Fields

For $a_p \left|0\right\rangle = \frac{1}{\sqrt{2\omega_p}} \left|p\right\rangle$ and $\left[a_p, a_k^\dagger\right] = (2\pi)^3 \delta^3(\vec{p} - \vec{k})$

$$A_\mu(x) = \sum_{a=1}^{2,3} \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left[a^a(k)\varepsilon_\mu^a(k)e^{-ikx} + a^{a\dagger}(k)\varepsilon_\mu^{a*}(k)e^{ikx}\right] \tag{1}$$

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left[a(k)e^{-ikx} + b^\dagger(k)e^{ikx}\right] \tag{2}$$

$$\psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} \left[a(p,s)u_a(p,s)e^{-ipx} + b^\dagger(p,s)v_a(p,s)e^{ipx}\right] \tag{3}$$

$$\bar{\psi}_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} \left[a^\dagger(p,s)\bar{u}_a(p,s)e^{-ipx} + b(p,s)\bar{v}_a(p,s)e^{ipx}\right] \tag{4}$$

Cross Section

$$d\Gamma = \frac{1}{2E_P} \frac{d^3k_1}{(2\pi)^3\, 2E(k_1)} \cdots \frac{d^3k_n}{(2\pi)^3\, 2E(k_n)} (2\pi)^4 \delta^4 \left(P - \sum_{i=1}^n k_i\right) |\mathcal{M}|^2 \tag{5}$$

$$d\sigma = \frac{1}{|\mathbf{v_1} - \mathbf{v_2}|} \frac{1}{2E(p_1)\, 2E(p_2)} \frac{d^3k_1}{(2\pi)^3\, 2E(k_1)} \cdots \frac{d^3k_n}{(2\pi)^3\, 2E(k_n)} (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{i=1}^n k_i\right) |\mathcal{M}|^2$$

$$\equiv \frac{1}{4E(p_1)E(p_2)|\mathbf{v_1} - \mathbf{v_2}|} \int |\mathcal{M}|^2 \text{ dLIPS} \tag{6}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 E_{CM}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathcal{M}|^2 \tag{7}$$

Field Theory

$$\langle\Omega| T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle = \frac{\langle 0| T\left\{\phi(x_1)...\phi(x_n)e^{i\int d^4x\mathcal{L}_{int}[\phi_0]}\right\}|0\rangle}{\langle 0| T\left\{e^{i\int d^4x\mathcal{L}_{int}[\phi_0]}\right\}|0\rangle} \tag{8}$$

$$T\left\{\psi(0)\bar{\psi}(x)\right\}_{Boson} = \psi(0)\bar{\psi}(x)\theta(-t) + \bar{\psi}(x)\psi(0)\theta(t) \tag{9}$$

$$T\left\{\psi(0)\bar{\psi}(x)\right\}_{Fermion} = \psi(0)\bar{\psi}(x)\theta(-t) - \bar{\psi}(x)\psi(0)\theta(t) \tag{10}$$

$$e^{i\omega_p t}\theta(-t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega\ \frac{e^{i\omega t}}{\omega - (\omega_p - i\epsilon)} \tag{11}$$

$$e^{-i\omega_p t}\theta(t) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega\ \frac{e^{i\omega t}}{\omega - (-\omega_p + i\epsilon)} \tag{12}$$

$$\langle 0| T\left\{\phi(x)\phi^\dagger(y)\right\}_B|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{ip(x-y)} \tag{13}$$

$$\langle 0| T\left\{\psi(x)\bar{\psi}(y)\right\}_F|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}e^{ip(x-y)} \tag{14}$$

LSZ

$$\langle f| S|i\rangle = \sqrt{2^n\omega_{p_1}...\omega_{p_n}}\langle\Omega| a_{p_3}(\infty)...a_{p_n}(\infty)a^\dagger_{p_1}(-\infty)a^\dagger_{p_2}(-\infty)|\Omega\rangle$$

$$= \sqrt{2^n\omega_{p_1}...\omega_{p_n}}\langle\Omega| T\left\{[a_{p_3}(\infty) - a_{p_3}(-\infty)]...[a_{p_n}(\infty) - a_{p_n}(-\infty)]...\left[a^\dagger_{p_2}(-\infty) - a^\dagger_{p_2}(\infty)\right]\right\}|\Omega\rangle$$

$$= \left[i\int d^4x_1 e^{-ip_i x_1}(p_i^2 + m^2)\right]...\left[i\int d^4x_n e^{ip_f x_n}(p_f^2 + m^2)\right]\langle\Omega| T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle \tag{15}$$

"In" States $\rightarrow i\int d^4x_1 e^{-ip_i x_1}(p_i^2 + m^2)\phi(x_1) = \sqrt{2\omega_p}\left[a^\dagger_{p_1}(-\infty) - a^\dagger_{p_1}(\infty)\right]$

"Out" States $\rightarrow i\int d^4x_1 e^{ip_f x_n}(p_f^2 + m^2)\phi(x_n) = \sqrt{2\omega_p}\left[a_{p_n}(\infty) - a_{p_n}(-\infty)\right]$

Lippman-Schwinger

$$\Box_x\langle\phi_x\phi_1...\phi_n\rangle = \langle\mathcal{L}_{int}[\phi_x]\phi_1...\phi_n\rangle - i\sum_j \delta(x - x_j)\langle\phi_1...\phi_{j-1}\phi_{j+1}...\phi_n\rangle \tag{16}$$

Matrices

Dirac Basis

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma_i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Weyl Basis

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma_i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\sigma^\mu \equiv (1, \vec{\sigma}) \quad \bar{\sigma}^\mu \equiv (1, -\vec{\sigma}) \tag{17}$$

$$\psi_L = \left(\frac{1-\gamma^5}{2} \right) \psi \equiv \hat{\mathcal{P}}_L \psi \quad \psi_R = \left(\frac{1+\gamma^5}{2} \right) \psi \equiv \hat{\mathcal{P}}_R \tag{18}$$

$$S^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2} \epsilon^{ijk} \Sigma^k \qquad S^{0i} = K^i = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$V^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} \tag{19}$$

$$J^1 = i \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -1 \\ & & 1 & \end{pmatrix} \quad J^2 = i \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ & -1 & & 0 \end{pmatrix} \quad J^3 = i \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix}$$

$$K^1 = i \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad K^2 = i \begin{pmatrix} 0 & & -1 & \\ & 0 & & \\ -1 & & 0 & \\ & & & 0 \end{pmatrix} \quad K^3 = i \begin{pmatrix} 0 & & & -1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 0 \end{pmatrix}$$

$$u_1(p) = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ \sqrt{E+p_z} \\ 0 \end{pmatrix} \quad u_2(p) = \begin{pmatrix} 0 \\ \sqrt{E-p_z} \\ 0 \\ \sqrt{E+p_z} \end{pmatrix} \quad v_1(p) = \begin{pmatrix} \sqrt{E-p_z} \\ 0 \\ -\sqrt{E+p_z} \\ 0 \end{pmatrix} \quad v_2(p) = \begin{pmatrix} 0 \\ \sqrt{E-p_z} \\ 0 \\ -\sqrt{E+p_z} \end{pmatrix}$$

Algebraic Relations

$$\left[a_p, a_k^\dagger\right]_B = (2\pi)^3\delta^3(\vec{p}-\vec{k}) \tag{20}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad [J_i, K_j] = i\epsilon_{ijk}K_k \quad [K_i, K_j] = -i\epsilon_{ijk}J_k \tag{21}$$

$$[V^{\mu\nu}, V^{\rho\sigma}] = i\left(g^{\nu\rho}V^{\mu\sigma} + g^{\mu\sigma}V^{\nu\rho} - g^{\mu\rho}V^{\nu\sigma} - g^{\nu\sigma}V^{\mu\rho}\right) \tag{22}$$

$$J_i^\pm = \frac{1}{2}\left(J_i \pm iK_i\right) \tag{23}$$

$$[\gamma^\mu, S^{\rho\sigma}] = i\left(\delta^{\rho\mu}\delta^\sigma_\nu - \delta^\rho_\nu\delta^{\sigma\mu}\right)\gamma^\nu \equiv \left(\mathcal{J}^{\rho\sigma}\right)^\mu_\nu \gamma^\nu \tag{24}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \frac{1}{2}\sigma^{\mu\nu} \tag{25}$$

$$[iD_\mu, iD_\nu] = iqF_{\mu\nu} \to -ieF_{\mu\nu} \tag{26}$$

$$D_\mu \equiv \partial_\mu + iqA_\mu \to \partial_\mu - ieA_\mu \tag{27}$$

$$\left[J_i^+, J_j^-\right] = 0 \quad \left[J_i^\pm, J_j^\pm\right] = i\epsilon_{ijk}J_k^\pm \tag{28}$$

Since only two relations are non-zero, the so(1,3) representation is reducible to two su(2) representations. i.e. so(1.3) $\mapsto$ su(2) $\oplus$ su(2)

$$\bar{u}_s u_{s'} = 2m\delta_{ss'} \quad u_s^\dagger u_{s'} = 2E\delta_{ss'} \tag{29}$$

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m \quad \sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \not{p} - m \tag{30}$$

Lorentz Transformation for Fermions

$$\Lambda_{\frac{1}{2}}(\theta) = e^{i\theta_{\mu\nu}S^{\mu\nu}} \tag{31}$$

$$\Lambda_{\frac{1}{2}}^{-1}(\theta) = \left(\gamma^0\Lambda_{\frac{1}{2}}(\theta)\gamma^0\right)^\dagger = \gamma^0\Lambda_{\frac{1}{2}}^\dagger(\theta)\gamma^0 = e^{-i\theta_{\mu\nu}S^{\mu\nu}} \tag{32}$$

Misc.

$$W^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{33}$$

$$T_{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - g_{\mu\nu}\mathcal{L} \tag{34}$$

$$E = \int d^3x\, \mathcal{E} = \int d^3x\, T_{00} \tag{35}$$

$$V^\alpha \to \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu} \left(\mathcal{J}^{\mu\nu}\right)^\alpha_\beta\right) V^\beta \tag{36}$$

$$\sigma^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{2}\left(\sigma^{\mu\nu} - \sigma^{\nu\mu}\right)\partial_\mu\partial_\nu = \frac{1}{2}\left(\sigma^{\mu\nu}\partial_\mu\partial_\nu - \sigma^{\nu\mu}\partial_\nu\partial_\mu\right) = 0 \tag{37}$$

$$\epsilon^{ijk}\epsilon_{lmn} = \det \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{38}$$

Suppose that $D(g)$ is some operatorwithin group G where $D(g_1)\left|g_2\right\rangle = \left|g_1 g_2\right\rangle$ etc. and $D(g) \to D'(g) = S^{-1}D(g)S$, then a representation is said to be redicible if it has an invariant subspace, i.e. that the action of any $D(g)$ on any vector in the subspace is still in the subspace. In terms of a projection operator $\hat{P}$ onto the subspace,

$$\hat{P}D(g)\hat{P} = D(g)\hat{P}\ \forall g \in G \tag{39}$$

For $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu$ there are $2j+1 = 3$ DOF. Using $\partial_\mu A^\mu = 0$ brings us down to two DOF. There is no gauge invariance, but there are three polarization vectors: $\epsilon^1 = (0,1,0,0),\ \epsilon^2 = (0,0,1,0),\ \epsilon^3 = (p_z/m, 0, 0, E/m)$ where $p^\mu = (E,0,0,p_z)$. If we let $p^\mu = (m,\vec{0})$ i.e the "Little Group" we can only have elements which leave the momentum alone under boosts. Therefore, $\epsilon^1 \to \epsilon^1,\ \epsilon^2 \to \epsilon^2,\ \epsilon^3 \to (0,0,0,1)$ and so $p^\mu \epsilon_\mu = 0$ and the square of all polarizations is -1. This is SO(3) and it is finite dimensional since the momentum is zero.

For $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ there is gauge invariance ($A_\mu \to A_\mu + \partial_\mu \alpha$). We can use this to remove the $\epsilon^0 = (1,0,0,0)$ polarization and bring us down to two DOF. We have $p^\mu = E(1,0,0,1),\ \epsilon^1 = (0,1,0,0),\ \epsilon^2 = (0,0,1,0)$. These are part of ISO(2) and for spin J have $2J$ DOF. Since A^μ integrates over all momentum, then this representation is infinite dimensional.