MATH5802M
Chapter 8: DFT for time series

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8.1 Spectral density and autocovariance

Let $X_0, X_1, \ldots, X_{n-1}$ be a time series. Then

1. Recall $\widehat{X}_0 = \sqrt{n}\tilde{X}$, where $\tilde{X}$ is the sample mean.
2. If $\tilde{X} = 0$, we obtain

$$\mathrm{var}(X) \approx \int_0^1 I(f)\,\mathrm{d}f. \tag{1}$$

Then $\int_a^b I(f_j)\,\mathrm{d}f$ quantifies the variance contributed by the frequency range $[a, b]$

Link to autocovariance

3. If g_k is the lag-k sample autocovariance of X, then

$$I(f_j) = \sqrt{n}\, \widehat{g}_j.$$

If X is periodic, we have $g_{n-k} = g_k$ Therefore,

$$I(f_j) = g_0 + 2 \sum_{1 \leqslant k < \frac{n}{2}} g_k \cos(2\pi f_j k) \quad + \quad g_{\frac{n}{2}}(-1)^j \quad \text{if } n \text{ even.}$$

The spectral density I is an estimate for the *power spectrum*

$$\mathcal{I}(f) = \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(2\pi f k).$$

8.2 Spectral density of white noise

Let $X_t \sim N(0, \sigma^2)$, i.i.d. Then $\widehat{X}_j$ is (complex) normally distributed with $\mathbb{E}(\operatorname{Re} \widehat{X}_j) = \mathbb{E}(\operatorname{Im} \widehat{X}_j) = 0$.

For the variances and covariances, we get

$$\operatorname{var}(\operatorname{Re} \widehat{X}_j) = \begin{cases} \sigma^2 & \text{if } j \in \{0, \frac{n}{2}\} \\ \frac{\sigma^2}{2} & \text{if } j \notin \{0, \frac{n}{2}\}, \end{cases}$$

$$\operatorname{var}(\operatorname{Im} \widehat{X}_j) = \begin{cases} 0 & \text{if } j \in \{0, \frac{n}{2}\} \\ \frac{\sigma^2}{2} & \text{if } j \notin \{0, \frac{n}{2}\}, \end{cases}$$

$$\operatorname{cov}(\operatorname{Re} \widehat{X}_j, \operatorname{Im} \widehat{X}_j) = 0.$$

Also, $\widehat{X}_j, \widehat{X}_k$ are uncorrelated for $j \neq k$.

j	$\operatorname{Re}\widehat{X}_j$	$\operatorname{Im}\widehat{X}_j$	$\mathbb{E}\{I(f_j)\}$	$I(f_j)$
0	$N(0,\sigma^2)$	0	σ^2	$\sigma^2\chi_1^2$
$1 \le j < \frac{n}{2}$	$N(0,\frac{\sigma^2}{2})$	$N(0,\frac{\sigma^2}{2})$	$\frac{\sigma^2}{2}+\frac{\sigma^2}{2}=\sigma^2$	$\frac{\sigma^2}{2}\chi_2^2$
$\frac{n}{2}$	$N(0,\sigma^2)$	0	σ^2	$\sigma^2\chi_1^2$ only for even n

On average, all frequencies contribute the same (hence the name "white noise").

8.3 Spectral density of AR(p)

Let X be given by

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_p X_{t-p} + \varepsilon_t,$$

where $\varepsilon_t \sim N(0,\sigma^2)$ i.i.d. What is the DFT of X?

DFT of AR(p)

Recall

$$Y_t = X_{t-1} \;\Rightarrow\; \widehat{Y}_j = e^{-2\pi i f_j} \widehat{X}_j,$$

and if $Z_t = X_t + Y_t$ then

$$\widehat{Z}_j = a\widehat{X}_j + b\widehat{Y}_j.$$

Hence we obtain

$$\widehat{X}_j = \frac{\widehat{\varepsilon}_j}{1 - \sum_{k=1}^{p} \alpha_k e^{-2\pi i f_j \cdot k}}.$$

and

$$\mathbb{E}[I(f_j)] = \frac{\sigma^2}{\left|1 - \sum_{k=1}^{p} \alpha_k e^{-2\pi i f_j \cdot k}\right|^2}.$$

AR(1), $X_t = \alpha X_{t-1} + \varepsilon_t$

$$\mathbb{E}[I(f_j)] = \frac{\sigma^2}{1 - 2\alpha\cos(2\pi f_j) + \alpha^2}$$