

1 Monic arrows

Monic arrows are an abstraction of injective functions.

Definition 1.1. An arrow $f : a \rightarrow b$ in a category C is *monic* if for any g_1, g_2 with codomain a , the implication

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

holds. Or, if the diagram

$$c \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} a \xrightarrow{f} b$$

commutes, then $g_1 = g_2$.

Definition 1.2. (Alternatively, Riehl pg. 11) An arrow $f : a \rightarrow b$ is *monic* iff for any C -object c , post-composition with f defines an injection $f_* : C(c, a) \rightarrow C(c, b)$. (Here $C(x, y)$ is the set of C -arrows from x to y .)

For both exercises in this section, take the situation to be as follows:

$$s \begin{array}{c} \xrightarrow{h_1} \\ \xrightarrow{h_2} \end{array} a \xrightarrow{f} b \xrightarrow{g} c$$

Where f and g are fixed, and s, h_1, h_2 are ‘any such’ objects/arrows.

Exercise 1.1.

Suppose that f and g are both monic, and that $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Since g is monic, that implies $f \circ h_1 = f \circ h_2$. But since f is monic, that implies $h_1 = h_2$. So using associativity and collapsing the chain of implication gives

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \implies h_1 = h_2.$$

Conclude $g \circ f$ is monic.

Exercise 1.2.

Now suppose that $g \circ f$ is monic. If $f \circ h_1 = f \circ h_2$ then clearly $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Then $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ and since $g \circ f$ is monic, $h_1 = h_2$. So

$$f \circ h_1 = f \circ h_2 \implies h_1 = h_2,$$

meaning f is monic.

2 Epic arrows

Definition 2.1. If f is *epic* then commutativity of

$$a \xrightarrow{f} b \begin{array}{c} \xrightarrow{g_1} \\ \xleftarrow{g_2} \end{array} c$$

implies $g_1 = g_2$.

Definition 2.2. (Alternatively, Riehl pg. 11) An arrow $f : a \rightarrow b$ is *epic* iff for any C -object c , pre-composition with f defines an injection $f^* : C(b, c) \rightarrow C(a, c)$. (Here $C(x, y)$ is the set of C -arrows from x to y .)

Dually to the exercises proven in the previous section we have

Fact 2.3. If $f : a \rightarrow b$ and $g : b \rightarrow c$ are *epic*, then $g \circ f : a \rightarrow c$ is *epic*.

Fact 2.4. If $g \circ f : a \rightarrow c$ is *epic*, then $g : b \rightarrow c$ is *epic*.

3 Iso arrows

Definition 3.1. An arrow $f : a \rightarrow b$ is *iso* if there exists another arrow $f^{-1} : b \rightarrow a$ such that

$$f \circ f^{-1} = 1_b$$

and

$$f^{-1} \circ f = 1_a.$$

This diagram commutes when the identity loops are included:

$$a \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{f^{-1}} \end{array} b$$

Fact 3.2. If an arrow is *iso* then it is *epic* and *monic*, but the converse isn't necessarily true. The converse is true in **Set** and any *Topos*.

Exercise 3.1. For any object a , the identity morphism 1_a is an inverse to itself and therefore is *iso*. Simply because

$$1_a \circ 1_a = 1_a.$$

$$a \begin{array}{c} \xrightarrow{1_a} \\ \xleftarrow{1_a} \end{array} a$$

Exercise 3.2. If $f : a \rightarrow b$ is *iso* then we can retrieve f^{-1} and then plug it right into the definition and find

$$f^{-1} \circ f = 1_a$$

and

$$f \circ f^{-1} = 1_b,$$

indicating that f^{-1} is iso.

$$\begin{array}{ccc} & f^{-1} & \\ a & \xrightarrow{\quad} & b \\ & f & \end{array}$$

Exercise 3.3. With $f : a \rightarrow b$ and $g : b \rightarrow c$ both iso, the situation looks like the following:

$$\begin{array}{ccccc} & f & & g & \\ a & \xrightarrow{\quad} & b & \xrightarrow{\quad} & c \\ & f^{-1} & & g^{-1} & \end{array}$$

Now we find that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_b \circ f = f^{-1} \circ f = 1_a$$

and

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_b \circ g^{-1} = g \circ g^{-1} = 1_c.$$

Thus $(f^{-1} \circ g^{-1})$ acts as an inverse to $g \circ f$, and $g \circ f$ is iso.

4 Isomorphic objects

Definition 4.1. Two C -objects a and b are *isomorphic*, or

$$a \cong b$$

if there exists an iso C -arrow

$$f : a \rightarrow b.$$

Definition 4.2. A category C is *skeletal* if $a \cong b$ implies $a = b$.

Exercise 4.1.

We wish to show that object isomorphism is an equivalence relation, or that it's reflexive, symmetric, and transitive. Fortunately the exercises from section 3 correspond exactly to these properties.

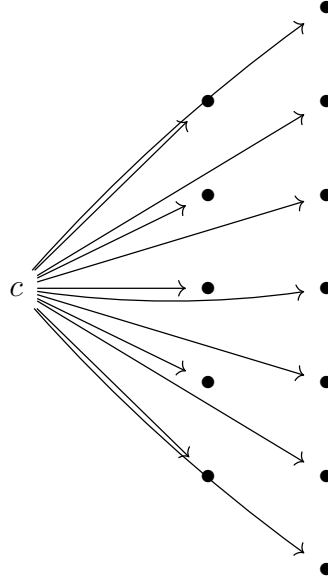
- (i) $a \cong a$ since 1_a is iso.
- (ii) If $a \cong b$ then some $f : a \rightarrow b$ is iso, and therefore $f^{-1} : b \rightarrow a$ is iso and $b \cong a$.
- (iii) If $a \cong b$ and $b \cong c$ then we have iso arrows $f : a \rightarrow b$ and $g : b \rightarrow c$. Then $g \circ f$ is iso, and $a \cong c$.

Exercise 4.2.

Suppose a and b are two **Finord**-objects such that $a \cong b$. Then there is some $f : a \rightarrow b$ that is iso. Since **Finord** is a subcategory of **Set**, iso arrows correspond to bijective functions. Then a and b must have the same cardinality, but by the definition of **Finord** distinct objects have distinct cardinalities. So $a = b$ and **Finord** is skeletal.

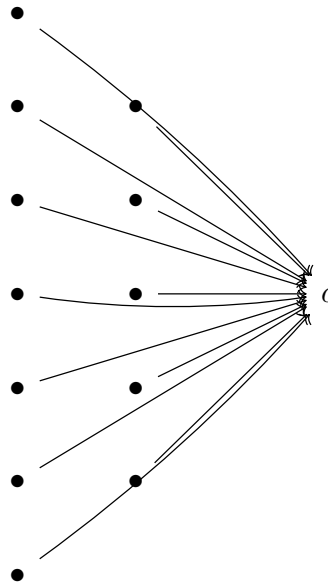
5 Initial objects

Definition 5.1. An object c is *initial* if for every C -object a , there is exactly one arrow $f : c \rightarrow a$.

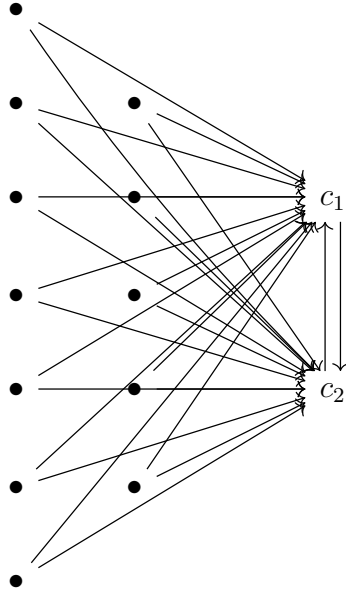


6 Terminal objects

Definition 6.1. An object c is *terminal* if for every C -object a , there is exactly one arrow $f : a \rightarrow c$.



Exercise 6.1. Let c_1 and c_2 be terminal C -objects.



By terminality there is a unique arrow $f_1 : c_1 \rightarrow c_2$ and a unique arrow $f_2 : c_2 \rightarrow c_1$. Then by the category axiom, $f_2 \circ f_1 : c_1 \rightarrow c_1$ and $f_1 \circ f_2 : c_2 \rightarrow c_2$ must exist. But again by terminality, there is a unique arrow $1_{c_1} : c_1 \rightarrow c_1$ and $1_{c_2} : c_2 \rightarrow c_2$, so the composition of f_1 and f_2 must give the identity. Conclude $c_1 \cong c_2$.

Exercise 6.2. (i) Terminal objects in \mathbf{Set}^2 are of the form $\langle \{e_1\}, \{e_2\} \rangle$, or pairs of singleton sets.

(ii) Terminal objects in \mathbf{Set}^\rightarrow are arrows with singleton sets as domain and codomain.

(iii) The terminal object in the poset (n, \leq) is the maximal element n , since $m \leq n$ for every m .

Exercise 6.3. Suppose $f : 1 \rightarrow a$ has its domain 1 a terminal object, and g_1, g_2 are any two parallel arrows from $c \rightarrow 1$.

$$c \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} 1 \xrightarrow{f} a$$

Well, since 1 is terminal the arrow from $c \rightarrow 1$ is unique and we see that $g_1 = g_2$, so regardless of whether $g_1 \circ f = g_2 \circ f$ holds (which it does), we can conclude f is monic.

7 Duality

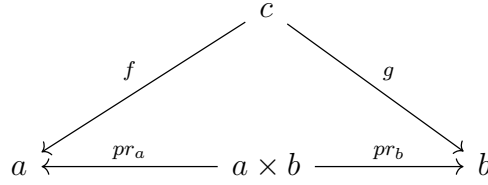
Any category can be turned into its opposite category. So any statement about a category can be dualized with all the arrows reversed.

8 Products

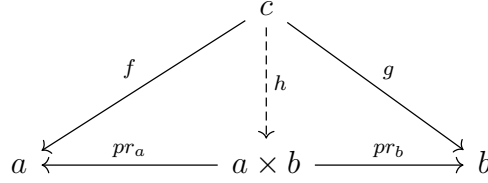
Definition 8.1. Given C -objects a and b , a *product* is a C -object $a \times b$ and 2 C -arrows pr_a, pr_b .

$$a \xleftarrow{pr_a} a \times b \xrightarrow{pr_b} b$$

For any c, f, g configured as follows



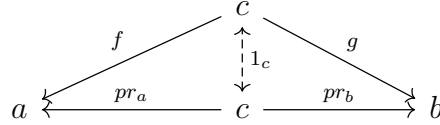
f and g determine a unique $h : c \rightarrow (a \times b)$ so that



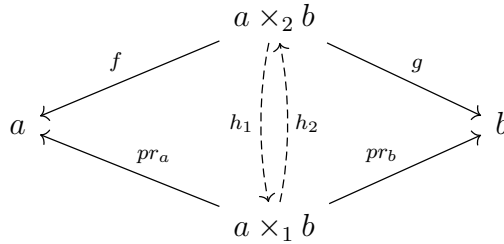
commutes. This is denoted

$$c := \langle f, g \rangle.$$

Fact 8.2. If c is a product $a \times b$, any arrow $f : c \rightarrow c$, f must be the identity 1_c . First observe that the identity must exist. Then plug c into definition 8.1 to see that f must be the unique arrow with that domain and codomain.



Fact 8.3. Any two products of a and b , say $a \times_1 b$ and $a \times_2 b$, are isomorphic to each other. Consider that in the diagram



h_1 and h_2 are uniquely determined by symmetric applications of definition 8.1. But by fact 8.2, composition of h_1 and h_2 must give identities.