MATH3802
Chapter 2: Stationary processes

Peter Thwaites

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Stochastic processes

- To get a model of a time series, we consider the X_t to be random variables. The collection $\{X_t\}_{t \in \mathbb{N}}$ is called a *stochastic process*.

Summary statistics

Definition 2.1 For a stochastic process $\{X_t\}$, we use the following definitions:

mean:

$$\mu(t) = \mathbb{E}(X_t);$$

variance:

$$\sigma^2(t) = \text{var}(X_t);$$

auto-covariance:

$$\gamma(s, t) = \text{cov}(X_s, X_t) = \mathbb{E}[(X_s - \mu(s))(X_t - \mu(t))];$$

auto-correlation:

$$\rho(s, t) = \text{corr}(X_s, X_t) = \frac{\gamma(s, t)}{\sqrt{\sigma^2(s)\sigma^2(t)}}.$$

Stationarity

Definition 2.2 ((Strong) stationarity) The stochastic process $\{X_t\}$ is *stationary* if $\{X_1, \dots, X_n\}$ and $\{X_{1+k}, \dots, X_{n+k}\}$ have the same distribution for all $k, n \in \mathbb{N}$.

Remarks:

1. If $\{X_t\}$ is stationary, then X_k has the same distribution as X_1 for all $k \in \mathbb{N}$. In particular, if the first two moments are finite then $\mu(t) = \mathbb{E}(X_t) = \mu$ and $\sigma^2(t) = \text{var}(X_t) = \sigma^2$ for all t. Therefore, a process with trend or seasonal effects cannot be stationary.
2. For stationary processes, sometimes it is useful to consider $t \in \mathbb{Z}$ instead of $t \in \mathbb{N}$.

Weak Stationarity

Definition 2.3 (Weak stationarity) A stochastic process $\{X_t\}$ is *weakly stationary* or *second order stationary* if

1. $\mu(t)$ is constant, i.e. $\mu(t) = \mu$ for all t;
2. $\gamma(s, t)$ only depends on the times t, s via the *lag* $t - s$, i.e. $\gamma(s, t) = \gamma_{t-s}$.

Note that $\sigma^2(t) = \sigma^2 = \gamma_0$.

Examples

Example 2.1 (White noise) Let $X_t \sim N(0, 1)$, i.i.d. Then $\mu(t) = 0$, $\sigma^2(t) = 1$, $\gamma(s, t) = 0$ for all $s, t \in \mathbb{N}$. Clearly $\{X_t\}$ is weakly stationary.

Indeed, the collection $\{X_{1+k}, \ldots, X_{n+k}\}$ is an i.i.d. collection of $N(0, 1)$ r.v.s for any shift $k \in \mathbb{N}$, so $\{X_t\}$ is strongly stationary.

Example 2.2 (Symmetric random walk) Let $X_0 = 0$ and
$X_t = \sum_{i=1}^{t} Z_i$ where Z_i takes values $+1, -1$ with probability $1/2$ each.

Then $\mu(t) = \sum_{k=1}^{t} \mathbb{E}(Z_i) = 0$ but $\sigma^2(t) = \sum_{k=1}^{t} \text{var}(Z_k) = t$, which is not constant so $\{X_t\}$ is neither weakly nor strongly stationary.

Exercise: Show that $\gamma(s,t) = s$ *and* $\rho(s,t) = \sqrt{s/t}$.

Remarks

1. Every stationary process is weakly stationary since, for all t,
$$\mu(t) = \mathbb{E}(X_t) = \mathbb{E}(X_1)$$
$$\gamma(t, t+k) = \text{cov}(X_t, X_{t+k}) = \text{cov}(X_1, X_k).$$
Therefore, both quantities are independent of t.
2. If $\{X_t\}$ is weakly stationary,
$$\gamma_0 = \text{cov}(X_t, X_t) = \text{var}(X_t) = \sigma^2 \ \forall t$$
$$\gamma_k = \text{cov}(X_t, X_{t+k}) = \text{cov}(X_{t+k}, X_t) = \gamma_{t-(t+k)} = \gamma_{-k},$$
so it is enough to consider lags $k \geqslant 0$.
Also note that
$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\sigma^2(s)\sigma^2(t)}} = \frac{\gamma_{t-s}}{\sqrt{\gamma_0\gamma_0}} = \frac{\gamma_{t-s}}{\gamma_0} =: \rho_{t-s},$$
and hence $\rho(t,t) = \rho_0 = 1$.

Definition 2.4 (Autocorrelation function) The quantity ρ_k is called the *lag k autocorrelation*; ρ_k as a function of k is the *autocorrelation function (acf)*. A plot of ρ_k against k is called a *correlogram*

Note that, strictly speaking, it only makes sense to compute the acf for weakly stationary processes. However, we shall see that we can use the acf to diagnose whether a process is stationary or not.

Estimation of μ, γ_k, ρ_k

Normally we would estimate covariances by using

$$\mathrm{cov}(X, Y) \approx \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$$

where X_i, Y_i are i.i.d. copies of X, Y. For a time series, we typically have only one *realisation* $X_1, X_2, \ldots, X_n$, so we can't estimate $\mathrm{cov}(X_1, X_2)$ as above. However, if $\{X_t\}$ is (weakly) stationary, we can use

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \approx \mu(X_t)$$

$$\widehat{\gamma}_k = \frac{1}{n-k}\sum_{i=1}^{n-k}(X_i - \widehat{\mu})(X_{i+k} - \widehat{\mu}) \approx \gamma_k$$

$$\widehat{\rho}_k = \widehat{\gamma}_k / \widehat{\gamma}_0.$$