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Chapter 3: Arrows Instead of Epsilon

1 Monic arrows

Monic arrows are an abstraction of injective functions.

> **Def:** An arrow $f : a \to b$ in a category C is *monic* if for any g_1, g_2 with codomain a, the implication
>
> $$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$
>
> holds. Or, if the diagram
>
> $$\begin{tikzcd} c \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$
>
> commutes, then $g_1 = g_2$.

> **Def:** *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *monic* iff for any C-object c, post-composition with f defines an injection $f_* : C(c, a) \to C(c, b)$. (Here $C(x, y)$ is the set of C-arrows from x to y.)

Exercises

For both exercises in this section, take the situation to be as follows:

$$\begin{tikzcd} s \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & a \arrow[r, "f"] & b \arrow[r, "g"] & c \end{tikzcd}$$

Where f and g are fixed, and s, h_1, h_2 are 'any such' objects/arrows.

1. Suppose that f and g are both monic, and that $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Since g is monic, that implies $f \circ h_1 = f \circ h_2$. But since f is monic, that implies $h_1 = h_2$. So using associativity and collapsing the chain of implication gives

   $$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \implies h_1 = h_2.$$

   Conclude $g \circ f$ is monic.

2. Now suppose that $g \circ f$ is monic. If $f \circ h_1 = f \circ h_2$ then clearly $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Then $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ and since $g \circ f$ is monic, $h_1 = h_2$. So

   $$f \circ h_1 = f \circ h_2 \implies h_1 = h_2,$$

   meaning f is monic.

2 Epic arrows

Def: If f is *epic* then commutativity of

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & c \end{tikzcd}$$

implies $g_1 = g_2$.

Def: *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *epic* iff for any C-object c, pre-composition with f defines an injection $f^* : C(b, x) \to C(a, c)$. (Here $C(x, y)$ is the set of C-arrows from x to y.)

Dually to the exercises proven in the previous section we have

Fact 1. *If $f : a \to b$ and $g : b \to c$ are epic, then $g \circ f : a \to c$ is epic.*

Fact 2. *If $g \circ f : a \to c$ is epic, then $g : b \to c$ is epic.*

3 Iso arrows

Def: An arrow $f : a \to b$ is *iso* if there exists another arrow $f^{-1} : b \to a$ such that

$$f \circ f^{-1} = 1_b$$

and

$$f^{-1} \circ f = 1_a.$$

This diagram commutes when the identity loops are included:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Fact 3. *If an arrow is iso then it is epic and monic, but the converse isn't necessarily true. The converse is true in **Set** and any Topos.*

Exercises

1. For any object a, the identity morphism 1_a is an inverse to itself and therefore is iso. Simply because

$$1_a \circ 1_a = 1_a.$$

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] & a \arrow[l, bend left, "1_a"] \end{tikzcd}$$

2. If $f : a \to b$ is iso then we can retrieve f^{-1} and then plug it right into the definition and find

$$f^{-1} \circ f = 1_a$$

and
$$f \circ f^{-1} = 1_b,$$
indicating that f^{-1} is iso.

$$\begin{tikzcd} a \arrow[r, bend left, "f^{-1}"] & b \arrow[l, bend left, "f"] \end{tikzcd}$$

3. With $f : a \to b$ and $g : b \to c$ both iso, the situation looks like the following:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & c \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

   Now we find that
$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_b \circ f = f^{-1} \circ f = 1_a$$
   and
$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_b \circ g^{-1} = g \circ g^{-1} = 1_c.$$
   Thus $(f^{-1} \circ g^{-1})$ acts as an inverse to $g \circ f$, and $g \circ f$ is iso.

4 Isomorphic objects

Def: Two C-objects a and b are *isomorphic*, or
$$a \cong b$$
if there exists an iso C-arrow
$$f : a \to b.$$

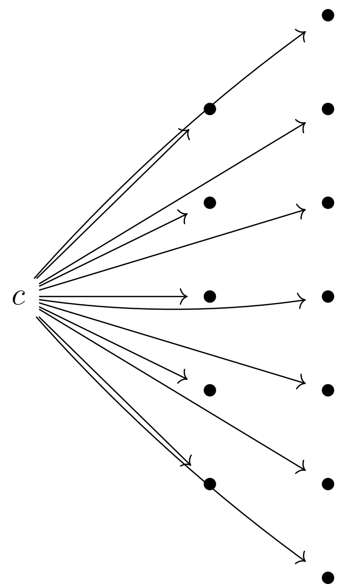
Def: A category C is *skeletal* if $a \cong b$ implies $a = b$.

Exercises

1. We wish to show that object isomorphism is an equivalence relation, or that it's reflexive, symmetric, and transitive. Fortunately the exercises from section 3 correspond exactly to these properties.
   (i) $a \cong a$ since 1_a is iso.
   (ii) If $a \cong b$ then some $f : a \to b$ is iso, and therefore $f^{-1} : b \to a$ is iso and $b \cong a$.
   (iii) If $a \cong b$ and $b \cong c$ then we have iso arrows $f : a \to b$ and $g : b \to c$. Then $g \circ f$ is iso, and $a \cong c$.
2. Suppose a and b are two **Finord**-objects such that $a \cong b$. Then there is some $f : a \to b$ that is iso. Since **Finord** is a subcategory of **Set**, iso arrows correspond to bijective functions. Then a and b must have the same cardinality, but by the definition of **Finord** distinct objects have distinct cardinalities. So $a = b$ and **Finord** is skeletal.

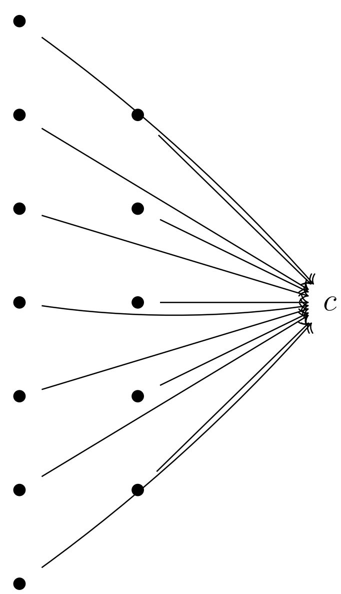
5 Initial objects

Def: An object c is *initial* if for every C-object a, there is exactly one arrow $f : c \to a$.



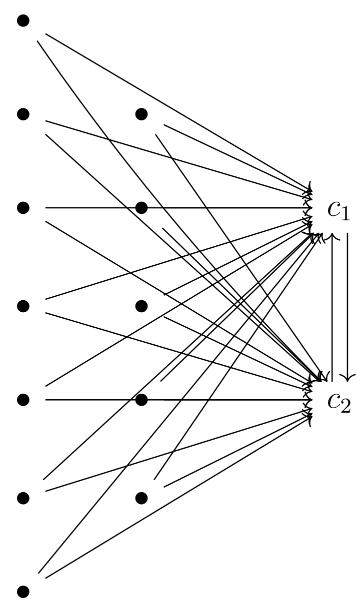
6 Terminal objects

Def: An object c is *terminal* if for every C-object a, there is exactly one arrow $f : a \to c$.



Exercises

1. Let c_1 and c_2 be terminal $C - objects$.



   By terminality there is a unique arrow $f_1 : c_1 \to c_2$ and a unique arrow $f_2 : c_2 \to c_1$. Then by the category axiom, $f_2 \circ f_1 : c_1 \to c_1$ and $f_1 \circ f_2 : c_2 \to c_2$ must exist. But again by terminality, there is a unique arrow $1_{c_1} : c_1 \to c_1$ and $1_{c_2} : c_2 \to c_2$, so the composition of f_1 and f_2 must give the identity. Conclude $c_1 \cong c_2$.

2. (i) Terminal objects in $\mathbf{Set}^2$ are of the form $\langle \{e_1\}, \{e_2\} \rangle$, or pairs of singleton sets.

   (ii) Terminal objects in $\mathbf{Set}^{\to}$

   (iii) The terminal object in the poset $(n, \leq)$ is the maximal element n, since $m \leq n$ for every m.

3. Suppose $f : 1 \to a$ has its domain 1 a terminal object, and g_1, g_2 are any two parallel arrows from $c \to 1$.

$$\begin{tikzcd} c \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & 1 \arrow[r, "f"] & a \end{tikzcd}$$

   Well, since 1 is terminal the arrow from $c \to 1$ is unique and we see that $g_1 = g_2$, so regardless of whether $g_1 \circ f = g_2 \circ f$ holds (which it does), we can conclude f is monic.

7 Duality

Any category can be turned into its opposite category. So any statement about a category can be dualized with all the arrows reversed.

8 Products

Def: Given C-objects a and b, a *product* is a C-object $a \times b$ and 2 C-arrows pr_a, pr_b.

$$a \xleftarrow{pr_a} a \times b \xrightarrow{pr_b} b$$

For any c, f, g configured as follows

$$\begin{tikzcd}
 & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

f and g determine a unique $h : c \to (a \times b)$ so that

$$\begin{tikzcd}
 & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "h"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

commutes. This is denoted

$$c := \langle f, g \rangle.$$

Fact 4. *Any two products of a and b, say $a \times_1 b$ and $a \times_2 b$, are isomorphic to each other. Consider that in the diagram*

$$\begin{tikzcd}
 & a \times_2 b \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, bend right, "h_1"'] & \\
a & & b \\
 & a \times_1 b \arrow[ul, "pr_a"] \arrow[ur, "pr_b"'] \arrow[uu, dashed, bend right, "h_2"'] &
\end{tikzcd}$$

h_1 and h_2 are uniquely determined. But the arrow from a product to itself must also be unique, so the composition of h_1 and h_2 must give identities (depending on the order).