Chapter 4

Introducing Topoi

4.1 Subobjects

Definition 4.1.1. A *subobject* of d is an equivalence class of monic arrows into d.

We are using a new notion of equivalence. Start with the idea that a subobject is simply a monic arrow into d. Now write that $f \subseteq g$ *iff* these is an arrow $h : a \to b$ making

$$\begin{tikzcd}
a \arrow[dr, tail, "f"] \arrow[dd, "h"'] & \\
& d \\
b \arrow[ur, tail, "g"'] &
\end{tikzcd}$$

commute.

Now say that $f \cong g$ *iff* $f \subseteq g$ and $g \subseteq f$.

$$\begin{tikzcd}
a \arrow[dr, tail, "f"] \arrow[dd, bend right, "i"'] & \\
& d \\
b \arrow[ur, tail, "g"'] \arrow[uu, bend right, "h"'] &
\end{tikzcd}$$

Here, $g \circ h = f$ and $f \circ i = g$

Exercise 4.1.1. We wish to show that in the above diagram, a and b are isomorphic. Substituting each of our 'commuting' equations into the other in turn yields

$$f \circ i \circ h = f,$$

$$g \circ h \circ i = g.$$

Since both f and g are monic, left-cancel to get

$$i \circ h = 1_a,$$

$$h \circ i = 1_b.$$

Conclude that h and i are monic and mutual inverses. Thus a and b are isomorphic.

Exercise 4.1.2. Now we wish to show that $\cong$ in this sense is indeed an equivalence relation.

1. *reflexivity:* $f \cong f$ for $f : a \to b$ since there is an arrow $1_a : a \to a$, and the relationship $a \subseteq a$ is reflexive.

2. *symmetry:* If $f \cong g$ then $f \subseteq g$ and $g \subseteq f$. So by the symmetry of the word 'and' we have that $g \cong f$.

3. *transitivity:* Suppose $f \cong g$ and if $g \cong k$. Then from the first relation we know $f \subseteq g$ and $g \subseteq f$. From the second we know $g \subseteq k$ and $k \subseteq g$.

   Since $f \subseteq g$ and $g \subseteq k$, $f \subseteq k$. And since $k \subseteq g$ and $g \subseteq f$, $k \subseteq f$.

   Since $f \subseteq k$ and $k \subseteq f$, conclude $f \cong k$.

Exercise 4.1.3. Suppose that $[f] = [f']$ and $[g] = [g']$.

($\Rightarrow$) Suppose $f \subseteq g$. Since $f \cong f'$, $f' \subseteq f$. Then $f' \subseteq g'$

($\Leftarrow$) Suppose $f' \subseteq g'$. Since $f \cong f'$, $f \subseteq f'$ Then $f \subseteq g$.

Conclude $f \subseteq g$ iff $f' \subseteq g'$. In other words $\subseteq$ is stable under $\cong$.

Exercise 4.1.4.

$$Sub(D) \cong P(D)$$

Definition 4.1.2. The *naming arrow* for f

Exercise 4.1.5. For any element $x : 1 \to a$ of a,

$$ev \circ \langle \ulcorner f \urcorner, x \rangle = f \circ x$$

4.2 Classifying Subobjects

We are seeking to generalize the situation where a subset $A \subseteq D$ has a characteristic function $\chi_A : D \to \{0, 1\}$, where

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

In fact the functions from $D \to \{0, 1\}$ are in bijection with all possible subsets of D. Then given an $f : D \to \{0, 1\}$ we define A_f as the subset determined by f, and then we have

$$\begin{tikzcd} A_f \arrow[r, hook] \arrow[d, "!"'] & D \arrow[d, "f"] \\ 1 \arrow[r, "true"'] & \{0,1\} \end{tikzcd}$$

is a pullback.

Definition 4.2.1. If C is a category with a terminal object 1, then a *subobject classifier* consist of an object Ω, and an arrow $true : 1 \to \Omega$.

The Ω axiom is that for any monic $f : a \to d$ there is a unique arrow $\chi_f : d \to \Omega$ so that

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "!"'] & d \arrow[d, "\chi_f"] \\ 1 \arrow[r, "true"'] & \Omega \end{tikzcd}$$

is a pullback square.

The arrow χ_f is the *characteristic arrow* of f.

Definition 4.2.2. For any a there is a unique arrow $! : a \to 1$. The composite $true \circ !$ yields an arrow denoted $true_a$:

$$\begin{tikzcd} a \arrow[r, "!"] \arrow[dr, "true_a"'] & 1 \arrow[d, "true"] \\ & \Omega \end{tikzcd}$$

Exercise 4.2.1. Plugging $true : 1 \to \Omega$ into definition 4.2.1 gives:

$$\begin{tikzcd} 1 \arrow[r, tail, "true"] \arrow[d, "!"'] & \Omega \arrow[d, "\chi_{true}"] \\ 1 \arrow[r, "true"'] & \Omega \end{tikzcd}$$

and we see that ! must be the identity $1_1 : 1 \to 1$. So for the diagram to be a pullback square we must have

$$\chi_{true} \circ true = true \circ 1_1 = true$$

and therefore

$$\chi_{true} = 1_\Omega.$$

Exercise 4.2.2. Plugging $1_\Omega : \Omega \to \Omega$ into definition 4.2.1 gives

$$\begin{tikzcd} \Omega \arrow[r, tail, "true"] \arrow[d, "!"'] & \Omega \arrow[d, "\chi_{1_\Omega}"] \\ 1 \arrow[r, "true"'] & \Omega \end{tikzcd}$$

To be a pullback we must have $\chi_{1_\Omega} \circ 1_\Omega = true \circ !$. Or

$$\chi_{1_\Omega} = true_\Omega$$

using the notation of definition 4.2.2.

Exercise 4.2.3. For any $f : a \to b$, since $!_a : a \to 1$ is unique:

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "!_a"'] & & b \arrow[dl, "!_b"] \\ & 1 & \end{tikzcd}$$

We must have $!_b \circ f = !_a$. Thus

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "!_a"'] & & b \arrow[dl, "!_b"] \\ & 1 \arrow[d, "true"] & \\ & \Omega & \end{tikzcd}$$

commutes, or $true \circ !_b \circ f = true \circ !_a$. Therefore using definition 4.2.2,

$$true_b \circ f = true_a.$$

4.3 Definition of Topos

Definition 4.3.1. An *elementary topos* is a category which

1. is finitely complete.
2. is finitely co-complete.
3. has exponentiation.
4. has a subobject classifier.

Fact 4.3.2. *More succinctly, a topos can be defined as a Cartesian closed category with a subobject classifier.*