Chapter 1

Mathematics = Set Theory?

1.1 Where Set Theory Cracks

If it starts to feel like you're working with a system that's *too* powerful, try this: turn the system on itself.

Maybe you'll feed the system to itself, as in the proof of the undecidability of the Halting Problem. Maybe you'll reduce its power to a handful of (say) 3 properties and show that any 2 precludes the possiblity of the 3rd:

- the CAP theorems for distributed databases
- Kleinberg's impossibility theorem on clustering algorithms
- Arrow's impossibility theorem on social choice

In this case we simply hold up the mirror to the system. If we can ask for the set of all sets, we can quickly reach a dead end in the form of Russell's paradox.

We have two different trap doors that let us squeeze tighter and keep moving forward.

Zermelo–Fraenkel set theory tightens the standard on set comprehensions. And it makes sense. It's hard to imagine what the first line of 'bad' Python would do:

```
bad_set = {a if p(a)}
good_set = {b for b in source_set if p(b)}
```

The von Neumann–Bernays–Gödel (NBG) adds to ZF an abstract notion of 'class' allowing us to talk about the 'class of all sets'. Both provide a strong foundation for modern math, but category theory has emerged as a more abstract foundation.

Chapter 2

What Categories Are

2.1 Functions as Sets

We are only going to wallow in the swamp of Set Theory for a brief moment longer. Given a function $f : A \to B$ we can derive some related sets.

Definition 2.1.1. The *relation*

$$\hat{f} := \{\langle a, f(a)\rangle \mid a \in A\}.$$

Definition 2.1.2. The *image set*

$$f(A) := \{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

Equivalently

$$f(A) := \left\{b \mid \langle a, b\rangle \in \hat{f} \text{ for some } a \in A\right\}.$$

2.2 Composition

The power of composition is that it can't be resisted. Say I hand you an $f : A \to B$ and a $G : B \to C$. Then there is a clear procedure for getting from A to C.

(1) Take your $a \in A$.

(2) Apply f to a, yielding $f(a)$ which is some $b \in B$.

(3) Apply g to b, yielding $g(b)$, which is some $c \in C$

This procedure yields the

Definition 2.2.1. *Function composition*

$$g \circ f.$$

Fact 2.2.2. *Functions are associative, or*

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Exercise 2.2.1. Convince yourself that functions are associative.

2.2.1 Identities

Given any set B there is an important function that comes for free. We call it $1_B : B \to B$ and it's given by

$$1_B(b) = b$$

for any $b \in B$.

Fact 2.2.3. *Identities are absorbed left and right. In other words, if we have $f : A \to B$ and a $G : B \to C$, then*

$$1_B \circ f = f$$

and

$$g \circ 1_B = g.$$

That gives us our first commutative diagram, for which any path is equivalent:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] \arrow[uu, leftrightarrow, "1_B" description] & \end{tikzcd}$$

2.3 Category Axioms

We are ready for the abstract view of the above.

Definition 2.3.1. A *category* has

1. a collection of objects:
$$a, b, c.$$
2. a collection of arrows, each with specific domain and codomain:
$$f : a \to b,$$
$$g : b \to c.$$
3. an associative composition operation that yields a unique arrow 'skipping' aligned domains and codomains:
$$g \circ f : a \to c.$$
4. an identity arrow for each object:
$$f \circ 1_A = f$$
$$1_B \circ f = f$$
$$g \circ 1_B = g$$
$$1_C \circ g = g$$

For now we will assume that the collection of arrows between any two objects is 'only' a set's worth. In other words we will only deal with *locally small* categories for now. (Riehl Page 7)

Fact 2.3.2. *Diagrams are a convenient way to present categories:*

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

But you must keep in mind there are some implicit things not being shown here. The above is a compact verion of the following:

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f" description] & B \arrow[r, "g"'] & C \\ A \arrow[u, leftrightarrow, "1_A" description] \arrow[r, "f"] \arrow[rr, bend right, "g \circ f" description] & B \arrow[u, leftrightarrow, "1_B" description] \arrow[r, "g"] & C \arrow[u, leftrightarrow, "1_C" description] \end{tikzcd}$$

Fact 2.3.3. *Saying a diagram 'commutes' is a convenient way to present equivalent compositions of arrows. For example, if*

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "h"] & B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes then we are saying that

$$g \circ f = h.$$

Definition 2.3.4. Given two objects a, b in a category C, we define the *hom-set*

$$C(a, b) \quad \text{or} \quad \hom(a, b)$$

as the set of all arrows from a to b.

2.4 Comma Categories

Definition 2.4.1. The comma-category $C \downarrow a$ is formed from any category C and any C-object a. Its objects are all the C-arrows with codomain a. (IE, $f_1 : c_1 \to a$ and $f_2 : c_2 \to a$). Its arrows are all C-arrows between the objects' domains, that commute with the 'object arrows'. (IE, $g : c_1 \to c_2$ so that $f_1 = f_2 \circ g$). So $C \downarrow a$ looks like this:

$$f_1 \xrightarrow{g} f_2$$

and indicates that this diagram commutes in the original category:

$$\begin{tikzcd} c_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & c_2 \arrow[dl, "f_2"] \\ & a & \end{tikzcd}$$

(TODO: Verify category axioms)

Example. Take C to be the preorder on natural numbers, and a to be a given number. For example, let $a = 3$.

$$1 \longrightarrow 2 \longrightarrow \mathbf{3} \longrightarrow 4 \longrightarrow 5 \longrightarrow \dots$$

Objects in $C \downarrow 3$ are statements of '$n \leq 4$' relationships. Arrows in $C \downarrow 3$ from '$m \leq 3$' to '$n \leq 3$' are '$m \leq n$'.

$$\begin{tikzcd} (1 \leq 3) \arrow[r, "1\leq 2"'] \arrow[rr, bend left, "1\leq 3"] & (2 \leq 3) \arrow[r, "2\leq 3"'] & (3 \leq 3) \end{tikzcd}$$

Example. Recall **Matr(k)** has the natural numbers $\mathbb{N}$ as objects, and $(n \times m)$ matrices as arrows from $m \to n$.

Then **Matr(k)** $\downarrow 3$ has as objects all $3 \times n$ matrices where $n \in \mathbb{N}$.

Then if object A is a $3 \times m$ matrix, and object B is a $3 \times n$ matrix, then an arrow $C : A \to B$ is an $n \times m$ matrix such that $A = BC$.

Thus the situation in **Matr(k)** $\downarrow 3$ is

$$A_{3\times m} \xrightarrow{C_{n\times m}} B_{3\times n}$$

indicating that this diagram commutes in **Matr(k)**:

$$\begin{tikzcd} m \arrow[rr, "C_{n\times m}"] \arrow[dr, "A_{3\times m}"'] & & n \arrow[dl, "B_{3\times n}"] \\ & 3 & \end{tikzcd}$$

Chapter 3

Arrows Instead of Epsilon

3.1 Monic arrows

Monic arrows are an abstraction of injective functions.

Definition 3.1.1. An arrow $f : a \to b$ in a category C is *monic* if for any g_1, g_2 with codomain a, the implication

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

holds. Or, if the diagram

$$\begin{tikzcd} c \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

commutes, then $g_1 = g_2$.

Definition 3.1.2. *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *monic* iff for any C-object c, post-composition with f defines an injection $f_* : C(c, a) \to C(c, b)$. (Here $C(x, y)$ is the set of C-arrows from x to y.)

For both exercises in this section, take the situation to be as follows:

$$\begin{tikzcd} s \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & a \arrow[r, "f"] & b \arrow[r, "g"] & c \end{tikzcd}$$

Where f and g are fixed, and s, h_1, h_2 are 'any such' objects/arrows.

Exercise 3.1.1.

Suppose that f and g are both monic, and that $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Since g is monic, that implies $f \circ h_1 = f \circ h_2$. But since f is monic, that implies $h_1 = h_2$. So using associativity and collapsing the chain of implication gives

$$(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \implies h_1 = h_2.$$

Conclude $g \circ f$ is monic.

Exercise 3.1.2.

Now suppose that $g \circ f$ is monic. If $f \circ h_1 = f \circ h_2$ then clearly $g \circ (f \circ h_1) = g \circ (f \circ h_2)$. Then $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ and since $g \circ f$ is monic, $h_1 = h_2$. So

$$f \circ h_1 = f \circ h_2 \implies h_1 = h_2,$$

meaning f is monic.

3.2 Epic arrows

Definition 3.2.1. If f is *epic* then commutativity of

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & c \end{tikzcd}$$

implies $g_1 = g_2$.

Definition 3.2.2. *(Alternatively, Riehl pg. 11)* An arrow $f : a \to b$ is *epic* iff for any C-object c, pre-composition with f defines an injection $f^* : C(b, x) \to C(a, c)$. (Here $C(x, y)$ is the set of C-arrows from x to y.)

Dually to the exercises proven in the previous section we have

Fact 3.2.3. *If $f : a \to b$ and $g : b \to c$ are epic, then $g \circ f : a \to c$ is epic.*

Fact 3.2.4. *If $g \circ f : a \to c$ is epic, then $g : b \to c$ is epic.*

3.3 Iso arrows

Definition 3.3.1. An arrow $f : a \to b$ is *iso* if there exists another arrow $f^{-1} : b \to a$ such that

$$f \circ f^{-1} = 1_b$$

and

$$f^{-1} \circ f = 1_a.$$

This diagram commutes when the identity loops are included:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Fact 3.3.2. *If an arrow is iso then it is epic and monic, but the converse isn't necessarily true. The converse* is *true in **Set** and any Topos.*

Exercise 3.3.1. For any object a, the identity morphism 1_a is an inverse to itself and therefore is iso. Simply because

$$1_a \circ 1_a = 1_a.$$

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] & a \arrow[l, bend left, "1_a"] \end{tikzcd}$$

Exercise 3.3.2. If $f : a \to b$ is iso then we can retrieve f^{-1} and then plug it right into the definition and find

$$f^{-1} \circ f = 1_a$$

and

$$f \circ f^{-1} = 1_b,$$

indicating that f^{-1} is iso.

$$\begin{tikzcd} a \arrow[r, bend left, "f^{-1}"] & b \arrow[l, bend left, "f"] \end{tikzcd}$$

Exercise 3.3.3. With $f : a \to b$ and $g : b \to c$ both iso, the situation looks like the following:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & c \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

Now we find that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_b \circ f = f^{-1} \circ f = 1_a$$

and

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_b \circ g^{-1} = g \circ g^{-1} = 1_c.$$

Thus $(f^{-1} \circ g^{-1})$ acts as an inverse to $g \circ f$, and $g \circ f$ is iso.

3.4 Isomorphic objects

Definition 3.4.1. Two C-objects a and b are *isomorphic*, or

$$a \cong b$$

if there exists an iso C-arrow

$$f : a \to b.$$

Definition 3.4.2. A category C is *skeletal* if $a \cong b$ implies $a = b$.

Exercise 3.4.1.

We wish to show that object isomorphism is an equivalence relation, or that it's reflexive, symmetric, and transitive. Fortunately the exercises from section 3 correspond exactly to these properties.

(i) $a \cong a$ since 1_a is iso.

(ii) If $a \cong b$ then some $f : a \to b$ is iso, and therefore $f^{-1} : b \to a$ is iso and $b \cong a$.

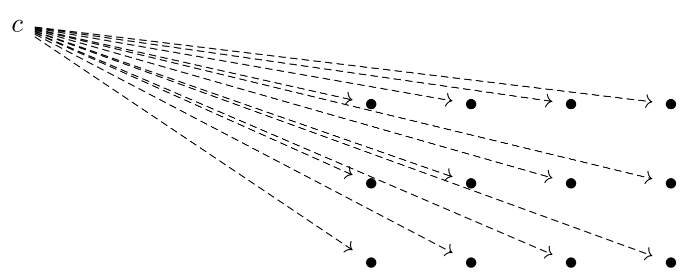
(iii) If $a \cong b$ and $b \cong c$ then we have iso arrows $f : a \to b$ and $g : b \to c$. Then $g \circ f$ is iso, and $a \cong c$.

Exercise 3.4.2.

Suppose a and b are two **Finord**-objects such that $a \cong b$. Then there is some $f : a \to b$ that is iso. Since **Finord** is a subcategory of **Set**, iso arrows correspond to bijective functions. Then a and b must have the same cardinality, but by the definition of **Finord** distinct objects have distinct cardinalities. So $a = b$ and **Finord** is skeletal.

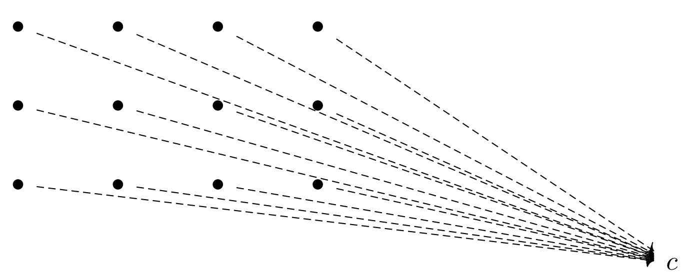
3.5 Initial objects

Definition 3.5.1. An object c is *initial* if for every C-object a, there is exactly one arrow $f : c \to a$.

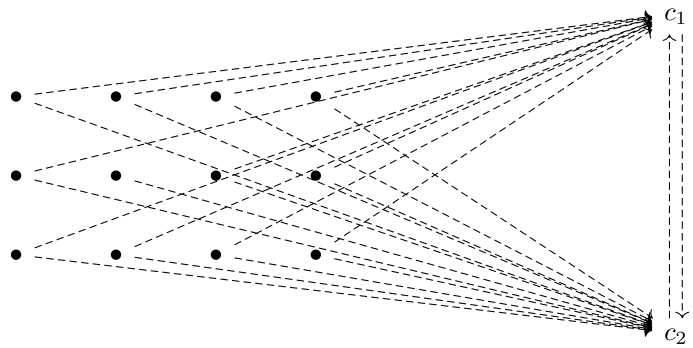


3.6 Terminal objects

Definition 3.6.1. An object c is *terminal* if for every C-object a, there is exactly one arrow $f : a \to c$.



Exercise 3.6.1. Let c_1 and c_2 be terminal $C-objects$.



By terminality there is a unique arrow $f_1 : c_1 \to c_2$ and a unique arrow $f_2 : c_2 \to c_1$. Then by the category axiom, $f_2 \circ f_1 : c_1 \to c_1$ and $f_1 \circ f_2 : c_2 \to c_2$ must exist. But again by terminality, there is a unique arrow $1_{c_1} : c_1 \to c_1$ and $1_{c_2} : c_2 \to c_2$, so the composition of f_1 and f_2 must give the identity. Conclude $c_1 \cong c_2$.

Exercise 3.6.2. (i) Terminal objects in $\mathbf{Set}^2$ are of the form $\langle \{e_1\}, \{e_2\} \rangle$, or pairs of singleton sets.

(ii) Terminal objects in $\mathbf{Set}^{\to}$ are arrows with singleton sets as domain and codomain.

(iii) The terminal object in the poset $(n, \leq)$ is the maximal element n, since $m \leq n$ for every m.

Exercise 3.6.3. Suppose $f : 1 \to a$ has its domain 1 a terminal object, and g_1, g_2 are any two parallel arrows from $c \to 1$.

$$\begin{tikzcd} c \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & 1 \arrow[r, "f"] & a \end{tikzcd}$$

Well, since 1 is terminal the arrow from $c \to 1$ is unique and we see that $g_1 = g_2$, so regardless of whether $g_1 \circ f = g_2 \circ f$ holds (which it does), we can conclude f is monic.

3.7 Duality

Any category can be turned into its opposite category. So any statement about a category can be dualized with all the arrows reversed.

3.8 Products

Definition 3.8.1. Given C-objects a and b, a *product* is a C-object $a \times b$ and 2 C-arrows pr_a, pr_b.

$$\begin{tikzcd} a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

For any c, f, g configured as follows

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

f and g determine a unique $h : c \to (a \times b)$ so that

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "h"] & \\ a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b \end{tikzcd}$$

commutes. This is denoted

$$h := \langle f, g \rangle.$$

Fact 3.8.2. *If c is a product $a \times b$, any arrow $f : c \to c$, f must be the identity 1_c. First observe that the identity must exist. Then plug c into definition 3.8.1 to see that f must be the unique arrow with that domain and codomain.*

$$\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, leftrightarrow, "1_c"] & \\
a & c \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Fact 3.8.3. *Any two products of a and b, say $a \times_1 b$ and $a \times_2 b$, are isomorphic to each other. Consider that in the diagram*

$$\begin{tikzcd}
& a \times_2 b \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, bend right, "h_1"'] & \\
a & & b \\
& a \times_1 b \arrow[ul, "pr_a"] \arrow[ur, "pr_b"'] \arrow[uu, dashed, bend right, "h_2"'] &
\end{tikzcd}$$

h_1 and h_2 are uniquely determined by symmetric applications of definition 3.8.1. But by fact 3.8.2, composition of h_1 and h_2 must give identities.

Exercise 3.8.1. The fact that $\langle pr_a, pr_b \rangle = 1_{a\times b}$ follows as a special case of fact 3.8.2, by plugging in the projection functions.

$$\begin{tikzcd}
& a \times b \arrow[dl, "pr_a"'] \arrow[dr, "pr_b"] \arrow[d, dashed, leftrightarrow, "1_{a\times b}"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Exercise 3.8.2. Suppose we have parallel $f, k : c \rightrightarrows a$ and $g, h : c \rightrightarrows b$ and $p : c \to a \times b$ such that $p = \langle f, g \rangle = \langle k, h \rangle$. Then $f = pr_a \circ p$ and $k = pr_a \circ p$. It doesn't take any special cancellation rules to see that identically $f = k$. Similarly $g = h$.

$$\begin{tikzcd}
& c \arrow[dl, bend right, "f"'] \arrow[dl, "k"] \arrow[dr, bend left, "g"] \arrow[dr, "h"'] \arrow[d, dashed, "p"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Exercise 3.8.3. Suppose the situation is as follows.

$$\begin{tikzcd}
& d \arrow[d, "h"] & \\
& c \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] \arrow[d, dashed, "\langle f,g\rangle"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

Then by compositionality there must exist $h \circ f : d \to a$ and $g \circ f : d \to b$. There also must exist $h \circ \langle f, g \rangle : d \to a \times b$. By collapsing the diagram to

$$\begin{tikzcd}
& d \arrow[dl, bend right, "f\circ h"'] \arrow[dr, bend left, "g\circ h"] \arrow[d, dashed, "\langle f\circ h, g\circ h\rangle"] & \\
a & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] & b
\end{tikzcd}$$

we see the arrow from $d \to a \times b$ must be unique, and therefore

$$\langle f \circ h, g \circ h \rangle = h \circ \langle f, g \rangle.$$

Exercise 3.8.4. Suppose a category C has a terminal object t, and products. Let a be a C-object and consider the product $a \times t$.

$$
\begin{tikzcd}
a & a \times t \arrow[l] \arrow[r] & t
\end{tikzcd}
$$

Plugging a into the product definition using the arrows given by 1_a and ! (the unique arrow to t) yields the unique arrow $h = \langle 1_a, ! \rangle$.

$$
\begin{tikzcd}
 & a \arrow[dl, "1_a"'] \arrow[d, dashed, "h"] \arrow[dr, "!"] & \\
a & a \times t \arrow[l, "pr_a"] \arrow[r] & t
\end{tikzcd}
$$

By definition we have that $pr_a \circ h = 1_a$. And since $h \circ pr_a$ maps $a \times t \to a \times t$ it follows from fact 3.8.2 that $h \circ pr_a = 1_{a \times t}$. Given these two iso arrows we conclude

$$a \cong a \times t.$$

Example. As a specific example of exercise 3.8.4, let's work in **Set** where A is any set, and our terminal set t is any singleton set $\{b\}$.

$$
\begin{tikzcd}
 & A \arrow[dl, "1_a"'] \arrow[d, dashed, "h"] \arrow[dr, "!"] & \\
A & A \times \{b\} \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & \{b\}
\end{tikzcd}
$$

Now we can explicitly say what all of our functions do:

$$1_a(x) = x$$

$$!(x) = b$$

$$pr_a \left(\langle a, b \rangle \right) = a$$

$$pr_b \left(\langle a, b \rangle \right) = b$$

$$h(x) = \langle x, b \rangle$$

The fact that $pr_a(h(x)) = x$ and $h(pr_a(\langle a, b \rangle)) = \langle a, b \rangle$ gives the isomorphism

$$A \cong A \times \{b\}.$$

Intuitively, elements in A can be placed in one-to-one correspondence with elements in $A \times \{b\}$ by simply sending x to the tuple $\langle x, b \rangle$.

Definition 3.8.4. Given two products $a \times b$ and $c \times d$, and arrows $f : a \to c$ and $g : b \to d$

$$
\begin{tikzcd}
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "pr_c"'] \arrow[r, "pr_d"] & d
\end{tikzcd}
$$

the unique *product arrow* $(f \times g) : a \times b \to c \times d$ is found as $\langle f \circ pr_a, g \circ pr_b \rangle$.

$$\begin{tikzcd}
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] \arrow[d, dashed, "f \times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "pr_c"] \arrow[r, "pr_d"'] & d
\end{tikzcd}$$

Exercise 3.8.5. In the reflexive case we consider the product arrow $1_a \times 1_b$.

$$\begin{tikzcd}
a \arrow[d, "1_a"] & a \times b \arrow[l, "pr_a"'] \arrow[r, "pr_b"] \arrow[d, dashed, "1_a \times 1_b"] & b \arrow[d, "1_b"] \\
a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b
\end{tikzcd}$$

By definition we have

$$1_a \times 1_b = \langle 1_a \circ pr_a, 1_b \circ pr_b \rangle = \langle pr_a, pr_b \rangle.$$

Then applying exercise 3.8.1 from here gives the desired result

$$1_a \times 1_b = 1_{a \times b}.$$

Exercise 3.8.6. The isomorphism $a \times b \cong b \times a$ follows by plugging each object into definition 3.8.1 in relation to the other. In this way the two unique (dashed) arrows are found:

$$\begin{tikzcd}
a \arrow[drr] & a \times b \arrow[l] \arrow[r] \arrow[d, dashed, bend right] & b \arrow[dll] \\
b & b \times a \arrow[l] \arrow[r] \arrow[u, dashed, bend right] & a
\end{tikzcd}$$

and fact 3.8.2 tells us that they are iso.

Exercise 3.8.7. In this exercise we wish to show that products are associative up to isomorphism. So given C-objects a, b, c, form the products $(a \times b) \times c$ and $a \times (b \times c)$. Here they are with the relevant projection arrows:

$$
\begin{tikzcd}
& (a \times b) \arrow[dl] \arrow[r] & b & (b \times c) \arrow[l] \arrow[dr] & \\
a & & & & c \\
& (a \times b) \times c \arrow[uu] \arrow[urrr] & & a \times (b \times c) \arrow[uu] \arrow[ulll] &
\end{tikzcd}
$$

We'll use the product definition side-by-side, noting that composition of projections gives us our arrows $a \times (b \times c) \to b$ and $(a \times b) \times c \to b$:

$$
\begin{tikzcd}
& a \times (b \times c) \arrow[dl] \arrow[d, dashed] \arrow[dr] & & (a \times b) \times c \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
a & a \times b \arrow[l] \arrow[r] & b & b \times c \arrow[l] \arrow[r] & c
\end{tikzcd}
$$

The definition yields unique arrows $a \times (b \times c) \to b \times c$ and $(a \times b) \times c \to a \times b$. Using these arrows along-side the 'first-order' projection arrows, we use the product definition again to find the unique arrows

$$
\begin{tikzcd}
& a \times (b \times c) \arrow[dl, dashed] \arrow[dr] \arrow[dd, dashed, bend left] & \\
a \times b & & b \times c \\
& (a \times b) \times c \arrow[ul] \arrow[ur, dashed] \arrow[uu, dashed, bend left] &
\end{tikzcd}
$$

Using fact 3.8.2 gives that $(a \times b) \times c \cong a \times (b \times c)$.

Exercise 3.8.8. Consider the situation of a pair of arrows to the codomain objects of the product arrow's constituent arrows:

(i)

$$\begin{tikzcd}
 & e \arrow[dl, "h"'] \arrow[d, dashed, "\langle h,k\rangle"] \arrow[dr, "k"] & \\
a \arrow[d, "f"] & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] \arrow[d, dashed, "f\times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

We can use composition to collapse a and b out of the picture:

$$\begin{tikzcd}
 & e \arrow[dl, "f\circ h"'] \arrow[d, "(f\times g)\circ\langle h,k\rangle"] \arrow[dr, "g\circ k"] & \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

and thus find that $(f \times g) \circ \langle h, k\rangle$ is the unique arrow determined by $f \circ h$ and $g \circ k$.

(ii) In the situation where we can place two product arrows end-to-end:

$$\begin{tikzcd}
e \arrow[d, "h"'] & e \times e' \arrow[l] \arrow[r] \arrow[d, dashed, "h\times k"] & e' \arrow[d, "k"] \\
a \arrow[d, "f"] & a \times b \arrow[l] \arrow[r] \arrow[d, dashed, "f\times g"] & b \arrow[d, "g"] \\
c & c \times d \arrow[l] \arrow[r] & d
\end{tikzcd}$$

we again use composition to collapse the middle level

$$\begin{tikzcd}
e & e \times e' \arrow[l] \arrow[r] \arrow[d, dashed, "(f\times g)\circ(h\times k)"] & e' \arrow[d, "g\circ k"] \\
c & c \times d \arrow[l] \arrow[r] & d
\arrow[from=1-1, to=2-1, "f\circ h"]
\end{tikzcd}$$

Giving (more details needed?)

$$(f \times g) \circ (h \times k) = (f \circ g) \times (h \circ k).$$

3.8.1 Finite Products

Definition 3.8.5. Given a C-object a, the *finite product* (for some integer m) consists of object a_m, and m projection arrows pr_a^m.

$$\begin{tikzcd}
a \\
a^m \arrow[u, bend left=40, "pr"] \arrow[u, bend left=15, "pr_a^2"] \arrow[u, phantom, "\cdots"] \arrow[u, bend right=40, "pr_a^m"']
\end{tikzcd}$$

So that for any m parallel arrows $f_i : c \to a$, there is a unique arrow $\langle f_1, \ldots, f_m \rangle : c \to a^m$ making

$$\begin{tikzcd}
c \arrow[d, bend right=40, "f_1"'] \arrow[d, bend right=15, "f_2"] \arrow[d, bend left=10, "f_m"] \arrow[dd, dashed, bend left=60, "\langle f_1,\ldots,f_m\rangle"] \\
a \\
a^m \arrow[u, bend left=40, "pr"] \arrow[u, bend left=15, "pr_a^2"] \arrow[u, bend right=10, "pr_a^m"']
\end{tikzcd}$$

commute.

Definition 3.8.6. A more *general finite product* of m (not necessarily different) C-objects $a_1 \times a_2 \times \ldots \times a_m$ consists of a C-object and m projection arrows:

$$\begin{tikzcd}
a_1 & a_2 \arrow[rr, dotted, no head] & & a_m \\
& & a_1 \times \ldots \times a_m \arrow[ull, "pr_{a_1}"] \arrow[ul, "pr_{a_2}"'] \arrow[ur, "pr_{a_m}"'] &
\end{tikzcd}$$

So that for any c-object with m arrows $f_1 : c \to a_1$, $f_2 : c \to a_2$, etc, there is a unique arrow $\langle f_1, f_2, \ldots, f_m \rangle$ making

$$\begin{tikzcd}
& & c \arrow[dll, "f_1"'] \arrow[dl, "f_2"] \arrow[dr, "f_m"] \arrow[dd, dashed, "\langle f_1, f_2, \ldots, f_m \rangle"] & \\
a_1 & a_2 \arrow[rr, dotted, no head] & & a_m \\
& & a_1 \times \ldots \times a_m \arrow[ull, "pr_{a_1}"] \arrow[ul, "pr_{a_2}"'] \arrow[ur, "pr_{a_m}"'] &
\end{tikzcd}$$

commute.

Definition 3.8.7. A *general product arrow* is given by a family of m mappings between the components of two general products.

$$\begin{tikzcd}
& & b_1 \times b_2 \times \ldots \times b_m \arrow[dll, "pr_{b_1}"'] \arrow[dl, "pr_{b_2}"] \arrow[dr, "pr_{b_m}"] & \\
b_1 & b_2 \arrow[rr, dotted, no head] & & b_m \\
a_1 \arrow[u, "f_1"] & a_2 \arrow[u, "f_2"] \arrow[rr, dotted, no head] & & a_m \arrow[u, "f_m"'] \\
& & a_1 \times \ldots \times a_m \arrow[ull, "pr_{a_1}"] \arrow[ul, "pr_{a_2}"'] \arrow[ur, "pr_{a_m}"'] \arrow[uuu, dashed, "f_1 \times f_2 \cdots \times f_m"'] &
\end{tikzcd}$$

The product arrow $f_1 \times f_2 \cdots \times f_m$ is the unique arrow found by using

$$\langle f_1 \circ pr_{a_1}, f_2 \circ pr_{a_2}, \ldots, f_m \circ pr_{a_m} \rangle$$

in the general product definiton of $b_1 \times b_2 \times \ldots \times b_m$.

3.9 Co-products

Definition 3.9.1. A *co-product* of C-objects a and b is given by a a C-object denoted $a + b$, and injection functions i_a and i_b.

$$a \xrightarrow[i_a]{} a + b \xleftarrow[i_b]{} b$$

For any $f : a \to c$ and $g : b \to c$, there is a unique arrow $[f, g] : (a + b) \to c$ so that

$$\begin{tikzcd} a \arrow[r, "i_a"'] \arrow[dr, "f"'] & a+b \arrow[d, dashed, "{[f,g]}"] & b \arrow[l, "i_b"] \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes.

Exercise 3.9.1. In **Set** we are told that the co-product $A + B$ is the disjoint union, with i_A and i_B being the disjoint identity function (Ie, $i_A(x) = (x, 0)$ for $x \in A$, and $i_B(x) = (x, 1)$ for $x \in B$. Now suppose we have functions $f : A \to C$ and $g : B \to C$.

$$\begin{tikzcd} A \arrow[r, "i_A"'] \arrow[dr, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "i_B"] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

We can find $[f, g] : A + B \to C$ making this diagram commute by using the rule

$$[f, g](\langle x, y\rangle) = \begin{cases} f(x) & y = 0 \\ g(x) & y = 1. \end{cases}$$

To see that it $[f, g]$ is unique, notice that as given, $[f, g] \circ i_A = f$ iff $[f, g](i_A(x)) = f(x)$ for all $x \in A$. Similarly on the B side - there is no other way to recover the action of f and g out of $A + B$.

Exercise 3.9.2. If $A \cup B = \emptyset$ then we notice that $A \cup B$ satisfies the definition of co-product

$$\begin{tikzcd} A \arrow[r, "i_A"'] \arrow[dr, "f"'] & A \cup B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "i_B"] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

where i_A and i_B are the inclusion functions, and

$$[f, g] = \begin{cases} f(x) & x \in A \\ g(x) & x \in B, \end{cases}$$

which is well defined because any x is in either A or B but not both. Then applying the dual of fact 3.8.3, co-products are isomorphic and therefore $A \cup B \cong A + B$.

Definition 3.9.2. Given two co-products $a + b$ and $c + d$, and arrows $f : a \to c$ and $g : b \to d$

$$\begin{tikzcd} a \arrow[r, "i_a"'] \arrow[d, "f"] & a+b & b \arrow[l, "i_b"] \arrow[d, "g"] \\ c \arrow[r, "i_c"'] & c+d & d \arrow[l, "i_d"] \end{tikzcd}$$

the *co-product arrow* $f + g$ is found by using $[i_c \circ f, i_d \circ g]$ in the co-product definition of $c + d$.

$$\begin{tikzcd} a \arrow[r, "i_a"'] \arrow[d, "f"] & a+b \arrow[d, dashed, "f+g"] & b \arrow[l, "i_b"] \arrow[d, "g"] \\ c \arrow[r, "i_c"'] & c+d & d \arrow[l, "i_d"] \end{tikzcd}$$

3.10 Equalizers

Definition 3.10.1. An arrow i *equalizes* f and g if they are laid out as follows

$$\begin{tikzcd} e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

where $f \circ i = g \circ i$. Additionaly we demand the limiting property: if another e^* and i^* work as above, there is a unique arrow $e^* \to e$ making

$$\begin{tikzcd} e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\ e^* \arrow[u, dashed] \arrow[ur, "i^*"'] & & \end{tikzcd}$$

commute.

Fact 3.10.2. *Every equalizer is monic.*

Fact 3.10.3. *An epic equalizer is iso.*

$$\begin{tikzcd} e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\ a \arrow[u, dashed] \arrow[ur, "1_a"'] & & \end{tikzcd}$$

Exercise 3.10.1. Working in **Set**, we wish to show that monics are equalizers. Suppose we have some injective function $i : E \to A$:

$$\begin{tikzcd} E \arrow[r, tail, "i"] & A \end{tikzcd}$$

We seek functions which i is an equalizer. Let $f, g : A \rightrightarrows \{0,1\}$ be given by

$$f(x) = 1$$

$$g(x) = \begin{cases} 1 & x \in i(E) \\ 0 & x \notin i(E). \end{cases}$$

Now clearly $g(i(x)) = f(i(x))$ for all $x \in E$, so $i \circ f = i \circ g$. Supposing that there is another $i^* : C \to A$ such that $i^* \circ f = i^* \circ g$. We must have ...? TODO: Finish showing universal property

$$\begin{tikzcd} E \arrow[r, tail, "i"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & \{0,1\} \\ C \arrow[u, dashed] \arrow[ur, "i^*"'] & & \end{tikzcd}$$

Exercise 3.10.2. Working in a poset, suppose that i equalizes f and g. Recall that any 2 parallel arrows are equal. Then in particular $f \circ i = g \circ i$ as follows:

$$\begin{tikzcd} e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

But plugging a and 1_a into the definition for equalizer, since $f \circ 1_a = f = g = g \circ 1_a$ we retrieve the unique arrow $a \to e$.

$$\begin{tikzcd}
e \arrow[r, "i"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
a \arrow[u, dashed] \arrow[ur, "1_a"'] & &
\end{tikzcd}$$

Since we have arrows $a \to e$ and $e \to a$ we simply apply the antisymmetric property of posets to determine $e = a$, and the single-arrow property to determine that $i = 1_a$.

3.11 Limits and co-limits

Definition 3.11.1. A *diagram* is informally a 'view' of some objects and arrows within a category. We haven't looked at functors yet but formally, a diagram in a category C is a functor $F : J \to C$ where J is an indexing category. (Riehl pg 38)

Definition 3.11.2. Given a diagram D, a a D*-cone* consists of a C-object c together with component arrows $f_i : c_i \to d_i$ for each $d_i \in d$ that commute with any arrow g in D.

$$\begin{tikzcd}
& c \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\
d_i \arrow[rr] & & d_j
\end{tikzcd}$$

Definition 3.11.3. Given a diagram D, a *limit* for D is a D-cone such that any other D-cone factors through uniquely.

$$\begin{tikzcd}
c \arrow[dr, "f_i"'] \arrow[drr, "f_j"] & & & c' \arrow[lll, dashed] \arrow[dll, "f'_i"'] \arrow[dl, "f'_j"] \\
& d_i \arrow[r] & d_j &
\end{tikzcd}$$

We can apply cones to define some earlier objects in a more succinct way.

Definition 3.11.4. The *product* of a and b is a limit for the arrow-less diagram

$$a \qquad\qquad b.$$

Definition 3.11.5. An *equalizer* of f and g is a limit for the diagram

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b
\end{tikzcd}$$

Definition 3.11.6. A *terminal object* is a limit for the empty diagram.

Exercise 3.11.1. Dualize to reach definitions for *co-cones* and *co-limits*.

3.12 Co-equalizers

Definition 3.12.1. A *co-equalizer* of $f, g : a \rightrightarrows b$ is a co-limit for the diagram

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

Let's work backwards from that definition and try to reach something similar to definition 3.10.1 for equalizers! A co-cone over the above diagram consist of an object c, and arrows $f_a : a \to c$ and $f_b : b \to c$ such that

$$\begin{tikzcd} a \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "f_a"'] & & b \arrow[dl, "f_b"] \\ & c & \end{tikzcd}$$

commutes. But that means that $f_b \circ f = f_a$ and $f_b \circ g = f_a$. So we can simply drop the f_a and demand an f_b making

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "f_b"] & c \end{tikzcd}$$

commute. That gave us our co-cone, but what we want is a co-limit: *the* co-cone that is universal among co-cones. So if we have a c' and f'_b such that $f'_b \circ f = f'_b \circ g$, then c must must factor through c' in a unique way:

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "f_b"] \arrow[dr, "f'_b"'] & c \arrow[d, dashed] \\ & & c' \end{tikzcd}$$

3.13 Pullbacks

Definition 3.13.1. Given $f : a \to c$ and $g : b \to c$, a *pullback* is a limit for the diagram

$$\begin{tikzcd} a \arrow[r, "f"] & c & b \arrow[l, "g"'] \end{tikzcd}$$

A cone for this diagram takes the form of an object p with arrows $f' : p \to b$ and $g' : p \to a$ so that

$$\begin{tikzcd} p \arrow[r, "f'"'] \arrow[d, "g'"] & b \arrow[d, "g"'] \\ a \arrow[r, "f"] & c \end{tikzcd}$$

commutes. And the limit means that any cone factors through:

$$\begin{tikzcd} p' \arrow[dr, dashed] \arrow[drr, "f''"] \arrow[ddr, "g''"'] & & \\ & p \arrow[r, "f'"'] \arrow[d, "g'"] & b \arrow[d, "g"'] \\ & a \arrow[r, "f"] & c \end{tikzcd}$$

Exercise 3.13.1. Suppose

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d] & b \arrow[d] \\ c \arrow[r, "f"] & d \end{tikzcd}$$

is a pullback square, and f is monic. Now suppose we have parallel $h_1, h_2 : e \rightrightarrows a$

$$\begin{tikzcd} e \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & a \arrow[r, "g"] \arrow[d, "f'"] & b \arrow[d, "g'"] \\ & c \arrow[r, "f"] & d \end{tikzcd}$$

so that $g \circ h_1 = g \circ h_2$. Then

$$g' \circ g \circ h_1 = g' \circ g \circ h_2.$$

From the original pullback we have $g' \circ g = f \circ f'$, so substituting on both sides gives $f \circ f' \circ h_1 = f \circ f' \circ h_2$. Since f is monic, $f' \circ h_1 = f' \circ h_2$.

But also, substituting on one side gives $g' \circ g \circ h_1 = f \circ f' \circ h_1$. Then

$$\begin{tikzcd} e \arrow[r, "g \circ h_1 = g \circ h_2"] \arrow[d, "f' \circ h_1 = f' \circ h_2"] & b \arrow[d, "g'"] \\ c \arrow[r, "f"] & d \end{tikzcd}$$

commutes, meaning it must factor uniquely through our original pullback. Thus the arrow $e \to a$ is unique, meaning $h_1 = h_2$. Conclude f is monic.

3.14 Pushouts

Definition 3.14.1. Given arrows $f : a \to b$ and $g : a \to c$, a *pushout* is a colimit for the diagram

$$b \xleftarrow{f} a \xrightarrow{g} c$$

The co-limit, or universal co-cone for the above diagram consists of an object d and arrows $f' : c \to d$ and $g' : b \to d$ so that $g' \circ f = f' \circ g$. Further, if there is a d', f'', g'' behaving the same way, then d will factor uniquely through it.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "g'"] \arrow[ddr, "g''"] & \\ c \arrow[r, "f'"] \arrow[drr, "f''"'] & d \arrow[dr, dashed] & \\ & & d' \end{tikzcd}$$

3.15 Completeness

Definition 3.15.1. A category is *complete* if every diagram has a limit.

Definition 3.15.2. A category if *finitely complete* if every *finite* diagram has a limit.

Fact 3.15.3. *If C has a terminal object and a pullback for each pair of arrows with common codomain, then C is finitely complete.*

Exercise 3.15.1. Working in a category with pullbacks and products, we demonstrate how to constuct equalizers. Given a parallel pair $f, g : a \rightrightarrows b$, first form the product:

$$
\begin{tikzcd}
a \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & b \\
& a \times b \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Then we can find the two product arrows $f' = \langle 1_a, f\rangle$ and $g' = \langle 1_a, g\rangle$.

$$
\begin{tikzcd}
& a \arrow[dl, "1_a"'] \arrow[d, dashed, shift right, "\langle 1_a, f\rangle"'] \arrow[d, dashed, shift left, "\langle 1_a, g\rangle"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & \\
a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b
\end{tikzcd}
$$

Notice that we get the following facts which you can wager will be made use of shortly:

$$pr_a \circ \langle 1_a, f\rangle = 1_a,$$

$$pr_a \circ \langle 1_a, g\rangle = 1_a,$$

$$pr_b \circ \langle 1_a, f\rangle = f,$$

$$pr_b \circ \langle 1_a, g\rangle = g.$$

Since $\langle 1_a, f\rangle$ and $\langle 1_a, g\rangle$ share a codomain we can form the pullback:

$$
\begin{tikzcd}
d \arrow[r, "q"] \arrow[d, "p"'] & a \arrow[d, "\langle 1_a, g\rangle"] \\
a \arrow[r, "\langle 1_a, f\rangle"] & a \times b
\end{tikzcd}
$$

with $\langle 1_a, f\rangle \circ p = \langle 1_a, g\rangle \circ q$. We can tack on the projection functions to move forward:

$$
\begin{tikzcd}
d \arrow[r, "q"] \arrow[d, "p"'] & a \arrow[d, "\langle 1_a, g\rangle"] & \\
a \arrow[r, "\langle 1_a, f\rangle"] & a \times b \arrow[dl, "pr_b"] \arrow[dr, "pr_a"'] & \\
b & & a
\end{tikzcd}
$$

First,

$$pr_a \circ \langle 1_a, f\rangle \circ p = pr_a \circ \langle 1_a, g\rangle \circ q.$$

Substituting from above gives

$$1_a \circ p = 1_a \circ q,$$

or $p = q$. So we can dispense with q from here on. Let's tack on the other projection function:

$$pr_b \circ \langle 1_a, f\rangle \circ p = pr_b \circ \langle 1_a, g\rangle \circ p.$$

Substituting from above,

$$f \circ p = g \circ p.$$

Therefore p is an equalizer of f and g.

Exercise 3.15.2. Working in a category with products and equalizers, we demonstrate how to construct pullbacks.

Given $f : a \to c$ and $g : b \to c$, form the product $a \times b$ and the parallel functions $f \circ pr_a, g \circ pr_b : a \times b \rightrightarrows c$:

$$\begin{tikzcd}
& a \arrow[dr, "f"] & \\
a \times b \arrow[ur, "pr_a"] \arrow[dr, "pr_b"'] & & c \\
& b \arrow[ur, "g"'] &
\end{tikzcd}$$

Then there is an equalizer $i : e \to a \times b$ such that $(f \circ pr_a) \circ i = (g \circ pr_b) \circ i$:

$$\begin{tikzcd}
e \arrow[r, "i"] & a \times b \arrow[r, shift left, "f\circ pr_a"] \arrow[r, shift right, "g\circ pr_b"'] & c
\end{tikzcd}$$

But then $f \circ (pr_a \circ i) = g \circ (pr_b \circ i)$ meaning

$$\begin{tikzcd}
e \arrow[r, "pr_b\circ i"] \arrow[d, "pr_a\circ i"'] & b \arrow[d, "g"] \\
a \arrow[r, "f"'] & c
\end{tikzcd}$$

commutes. To see universality, suppose there is another e', i_a, i_b such that $f \circ i_a = g \circ i_b$, or

$$\begin{tikzcd}
e' \arrow[drr, bend left, "i_b"] \arrow[ddr, bend right, "i_a"'] & & \\
& e \arrow[r, "pr_b\circ i"] \arrow[d, "pr_a\circ i"'] & b \arrow[d, "g"] \\
& a \arrow[r, "f"'] & c
\end{tikzcd}$$

commutes. Then we use the product $a \times b$ to retrieve the unique function $\langle i_a, i_b \rangle$:

$$\begin{tikzcd}
& e' \arrow[dl, "i_a"'] \arrow[d, dashed, "\langle i_a,i_b\rangle"] \arrow[dr, "i_b"] & \\
a & a \times b \arrow[l] \arrow[r] & b
\end{tikzcd}$$

Here $i_a = pr_a \circ \langle i_a, i_b \rangle$ and $i_b = pr_b \circ \langle i_a, i_b \rangle$. But since $f \circ i_a = g \circ i_b$ we have $f \circ pr_a \circ \langle i_a, i_b \rangle = g \circ pr_b \circ \langle i_a, i_b \rangle$, meaning that $\langle i_a, i_b \rangle$ functions as an equalizer:

$$\begin{tikzcd}
e' \arrow[d, dashed, "h"'] \arrow[dr, "\langle i_a,i_b\rangle"] & & \\
e \arrow[r, "i"] & a \times b \arrow[r, shift left, "f\circ pr_a"] \arrow[r, shift right, "g\circ pr_b"'] & c
\end{tikzcd}$$

So we get the unique factorization $h : e' \to e$ from the equalizer's universal property, and plug it straight back into our commutative square to see that we really have the universal pullback:

$$\begin{tikzcd}
e' \arrow[drr, bend left, "i_b"] \arrow[ddr, bend right, "i_a"'] \arrow[dr, dashed, "h"] & & \\
& e \arrow[r, "pr_b\circ i"] \arrow[d, "pr_a\circ i"'] & b \arrow[d, "g"] \\
& a \arrow[r, "f"'] & c
\end{tikzcd}$$

3.16 Exponentials

In **Set**, think of the exponential object B^A as all functions from A to B. Then we can apply an element $f \in B^A$ to take $a \in A$ to some $b \in B$ (namely $f(a)$). In other words we have the evaluation function $ev : B^A \times A \to B$ that takes $\langle f, a\rangle$ to $f(a)$. Furthermore, ev has a universal property.

The action of any function $g : C \times A \to B$ determines a unique function in B^A - but there is an in-between step. Namely, for any particular c, we first 'curry' g - Leave c fixed and evaluate $g(c, a)$ for every $a \in A$. Then g (*for that c only*) is reduced to a function $g_c : A \to B$, or in other words a member of B^A. Now we can lift the g_c's to get a function $\hat{g} : C \to B^A$. Simply let

$$\hat{g}(c) = g_c.$$

This is the only choice so that

$$ev(\langle \hat{g}(c), a\rangle) = g(\langle c, a\rangle)$$

for any $c \in C$ and $a \in A$.

I'm sure you are crying out for diagrams by this point, so let's get to the general definition:

Definition 3.16.1. If our category C has *exponentiation* then given any two objects a and b, there exists an object b^a and an arrow $ev : b^a \times a \to b$, so that any c and $g : c \times a \to b$, there is a unique $\hat{g} : c \to b^a$:

$$\begin{tikzcd}
b^a & b^a \times a \arrow[l] \arrow[r] \arrow[drr, "ev"] & a & \\
 & & & b \\
c \arrow[uu, dashed, "\hat{g}"] & c \times a \arrow[l] \arrow[r] \arrow[urr, "g"'] & a &
\end{tikzcd}$$

so that when we form the product arrow $\hat{g} \times 1_a$, we have $ev \circ (\hat{g} \times 1_a) = g$:

$$\begin{tikzcd}
b^a & b^a \times a \arrow[l] \arrow[rr] \arrow[dr, "ev"] & & a \\
 & & b & \\
c \arrow[uu, dashed, "\hat{g}"] & c \times a \arrow[l] \arrow[rr] \arrow[uu, dashed, "\hat{g}\times 1_a"'] \arrow[ur, "g"'] & & a \arrow[uu, "1_a"']
\end{tikzcd}$$

Definition 3.16.2. In the context of the last definition, the two arrows g and $\hat{g}$ are *exponential adjoints*.

Fact 3.16.3. *Furthermore there is a one-to-one correspondence (bijection) between arrows*

$$(c \times a) \to b \quad \textit{and} \quad c \to b^a.$$

In other words

$$C(c \times a, b) \cong C(c, b^a).$$

$$\begin{tikzcd}
b & & b^a \\
c \times a \arrow[u, "g"] & \Longleftrightarrow & c \arrow[u, "\hat{g}"']
\end{tikzcd}$$

Let's examine the claim that we've got a bijection from $C(c \times a, b) \to C(c, b^a)$. Given $\hat{g}, \hat{h} : c \to b^a$, if $\hat{g} = \hat{h}$ then we have $ev \circ (\hat{h} \times 1_a) = ev \circ (\hat{g} \times 1_a)$:

$$\begin{tikzcd}
b^a & b^a \times a \arrow[l] \arrow[dr, "ev"] & \\
 & & b \\
c \arrow[uu, bend left, "\hat{g}"] \arrow[uu, bend right, "\hat{h}"'] & c \times a \arrow[l] \arrow[uu, bend left, "\hat{g}\times 1_a"] \arrow[uu, bend right, "\hat{h}\times 1_a"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Thus the assignment is injective.

Now for any given $\hat{g} : c \to b^a$, we can form the arrow $ev \circ (\hat{g} \times 1_a) : c \times a \to b$. Let's call it h.

$$
\begin{tikzcd}
b^a & b^a \times a \arrow[l] \arrow[dr, "ev"] & \\
 & & b \\
c \arrow[uu, "\hat{g}"] & c \times a \arrow[l] \arrow[uu, "\hat{g}\times 1_a"] \arrow[ur, "h=ev\circ(\hat{g}\times 1_a)"'] &
\end{tikzcd}
$$

But then from the definition of exponentiation, h determines a unique *$\hat{h} : c \to b^a$. Therefore $\hat{h} = \hat{g}$ and the assignment is surjective.*

Let's state a few facts that hold true in a Cartesian Closed category with an initial object 0 and terminal object 1.

Fact 3.16.4.

$$0 \cong 0 \times a$$

for any a. This follows from applying fact 3.16.3 for any object b:

$$C(0, b^a) \cong C(0 \times a, b)$$

Since $C(0, b^a)$ has 1 member, so does $C(0 \times a, b)$. But since b was arbitrary, $0 \times a$ is initial. Lastly because of exercise 3.6.1 we get $0 \cong 0 \times a$.

Fact 3.16.5. *If there exists an arrow $a \to 0$, then $a \cong 0$.*

Take $f : a \to 0$ and form the product arrow $\langle f, 1_a \rangle : a \to 0 \times a$:

$$
\begin{tikzcd}
 & a \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,1_a\rangle"] \arrow[dr, "1_a"] & \\
0 & 0 \times a \arrow[l, "pr_0"] \arrow[r, "pr_a"'] & a
\end{tikzcd}
$$

By fact 3.8.2 we see that $\langle f, 1_a \rangle \circ pr_a = 1_{0\times a}$, so

$$0 \times a \cong a,$$

and by fact 3.16.4,

$$a \cong 0.$$

Fact 3.16.6. *If $0 \cong 1$ then all objects are isomorphic to each other.*

Any object has an arrow to 1, and since 0cong1 it will have an arrow to 0. Then due to fact 3.16.5, every object is isomorphic to 0.

Fact 3.16.7. *An arrow $f : 0 \to a$ for any a is monic.*

Suppose we have $g, h : b \rightrightarrows 0$ with $f \circ g = f \circ h$:

$$
\begin{tikzcd}
b \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & 0 \arrow[r, "f"] & a
\end{tikzcd}
$$

By fact 3.16.5, $b \cong 0$, meaning the arrow from $b \to 0$ is unique and $g = h$.

Exercise 3.16.1.

1. $$a^1 \cong a$$

   fact 3.16.3 gives for a^1 and any c that
   $$C(c \times 1, a) \cong C(c, a^1).$$

   In particular, plugging in a gives
   $$C(a \times 1, a) \cong C(a, a^1).$$

   TODO: Why does the bijection preserve isos?

2.
$$a^0 \cong 1$$

fact 3.16.3 gives for a^0 and any c that
$$C(c \times 0, a) \cong C(c, a^0).$$

From fact 3.16.4 we know that $c \times 0 \cong 0$, so $C(c \times 0, a)$ has 1 element, and $C(c, a^0)$ has one element. Since c was arbitrary we get that a^0 is terminal and therefore isomorphic to 1.

3.
$$1^a \cong 1$$

fact 3.16.3 gives for 1^a and any c that
$$C(c \times a, 1) \cong C(c, 1^a).$$

Since 1 is terminal, $C(c \times a, 1)$ has one member. Thus $C(c, 1^a)$ has one member, meaning 1^a is terminal and $1^a \cong 1$.

Chapter 4

Introducing Topoi

4.1 Subobjects

Definition 4.1.1. A *subobject* of d is an equivalence class of monic arrows into d.

We are using a new notion of equivalence. Start with the idea that a subobject is simply a monic arrow into d. Now write that $f \subseteq g$ *iff* these is an arrow $h : a \to b$ making

$$\begin{tikzcd} a \arrow[dd, "h"'] \arrow[dr, tail, "f"] & \\ & d \\ b \arrow[ur, tail, "g"'] & \end{tikzcd}$$

commute.

Now say that $f \cong g$ *iff* $f \subseteq g$ and $g \subseteq f$.

$$\begin{tikzcd} a \arrow[dd, bend right, "i"'] \arrow[dr, tail, "f"] & \\ & d \\ b \arrow[uu, bend right, "h"'] \arrow[ur, tail, "g"'] & \end{tikzcd}$$

Here, $g \circ h = f$ and $f \circ i = g$

Exercise 4.1.1. We wish to show that in the above diagram, a and b are isomorphic. Substituting each of our 'commuting' equations into the other in turn yields

$$f \circ i \circ h = f,$$

$$g \circ h \circ i = g.$$

Since both f and g are monic, left-cancel to get

$$i \circ h = 1_a,$$

$$h \circ i = 1_b.$$

Conclude that h and i are monic and mutual inverses. Thus a and b are isomorphic.

Exercise 4.1.2. Now we wish to show that $\cong$ in this sense is indeed an equivalence relation.

1. *reflexivity:* $f \cong f$ for $f : a \to b$ since there is an arrow $1_a : a \to a$, and the relationship $a \subseteq a$ is reflexive.

2. *symmetry:* If $f \cong g$ then $f \subseteq g$ and $g \subseteq f$. So by the symmetry of the word 'and' we have that $g \cong f$.

3. *transitivity:* Suppose $f \cong g$ and if $g \cong k$. Then from the first relation we know $f \subseteq g$ and $g \subseteq f$. From the second we know $g \subseteq k$ and $k \subseteq g$.

   Since $f \subseteq g$ and $g \subseteq k$, $f \subseteq k$. And since $k \subseteq g$ and $g \subseteq f$, $k \subseteq f$.

   Since $f \subseteq k$ and $k \subseteq f$, conclude $f \cong k$.

Exercise 4.1.3. Suppose that $[f] = [f']$ and $[g] = [g']$.

($\Rightarrow$) Suppose $f \subseteq g$. Since $f \cong f'$, $f' \subseteq f$. Then $f' \subseteq g'$

($\Leftarrow$) Suppose $f' \subseteq g'$. Since $f \cong f'$, $f \subseteq f'$ Then $f \subseteq g$.

Conclude $f \subseteq g$ iff $f' \subseteq g'$. In other words $\subseteq$ is stable under $\cong$.

Exercise 4.1.4. In **Set**, $Sub(D) \cong P(D)$.

First, observe that if f_1, f_2 are monic arrows (or equivalently, injective set functions) in a subobject $f \in Sub(D)$, they must have identical images. For suppose they don't. Then there is no h that can make

$$\begin{tikzcd} A \arrow[dr, "f_1"] & \\ & D \\ B \arrow[uu, "h"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

commute.

Then each $f \in Sub(D)$ determines a subset $C \subseteq D$, under the image of any particular member. For example, in the above diagram we have $f_1(A) = f_2(B)$.

On the other hand, given a subset $C \subseteq D$, the inclusion function $i_C : C \to D$ is monic, and therefore $[i]$ is a subobject of D.

Furthermore we have $[i] = [f_1] = [f_2]$. Thus the image function and the inclusion function are inverses *up to isomorphism*, so

$$Sub(D) \cong P(D).$$

Definition 4.1.2. The *naming arrow* for f:

$$\begin{tikzcd} & & b \\ b^a & b^a \times a \arrow[l] \arrow[r] \arrow[ur, "ev"] & a \arrow[u, "f"'] \\ 1 \arrow[u, "\lceil f \rceil"] & 1 \times a \arrow[l, "pr_1"'] \arrow[r, "pr_a"] \arrow[u, "\lceil f \rceil \times 1_a"] & a \arrow[u, "1_a"'] \end{tikzcd}$$

Exercise 4.1.5. For any element $x : 1 \to a$ of a,

$$ev \circ \langle \lceil f \rceil, x \rangle = f \circ x$$

4.2 Classifying Subobjects

We are seeking to generalize the situation where a subset $A \subseteq D$ has a characteristic function $\chi_A : D \to \{0, 1\}$, where

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

In fact the functions from $D \to \{0, 1\}$ are in bijection with all possible subsets of D. Then given an $f : D \to \{0, 1\}$ we define A_f as the subset determined by f, and then we have

$$\begin{tikzcd} A_f \arrow[r, hook] \arrow[d, "!"'] & D \arrow[d, "f"] \\ 1 \arrow[r, "true"'] & \{0,1\} \end{tikzcd}$$

is a pullback.

Definition 4.2.1. If C is a category with a terminal object 1, then a *subobject classifier* consist of an object Ω, and an arrow $true : 1 \to \Omega$.

The Ω axiom is that for any monic $f : a \to d$ there is a unique arrow $\chi_f : d \to \Omega$ so that

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "!"'] & d \arrow[d, "\chi_f"] \\ 1 \arrow[r, "true"'] & \Omega \end{tikzcd}$$

is a pullback square.

The arrow χ_f is the *characteristic arrow* of f.

Definition 4.2.2. For any a there is a unique arrow $! : a \to 1$. The composite $true \circ !$ yields an arrow denoted $true_a$:

$$\begin{tikzcd} a \arrow[r, "!"] \arrow[dr, "true_a"'] & 1 \arrow[d, "true"] \\ & \Omega \end{tikzcd}$$

Exercise 4.2.1. Plugging $true : 1 \to \Omega$ into definition 4.2.1 gives:

$$\begin{tikzcd} 1 \arrow[r, tail, "true"] \arrow[d, "!"'] & \Omega \arrow[d, "\chi_{true}"] \\ 1 \arrow[r, "true"'] & \Omega \end{tikzcd}$$

and we see that ! must be the identity $1_1 : 1 \to 1$. So for the diagram to be a pullback square we must have

$$\chi_{true} \circ true = true \circ 1_1 = true$$

and therefore

$$\chi_{true} = 1_\Omega.$$

Exercise 4.2.2. Plugging $1_\Omega : \Omega \to \Omega$ into definition 4.2.1 gives

$$\begin{tikzcd} \Omega \arrow[r, tail, "true"] \arrow[d, "!"'] & \Omega \arrow[d, "\chi_{1_\Omega}"] \\ 1 \arrow[r, "true"'] & \Omega \end{tikzcd}$$

To be a pullback we must have $\chi_{1_\Omega} \circ 1_\Omega = true \circ !$. Or

$$\chi_{1_\Omega} = true_\Omega$$

using the notation of definition 4.2.2.

Exercise 4.2.3. For any $f : a \to b$, since $!_a : a \to 1$ is unique:

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "!_a"'] & & b \arrow[dl, "!_b"] \\ & 1 & \end{tikzcd}$$

We must have $!_b \circ f = !_a$. Thus

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dr, "!_a"'] & & b \arrow[dl, "!_b"] \\
& 1 \arrow[d, "true"] & \\
& \Omega &
\end{tikzcd}$$

commutes, or $true \circ !_b \circ f = true \circ !_a$. Therefore using definition 4.2.2,

$$true_b \circ f = true_a.$$

4.3 Definition of Topos

Definition 4.3.1. An *elementary topos* is a category which

1. is finitely complete.
2. is finitely co-complete.
3. has exponentiation.
4. has a subobject classifier.

Fact 4.3.2. *More succinctly, a topos can be defined as a Cartesian closed category with a subobject classifier.*

4.4 First Examples

4.5 Bundles and Sheaves

Exercise 4.5.1. We wish to show that

$$\begin{tikzcd}
A \arrow[rr, hook, "k"] \arrow[dd, "f"'] \arrow[dr, "f"] & & B \arrow[dl, "g"'] \arrow[dd, "\chi_k"] \\
& I & \\
I \arrow[rr, "T"'] \arrow[ur, "1_I"] & & 2 \times I \arrow[ul, "pr_I"']
\end{tikzcd}$$

is a pullback. Note that $1_I : I \to I$ is the terminal object in $Bn(I)$, and $f : A \to I$ is the unique arrow from itself (as an object) to the terminal object. So the desired outcome is to show that $\chi_k \circ k = T \circ f$.

Recall the definitions of $T : I \to 2 \times I$ and $\chi_k : B \to 2 \times I$ as product arrows:

$$T = \langle true \circ !_I, 1_I \rangle,$$

$$\chi_k = \langle \chi_A, g \rangle :$$

$$
\begin{tikzcd}
 & & A \arrow[d, "f"] & & \\
 & & I \arrow[dl, "!_I"'] \arrow[dd, dashed, "T"] \arrow[ddrr, "1_I"] & & \\
 & 1 \arrow[dl, "true"'] & & & \\
2 & & 2 \times I \arrow[ll] \arrow[rr] & & I \\
 & & B \arrow[ull, "\chi_A"] \arrow[u, dashed, "\chi_k"] \arrow[urr, "g"'] & & \\
 & & A \arrow[u, "k"'] & &
\end{tikzcd}
$$

Because of exercise 3.8.3 then we have

$$T \circ f = \langle true \circ !_I \circ f, 1_I \circ f \rangle,$$

$$\chi_k \circ k = \langle \chi_A \circ k, g \circ k \rangle.$$

By virtue of being an inclusion (and an arrow in $Bn(I)$) we have

$$
\begin{tikzcd}
A \arrow[rr, hook, "k"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
 & I &
\end{tikzcd}
$$

commutes, or $f = g \circ k$. And since

$$
\begin{tikzcd}
A \arrow[r, hook, "k"] \arrow[d, "!_I"'] & B \arrow[d, "\chi_A"] \\
1 \arrow[r, "true"'] & 2
\end{tikzcd}
$$

is a pullback in **Set** we get that $\chi_A \circ k = true \circ !_I$.

Lastly, observe that

$$true \circ !_I \circ f = true \circ !_A$$

since there is only one arrow $A \to 1$:

$$
\begin{tikzcd}
 & & A \arrow[dl, "!_A"'] \arrow[d, "f"] \\
 & 1 \arrow[dl, "true"'] & I \arrow[l, "!_I"] \\
2 & &
\end{tikzcd}
$$

Now putting it all together gives

$$\langle \chi_A \circ k, g \circ k \rangle = \langle true \circ !_I \circ f, 1_I \circ f \rangle,$$

or

$$\langle \chi_A, g \rangle \circ k = \langle true \circ !_I, 1_I \rangle \circ f,$$

or

$$T \circ f = \chi_k \circ k,$$

as was to be shown.