Useful Mathematics

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Expansions and Sums

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n} \approx x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) \tag{1}$$

$$(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + O(x^4) \tag{2}$$

$$(1+x)^{-1/2} \approx 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + O(x^4) \tag{3}$$

$$f(\mathbf{r}'_\alpha) = f(0) + \sum_i^{\infty} x'_{\alpha,i}\left(\frac{\partial f(\mathbf{r}'_\alpha)}{\partial x'_{\alpha,i}}\right) + \frac{1}{2}\sum_i^{\infty}\sum_j^{\infty} x'_{\alpha,i}x'_{\alpha,j}\left(\frac{\partial^2 f(\mathbf{r}'_\alpha)}{\partial x'_{\alpha,i}\partial x'_{\alpha,j}}\right) \tag{4}$$

$$\sum_{k=a}^{n} k = \frac{(n-a+1)(n+a)}{2} \tag{5}$$

$$\sum_{k=a}^{n} 2k+1 = (n+a+1)(n-a+1) \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

$$\sum_{k=m}^{n} x^k = \frac{m^m - x^n}{1-x} \tag{8}$$

$$\sum_{n}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x \tag{9}$$

$$\sum_{n}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x \tag{10}$$

Dirac Delta Function

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)] \tag{11}$$

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{\rho}\delta(\rho - \rho')\delta(\theta - \theta')\delta(z - z') \tag{12}$$

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{r^2}\delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \tag{13}$$

$$\delta(x) = \lim_{a\to\infty} \frac{\sin^2(ax)}{\pi a x^2} \tag{14}$$

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\sin(x/\epsilon)}{\pi x} \tag{15}$$

$$\delta(x) = \lim_{\sigma\to 0} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \tag{16}$$

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{\left|\frac{\partial g(x_i)}{\partial x_1}\right|} \tag{17}$$

$$\int_{-\infty}^{\infty} f(x)\delta[g(x)]\, dx = \sum_i \frac{f(x_i)}{\left|\frac{\partial g(x_i)}{\partial x_1}\right|} \tag{18}$$

Trig Relations

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v \tag{19}$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v \tag{20}$$

$$\sin u \pm \sin v = 2\sin\left(\frac{u \pm v}{2}\right)\cos\left(\frac{u \mp v}{2}\right) \tag{21}$$

$$\cos u + \cos v = 2\cos\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right) \tag{22}$$

$$\cos u - \cos v = -2\sin\left(\frac{u + v}{2}\right)\sin\left(\frac{u - v}{2}\right) \tag{23}$$

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)] \tag{24}$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)] \tag{25}$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)] \tag{26}$$

$$\sin(2u) = 2 \sin u \cos u \tag{27}$$

$$\cos(2u) = 1 - 2 \sin^2 u \tag{28}$$

Vector Operations

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{29}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{30}$$

$$\nabla \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A} \tag{31}$$

$$\nabla \times (\psi \mathbf{A}) = (\nabla \psi) \times \mathbf{A} + \psi \nabla \times \mathbf{A} \tag{32}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{33}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{34}$$

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\sigma \cdot (\mathbf{A} \times \mathbf{B}) \tag{35}$$

Cylindrical Operations

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\mathbf{e_1} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\mathbf{e_2} + \frac{\partial\psi}{\partial z}\mathbf{e_3} \tag{36}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z} \tag{37}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right)\mathbf{e_1} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right)\mathbf{e_2} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)\mathbf{e_3} \tag{38}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{39}$$

Spherical Operations

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e_1} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e_2} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e_3} \tag{40}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi} \tag{41}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]\mathbf{e_1} + \left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right]\mathbf{e_2} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]\mathbf{e_3} \tag{42}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{43}$$

Gamma Function

$$\Gamma(1/2+n) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \tag{44}$$

$$\Gamma(1/2-n) = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi} \tag{45}$$

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}[\frac{1}{\epsilon} + (1 + \frac{1}{2} + \cdots \frac{1}{n} - \gamma) + O(\epsilon)] \tag{46}$$

$$\Gamma(\epsilon/2) \approx \frac{2}{\epsilon} - \gamma + O(\epsilon) \tag{47}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \tag{48}$$

$$\Gamma(1+z) = z\Gamma(z) \tag{49}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{50}$$

Integrals

$$\int a^x \, dx = \frac{a^x}{\ln a} \tag{51}$$

$$\int \tanh(ax) \, dx = \frac{1}{a} \ln|\cosh(ax)| \tag{52}$$

$$\int \cosh(ax) \, dx = \frac{1}{a} \sinh(ax) \tag{53}$$

$$\int \sinh(ax) \, dx = \frac{1}{a} \cosh(ax) \tag{54}$$

$$\int \sin(ax)\sin(bx) \, dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)} \tag{55}$$

$$\int_0^\pi \sin(ax)\sin(bx) \, dx = \frac{\pi}{2}\delta_{ab} \tag{56}$$

$$\int \cos(ax)\cos(bx) \, dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b)} \tag{57}$$

$$\int \cos(ax)\sin(bx) \, dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)} \tag{58}$$

$$\int_0^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2} \tag{59}$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\ dx = \frac{\pi}{2} \tag{60}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left[x+\sqrt{x^2+a^2}\right] \tag{61}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = -\tan^{-1}\left[\frac{x\sqrt{x^2-a^2}}{x^2-a^2}\right] \tag{62}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) \tag{63}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \tag{64}$$

$$\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\ dx = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2-4ac}{4a}\right] \tag{65}$$

$$\int_0^\infty \frac{x}{e^x-1}\ dx = \frac{\pi^2}{6} \tag{66}$$

$$\int_0^\infty \frac{x^2}{e^x-1}\ dx = 2\zeta(3) \tag{67}$$

$$\int_0^\infty \frac{x^3}{e^x-1}\ dx = \frac{\pi^4}{15} \tag{68}$$

$$\int_{-\infty}^\infty D\varphi(x)\ e^{x\cdot A\cdot x+iJx} = \left(\frac{(2\pi i)^N}{\det|A|}\right)^{1/2} e^{-\frac{i}{2}J\cdot A^{-1}\cdot J} \tag{69}$$

$$\int D\varphi\ e^{-\frac{1}{2}\varphi\cdot K\varphi - V(\varphi)+J\cdot\varphi} = e^{-V(\delta/\delta J)}e^{\frac{1}{2}J\cdot K^{-1}\cdot J} \tag{70}$$

$$\int_0^{\pi/2}(\sin\theta)^{2n-1}(\cos\theta)^{2m-1}\ d\theta = \frac{1}{2}\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)} = \int_0^\infty t^{2n-1}(1+t^2)^{-(n+m)}\ dt \tag{71}$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2+2pq-m^2)^\alpha} = \frac{i\ (-1)^{d/2}\ \pi^{d/2}}{(2\pi)^d}\ \frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\ \frac{1}{(-q^2-m^2)^{\alpha-d/2}} \tag{72}$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^2}{(p^2+2pq-m^2)^\alpha} = \frac{i\,(-1)^{d/2}\,\pi^{d/2}}{(2\pi)^d\,\Gamma(\alpha)}\,\frac{\left[q^2\Gamma\left(\alpha-\frac{d}{2}\right)+\frac{d}{2}(-q^2-m^2)\Gamma\left(\alpha-1-\frac{d}{2}\right)\right]}{(-q^2-m^2)^{\alpha-d/2}} \tag{73}$$

$$\int_0^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\epsilon)^3} = \frac{-i}{32\pi^2m^2} \tag{74}$$

$$\int_0^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\epsilon)^2} = \frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}\right)-1+\cdots\right] \tag{75}$$

$$\int_0^\Lambda \frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2-m^2+i\epsilon)^2} = \frac{-i}{16\pi^2}\left[\Lambda^2-2m^2\ln\left(\frac{\Lambda^2}{m^2}\right)+m^2+\cdots\right] \tag{76}$$

$$\frac{1}{xy} = \int_0^1 \frac{d\alpha}{[x\alpha+y(1-\alpha)]^2} \tag{77}$$

$$\frac{1}{xyz} = 2\int_0^1\int_0^{1-\alpha}\frac{d\alpha\,d\beta}{[z+\alpha(x-z)+\beta(y-z)]^3} \tag{78}$$

$$\frac{1}{2}(-i\lambda)^2i^2\int_0^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(k-k')^2-m^2+i\epsilon} = \frac{i\lambda}{32\pi^2}\int_0^1 \ln\left(\frac{\Lambda^2}{m^2-\alpha(1-\alpha)k'^2-i\epsilon}\right)\,d\alpha \tag{79}$$

Particle Physics

$$(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{A}) = \mathbf{A}\cdot\mathbf{B}+i\sigma\cdot(\mathbf{A}\times\mathbf{B}) \tag{80}$$

$$\sigma_i\sigma_j = \delta_{ij}+i\epsilon_{ijk}\sigma_k \tag{81}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{82}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{83}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{84}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{85}$$

$$\gamma^5 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{86}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \tag{87}$$

$$s = (p_1 + p_2)^2 = 4E^2 = (p_3 + p_4)^2 \tag{88}$$

$$t = (p_1 - p_2)^2 = -2p^2(1 - \cos\theta) = (p_4 - p_2)^2 \tag{89}$$

$$u = (p_1 - p_4)^2 = -2p^2(1 + \cos\theta) = (p_2 - p_3)^2 \tag{90}$$

Traces

$$Tr[\gamma^\mu(c_V - c_A\gamma^5)(\not{p}_1 + m_1)\gamma^\nu(c_V - c_A\gamma^5)(\not{p}_2 + m_2)] = \tag{91}$$

$$4c_V^2\{(1+\epsilon^2)[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2)] - 2i\epsilon^{\mu\lambda\nu\sigma}p_{1\lambda}p_{2\sigma} + m_1 m_2 g^{\mu\nu}(1-\epsilon^2)\} \tag{92}$$

$$Tr[\gamma^\mu \ \not{p}_1\gamma^\nu \ \not{p}_2] = 4[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1 \cdot p_2)] \tag{93}$$

$$Tr[\gamma^\mu \ \not{p}_1\gamma^\nu \ \not{p}_2] \ Tr[\gamma_\mu \ \not{p}_3\gamma_\nu \ \not{p}_4] = 32[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)] \tag{94}$$

$$Tr[\gamma^\mu \ \not{p}_1\gamma^\nu\gamma^5 \ \not{p}_2] \ Tr[\gamma_\mu \ \not{p}_3\gamma_\nu\gamma^5 \ \not{p}_4] = 32[(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3)] \tag{95}$$

$$Tr[\gamma^\mu(1-\gamma^5) \ \not{p}_1\gamma^\nu(1-\gamma^5) \ \not{p}_2] \ Tr[\gamma_\mu(1-\gamma^5) \ \not{p}_3\gamma_\nu(1-\gamma^5) \ \not{p}_4] = 256(p_1 \cdot p_3)(p_2 \cdot p_4) \tag{96}$$

$$Tr[\gamma^\mu\gamma^\nu] = g^{\mu\nu}f(d) \qquad f(4) = 4 \tag{97}$$

$$Tr[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = f(d)\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right) \tag{98}$$

$$Tr[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = -4i\epsilon^{\mu\nu\lambda\sigma} \tag{99}$$

$$\gamma^\mu \not{p}\gamma_\mu = (2-d) \not{p} \tag{100}$$

$$\gamma^\mu \not{p} \not{q}\gamma_\mu = 4p\cdot q - \not{p} \not{q}(4-d) \tag{101}$$

$$\gamma^\mu \not{p} \not{q} \not{r}\gamma_\mu = (d-6) \not{r} \not{q} \not{p} + 2(4-d)[(p\cdot q) \not{r} + (q\cdot r) \not{p} - (p\cdot r) \not{q}] \tag{102}$$

Legendre Polynomials

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell = \sum_{j=0}^{[\ell/2]} \frac{(-1)^j(2\ell-2j)!x^{\ell-2j}}{2^j(\ell-j)!(\ell-2j)!j!} \tag{103}$$

$$\int_{-1}^1 P_\ell(x)P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell,\ell'} \tag{104}$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_\ell \frac{x_<^\ell}{x_>^{\ell+1}} P_l(\cos\theta) \tag{105}$$

$$(2\ell+1)\int_{-1}^1 P_\ell(x)\, dx = P_{\ell+1}(x) - P_{\ell-1}(x) \tag{106}$$

$$P_\ell(x) = \begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2-1) \\ P_3(x) = \frac{1}{2}(5x^3-3x) \\ P_4(x) = \frac{1}{8}(35x^4-30x^2+3) \end{cases} \tag{107}$$

Associated Legendre Polynomials

$$P_\ell^m = (x^2-1)^{m/2}\frac{d^m}{dx^m}P_\ell(x) \tag{108}$$

$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x) \tag{109}$$

$$\int_{-1}^{1} P_\ell^m(x)P_{\ell'}^m(x)\,dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell,\ell'} \tag{110}$$

$$P_\ell^m(x) = \begin{cases} P_1^1(x) = -1(1-x^2)^{1/2} \\ P_2^1(x) = -3x(1-x^2)^{1/2} \\ P_2^2(x) = 3(1-x^2) \end{cases} \tag{111}$$

Spherical Harmonics

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi\quad(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi} \tag{112}$$

$$Y_\ell^{*m}(\theta,\phi) = (-1)^m Y_\ell^{-m}(\theta,\phi) \tag{113}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi)Y_{\ell'}^{*m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \delta_{\ell,\ell'}\delta_{m,m'} \tag{114}$$

$$Y_\ell^m(\theta,\phi) = \begin{cases} Y_0^0 = \sqrt{\frac{1}{4\pi}} \\ Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \\ Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \\ Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \\ Y_2^{\pm 2} = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi} \end{cases} \tag{115}$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{*m}(\theta',\phi') Y_\ell^m(\theta,\phi)\; d\Omega' \tag{116}$$

Bessel Functions

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\;(n+k)!}\left(\frac{z}{2}\right)^{n+2k} \quad \text{for } n = \text{integer} \tag{117}$$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\;\Gamma(\nu+k+1)!}\left(\frac{z}{2}\right)^{\nu+2k} \quad \text{for } \nu \neq \text{integer} \tag{118}$$

$$J_\nu(z)|_{z\to\infty} \to \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{119}$$

$$J_n(z)|_{z\to 0} \to \frac{1}{n!}\left(\frac{z}{2}\right)^n \tag{120}$$

$$\int_0^a J_\nu\left(k_{\nu m}\frac{\rho}{a}\right) J_\nu\left(k_{\nu n}\frac{\rho}{a}\right)\; d\rho = \frac{a^2}{2}\left[J_{\nu+1}(k_{\nu m})\right]^2 \delta_{nm} \tag{121}$$

$$\int_0^\infty J_\nu(k'\rho) J_\nu(k\rho)\rho\; d\rho = \frac{1}{k}\delta(k-k') \tag{122}$$

Electrodynamics

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\ d^3x' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right]\ da' \tag{123}$$

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\left(A_{mn}\sin m\phi + B_{mn}\cos m\phi\right) \tag{124}$$

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)P_\ell(\cos\theta) \tag{125}$$

$$\Phi(r,\theta,\phi) = \sum_{m=-\ell}^{\ell}\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)Y_\ell^m(\theta,\phi) \tag{126}$$

$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0}\sum_{\ell,m}\frac{1}{2\ell+1}\left[\int Y^*_{\ell m}(\theta',\phi'){r'}^{\ell}\rho(\mathbf{x}')d^3x'\right]\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \tag{127}$$

$$q_{lm} = \int Y^*_{\ell m}(\theta',\phi'){r'}^{\ell}\rho(\mathbf{x}')d^3x \qquad \text{(multipole moments)} \tag{128}$$

$$W = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2\ d^3x = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\ d^3x = \frac{1}{8\pi\epsilon_0}\iint \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\ d^3x\ d^3x' \tag{129}$$

$$\begin{aligned} \nabla\cdot\mathbf{B} = 0 \qquad & \nabla\times\mathbf{E} + \tfrac{\partial\mathbf{B}}{\partial t} = 0 \\ \nabla\cdot\mathbf{D} = \rho \qquad & \nabla\times\mathbf{H} - \tfrac{\partial\mathbf{D}}{\partial t} = \mathbf{J} \end{aligned} \tag{130}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{131}$$

$$\int \frac{dx'}{\left[(x-x')^2+\rho^2\right]^{1/2}} = -\ln\left[(x-x') + \sqrt{\rho^2+(x-x')^2}\right] \tag{132}$$

$$\int \frac{dx'}{\left[(x-x')^2+\rho^2\right]^{3/2}} = \frac{x'-x}{\rho^2\sqrt{(x-x')^2+\rho^2}} \tag{133}$$

$$\int \frac{x'\ dx'}{\left[\rho^2+(x-x')^2\right]^{3/2}} = \frac{x'x-(\rho^2+x^2)}{\rho^2\sqrt{\rho^2+(x-x')^2}} \tag{134}$$

$$\frac{\mu_0 Ia}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\ d\phi'}{(a^2+r^2-2ar\sin\theta\cos\phi')^{1/2}} = \frac{\mu_0}{4\pi}\frac{4Ia}{\sqrt{a^2+r^2+2ar\sin\theta}}\left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right] \tag{135}$$

where $k^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}$

$$\int_V \nabla \cdot \mathbf{A}\ d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\ da \qquad \text{(Divergence theorem)} \tag{136}$$

$$\int_V \nabla \times \mathbf{A}\ d^3x = \int_S \mathbf{n} \times \mathbf{A}\ da \tag{137}$$

$$\int_V \left(\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi\right)\ d^3x = \int_S \phi\mathbf{n}\cdot\nabla\psi\ da \qquad \text{(Green's first identity)} \tag{138}$$

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\ d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\ da \qquad \text{(Green's second identity)} \tag{139}$$

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\ da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)} \tag{140}$$

Radial Function - $R_{nl}(r)$

$$R_{nl}(r) = \begin{cases} R_{10}(r) = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} \\ R_{20}(r) = \left(\frac{1}{2a_0}\right)^{3/2} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0} \\ R_{21}(r) = \left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}} \frac{r}{a_0} e^{-r/2a_0} \\ R_{30}(r) = 2\left(\frac{1}{3a_0}\right)^{3/2} \left(1 - \frac{2}{3}\frac{r}{a_0} + \frac{2}{27}\frac{r^2}{a_0^2}\right) e^{-r/3a_0} \\ R_{31}(r) = \left(\frac{1}{3a_0}\right)^{3/2} \frac{4\sqrt{2}}{3} \left(1 - \frac{1}{6}\frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \end{cases} \tag{141}$$

Quantum Mechanics

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{1}{m\omega}\hat{p}\right) \tag{142}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{143}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \tag{144}$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a}) \tag{145}$$

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \tag{146}$$

$$\hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \tag{147}$$

$$\left|n\right\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle \tag{148}$$

$$J_z\left|j\ m\right\rangle = \hbar m\left|j\ m\right\rangle \tag{149}$$

$$J_+\left|j\ m\right\rangle = \hbar\sqrt{(j-m)(j+m+1)}\left|j\ m+1\right\rangle \tag{150}$$

$$J_-\left|j\ m\right\rangle = \hbar\sqrt{(j+m)(j-m+1)}\left|j\ m-1\right\rangle \tag{151}$$

$$\hat{J}^2 = \frac{1}{2}\left(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\right) + \hat{J}_z^2 \tag{152}$$

$$\hat{J}^2\left|j, m\right\rangle = \hbar^2 j(j+1)\left|j, m\right\rangle \tag{153}$$

$$\hat{J}_z\left|j, m\right\rangle = \hbar m\left|j, m\right\rangle \tag{154}$$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{155}$$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \tag{156}$$

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\ dx = \sqrt{\pi} 2^n n! \delta_{mn} \tag{157}$$

$$H_n(x) = \begin{cases} H_0(x) = 1 \\ \\ H_1(x) = 2x \\ \\ H_2(x) = 4x^2 - 2 \\ \\ H_3(x) = 8x^3 - 12x \\ \\ H_4(x) = 16x^4 - 48x^2 + 12 \\ \end{cases} \tag{158}$$

$$K\left(\mathbf{x}'', t; \mathbf{x}', t_0\right) = \left\langle \mathbf{x}'' \right| \exp\left[\frac{-iH(t-t_0)}{\hbar}\right] \left| \mathbf{x}' \right\rangle \tag{159}$$

$$\psi(\mathbf{x}'', t) = \int\ d^3x'\ K\left(\mathbf{x}'', t; \mathbf{x}', t_0\right) \psi(\mathbf{x}', t_0) \tag{160}$$

$$\mathcal{D}^{(j)}_{m'm} = \left\langle j, m' \right| \exp\left(\frac{-i\mathbf{J}\cdot\hat{\mathbf{n}}\phi}{\hbar}\right) |j, m\rangle \tag{161}$$

$$\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) = \left\langle j, m' \right| \exp\left(\frac{-iJ_z\alpha}{\hbar}\right) \exp\left(\frac{-iJ_y\beta}{\hbar}\right) \exp\left(\frac{-iJ_z\gamma}{\hbar}\right) |j, m\rangle \tag{162}$$

$$= e^{-i(m'\alpha + m\gamma)} \left\langle j, m' \right| \exp\left(\tfrac{-iJ_y\beta}{\hbar}\right) |j, m\rangle = e^{-i(m'\alpha+m\gamma)} d^{(j)}_{m'm}$$

$$d^{(\ell)}_{00}(\beta)\left.\right|_{\beta=\theta} = P_\ell(\cos\theta) \tag{163}$$

$$d^{(j)}_{m'm}(\beta) = \sum_k (-1)^{k+m'-m} \tfrac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j-m'-k)!(j+m-k)!(k+m'-m)!k!} \times \left(\cos\tfrac{\beta}{2}\right)^{2j+m-m'-2k} \left(\sin\tfrac{\beta}{2}\right)^{m'-m+2k} \tag{164}$$

$$d^{(1/2)} = \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right) \\ \\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{pmatrix} \tag{165}$$

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix} \tag{166}$$

$$\begin{gathered} \sqrt{(j \mp m)(j \pm m + 1)}\, \langle j_1 j_2; m_1 m_2 |\, j_1 j_2; j, m \pm 1\rangle = \\ \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\, \langle j_1 j_2; m_1 \mp 1, m_2 |\, j_1 j_2; j, m\rangle + \\ \sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}\, \langle j_1 j_2; m_1, m_2 \mp 1 |\, j_1 j_2; j, m\rangle \end{gathered} \tag{167}$$