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Chapter 2: What Categories Are

1 Functions as Sets

Given a function $f : A \to B$ we can derive some related sets.

Definition 1.1. The *relation*

$$\hat{f} := \{\langle a, f(a)\rangle \mid a \in A\}.$$

Definition 1.2. The *image set*

$$f(A) := \{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

Equivalently

$$f(A) := \left\{b \mid \langle a, b\rangle \in \hat{f} \text{ for some } a \in A\right\}.$$

2 Composition

The power of composition is that it can't be resisted. Say I hand you an $f : A \to B$ and a $G : B \to C$. Then there is a clear procedure for getting from A to C.

(1) Take your $a \in A$.

(2) Apply f to a, yielding $f(a)$ which is some $b \in B$.

(3) Apply g to b, yielding $g(b)$, which is some $c \in C$

This procedure yields the

Definition 2.1. *Function composition*

$$g \circ f.$$

Fact 2.2. *Functions are associative, or*

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Exercise 2.1. Convince yourself that functions are associative.

2.1 Identities

Given any set B there is an important function that comes for free. We call it $1_B : B \to B$ and it's given by

$$1_B(b) = b$$

for any $b \in B$.

Fact 2.3. *Identities are absorbed left and right. In other words, if we have $f : A \to B$ and a $G : B \to C$, then*

$$1_B \circ f = f$$

and

$$g \circ 1_B = g.$$

That gives us our first commutative diagram, for which any path is equivalent:

$$\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[dr, "f"'] & 1_B \arrow[u] \arrow[d] & C \\
 & B \arrow[ur, "g"'] &
\end{tikzcd}$$

3 Category Axioms

We are ready for the abstract view of the above.

Definition 3.1. A *category* has

1. a collection of objects:
$$a, b, c.$$
2. a collection of arrows, each with specific domain and codomain:
$$f : a \to b,$$
$$g : b \to c.$$
3. an associative composition operation that yields a unique arrow 'skipping' aligned domains and codomains:
$$g \circ f : a \to c.$$
4. an identity arrow for each object:
$$f \circ 1_A = f$$
$$1_B \circ f = f$$
$$g \circ 1_B = g$$
$$1_C \circ g = g$$

Fact 3.2. *Diagrams are a convenient way to present categories:*

$$A \xrightarrow[f]{} B \xrightarrow[g]{} C$$

But you must keep in mind there are some implicit things not being shown here. The above is a compact *verion of the following:*

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, "g"'] & C \\
A \arrow[u, leftrightarrow, "1_A"] \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[u, leftrightarrow, "1_B"] \arrow[r, "g"] & C \arrow[u, leftrightarrow, "1_C"]
\end{tikzcd}$$

Fact 3.3. *Saying a diagram 'commutes' is a convenient way to present equivalent compositions of arrows. For example, if*

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left, "h"] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

commutes then we are saying that

$$g \circ f = h.$$

4 Comma Categories

Definition 4.1. The comma-category $C \downarrow a$ is formed from any category C and any C-object a. Its objects are all the C-arrows with codomain a. (IE, $f_1 : c_1 \to a$ and $f_2 : c_2 \to a$). Its arrows are all C-arrows between the objects' domains, that commute with the 'object arrows'. (IE, $g : c_1 \to c_2$ so that $f_1 = f_2 \circ g$). So $C \downarrow a$ looks like this:

$$f_1 \xrightarrow{g} f_2$$

and indicates that this diagram commutes in the original category:

$$\begin{tikzcd}
c_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & c_2 \arrow[dl, "f_2"] \\
& a &
\end{tikzcd}$$

(TODO: Verify category axioms)

Example. Take C to be the preorder on natural numbers, and a to be a given number. For example, let $a = 3$.

$$1 \longrightarrow 2 \longrightarrow \mathbf{3} \longrightarrow 4 \longrightarrow 5 \longrightarrow \dots$$

Objects in $C \downarrow 3$ are statements of '$n \leq 4$' relationships. Arrows in $C \downarrow 3$ from '$m \leq 3$' to '$n \leq 3$' are '$m \leq n$'.

$$\begin{tikzcd}
(1 \leq 3) \arrow[r, "1 \leq 2"'] \arrow[rr, bend left, "1 \leq 3"] & (2 \leq 3) \arrow[r, "2 \leq 3"'] & (3 \leq 3)
\end{tikzcd}$$

Example. Recall **Matr(k)** has the natural numbers $\mathbb{N}$ as objects, and $(n \times m)$ matrices as arrows from $m \to n$.

Then **Matr(k)** $\downarrow 3$ has as objects all $3 \times n$ matrices where $n \in \mathbb{N}$.

Then if object A is a $3 \times m$ matrix, and object B is a $3 \times n$ matrix, then an arrow $C : A \to B$ is an $n \times m$ matrix such that $A = BC$.

Thus the situation in $\mathbf{Matr(k)} \downarrow 3$ is

$$A_{3\times m} \xrightarrow{C_{n\times m}} B_{3\times n}$$

indicating that this diagram commutes in $\mathbf{Matr(k)}$:

$$\begin{tikzcd}
m \arrow[rr, "C_{n\times m}"] \arrow[dr, "A_{3\times m}"'] & & n \arrow[dl, "B_{3\times n}"] \\
& 3 &
\end{tikzcd}$$