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**Chapter 3: Arrows Instead of Epsilon**

# 1 Monic arrows

Monic arrows are an abstraction of injective functions.

**Definition 1.1.** An arrow $f : a \to b$ in a category $C$ is *monic* if for any $g_1, g_2$ with codomain $a$, the implication

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

holds. Or, if the diagram

$$\begin{tikzcd} c \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

commutes, then $g_1 = g_2$.

**Definition 1.2.** *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *monic* iff for any $C$-object $c$, post-composition with $f$ defines an injection $f_* : C(c, a) \to C(c, b)$. (Here $C(x, y)$ is the set of $C$-arrows from $x$ to $y$.)

For both exercises in this section, take the situation to be as follows:

$$\begin{tikzcd} s \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & a \arrow[r, "f"] & b \arrow[r, "g"] & c \end{tikzcd}$$

Where $f$ and $g$ are fixed, and $s, h_1, h_2$ are 'any such' objects/arrows.

**Exercise 1.1.**

Suppose that $f$ and $g$ are both monic, and that $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Since $g$ is monic, that implies $f \circ h_1 = f \circ h_2$. But since $f$ is monic, that implies $h_1 = h_2$. So using associativity and collapsing the chain of implication gives

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \implies h_1 = h_2.$$

Conclude $g \circ f$ is monic.

**Exercise 1.2.**

Now suppose that $g \circ f$ is monic. If $f \circ h_1 = f \circ h_2$ then clearly $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Then $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ and since $g \circ f$ is monic, $h_1 = h_2$. So

$$f \circ h_1 = f \circ h_2 \implies h_1 = h_2,$$

meaning $f$ is monic.

# 2 Epic arrows

**Definition 2.1.** If $f$ is *epic* then commutativity of

$$a \xrightarrow{f} b \rightrightarrows^{g_1}_{g_2} c$$

implies $g_1 = g_2$.

**Definition 2.2.** *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *epic* iff for any $C$-object $c$, pre-composition with $f$ defines an injection $f^* : C(b, x) \to C(a, c)$. (Here $C(x, y)$ is the set of $C$-arrows from $x$ to $y$.)

Dually to the exercises proven in the previous section we have

**Fact 2.3.** *If $f : a \to b$ and $g : b \to c$ are epic, then $g \circ f : a \to c$ is epic.*

**Fact 2.4.** *If $g \circ f : a \to c$ is epic, then $g : b \to c$ is epic.*

# 3 Iso arrows

**Definition 3.1.** An arrow $f : a \to b$ is *iso* if there exists another arrow $f^{-1} : b \to a$ such that

$$f \circ f^{-1} = 1_b$$

and

$$f^{-1} \circ f = 1_a.$$

This diagram commutes when the identity loops are included:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

**Fact 3.2.** *If an arrow is iso then it is epic and monic, but the converse isn't necessarily true. The converse* is *true in* ***Set*** *and any Topos.*

**Exercise 3.1.** For any object $a$, the identity morphism $1_a$ is an inverse to itself and therefore is iso. Simply because

$$1_a \circ 1_a = 1_a.$$

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] & a \arrow[l, bend left, "1_a"] \end{tikzcd}$$

**Exercise 3.2.** If $f : a \to b$ is iso then we can retrieve $f^{-1}$ and then plug it right into the definition and find

$$f^{-1} \circ f = 1_a$$

and

$$f \circ f^{-1} = 1_b,$$

indicating that $f^{-1}$ is iso.

$$
\begin{tikzcd}
a \arrow[r, bend left, "f^{-1}"] & b \arrow[l, bend left, "f"]
\end{tikzcd}
$$

**Exercise 3.3.** With $f : a \to b$ and $g : b \to c$ both iso, the situation looks like the following:

$$
\begin{tikzcd}
a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & c \arrow[l, bend left, "g^{-1}"]
\end{tikzcd}
$$

Now we find that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_b \circ f = f^{-1} \circ f = 1_a$$

and

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_b \circ g^{-1} = g \circ g^{-1} = 1_c.$$

Thus $(f^{-1} \circ g^{-1})$ acts as an inverse to $g \circ f$, and $g \circ f$ is iso.

# 4 Isomorphic objects

**Definition 4.1.** Two $C$-objects $a$ and $b$ are *isomorphic*, or

$$a \cong b$$

if there exists an iso $C$-arrow

$$f : a \to b.$$

**Definition 4.2.** A category $C$ is *skeletal* if $a \cong b$ implies $a = b$.

**Exercise 4.1.**

We wish to show that object isomorphism is an equivalence relation, or that it's reflexive, symmetric, and transitive. Fortunately the exercises from section 3 correspond exactly to these properties.

(i) $a \cong a$ since $1_a$ is iso.

(ii) If $a \cong b$ then some $f : a \to b$ is iso, and therefore $f^{-1} : b \to a$ is iso and $b \cong a$.

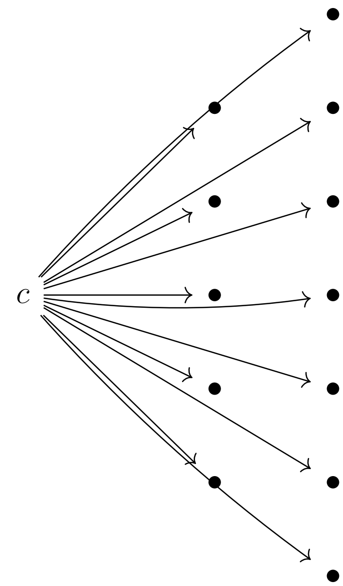
(iii) If $a \cong b$ and $b \cong c$ then we have iso arrows $f : a \to b$ and $g : b \to c$. Then $g \circ f$ is iso, and $a \cong c$.

**Exercise 4.2.**

Suppose $a$ and $b$ are two **Finord**-objects such that $a \cong b$. Then there is some $f : a \to b$ that is iso. Since **Finord** is a subcategory of **Set**, iso arrows correspond to bijective functions. Then $a$ and $b$ must have the same cardinality, but by the definition of **Finord** distinct objects have distinct cardinalities. So $a = b$ and **Finord** is skeletal.

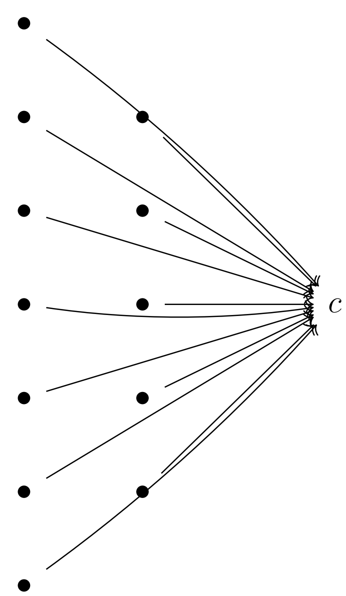
# 5 Initial objects

**Definition 5.1.** An object $c$ is *initial* if for every $C$-object $a$, there is exactly one arrow $f : c \to a$.

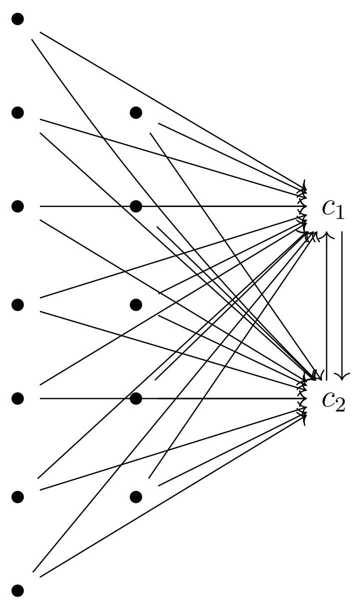



## 6 Terminal objects

**Definition 6.1.** An object $c$ is *terminal* if for every $C$-object $a$, there is exactly one arrow $f : a \to c$.



**Exercise 6.1.** Let $c_1$ and $c_2$ be terminal $C - objects$.




By terminality there is a unique arrow $f_1 : c_1 \to c_2$ and a unique arrow $f_2 : c_2 \to c_1$. Then by the category axiom, $f_2 \circ f_1 : c_1 \to c_1$ and $f_1 \circ f_2 : c_2 \to c_2$ must exist. But again by terminality, there is a unique arrow $1_{c_1} : c_1 \to c_1$ and $1_{c_2} : c_2 \to c_2$, so the composition of $f_1$ and $f_2$ must give the identity. Conclude $c_1 \cong c_2$.

**Exercise 6.2.** (i) Terminal objects in $\mathbf{Set}^2$ are of the form $\langle \{e_1\}, \{e_2\} \rangle$, or pairs of singleton sets.

(ii) Terminal objects in $\mathbf{Set}^{\to}$ are arrows with singleton sets as domain and codomain.

(iii) The terminal object in the poset $(n, \leq)$ is the maximal element $n$, since $m \leq n$ for every $m$.

**Exercise 6.3.** Suppose $f : 1 \to a$ has its domain 1 a terminal object, and $g_1, g_2$ are any two parallel arrows from $c \to 1$.

$$\begin{tikzcd} c \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & 1 \arrow[r, "f"] & a \end{tikzcd}$$

Well, since 1 is terminal the arrow from $c \to 1$ is unique and we see that $g_1 = g_2$, so regardless of whether $g_1 \circ f = g_2 \circ f$ holds (which it does), we can conclude $f$ is monic.

# 7 Duality

Any category can be turned into its opposite category. So any statement about a category can be dualized with all the arrows reversed.

# 8 Products

**Definition 8.1.** Given $C$-objects $a$ and $b$, a *product* is a $C$-object $a \times b$ and 2 $C$-arrows $pr_a, pr_b$.

$$\begin{tikzcd} a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

For any $c, f, g$ configured as follows

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

$f$ and $g$ determine a unique $h : c \to (a \times b)$ so that

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "h"] & \\ a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

commutes. This is denoted

$$c := \langle f, g \rangle.$$

**Fact 8.2.** *If $c$ is a product $a \times b$, any arrow $f : c \to c$, $f$ must be the identity $1_c$. First observe that the identity must exist. Then plug $c$ into definition 8.1 to see that $f$ must be the unique arrow with that domain and codomain.*

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, leftrightarrow, "1_c"] & \\ a & c \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

**Fact 8.3.** *Any two products of $a$ and $b$, say $a \times_1 b$ and $a \times_2 b$, are isomorphic to each other. Consider that in the diagram*

$$\begin{tikzcd} & a \times_2 b \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, bend right, "h_1"'] & \\ a & & b \\ & a \times_1 b \arrow[ul, "pr_a"] \arrow[ur, "pr_b"'] \arrow[uu, dashed, bend right, "h_2"'] & \end{tikzcd}$$

*$h_1$ and $h_2$ are uniquely determined by symmetric applications of definition 8.1. But by fact 8.2, composition of $h_1$ and $h_2$ must give identities.*

**Exercise 8.1.** The fact that $\langle pr_a, pr_b \rangle = 1_{a \times b}$ follows as a special case of fact 8.2, by plugging in the projection functions.

$$\begin{tikzcd} & a \times b \arrow[dl, "pr_a"'] \arrow[dr, "pr_b"] \arrow[d, dashed, leftrightarrow, "1_{a \times b}"] & \\ a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

**Exercise 8.2.** Suppose we have parallel $f, k : c \rightrightarrows a$ and $g, h : c \rightrightarrows b$ and $p : c \to a \times b$ such that $p = \langle f, g \rangle = \langle k, h \rangle$. Then $f = pr_a \circ p$ and $k = pr_a \circ p$. It doesn't take any special cancellation rules to see that identically $f = k$. Similarly $g = h$.

$$\begin{tikzcd} & c \arrow[dl, bend right, "f"'] \arrow[dl, "k"] \arrow[dr, bend left, "g"] \arrow[dr, "h"'] \arrow[d, dashed, "p"] & \\ a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

**Exercise 8.3.** Suppose the situation is as follows.

$$\begin{tikzcd}
& d \arrow[d, "h"'] & \\
& c \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] \arrow[d, dashed, "\langle f,g \rangle"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Then by compositionality there must exist $h \circ f : d \to a$ and $g \circ f : d \to b$. There also must exist $h \circ \langle f, g \rangle : d \to a \times b$. By collapsing the diagram to

$$\begin{tikzcd}
& d \arrow[dl, bend right, "f \circ h"'] \arrow[dr, bend left, "g \circ h"] \arrow[d, dashed, "\langle f \circ h, g \circ h \rangle"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

we see the arrow from $d \to a \times b$ must be unique, and therefore

$$\langle f \circ h, g \circ h \rangle = h \circ \langle f, g \rangle.$$

**Exercise 8.4.** Suppose a category $C$ has a terminal object $t$, and products. Let $a$ be a $C$-object and consider the product $a \times t$.

$$\begin{tikzcd}
a & a \times t \arrow[l] \arrow[r] & t
\end{tikzcd}$$

Plugging $a$ into the product definition using the arrows given by $1_a$ and ! (the unique arrow to $t$) yields the unique arrow $h = \langle 1_a, ! \rangle$.

$$\begin{tikzcd}
& a \arrow[dl, "1_a"'] \arrow[dr, "!"] \arrow[d, dashed, "h"] & \\
a & a \times t \arrow[l, "pr_a"] \arrow[r] & t
\end{tikzcd}$$

By definition we have that $pr_a \circ h = 1_a$. And since $h \circ pr_a$ maps $a \times t \to a \times t$ it follows from fact 8.2 that $h \circ pr_a = 1_{a \times t}$. Given these two iso arrows we conclude

$$a \cong a \times t.$$

**Example.** As a specific example of exercise 8.4, let's work in **Set** where $A$ is any set, and our terminal set $t$ is any singleton set $\{b\}$.

$$\begin{tikzcd}
& A \arrow[dl, "1_a"'] \arrow[dr, "!"] \arrow[d, dashed, "h"] & \\
A & A \times \{b\} \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & \{b\}
\end{tikzcd}$$

Now we can explicitly say what all of our functions do:

$$1_a(x) = x$$

$$!(x) = b$$
$$pr_a\left(\langle a, b\rangle\right) = a$$
$$pr_b\left(\langle a, b\rangle\right) = b$$
$$h(x) = \langle x, b\rangle$$

The fact that $pr_a(h(x)) = x$ and $h(pr_a(\langle a, b\rangle)) = \langle a, b\rangle$ gives the isomorphism

$$A \cong A \times \{b\}\,.$$

Intuitively, elements in $A$ can be placed in one-to-one correspondence with elements in $A \times \{b\}$ by simply sending $x$ to the tuple $\langle x, b\rangle$.

**Definition 8.4.** Given two products $a \times b$ and $c \times d$, and arrows $f : a \to c$ and $g : b \to d$

$$\begin{tikzcd}
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "pr_c"] \arrow[r, "pr_d"'] & d
\end{tikzcd}$$

the unique *product arrow* $(f \times g) : a \times b \to c \times d$ is found as $\langle f \circ pr_a, g \circ pr_b\rangle$.

$$\begin{tikzcd}
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] \arrow[d, dashed, "f \times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "pr_c"] \arrow[r, "pr_d"'] & d
\end{tikzcd}$$

**Exercise 8.5.** In the reflexive case we consider the product arrow $1_a \times 1_b$.

$$
\begin{tikzcd}
a \arrow[d, "1_a"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] \arrow[d, dashed, "1_a \times 1_b"] & b \arrow[d, "1_b"] \\
a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b
\end{tikzcd}
$$

By definition we have

$$1_a \times 1_b = \langle 1_a \circ pr_a, 1_b \circ pr_b \rangle = \langle pr_a, pr_b \rangle.$$

Then applying exercise 8.1 from here gives the desired result

$$1_a \times 1_b = 1_{a \times b}.$$

**Exercise 8.6.** The isomorphism $a \times b \cong b \times a$ follows by plugging each object into definition 8.1 in relation to the other. In this way the two unique (dashed) arrows are found:

$$
\begin{tikzcd}
a \arrow[drr] & a \times b \arrow[l] \arrow[r] \arrow[d, dashed, bend right] & b \arrow[dll] \\
b & b \times a \arrow[l] \arrow[r] \arrow[u, dashed, bend right] & a
\end{tikzcd}
$$

and fact 8.2 tells us that they are iso.

**Exercise 8.7.** In this exercise we wish to show that products are associative up to isomorphism. So given $C$-objects $a, b, c$, form the products $(a \times b) \times c$ and $a \times (b \times c)$. Here they are with the relevant projection arrows:

$$\begin{tikzcd}
 & (a \times b) \arrow[r] \arrow[dl] & b & (b \times c) \arrow[l] \arrow[dr] & \\
a & & & & c \\
 & (a \times b) \times c \arrow[uu] \arrow[urrr] & & a \times (b \times c) \arrow[uu] \arrow[ulll] &
\end{tikzcd}$$

We'll use the product definition side-by-side, noting that composition of projections gives us our arrows $a \times (b \times c) \to b$ and $(a \times b) \times c \to b$:

$$\begin{tikzcd}
 & a \times (b \times c) \arrow[dl] \arrow[d, dashed] \arrow[dr, "|" marking] & & (a \times b) \times c \arrow[dl, "|" marking] \arrow[d, dashed] \arrow[dr] & \\
a & a \times b \arrow[l] \arrow[r] & b & b \times c \arrow[l] \arrow[r] & c
\end{tikzcd}$$

The definition yields unique arrows $a \times (b \times c) \to b \times c$ and $(a \times b) \times c \to a \times b$. Using these arrows along-side the 'first-order' projection arrows, we use the product definition again to find the unique arrows

$$\begin{tikzcd}
 & a \times (b \times c) \arrow[dl, dashed] \arrow[dr] \arrow[dd, dashed, bend left] & \\
a \times b & & b \times c \\
 & (a \times b) \times c \arrow[ul] \arrow[ur, dashed] \arrow[uu, dashed, bend left] &
\end{tikzcd}$$

Using fact 8.2 gives that $(a \times b) \times c \cong a \times (b \times c)$.

**Exercise 8.8.** Consider the situation of a pair of arrows to the codomain objects of the product arrow's constituent arrows:

(i)

$$
\begin{tikzcd}
 & e \arrow[dl, "h"'] \arrow[d, dashed, "{\langle h,k\rangle}"] \arrow[dr, "k"] & \\
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] \arrow[d, dashed, "f\times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}
$$

We can use composition to collapse $a$ and $b$ out of the picture:

$$
\begin{tikzcd}
 & e \arrow[dl, "f\circ h"'] \arrow[d, "{(f\times g)\circ\langle h,k\rangle}"] \arrow[dr, "g\circ k"] & \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}
$$

and thus find that $(f \times g) \circ \langle h, k\rangle$ is the unique arrow determined by $f \circ h$ and $g \circ k$.

(ii) In the situation where we can place two product arrows end-to-end:

$$\begin{tikzcd}
e & e \times e' \arrow[l] \arrow[r] \arrow[d, dashed, "h\times k"] & e' \arrow[d, "k"] \\
a \arrow[from=1-1, "h"'] \arrow[d, "f"] & a \times b \arrow[l] \arrow[r] \arrow[d, dashed, "f\times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

we again use composition to collapse the middle level

$$\begin{tikzcd}
e \arrow[d, "f\circ h"] & e \times e' \arrow[l] \arrow[r] \arrow[d, dashed, "(f\times g)\circ(h\times k)"] & e' \arrow[d, "g\circ k"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

Giving (more details needed?)

$$(f \times g) \circ (h \times k) = (f \circ g) \times (h \circ k).$$

## 8.1 Finite Products

**Definition 8.5.** Given a $C$-object $a$, the *finite product* (for some integer $m$) consists of object $a_m$, and

$m$ projection arrows $pr_a^m$.

$$
\begin{tikzcd}
a \\
a^m \arrow[u, bend left=40, "pr"] \arrow[u, bend left=15, "pr_a^2"] \arrow[u, phantom, "\cdots"] \arrow[u, bend right=35, "pr_a^m"']
\end{tikzcd}
$$

So that for any $m$ parallel arrows $f_i : c \to a$, there is a unique arrow $\langle f_1, \dots, f_m \rangle : c \to a^m$ making

$$
\begin{tikzcd}
c \arrow[d, bend right=40, "f_1"'] \arrow[d, bend right=15, "f_2"] \arrow[d, phantom, "\cdots"] \arrow[d, bend left=10, "f_m"] \arrow[dd, dashed, bend left=50, "\langle f_1,\dots,f_m \rangle"] \\
a \\
a^m \arrow[u, bend left=40, "pr"] \arrow[u, bend left=15, "pr_a^2"] \arrow[u, bend right=10, "pr_a^m"']
\end{tikzcd}
$$

commute.

**Definition 8.6.** A more *general finite product* of $m$ (not necessarily different) $C$-objects $a_1 \times a_2 \times \ldots \times a_m$ consists of a $C$-object and $m$ projection arrows:

$$
\begin{tikzcd}
a_1 & a_2 \arrow[rr, dotted, no head] & & a_m \\
& a_1 \times \ldots \times a_m \arrow[ul, "pr_{a_1}"] \arrow[u, "pr_{a_2}"'] \arrow[urr, "pr_{a_m}"'] & &
\end{tikzcd}
$$

So that for any $c$-object with $m$ arrows $f_1 : c \to a_1$, $f_2 : c \to a_2$, etc, there is a unique arrow $\langle f_1, f_2, \ldots, f_m\rangle$ making

$$\begin{tikzcd}
 & & c \arrow[dll, "f_1"'] \arrow[dl, "f_2"] \arrow[dr, "f_m"] \arrow[dd, dashed, "\langle f_1,f_2,\ldots,f_m\rangle"] & \\
a_1 & a_2 \arrow[rr, dotted, no head] & & a_m \\
 & & a_1 \times \ldots \times a_m \arrow[ull, "pr_{a_1}"] \arrow[ul, "pr_{a_2}"'] \arrow[ur, "pr_{a_m}"'] &
\end{tikzcd}$$

commute.

**Definition 8.7.** A *general product arrow* is given by a family of $m$ mappings between the components of two general products.

$$\begin{tikzcd}
 & & b_1 \times b_2 \times \ldots \times b_m \arrow[dll, "pr_{b_1}"'] \arrow[dl, "pr_{b_2}"] \arrow[dr, "pr_{b_m}"] & \\
b_1 & b_2 \arrow[rr, dotted, no head] & & b_m \\
a_1 \arrow[u, "f_1"] & a_2 \arrow[u, "f_2"] \arrow[rr, dotted, no head] & & a_m \arrow[u, "f_m"'] \\
 & & a_1 \times \ldots \times a_m \arrow[ull, "pr_{a_1}"] \arrow[ul, "pr_{a_2}"'] \arrow[ur, "pr_{a_m}"'] \arrow[uuu, dashed, "f_1 \times f_2 \cdots \times f_m"'] &
\end{tikzcd}$$

The product arrow $f_1 \times f_2 \cdots \times f_m$ is the unique arrow found by using

$$\langle f_1 \circ pr_{a_1}, f_2 \circ pr_{a_2}, \ldots, f_m \circ pr_{a_m}\rangle$$

in the general product definiton of $b_1 \times b_2 \times \ldots \times b_m$.

# 9 Co-products

**Definition 9.1.** A *co-product* of $C$-objects $a$ and $b$ is given by a a $C$-object denoted $a+b$, and injection functions $i_a$ and $i_b$.

$$a \xrightarrow[i_a]{} a+b \xleftarrow[i_b]{} b$$

For any $f : a \to c$ and $g : b \to c$, there is a unique arrow $[f, g] : (a + b) \to c$ so that

$$\begin{tikzcd} a \arrow[r, "i_a"'] \arrow[dr, "f"'] & a+b \arrow[d, dashed, "{[f,g]}"] & b \arrow[l, "i_b"] \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes.

**Exercise 9.1.** In **Set** we are told that the co-product $A + B$ is the disjoint union, with $i_A$ and $i_B$ being the disjoint identity function (Ie, $i_A(x) = (x, 0)$ for $x \in A$, and $i_B(x) = (x, 1)$ for $x \in B$. Now suppose we have functions $f : A \to C$ and $g : B \to C$.

$$\begin{tikzcd} A \arrow[r, "i_A"'] \arrow[dr, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "i_B"] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

We can find $[f, g] : A + B \to C$ making this diagram commute by using the rule

$$[f,g](\langle x,y\rangle) = \begin{cases} f(x) & y = 0 \\ g(x) & y = 1. \end{cases}$$

To see that it $[f, g]$ is unique, notice that as given, $[f, g] \circ i_A = f$ iff $[f, g](i_A(x)) = f(x)$ for all $x \in A$. Similarly on the $B$ side - there is no other way to recover the action of $f$ and $g$ out of $A + B$.

**Exercise 9.2.** If $A \cup B = \emptyset$ then we notice that $A \cup B$ satisfies the definition of co-product

$$\begin{tikzcd} A \arrow[r, "i_A"'] \arrow[dr, "f"'] & A \cup B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "i_B"] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

where $i_A$ and $i_B$ are the inclusion functions, and

$$[f,g] = \begin{cases} f(x) & x \in A \\ g(x) & x \in B, \end{cases}$$

which is well defined because any $x$ is in either $A$ or $B$ but not both. Then applying the dual of fact 8.3, co-products are isomorphic and therefore $A \cup B \cong A + B$.

**Definition 9.2.** Given two co-products $a + b$ and $c + d$, and arrows $f : a \to c$ and $g : b \to d$

$$\begin{tikzcd} a \arrow[r, "i_a"'] \arrow[d, "f"] & a+b & b \arrow[l, "i_b"] \arrow[d, "g"] \\ c \arrow[r, "i_c"'] & c+d & d \arrow[l, "i_d"] \end{tikzcd}$$

the *co-product arrow* $f+g$ is found by using $[i_c \circ f, i_d \circ g]$ in the co-product definition of $c+d$.

$$\begin{tikzcd}
a \arrow[r, "i_a"'] \arrow[d, "f"] & a+b \arrow[d, dashed, "f+g"] & b \arrow[l, "i_b"'] \arrow[d, "g"] \\
c \arrow[r, "i_c"'] & c+d & d \arrow[l, "i_d"']
\end{tikzcd}$$

# 10 Equalizers

**Definition 10.1.** An arrow $i$ *equalizes* $f$ and $g$ if they are laid out as follows

$$\begin{tikzcd}
e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b
\end{tikzcd}$$

where $f \circ i = g \circ i$. Additionaly we demand the limiting property: if another $e^*$ and $i^*$ work as above, there is a unique arrow $e^* \to e$ making

$$\begin{tikzcd}
e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
e^* \arrow[u, dashed] \arrow[ur, "i^*"'] & &
\end{tikzcd}$$

commute.

**Fact 10.2.** *Every equalizer is monic.*

**Fact 10.3.** *An epic equalizer is iso.*

$$\begin{tikzcd}
e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
a \arrow[u, dashed] \arrow[ur, "1_a"'] & &
\end{tikzcd}$$

**Exercise 10.1.** Working in **Set**, we wish to show that monics are equalizers. Suppose we have some injective function $i : E \to A$:

$$\begin{tikzcd}
E \arrow[r, tail, "i"] & A
\end{tikzcd}$$

We seek functions which $i$ is an equalizer. Let $f, g : A \rightrightarrows \{0,1\}$ be given by

$$f(x) = 1$$

$$g(x) = \begin{cases} 1 & x \in i(E) \\ 0 & x \notin i(E). \end{cases}$$

Now clearly $g(i(x)) = f(i(x))$ for all $x \in E$, so $i \circ f = i \circ g$. Supposing that there is another $i^* : C \to A$ such that $i^* \circ f = i^* \circ g$. We must have ...?

$$\begin{tikzcd}
E \arrow[r, tail, "i"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & \{0,1\} \\
C \arrow[u, dashed] \arrow[ur, "i^*"'] & &
\end{tikzcd}$$

**Exercise 10.2.** Working in a poset, suppose that $i$ equalizes $f$ and $g$. Recall that any 2 parallel arrows are equal. Then in particular $f \circ i = g \circ i$ as follows:

$$\begin{tikzcd}
e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b
\end{tikzcd}$$

But plugging $a$ and $1_a$ into the definition for equalizer, since $f \circ 1_a = f = g = g \circ 1_a$ we retrieve the unique arrow $a \to e$.

$$\begin{tikzcd}
e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
a \arrow[u, dashed] \arrow[ur, "1_a"'] & &
\end{tikzcd}$$

Since we have arrows $a \to e$ and $e \to a$ we simply apply the antisymmetric property of posets to determine $e = a$, and the single-arrow property to determine that $i = 1_a$.

# 11 Limits and co-limits

**Definition 11.1.** A *diagram...*

**Definition 11.2.** Given a diagram $D$, a a *D-cone* consists of a $C$-object $c$ together with component arrows $f_i : c_i \to d_i$ for each $d_i \in d$ that commute with any arrow $g$ in $D$.

$$\begin{tikzcd} & c \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ d_i \arrow[rr] & & d_j \end{tikzcd}$$

**Definition 11.3.** Given a diagram $D$, a *limit* for $D$ is a $D$-cone such that any other $D$-cone factors through uniquely.

$$\begin{tikzcd} c \arrow[dr, "f_i"'] \arrow[drr, "f_j"] & & & c' \arrow[lll, dashed] \arrow[dll, "f'_i"'] \arrow[dl, "f'_j"] \\ & d_i \arrow[r] & d_j & \end{tikzcd}$$