n \times m)$ matrices as arrows from $m \rightarrow n$.

Then $\mathbf{Matr}(\mathbf{k}) \downarrow 3$ has as objects all $3 \times n$ matrices where $n \in \mathbb{N}$.

Then if object A is a $3 \times m$ matrix, and object B is a $3 \times n$ matrix, then an arrow $C : A \rightarrow B$ is an $n \times m$ matrix such that $A = BC$.

Thus the situation in $\mathbf{Matr}(\mathbf{k}) \downarrow 3$ is

$$A_{3 \times m} \xrightarrow{C_{n \times m}} B_{3 \times n}$$

indicating that this diagram commutes in $\mathbf{Matr}(\mathbf{k})$:

$$\begin{array}{ccc} m & \xrightarrow{C_{n \times m}} & n \\ & \searrow A_{3 \times m} & \swarrow B_{3 \times n} \\ & 3 & \end{array}$$