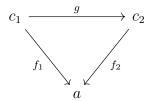
Topoi by Robert Goldblatt Notes by R. Riley Holmes Chapter 2: What Categories Are

Def: The comma-category $C \downarrow a$ is formed from any category C and any C-object a. Its objects are all the C-arrows with codomain a. (IE, $f_1: c_1 \to a$ and $f_2: c_2 \to a$). Its arrows are all C-arrows between the objects' domains, that commute with the 'object arrows'. (IE, $g: c_1 \to c_2$ so that $f_1 = f_2 \circ g$). So $C \downarrow a$ looks like this:

$$f_1 \xrightarrow{g} f_2$$

and indicates that this diagram commutes in the original category:



(TODO: Verify category axioms)

Example. Take C to be the preorder on natural numbers, and a to be a given number. For example, let a = 3.

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \dots$$

Objects in $C \downarrow 3$ are statements of 'n-less-than-3' relationships. Arrows in $C \downarrow 3$ from 'm-less-than-3' to 'n-less-than-3' are 'm-less-than-m'.

$$(1 \le 3) \xrightarrow{1 \le 3} (2 \le 3) \xrightarrow{2 \le 3} (3 \le 3)$$

Example. Recall Matr(k) has the natural numbers \mathbb{N} as objects, and $(n \times m)$ matrices as arrows from $m \to n$.

Then $Matr(k) \downarrow 3$ has as objects all $3 \times n$ matrices where $n \in \mathbb{N}$.

Then if object A is a $3 \times m$ matrix, and object C is a $3 \times n$ matrix, then an arrow $B: A \to C$ is an $m \times n$ matrix such that A = BC.

Thus the situation in $Matr(k) \downarrow 3$ is

$$A_{3\times m} \xrightarrow{B_{n\times m}} C_{3\times n}$$

indicating that this diagram commutes in Matr(k):

