MATH5802M
Chapter 6: Fitting sine waves

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with thanks to Stuart Barber for previous years' notes and slides

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Admin details

Lectures As 3802 plus Friday 9am, RSLT14.
Assessment 2.5 hour end of semester examination (80%) and an assessed practical (20%).
Trigonometry and complex numbers will be used a lot in spectral analysis.

6.1 Frequency

- In *spectral analysis*, a time series is analysed by decomposing it into sine waves of different amplitude and frequency.

  low frequency $\approx$ slowly changing trends
  high frequency $\approx$ rapid changes (noise).

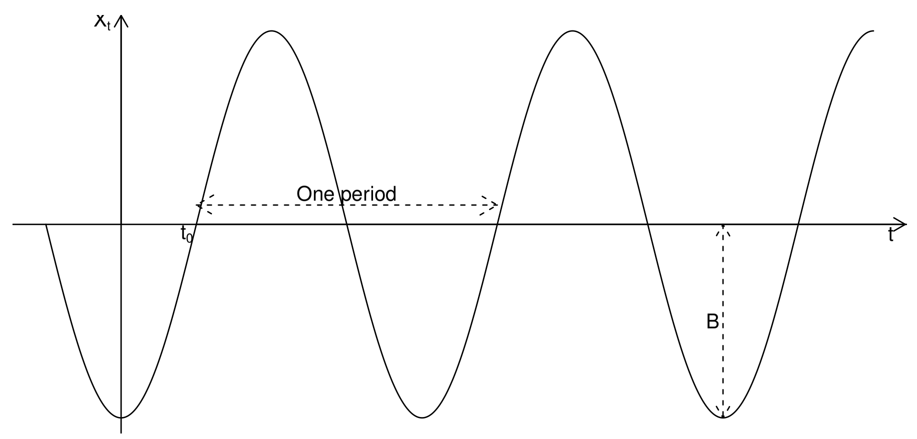
- If the data have a periodic component of known frequency, we can fit the following model:

$$X_t = B\sin(2\pi f(t - t_0)) + \mu + \varepsilon_t,$$

where

μ is a baseline or mean;
f is *frequency* (in cycles per unit time);
t_0 is the start time of one period;
B is the *amplitude*

Graphical interpretation



Example 6.1 If t is given in months since the start of measurements, then a period length of one year implies $f = 1/12$.

Parameterisation

- How many distinct parameters are there in

$$X_t = B\sin(2\pi f(t - t_0)) + \mu + \varepsilon_t?$$

- Use

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

to get

$$X_t = A'\cos(2\pi ft) + B'\sin(2\pi ft) + \mu + \varepsilon_t.$$

- Since any periodic component with frequency f can be written in this form, we can use this model for the sum of arbitrary many shifted sines/cosines with frequency f.
- Therefore we need just *two* parameters (A, B) per frequency.

6.2 Aliasing

- We have seen that only two parameters per frequency are required; frequency f is described by

$$A\cos(2\pi ft) + B\sin(2\pi ft). \tag{1}$$

A model with more parameters for frequency f cannot be distinguished from (1) by using observations.

- Since

$$A\cos(2\pi(-f)t) + B\sin(2\pi(-f)t) = A\cos(2\pi ft) - B\sin(2\pi ft),$$

the frequencies f and $-f$ are indistinguishable; they are *aliases* of each other. Hence we can assume $f \geqslant 0$.

- Assume that observations are given at times $0, \Delta, 2\Delta, 3\Delta, \ldots$, i.e. we have sampling frequency $1/\Delta$. A component with frequency f will be observed as

$$X_k = A\cos(2\pi f k\Delta) + B\sin(2\pi f k\Delta).$$

- The bigger f, the faster X_k oscillates, until $f = \frac{1}{2\Delta}$. Here, we have

$$X_k = A(-1)^k.$$

  This is the fastest possible oscillation.
- Now, $\frac{1}{2\Delta} + \varepsilon$ and $\frac{1}{2\Delta} - \varepsilon$ are aliases, so we can assume $0 \leqslant f \leqslant \frac{1}{2\Delta}$.

Definition 6.1(Nyquist frequency) The frequency $f = \frac{1}{2\Delta}$ is called the *Nyquist frequency* for the sampling frequency $1/\Delta$.