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Chapter 3: Arrows Instead of Epsilon

1 Monic arrows

Monic arrows are an abstraction of injective functions.

Definition 1.1. An arrow $f : a \to b$ in a category C is *monic* if for any g_1, g_2 with codomain a, the implication

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

holds. Or, if the diagram

$$\begin{tikzcd} c \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

commutes, then $g_1 = g_2$.

Definition 1.2. *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *monic* iff for any C-object c, post-composition with f defines an injection $f_* : C(c, a) \to C(c, b)$. (Here $C(x, y)$ is the set of C-arrows from x to y.)

For both exercises in this section, take the situation to be as follows:

$$\begin{tikzcd} s \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & a \arrow[r, "f"] & b \arrow[r, "g"] & c \end{tikzcd}$$

Where f and g are fixed, and s, h_1, h_2 are 'any such' objects/arrows.

Exercise 1.1.

Suppose that f and g are both monic, and that $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Since g is monic, that implies $f \circ h_1 = f \circ h_2$. But since f is monic, that implies $h_1 = h_2$. So using associativity and collapsing the chain of implication gives

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \implies h_1 = h_2.$$

Conclude $g \circ f$ is monic.

Exercise 1.2.

Now suppose that $g \circ f$ is monic. If $f \circ h_1 = f \circ h_2$ then clearly $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Then $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ and since $g \circ f$ is monic, $h_1 = h_2$. So

$$f \circ h_1 = f \circ h_2 \implies h_1 = h_2,$$

meaning f is monic.

2 Epic arrows

Definition 2.1. If f is *epic* then commutativity of

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & c \end{tikzcd}$$

implies $g_1 = g_2$.

Definition 2.2. *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *epic* iff for any C-object c, pre-composition with f defines an injection $f^* : C(b, x) \to C(a, c)$. (Here $C(x, y)$ is the set of C-arrows from x to y.)

Dually to the exercises proven in the previous section we have

Fact 2.3. *If $f : a \to b$ and $g : b \to c$ are epic, then $g \circ f : a \to c$ is epic.*

Fact 2.4. *If $g \circ f : a \to c$ is epic, then $g : b \to c$ is epic.*

3 Iso arrows

Definition 3.1. An arrow $f : a \to b$ is *iso* if there exists another arrow $f^{-1} : b \to a$ such that

$$f \circ f^{-1} = 1_b$$

and

$$f^{-1} \circ f = 1_a.$$

This diagram commutes when the identity loops are included:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Fact 3.2. *If an arrow is iso then it is epic and monic, but the converse isn't necessarily true. The converse* is *true in* ***Set*** *and any Topos.*

Exercise 3.1. For any object a, the identity morphism 1_a is an inverse to itself and therefore is iso. Simply because

$$1_a \circ 1_a = 1_a.$$

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] & a \arrow[l, bend left, "1_a"] \end{tikzcd}$$

Exercise 3.2. If $f : a \to b$ is iso then we can retrieve f^{-1} and then plug it right into the definition and find

$$f^{-1} \circ f = 1_a$$

and

$$f \circ f^{-1} = 1_b,$$

indicating that f^{-1} is iso.

$$\begin{tikzcd} a \arrow[r, bend left, "f^{-1}"] & b \arrow[l, bend left, "f"] \end{tikzcd}$$

Exercise 3.3. With $f : a \to b$ and $g : b \to c$ both iso, the situation looks like the following:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & c \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

Now we find that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_b \circ f = f^{-1} \circ f = 1_a$$

and

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_b \circ g^{-1} = g \circ g^{-1} = 1_c.$$

Thus $(f^{-1} \circ g^{-1})$ acts as an inverse to $g \circ f$, and $g \circ f$ is iso.

4 Isomorphic objects

Definition 4.1. Two C-objects a and b are *isomorphic*, or

$$a \cong b$$

if there exists an iso C-arrow

$$f : a \to b.$$

Definition 4.2. A category C is *skeletal* if $a \cong b$ implies $a = b$.

Exercise 4.1.

We wish to show that object isomorphism is an equivalence relation, or that it's reflexive, symmetric, and transitive. Fortunately the exercises from section 3 correspond exactly to these properties.

(i) $a \cong a$ since 1_a is iso.

(ii) If $a \cong b$ then some $f : a \to b$ is iso, and therefore $f^{-1} : b \to a$ is iso and $b \cong a$.

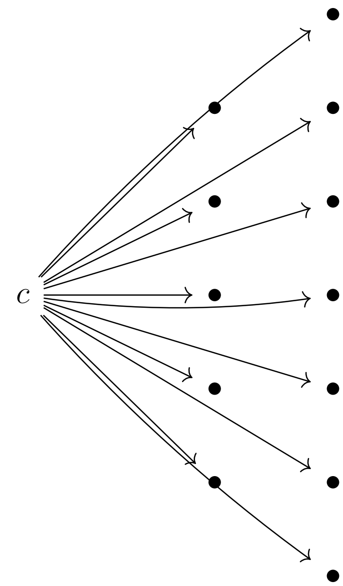
(iii) If $a \cong b$ and $b \cong c$ then we have iso arrows $f : a \to b$ and $g : b \to c$. Then $g \circ f$ is iso, and $a \cong c$.

Exercise 4.2.

Suppose a and b are two **Finord**-objects such that $a \cong b$. Then there is some $f : a \to b$ that is iso. Since **Finord** is a subcategory of **Set**, iso arrows correspond to bijective functions. Then a and b must have the same cardinality, but by the definition of **Finord** distinct objects have distinct cardinalities. So $a = b$ and **Finord** is skeletal.

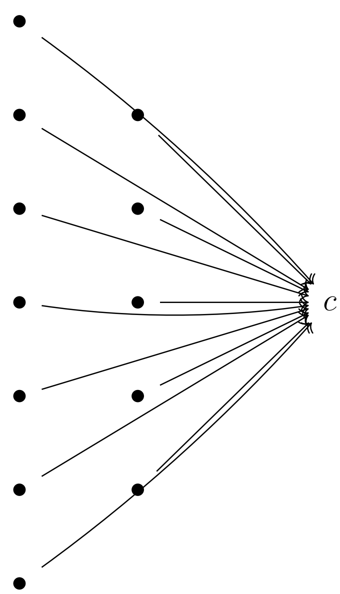
5 Initial objects

Definition 5.1. An object c is *initial* if for every C-object a, there is exactly one arrow $f : c \to a$.

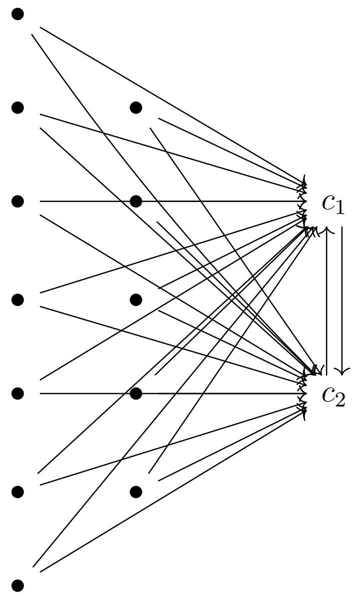


6 Terminal objects

Definition 6.1. An object c is *terminal* if for every C-object a, there is exactly one arrow $f : a \to c$.



Exercise 6.1. Let c_1 and c_2 be terminal $C - objects$.



By terminality there is a unique arrow $f_1 : c_1 \to c_2$ and a unique arrow $f_2 : c_2 \to c_1$. Then by the category axiom, $f_2 \circ f_1 : c_1 \to c_1$ and $f_1 \circ f_2 : c_2 \to c_2$ must exist. But again by terminality, there is a unique arrow $1_{c_1} : c_1 \to c_1$ and $1_{c_2} : c_2 \to c_2$, so the composition of f_1 and f_2 must give the identity. Conclude $c_1 \cong c_2$.

Exercise 6.2. (i) Terminal objects in $\mathbf{Set}^2$ are of the form $\langle \{e_1\}, \{e_2\} \rangle$, or pairs of singleton sets.

(ii) Terminal objects in $\mathbf{Set}^{\to}$ are arrows with singleton sets as domain and codomain.

(iii) The terminal object in the poset $(n, \leq)$ is the maximal element n, since $m \leq n$ for every m.

Exercise 6.3. Suppose $f : 1 \to a$ has its domain 1 a terminal object, and g_1, g_2 are any two parallel arrows from $c \to 1$.

$$\begin{tikzcd} c \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & 1 \arrow[r, "f"] & a \end{tikzcd}$$

Well, since 1 is terminal the arrow from $c \to 1$ is unique and we see that $g_1 = g_2$, so regardless of whether $g_1 \circ f = g_2 \circ f$ holds (which it does), we can conclude f is monic.

7 Duality

Any category can be turned into its opposite category. So any statement about a category can be dualized with all the arrows reversed.

8 Products

Definition 8.1. Given C-objects a and b, a *product* is a C-object $a \times b$ and 2 C-arrows pr_a, pr_b.

$$a \xleftarrow{pr_a} a \times b \xrightarrow{pr_b} b$$

For any c, f, g configured as follows

$$\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[dr, "g"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

f and g determine a unique $h : c \to (a \times b)$ so that

$$\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "h"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

commutes. This is denoted

$$c := \langle f, g \rangle.$$

Fact 8.2. *If c is a product $a \times b$, any arrow $f : c \to c$, f must be the identity 1_c. First observe that the identity must exist. Then plug c into definition 8.1 to see that f must be the unique arrow with that domain and codomain.*

$$\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, leftrightarrow, "1_c"] & \\
a & c \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Fact 8.3. *Any two products of a and b, say $a \times_1 b$ and $a \times_2 b$, are isomorphic to each other. Consider that in the diagram*

$$\begin{tikzcd}
& a \times_2 b \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, bend right, "h_1"'] & \\
a & & b \\
& a \times_1 b \arrow[ul, "pr_a"] \arrow[ur, "pr_b"'] \arrow[uu, dashed, bend right, "h_2"'] &
\end{tikzcd}$$

h_1 and h_2 are uniquely determined by symmetric applications of definition 8.1. But by fact 8.2, composition of h_1 and h_2 must give identities.

Exercise 8.1. The fact that $\langle pr_a, pr_b \rangle = 1_{a \times b}$ follows as a special case of fact 8.2, by plugging in the projection functions.

$$\begin{tikzcd}
& a \times b \arrow[dl, "pr_a"'] \arrow[dr, "pr_b"] \arrow[d, dashed, leftrightarrow, "1_{a \times b}"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Exercise 8.2. Suppose we have parallel $f, k : c \rightrightarrows a$ and $g, h : c \rightrightarrows b$ and $p : c \to a \times b$ such that $p = \langle f, g \rangle = \langle k, h \rangle$. Then $f = pr_a \circ p$ and $k = pr_a \circ p$. It doesn't take any special cancellation rules to see that identically $f = k$. Similarly $g = h$.

$$\begin{tikzcd}
& c \arrow[dl, bend right, "f"'] \arrow[dl, "k"] \arrow[dr, bend left, "g"] \arrow[dr, "h"'] \arrow[d, dashed, "p"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Exercise 8.3. Suppose the situation is as follows.

$$
\begin{tikzcd}
 & d \arrow[d, "h"'] & \\
 & c \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, bend left, "g"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}
$$

Then by compositionality there must exist $h \circ f : d \to a$ and $g \circ f : d \to b$. There also must exist $h \circ \langle f, g \rangle : d \to a \times b$. By collapsing the diagram to

$$
\begin{tikzcd}
 & d \arrow[dl, bend right, "f \circ h"'] \arrow[d, dashed, "\langle f \circ h, g \circ h \rangle"] \arrow[dr, bend left, "g \circ h"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}
$$

we see the arrow from $d \to a \times b$ must be unique, and therefore

$$\langle f \circ h, g \circ h \rangle = h \circ \langle f, g \rangle.$$

Exercise 8.4. Suppose a category C has a terminal object t, and products. Let a be a C-object and consider the product $a \times t$.

$$a \longleftarrow a \times t \longrightarrow t$$

Plugging a into the product definition using the arrows given by 1_a and ! (the unique arrow to t) yields the unique arrow $h = \langle 1_a, ! \rangle$.

$$
\begin{tikzcd}
 & a \arrow[dl, "1_a"'] \arrow[d, dashed, "h"] \arrow[dr, "!"] & \\
a & a \times t \arrow[l, "pr_a"] \arrow[r] & t
\end{tikzcd}
$$

By definition we have that $pr_a \circ h = 1_a$. And since $h \circ pr_a$ maps $a \times t \to a \times t$ it follows from fact 8.2 that $h \circ pr_a = 1_{a \times t}$. Given these two iso arrows we conclude

$$a \cong a \times t.$$

Example. As a specific example of exercise 8.4, let's work in **Set** where A is any set, and our terminal set t is any singleton set $\{b\}$.

$$
\begin{tikzcd}
 & A \arrow[dl, "1_a"'] \arrow[d, dashed, "h"] \arrow[dr, "!"] & \\
A & A \times \{b\} \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & \{b\}
\end{tikzcd}
$$

Now we can explicitly say what all of our functions do:

$$1_a(x) = x$$

$$!(x) = b$$
$$pr_a\left(\langle a, b\rangle\right) = a$$
$$pr_b\left(\langle a, b\rangle\right) = b$$
$$h(x) = \langle x, b\rangle$$

The fact that $pr_a(h(x)) = x$ and $h(pr_a(\langle a, b\rangle)) = \langle a, b\rangle$ gives the isomorphism

$$A \cong A \times \{b\}\,.$$

Intuitively, elements in A can be placed in one-to-one correspondence with elements in $A \times \{b\}$ by simply sending x to the tuple $\langle x, b\rangle$.

Definition 8.4. Given two products $a \times b$ and $c \times d$, and arrows $f : a \to c$ and $g : b \to d$

$$\begin{tikzcd}
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"'] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "pr_c"] \arrow[r, "pr_d"] & d
\end{tikzcd}$$

the unique *product arrow* $(f \times g) : a \times b \to c \times d$ is found as $\langle f \circ pr_a, g \circ pr_b\rangle$.

$$\begin{tikzcd}
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"'] \arrow[d, dashed, "f \times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "pr_c"] \arrow[r, "pr_d"] & d
\end{tikzcd}$$

Exercise 8.5. In the reflexive case we consider the product arrow $1_a \times 1_b$.

$$\begin{tikzcd}
a \arrow[d, "1_a"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] \arrow[d, dashed, "1_a \times 1_b"] & b \arrow[d, "1_b"] \\
a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b
\end{tikzcd}$$

By definition we have

$$1_a \times 1_b = \langle 1_a \circ pr_a, 1_b \circ pr_b \rangle = \langle pr_a, pr_b \rangle.$$

Then applying exercise 8.1 from here gives the desired result

$$1_a \times 1_b = 1_{a \times b}.$$

Exercise 8.6. The isomorphism $a \times b \cong b \times a$ follows by plugging each object into definition 8.1 in relation to the other. In this way the two unique (dashed) arrows are found:

$$\begin{tikzcd}
a \arrow[drr] & a \times b \arrow[l] \arrow[r] \arrow[d, dashed, bend right] & b \arrow[dll] \\
b & b \times a \arrow[l] \arrow[r] \arrow[u, dashed, bend right] & a
\end{tikzcd}$$

and fact 8.2 tells us that they are iso.

Exercise 8.7.

$$\begin{tikzcd}
 & (a \times b) \arrow[r] \arrow[dl] & b & (b \times c) \arrow[l] \arrow[dr] & \\
a & & & & c \\
 & (a \times b) \times c \arrow[uu] \arrow[urrr] & & a \times (b \times c) \arrow[uu] \arrow[ulll] &
\end{tikzcd}$$

Exercise 8.8. Consider the situation of a pair of arrows to the codomain objects of the product arrow's constituent arrows:

(i)

$$\begin{tikzcd}
 & e \arrow[dl, "h"'] \arrow[d, dashed, "\langle h{,}k \rangle"] \arrow[dr, "k"] & \\
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] \arrow[d, dashed, "f \times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

We can use composition to collapse a and b out of the picture:

$$\begin{tikzcd}
 & e \arrow[dl, "f \circ h"'] \arrow[d, "(f \times g) \circ \langle h{,}k \rangle"] \arrow[dr, "g \circ k"] & \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

and thus find that $(f \times g) \circ \langle h, k \rangle$ is the unique arrow determined by $f \circ h$ and $g \circ k$.

(ii) In the situation where we can place two product arrows end-to-end:

$$\begin{tikzcd}
e & e \times e' \arrow[l] \arrow[r] \arrow[d, dashed, "h\times k"] & e' \arrow[d, "k"] \\
a \arrow[d, "f"] & a \times b \arrow[l] \arrow[r] \arrow[d, dashed, "f\times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

we again use composition to collapse the middle level

$$\begin{tikzcd}
e \arrow[d, "f\circ h"] & e \times e' \arrow[l] \arrow[r] \arrow[d, dashed, "(f\times g)\circ(h\times k)"] & e' \arrow[d, "g\circ k"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

Giving (more details needed?)

$$(f \times g) \circ (h \times k) = (f \circ g) \times (h \circ k).$$

8.1 Finite Products

Definition 8.5. Given a C-object a, the *finite product* (for some integer m) consists of object a_m, and

m projection arrows pr_a^m.

$$\begin{tikzcd}
a \\
a^m \arrow[u, bend left=40, "pr"] \arrow[u, bend left=10, "pr_a^2"] \arrow[u, phantom, "\cdots"] \arrow[u, bend right=40, "pr_a^m"']
\end{tikzcd}$$

So that for any m parallel arrows $f_i : c \to a$, there is a unique arrow $\langle f_1, \ldots, f_m \rangle : c \to a^m$ making

$$\begin{tikzcd}
c \arrow[d, bend right=40, "f_1"'] \arrow[d, bend right=15, "f_2"] \arrow[d, phantom, "\cdots"] \arrow[d, bend left=5, "f_m"] \arrow[dd, dashed, bend left=60, "\langle f_1,\ldots,f_m \rangle"] \\
a \\
a^m \arrow[u, bend left=40, "pr"] \arrow[u, bend left=15, "pr_a^2"] \arrow[u, bend right=5, "pr_a^m"']
\end{tikzcd}$$

commute.

Definition 8.6. A more *general finite product* of m (not necessarily different) C-objects $a_1 \times a_2 \times \ldots \times a_m$ consists of a C-object and m projection arrows:

$$\begin{tikzcd}
a_1 & a_2 \arrow[rr, phantom, "\cdots"] & & a_m \\
& a_1 \times \ldots \times a_m \arrow[ul, "pr_{a_1}"] \arrow[u, "pr_{a_2}"'] \arrow[urr, "pr_{a_m}"'] & &
\end{tikzcd}$$

So that for any c-object with m arrows $f_1 : c \to a_1$, $f_2 : c \to a_2$, etc, there is a unique arrow $\langle f_1, f_2, \ldots, f_m \rangle$ making

$$
\begin{tikzcd}
 & & c \arrow[dll, "f_1"'] \arrow[dl, "f_2"] \arrow[dd, dashed, "{\langle f_1, f_2, \ldots, f_m \rangle}"] \arrow[dr, "f_m"] & \\
a_1 & a_2 \arrow[rr, dotted, no head] & & a_m \\
 & & a_1 \times \ldots \times a_m \arrow[ull, "pr_{a_1}"] \arrow[ul, "pr_{a_2}"'] \arrow[ur, "pr_{a_m}"'] &
\end{tikzcd}
$$

commute.

Definition 8.7. A *general product arrow* is given by a family of m mappings between the components of two general products.

$$
\begin{tikzcd}
 & & b_1 \times b_2 \times \ldots \times b_m \arrow[dll, "pr_{b_1}"'] \arrow[dl, "pr_{b_2}"] \arrow[dr, "pr_{b_m}"] & \\
b_1 & b_2 \arrow[rr, dotted, no head] & & b_m \\
a_1 \arrow[u, "f_1"] & a_2 \arrow[u, "f_2"] \arrow[rr, dotted, no head] & & a_m \arrow[u, "f_m"'] \\
 & & a_1 \times \ldots \times a_m \arrow[ull, "pr_{a_1}"] \arrow[ul, "pr_{a_2}"'] \arrow[ur, "pr_{a_m}"'] \arrow[uuu, dashed, "f_1 \times f_2 \cdots \times f_m"'] &
\end{tikzcd}
$$

The product arrow $f_1 \times f_2 \cdots \times f_m$ is the unique arrow found by using

$$\langle f_1 \circ pr_{a_1}, f_2 \circ pr_{a_2}, \ldots, f_m \circ pr_{a_m} \rangle$$

in the general product definiton of $b_1 \times b_2 \times \ldots \times b_m$.