MATH5802M
Chapter 9: Complications

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with thanks to Stuart Barber for previous years' notes and slides

February 2017

9.1 Inconsistency

- Recall if $\{\varepsilon_t\}$ is WN,

$$\text{Re}(\widehat{\varepsilon}_j) \sim N(0, \sigma_\varepsilon^2/2), \qquad \text{Im}(\widehat{\varepsilon}_j) \sim N(0, \sigma_\varepsilon^2/2),$$

  so

$$\text{var}[I_\varepsilon(f_j)] = \sigma_\varepsilon^4.$$

- Similarly, if $\{X_t\}$ is an AR(1) process, then

$$\text{var}[I_X(f_j)] = \kappa\sigma_\varepsilon^4$$

  for some constant κ.
- In general, $I(f)$ is an unbiased but inconsistent estimate of $\mathcal{I}(f)$.
- We consider two ways of solving this problem: truncating the autocovariance function and smoothing the raw periodogram.

Truncating the autocovariance function

- Recall

$$I(f_j) = g_0 + 2 \sum_{1 \leqslant k < n/2} g_k \cos(2\pi f_j k) + g_{n/2}(-1)^j, \qquad n \text{ even}$$

where g_k is the lag k sample autocovariance.

- Hence we might use

$$\mathcal{I}_T(f_j) = \lambda_0 g_0 + 2 \sum_{k=1}^{m} \lambda_k g_k \cos(2\pi f_j k), \qquad (1)$$

with $m < n/2$, for some suitable weights $\{\lambda_k\}$ called the *lag window*.

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└9.1 Inconsistency

- Two possible choices of lag window are

$$\text{Tukey} \quad \lambda_k = \frac{1}{2}\left[1 + \cos\left(\frac{\pi k}{m}\right)\right] \qquad k = 0, 1, \ldots, m$$

$$\text{Parzen} \quad \lambda_k = \begin{cases} 1 - 6\left(\frac{k}{m}\right)^2 + 6\left(\frac{k}{m}\right)^3 & k = 0, 1, \ldots, m/2 \\ 2\left(1 - \frac{k}{m}\right)^3 & m/2 + 1, \ldots, m. \end{cases}$$

Smoothing the raw periodogram

- Let $w = 2w^* + 1$ and define

$$\mathcal{I}_S(f_j) = \frac{1}{w} \sum_{k=-w^*}^{w^*} I(f_{j+k}). \tag{2}$$

- This estimate of $\mathcal{I}$ is consistent but biased.
- We need to balance
  - large w (small variance) with
  - small w (small bias).
- Smoothing the periodogram in this way is equivalent to using (1) with

$$\lambda_k = \begin{cases} 1 & k = 0 \\ \frac{\sin(wk\pi/n)}{m\sin(k\pi/n)} & k = 1, 2, \ldots, n-1. \end{cases}$$

Spectral window

- The estimates 1 and 2 can be compared using a quantity called the *spectral window* $K(f_j) = \widehat{\lambda}_j$ (the DFT of the lag window).
- The truncated estimate of the power spectrum is

$$\mathcal{I}_T(f_j) = \frac{1}{n} \sum_{j'=0}^{n-1} K(f_{j'}) I(f_j - f_{j'}).$$

- You can have a sharp cut-off in the frequency domain (Daniel smoothing) *or* in the time domain (truncation), but not both.

9.2 Spectral leakage

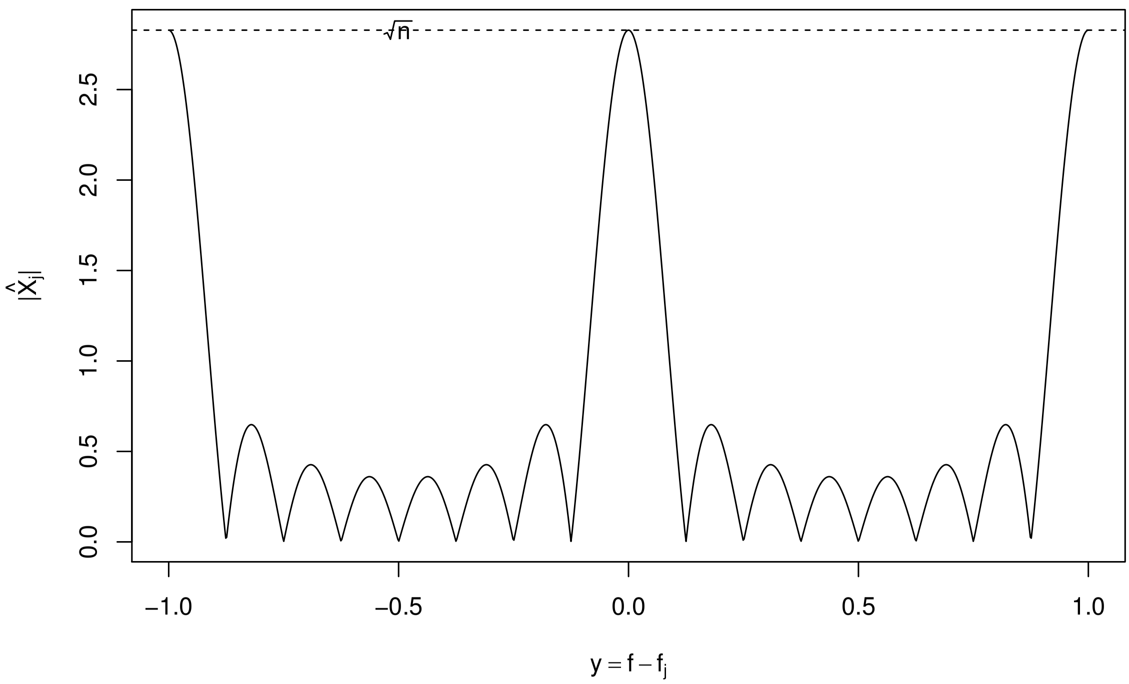
What if $X_t = e^{2\pi i f t}$ where f need not be a Fourier frequency? Then

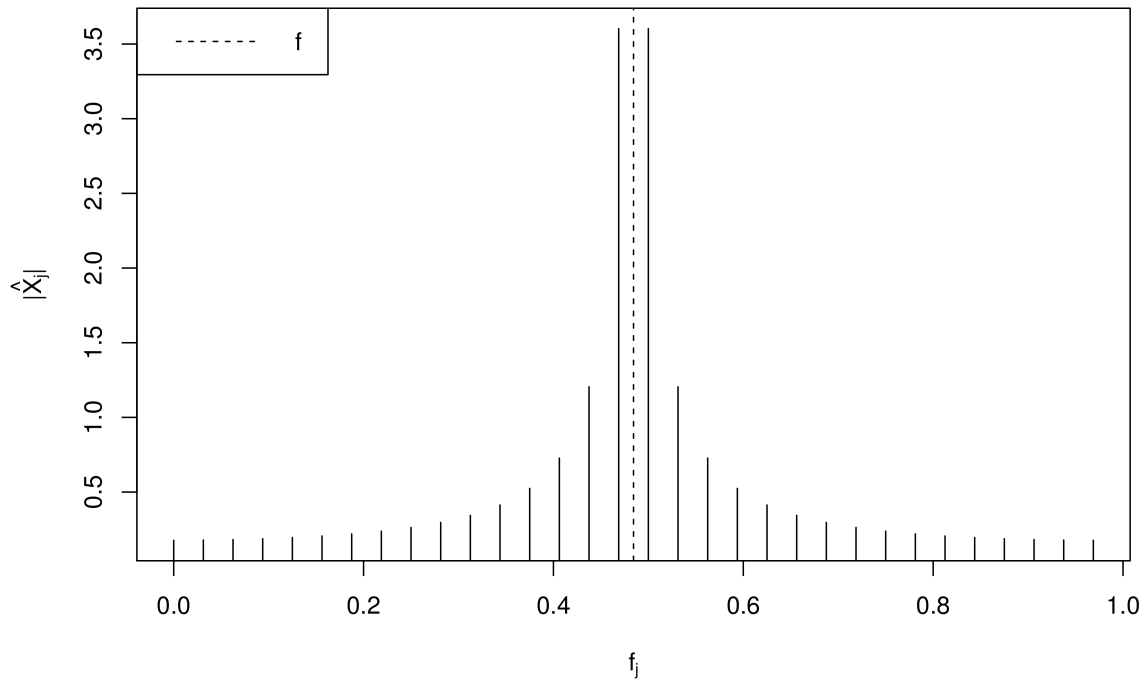
$$\widehat{X}_j = \begin{cases} \sqrt{n} & \text{if } f = f_j \\ 0 & \text{if } f = f_k, k \neq j \\ \frac{1}{\sqrt{n}} \frac{1-e^{2\pi(f-f_j)n}}{1-e^{2\pi i(f-f_j)}} & \text{else.} \end{cases}$$

In the third case,

$$|\widehat{X}_j| = \frac{1}{\sqrt{n}} \left| \frac{\sin[\pi n(f-f_j)]}{\sin[\pi(f-f_j)]} \right| .$$

If f is not a Fourier frequency, we will get contributions for several $\widehat{X}_j$. The worst case is $f = (j \pm \frac{1}{2})/n$.





9.3 Harmonics

What if there is a periodic component which is not a sine wave? Consider the contribution to $\widehat{X}_j$ from time t:

$$\delta_{j,t} = \frac{1}{\sqrt{n}} X_t e^{-2\pi i f_j t} \qquad 0 \leqslant j \leqslant n/2,$$

and to $\widehat{X}_{mj}$, $m = 2, 3, \ldots$

$$\delta_{mj,t} = \delta_{j,t} \left(e^{-2\pi i f_j t}\right)^{(m-1)} \qquad \begin{cases} j < mj < n/2 \\ 1 < m < n/(2j) \end{cases}$$

So if there is a non-trigonometric periodic component at frequency f_j, its effects will also be felt at frequencies $f_{2j}, f_{3j}, \cdots$. These are called the *harmonics* of the *fundamental frequency* f_j.