Topoi by Robert Goldblatt
Notes by R. Riley Holmes
Chapter 3: Arrows Instead of Epsilon

1 Monic arrows

Monic arrows are an abstraction of injective functions.

Def: An arrow $f : a \to b$ in a category C is *monic* if for any g_1, g_2 with codomain a, the implication

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

holds. Or, if the diagram

$$\begin{tikzcd} c \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

commutes, then $g_1 = g_2$.

Exercises

For both exercises in this section, take the situation to be as follows:

$$\begin{tikzcd} s \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & a \arrow[r, "f"] & b \arrow[r, "g"] & c \end{tikzcd}$$

Where f and g are fixed, and s, h_1, h_2 are 'any such' objects/arrows.

1. Suppose that f and g are both monic, and that $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Since g is monic, that implies $f \circ h_1 = f \circ h_2$. But since f is monic, that implies $h_1 = h_2$. So using associativity and collapsing the chain of implication gives

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \implies h_1 = h_2.$$

   Conclude $g \circ f$ is monic.

2. Now suppose that $g \circ f$ is monic. If $f \circ h_1 = f \circ h_2$ then clearly $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Then $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ and since $g \circ f$ is monic, $h_1 = h_2$. So

$$f \circ h_1 = f \circ h_2 \implies h_1 = h_2,$$

   meaning f is monic.

2 Epic arrows

Def: If f is *epic* then commutativity of

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & c \end{tikzcd}$$

implies $g_1 = g_2$.

3 Iso arrows

Def: An arrow $f : a \to b$ is *iso* if there exists another arrow $f^{-1} : b \to a$ such that

$$f \circ f^{-1} = 1_b$$

and

$$f^{-1} \circ f = 1_a.$$

This diagram commutes when the identity loops are included:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Fact 1. *If an arrow is iso then it is epic and monic, but the converse isn't necessarily true. The converse* is *true in* ***Set*** *and any Topos.*

3.1 Exercises

1. For any object a, the identity morphism 1_a is an inverse to itself and therefore is iso. Simply because

$$1_a \circ 1_a = 1_a.$$

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] & a \arrow[l, bend left, "1_a"] \end{tikzcd}$$

2. If $f : a \to b$ is iso then we can retrieve f^{-1} and then plug it right into the definition and find

$$f^{-1} \circ f = 1_a$$

and

$$f \circ f^{-1} = 1_b,$$

indicating that f^{-1} is iso.

$$\begin{tikzcd} a \arrow[r, bend left, "f^{-1}"] & b \arrow[l, bend left, "f"] \end{tikzcd}$$

3. With $f : a \to b$ and $g : b \to c$ both iso, the situation looks like the following:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & c \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

Now we find that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_b \circ f = f^{-1} \circ f = 1_a$$

and

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_b \circ g^{-1} = g \circ g^{-1} = 1_c.$$

Thus $(f^{-1} \circ g^{-1})$ acts as an inverse to $g \circ f$, and $g \circ f$ is iso.

4 Isomorphic objects

Def: Two C-objects a and b are *isomorphic*, or

$$a \cong b$$

if there exists an iso C-arrow

$$f : a \to b.$$

Def: A category C is *skeletal* if $a \cong b$ implies $a = b$.

4.1 Exercises

1. We wish to show that object isomorphism is an equivalence relation, or that it's reflexive, symmetric, and transitive. Fortunately the exercises from section 3 correspond exactly to these properties.

   (i) $a \cong a$ since 1_a is iso.

   (ii) If $a \cong b$ then some $f : a \to b$ is iso, and therefore $f^{-1} : b \to a$ is iso.

   (iii) If $a \cong b$ and $b \cong c$ then we have iso arrows $f : a \to b$ and $g : b \to c$. Then $g \circ f$ is iso, and $a \cong c$.

2. Suppose a and b are two **Finord**-objects such that $a \cong b$. Then there is some $f : a \to b$ that is iso. Since **Finord** is a subcategory of **Set**, iso arrows correspond to bijective functions. Then a and b must have the same cardinality, but by the definition of **Finord** distinct objects have distinct cardinalities. So $a = b$ and **Finord** is skeletal.