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Chapter 2: What Categories Are

Def: The comma-category $C \downarrow a$ is formed from any category C and any C-object a. Its objects are all the C-arrows with codomain a. (IE, $f_1 : c_1 \to a$ and $f_2 : c_2 \to a$). Its arrows are all C-arrows between the objects' domains, that commute with the 'object arrows'. (IE, $g : c_1 \to c_2$ so that $f_1 = f_2 \circ g$). So $C \downarrow a$ looks like this:

$$f_1 \xrightarrow{g} f_2$$

and indicates that this diagram commutes in the original category:

$$\begin{tikzcd} c_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & c_2 \arrow[dl, "f_2"] \\ & a & \end{tikzcd}$$

(TODO: Verify category axioms)

Example. *Take C to be the preorder on natural numbers, and a to be a given number. For example, let $a = 3$.*

$$1 \longrightarrow 2 \longrightarrow \boldsymbol{3} \longrightarrow 4 \longrightarrow 5 \longrightarrow \dots$$

Objects in $C \downarrow 3$ are statements of '$n \leq 4$' relationships. Arrows in $C \downarrow 3$ from '$m \leq 3$' to '$n \leq 3$' are '$m \leq n$'.

$$\begin{tikzcd} (1 \leq 3) \arrow[r, "1\leq 2"'] \arrow[rr, bend left, "1\leq 3"] & (2 \leq 3) \arrow[r, "2\leq 3"'] & (3 \leq 3) \end{tikzcd}$$

Example. *Recall **Matr(k)** has the natural numbers $\mathbb{N}$ as objects, and $(n \times m)$ matrices as arrows from $m \to n$.*

*Then **Matr(k)** $\downarrow 3$ has as objects all $3 \times n$ matrices where $n \in \mathbb{N}$.*

Then if object A is a $3 \times m$ matrix, and object B is a $3 \times n$ matrix, then an arrow $C : A \to B$ is an $n \times m$ matrix such that $A = BC$.

*Thus the situation in **Matr(k)** $\downarrow 3$ is*

$$A_{3\times m} \xrightarrow{C_{n\times m}} B_{3\times n}$$

*indicating that this diagram commutes in **Matr(k)**:*

$$\begin{tikzcd} m \arrow[rr, "C_{n\times m}"] \arrow[dr, "A_{3\times m}"'] & & n \arrow[dl, "B_{3\times n}"] \\ & 3 & \end{tikzcd}$$