M3/4/5 S4 Problem Sheet 4 Solutions: Poisson Processes

Question 6

Note that this question is about thinning of a Poisson process!
• Clearly $N_0^g \equiv 0$.
• Next we consider the independent increments property. We can argue as follows: We know that $\{N_t\}_{t\geq 0}$ is a Poisson process. At each time when an event occurs (i.e. when a car arrives), the event is (independently) classified as "green car" with probability p and as "not a green car" with probability $1-p$. Accordingly, we can define two independent stochastic processes $\{N_t^g\}_{t\geq 0}$ and $\{N_t^{ng}\}_{t\geq 0}$, where $N_t = N_t^g + N_t^{ng}$ for all $t \geq 0$. Both processes inherit the independent increments from $\{N_t\}_{t\geq 0}$.

• N^g also inherits the stationary increments from N. Finally, we study the law of the number of green cars at time t. Let $k \in \{0, 1, 2, \dots\}$. Then

$$\begin{aligned}
\mathbb{P}(N_t^g = k) &= \sum_{n=k}^{\infty} \mathbb{P}(N_t^g = k, N_t = n) \\
&= \sum_{n=k}^{\infty} \mathbb{P}(N_t^g = k | N_t = n)\mathbb{P}(N_t = n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \mathbb{P}(N_t = n) \\
&= e^{-\lambda t} \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(\lambda t)^n}{n!} \\
&= \left(\frac{p}{1-p}\right)^k \frac{e^{-\lambda t}}{k!} \sum_{n=k}^{\infty} \left(\frac{[(1-p)\lambda t]^n}{(n-k)!}\right) \\
&= \left(\frac{p}{1-p}\right)^k \frac{e^{-\lambda t}}{k!} [(1-p)\lambda t]^k \sum_{s=0}^{\infty} \left(\frac{[(1-p)\lambda t]^s}{s!}\right) \\
&= \frac{(\lambda p t)^k}{k!} e^{-\lambda p t}.
\end{aligned}$$

Hence we know that N^g has stationary increments and that $N_t^g \sim Poi(\lambda p t)$. Hence we can conclude that for $s < t$ we have $N_t^g - N_s^g \sim Poi(\lambda p(t-s))$, which completes the solution.

Question 7

Let $\phi_{Z_t}(u)$ (resp. $\phi_X(u)$) denote the characteristic function of Z_t (resp. X)

$$\begin{aligned}
\phi_{Z_t}(u) = \mathbb{E}\Big[\exp\{iuZ_t\}\Big] &= \mathbb{E}\Big[\exp\{iu\sum_{i=1}^{N_t} X_i\}\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[\exp\{iu\sum_{i=1}^{N_t} X_i\}\big|N_t\big]\Big] \\
&= \mathbb{E}\Big[\phi_x(u)^{N_t}\Big] \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t\phi_x(u))^n}{n!} \\
&= \exp\{\lambda t[\phi_x(u) - 1]\}.
\end{aligned}$$

Question 8

(a) The number, X, of customers arriving in 30 seconds has a Poisson distribution with parameter $10 \times 0.5 = 5$. So $\mathbb{P}(X > 5) = 1 - \mathbb{P}(X \leq 5) = 1 - \sum_{k=0}^{5} \mathbb{P}(X = k) = 0.3840$.
Alternatively you could argue as follows: We consider a Poisson process of rate $\lambda = 10$. Note that the time in this problem is measured in minutes. Hence 30s corresponds to a time of $t = 0.5$. Hence, we need to compute $\mathbb{P}(N_{0.5} > 5)$. We know that $N_{0.5} \sim Poi(0.5 \cdot 10)$. Then you can argue as before.

(b) Each type of customer arrives according to a Poisson process, independently of the other two types. In particular,

- customers of type A arrive according to a Poisson process $\{N^A\}$ of intensity $\lambda \cdot 0.6 = 6$,
- customers of type B arrive according to a Poisson process $\{N^B\}$ of intensity $\lambda \cdot 0.3 = 3$,
- and customers of type C arrive according to a Poisson process $\{N^C\}$ of intensity $\lambda \cdot 0.1 = 1$.

Thus

$$\begin{aligned}
\mathbb{P}(N_1^A = 6)\mathbb{P}(N_1^B = 3)\mathbb{P}(N_1^C = 1) &= 0.1606 \times 0.2240 \times 0.3679 \\
&= 0.0132.
\end{aligned}$$

(c) As 20 events are known to have occurred, time is irrelevant. The number X_C of type C customers is distributed as $\mathcal{B}in(20, 0.1)$ (Binomial distribution) so that

$$\mathbb{P}(X_C = 1) = \binom{20}{1}(0.1)(0.9)^{19} = 0.270.$$

(d) The probability that any customer requires a type A transaction is 0.6. Hence the probability that the first three customers are type A is $(0.6)^3 = 0.216$.

(e) Suppose that the required time is t. The probability that at least one type A customer has arrived by time t is $(1 - e^{-6t})$ and for type B it is $(1 - e^{-3t})$. Since the two Poisson processes are independent, we must find t such that

$$(1 - e^{-6t})(1 - e^{-3t}) = 0.9.$$

Solving this numerically gives $t = 0.794$ minutes.
Alternatively, using the language of Poisson processes, you could argue that we need to compute

$$\mathbb{P}(N_t^A \geq 1, N_t^B \geq 1).$$

Using the independence of the two processes, we get

$$\begin{aligned}
\mathbb{P}(N_t^A \geq 1, N_t^B \geq 1) &= \mathbb{P}(N_t^A \geq 1)\mathbb{P}(N_t^B \geq 1) \\
&= (1 - \mathbb{P}(N_t^A = 0))(1 - \mathbb{P}(N_t^B = 0)) \\
&= (1 - e^{-6t})(1 - e^{-3t}).
\end{aligned}$$

Note: When solving

$$(1 - e^{-6t})(1 - e^{-3t}) = 0.9$$

numerically, you will get the following three solutions: $0.794, -0.154, 0.127 - 1.047i$. We pick the only positive solution, i.e. $t = 0.794$.

Question 9

(a)

$$m(t) = \int_0^t \lambda(u)du = \int_0^t 10(1 + 2u)du = 10(t + t^2).$$

Five minutes is $\frac{5}{60} = \frac{1}{12}$ hours, so that the number of customers arriving between 10.00 and 10.05 is a Poisson process with rate m(1/12) = 0.9028. It follows that the probability that two customers have arrived by 10.05 is

$$\frac{m(1/12)^2}{2}\exp(-m(1/12)) = 0.1652.$$

(b) Converting to the units we need, we want to know the probability that 6 customers arrive in the interval $[0.75, 1.00]$. This will have a Poisson distribution with parameter $m(1) - m(0.75) = 6.875$. From which it follows that the probability that 6 customers will arrive is

$$\exp(-6.875)(6.875)^6/6! = 0.1515.$$

(c) The number of customers arriving between these times will have a Poisson distribution with parameter $m(2) - m(1) = 40$. Hence, for $X \sim Poi(40)$, we need to compute

$$\mathbb{P}(X > 50) = 1 - \mathbb{P}(X \leq 50) = 1 - \sum_{k=0}^{50} e^{-40}\frac{40^k}{k!}.$$

The probability that more than 50 arrive can be obtained by summing the relevant probabilities from a table of the Poisson distribution, parameter 40, or by using a calculator. Then we find that the corresponding probability is 0.0526.
Note: In the exam I will NOT ask you to sum up that many terms, but I expect you to understand how to solve the problem theoretically!

(d) The C.D.F to the time of the first arrival is $F(t) = 1 - \exp(-10(t+t^2))$. This can be shown as follows: Let X_1 denote the first inter–arrival time, i.e. the time until the first arrival. Recall that $N_t \sim Poi(m(t))$, where $m(t) = 10(t + t^2)$. Consider $t > 0$, then

$$\mathbb{P}(X_1 > t) = \mathbb{P}(\text{no events in } [0, t]) = \mathbb{P}(N_t = 0) = \exp(-m(t)).$$

Hence $F(t) = \mathbb{P}(X_1 \leq t) = 1 - \exp(-m(t))$.

The median of this is given by the value of t for which

$$F(t) = \mathbb{P}(X_1 \leq t) = 0.5,$$

yielding 0.065 (hours), so that the median time of arrival of the first customer is about 3.9 minutes after 10.

(e) The C.D.F of the time to the first arrival after 11 is $F(t) = 1 - \exp(-(m(t) - m(1)))$, with $m(t) - m(1) = 10(t + t^2) - 20$. Putting $F(t) = 0.95$ leads to $t = 1.097$ (hours, i.e. 65.8 minutes), so that there is a probability of 0.95 that at least one customer will have arrived after 11.00 and before 11.06.

Question 10

Let $X_t = \sum_{i=1}^{N_t} Y_i$, where the Y_i are i.i.d with P.M.F

$$p(y) = p(1-p)^{y-1}, \quad y = 1, 2 \ldots .$$

Then $\mathbb{E}(Y_1) = 1/p$ and $Var(Y_1) = (1-p)p^{-2}$. Now we use the results from the lecture notes, to deduce that the mean is given by

$$\mathbb{E}(X_t) = \mathbb{E}(N_t)\mathbb{E}(Y_1) = \lambda t/p.$$

For the variance, we get (using the results from the lecture notes)

$$\begin{aligned} Var(X_t) &= \mathbb{E}\Bigg[\left(\sum_{i=1}^{N_t} Y_i\right)^2\Bigg] - \left(\mathbb{E}\Bigg[\sum_{i=1}^{N_t} Y_i\Bigg]\right)^2 = \lambda t \mathbb{E}(Y_1^2) \\ &= \lambda t (Var(Y_1) + (\mathbb{E}(Y_1))^2) \\ &= \lambda t((1-p)p^{-2} + p^{-2}) = \frac{\lambda t(2-p)}{p^2}. \end{aligned}$$

However, the question does not specify this form of geometric distribution. If you use the alternative form

$$p(y) = p(1-p)^y \quad y = 0, 1 \ldots$$

you will get a different answer.