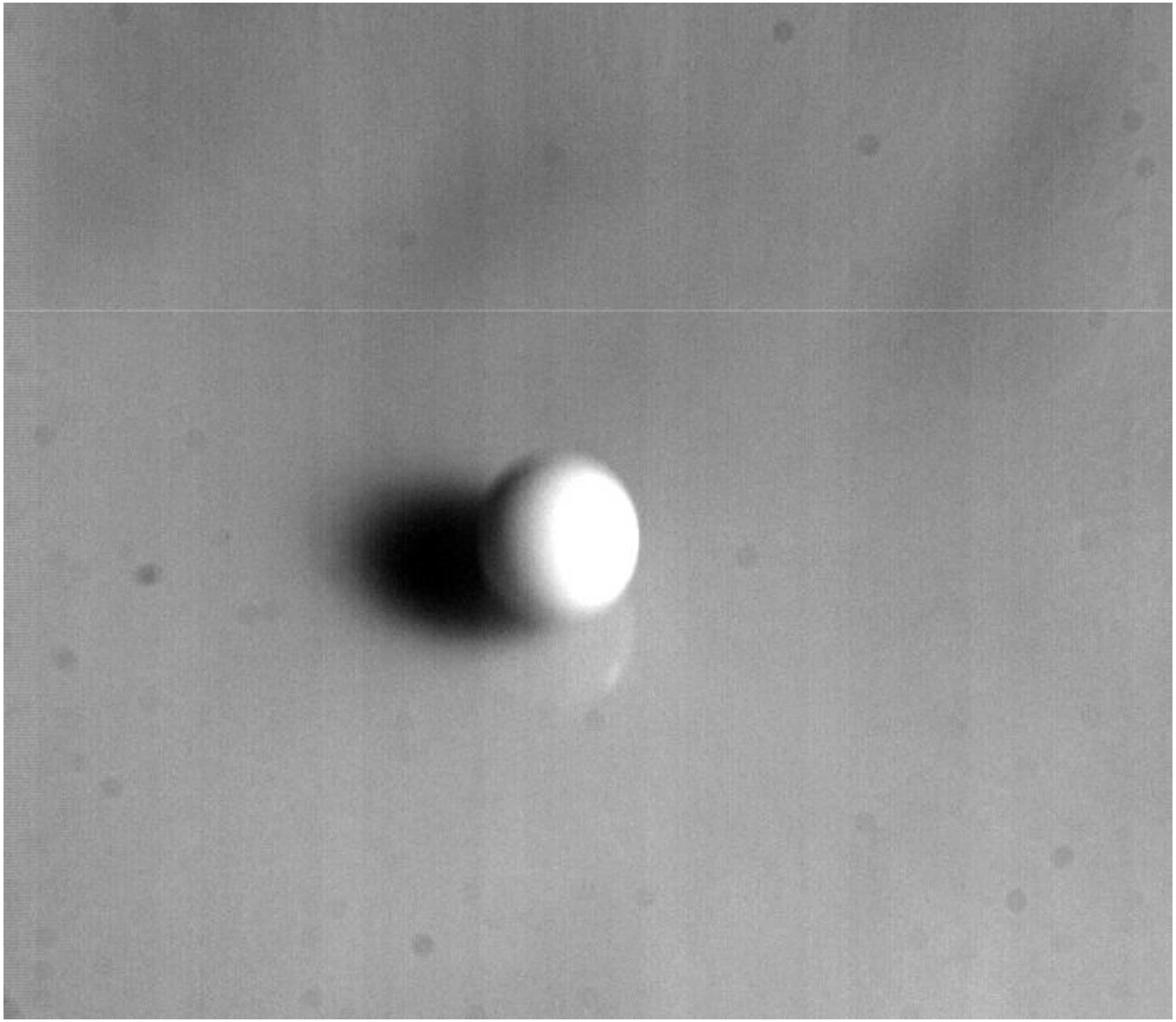
Experimental and theoretical study of pattern identification in physical systems with O(2) symmetry

Rory Hartong-Redden

adviser
Professor Rouslan Krechetnikov
Department of Mechanical Engineering

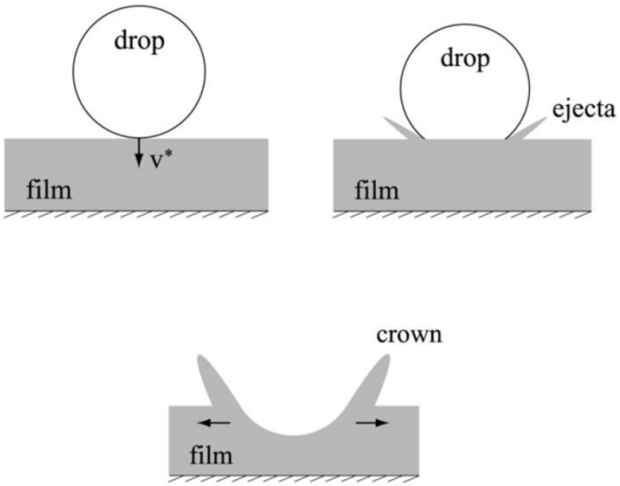
Motivation: the drop splash problem



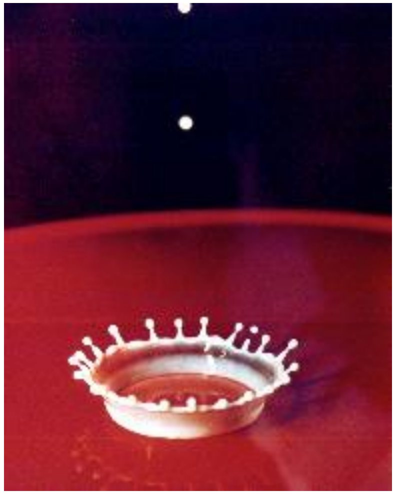
Motivation: the drop splash problem

1. Physical setup: liquid with known properties

2. Geometry: droplet height, diameter, film thickness



The three stages of crown formation. Taken from R. Krechetnikov, G.M. Homsy. "Crown forming instability in the drop splash problem." J. Colloid Interface Sci 331 (2009) 555-559.



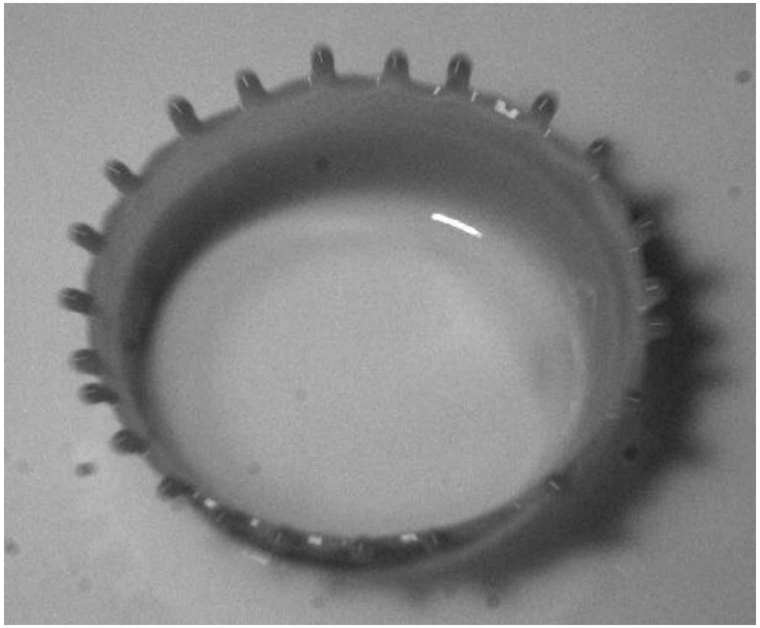
Edgerton's "milk coronet," 1936

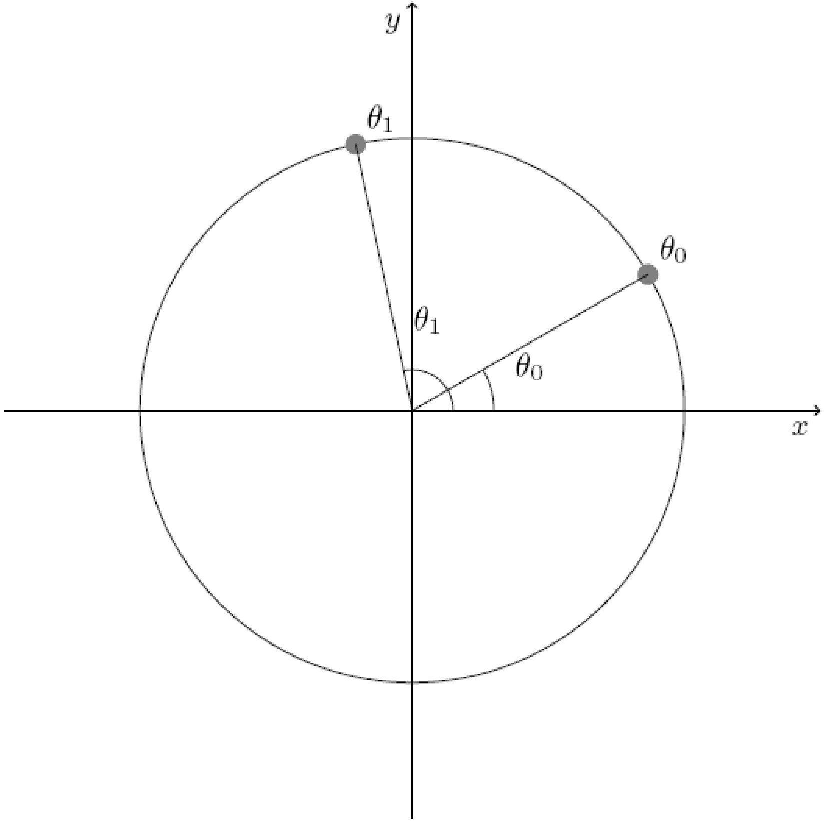
Experimental objective

Quantitative description of a rim

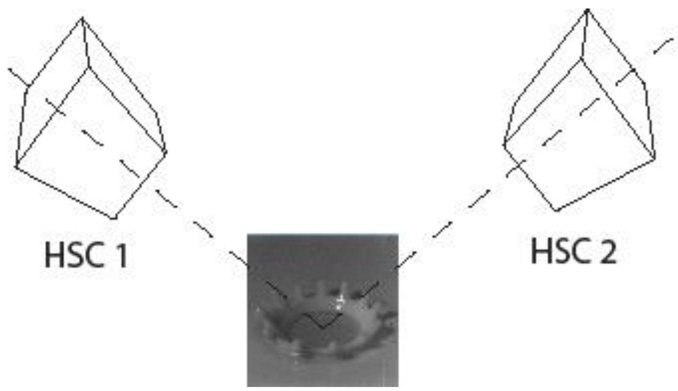
1. Spikes lie on a circle

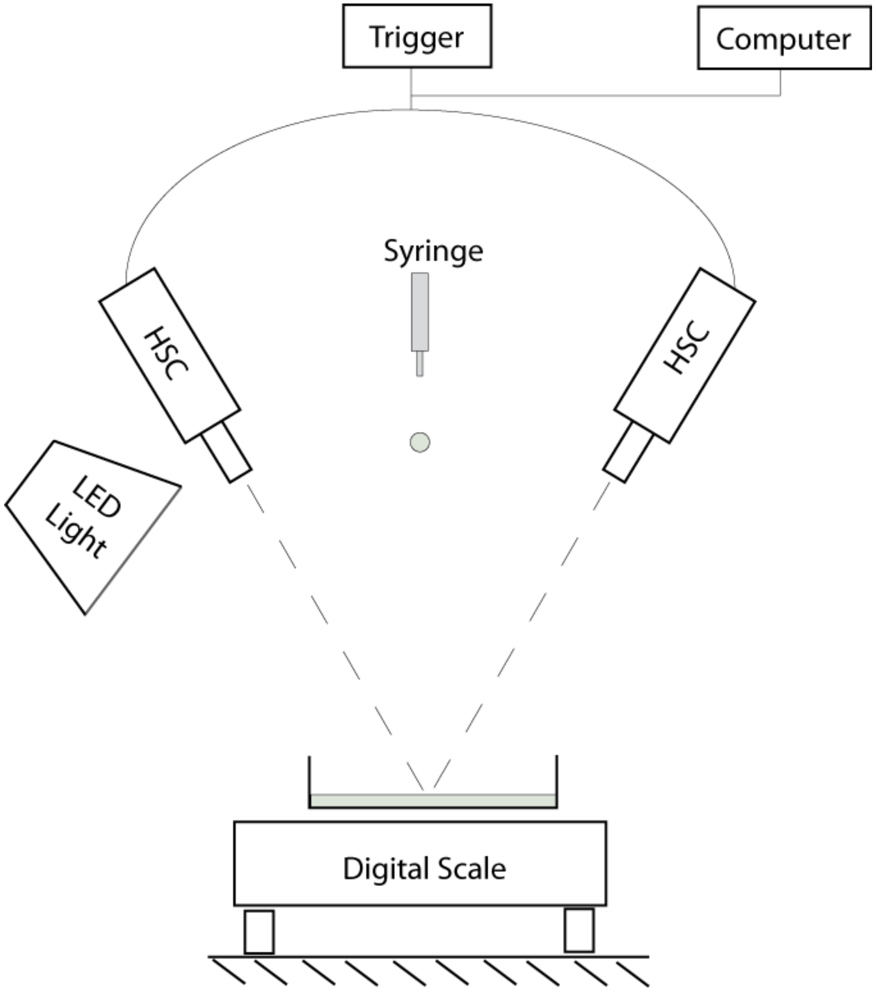
2. Each spike described by an angle





Experimental Setup





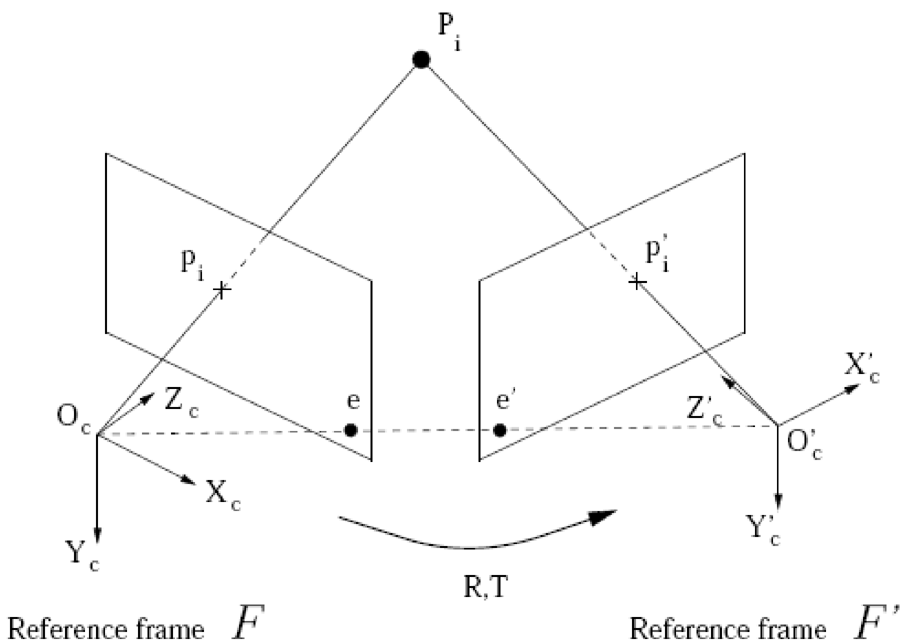
Components

- two high speed cameras
- precision balance
- pump and syringe
- triggering device (function generator)
- high intensity light (LED bulb)

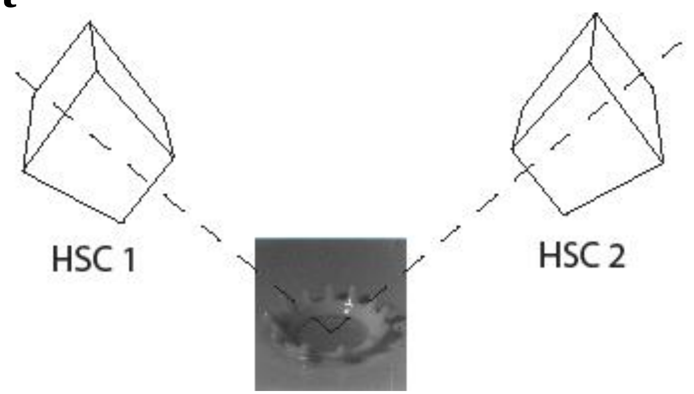
Stereo calibration and triangulation

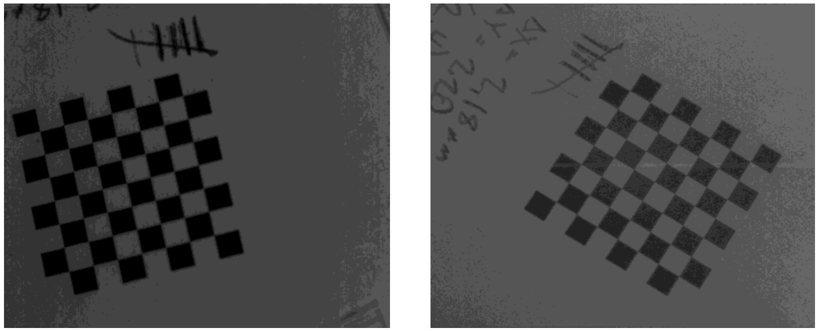
1. **Stereo triangulation – the determination of 3D points from a pair of coordinates on an image**

2. **Requires that cameras be calibrated – implemented using the "Camera Calibration Toolbox for Matlab" by J.Y. Bouguet**



Stereo triangulation requires knowledge of the rotation R and translation T between the left and right cameras. Taken from J.Y. Bouguet, "Visual methods for three-dimensional modeling." PhD thesis, Caltech, 1999.





Images used to calibrate each camera. Each checkerboard

Experimental discussion

Three inter-related experimental challenges

1. **Focus and depth of field**
   a. **f-number**
   b. **depth of field**

2. **Sufficient lighting and contrast**
   a. **position and quantity of lighting**
   b. **opacity of liquid**

3. **Controlled environment**
   a. **stable temperature**
   b. **positioning equipment**

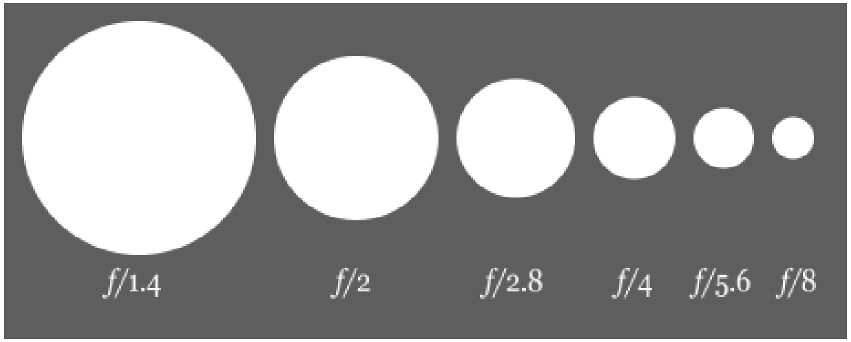
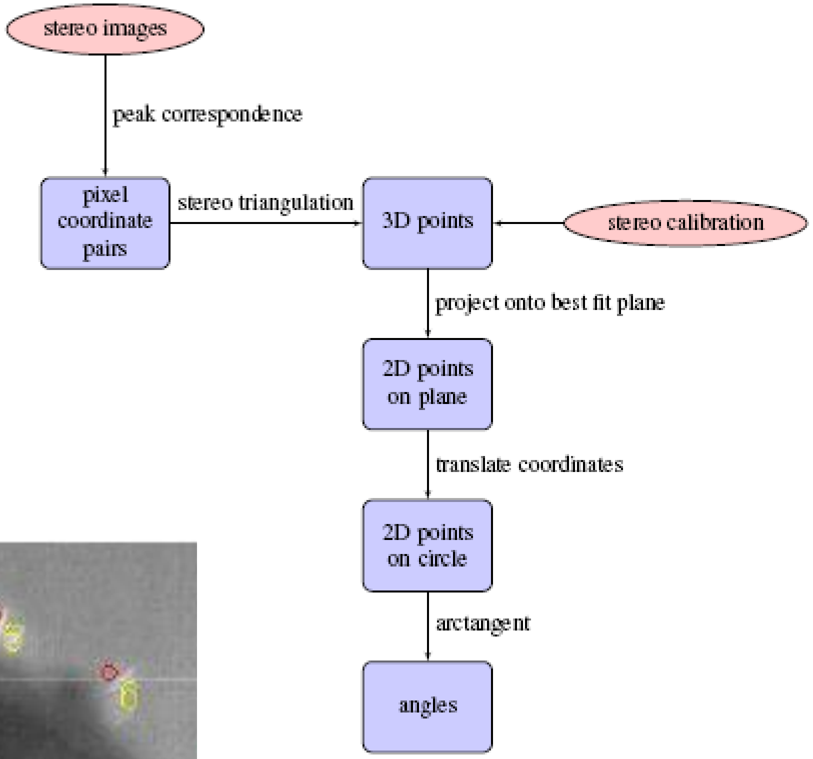
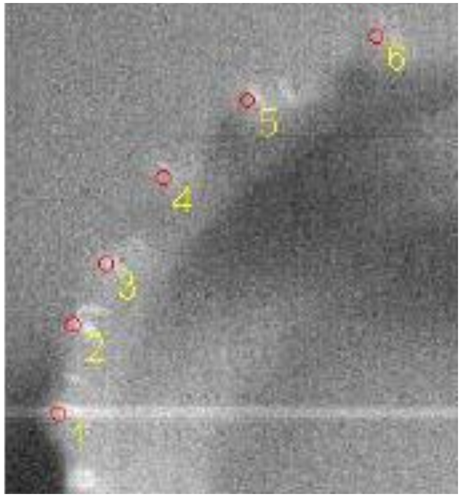


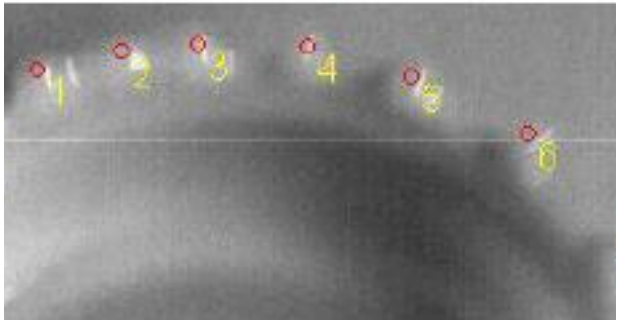
Diagram illustrating increasing f-numbers, taken from wikipedia

Data analysis procedure

1. Capture images of drop splash
2. **Identify corresponding crown peaks by hand**
3. Use stereo triangulation to extract 3D points
4. Project onto plane
5. Project onto circle
6. Project onto circle, calculate angles using atan2(y/x)



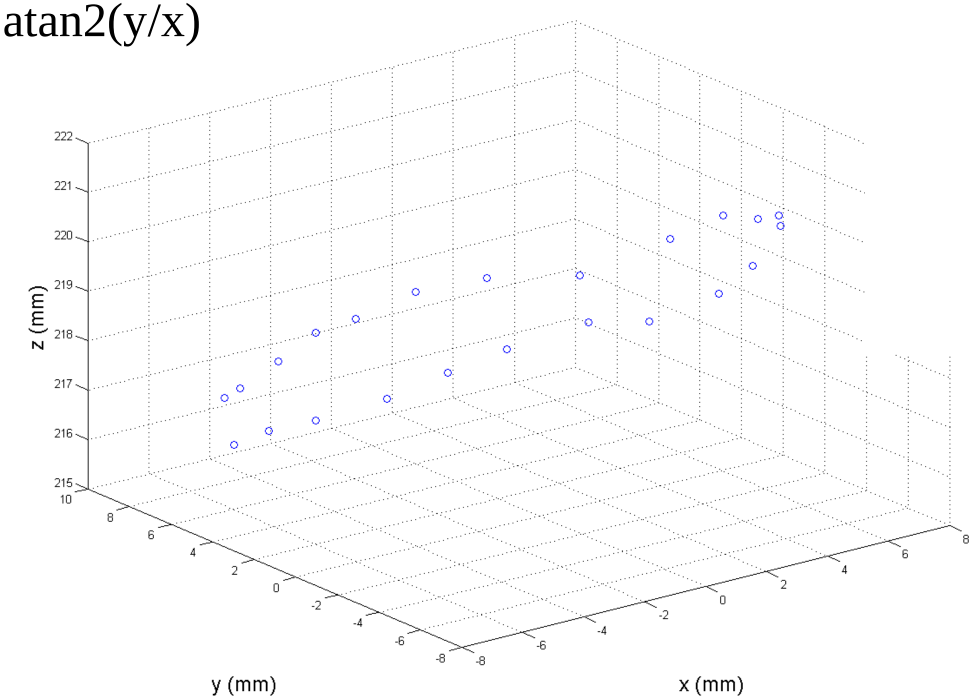




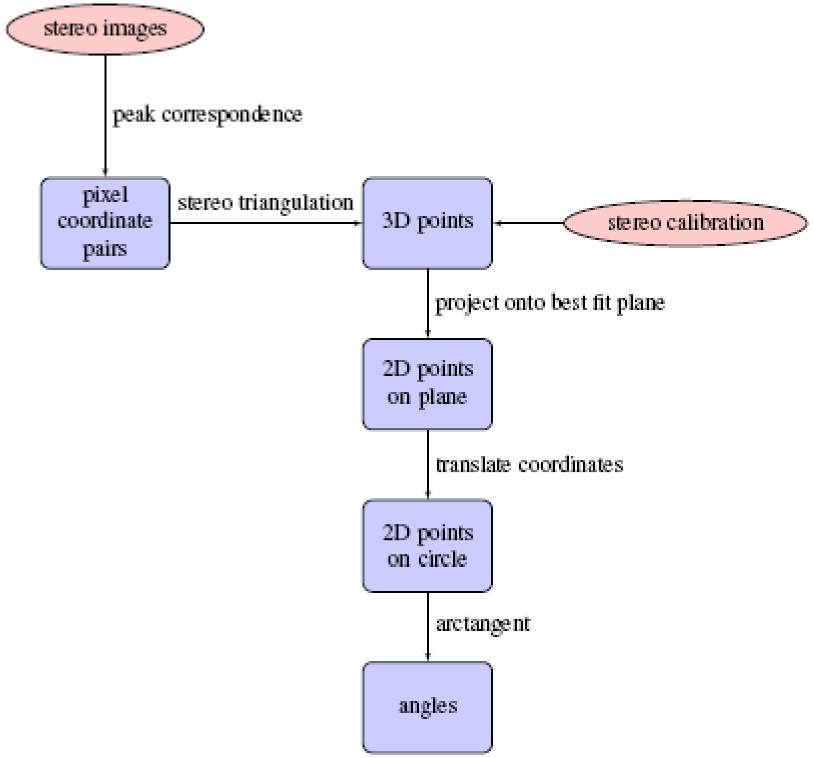
Corresponding spikes labeled by number in left and right images

Data analysis procedure

1. Capture images of drop splash
2. Identify corresponding crown peaks by hand
3. **Use stereo triangulation to extract 3D points**
4. Rotate into best fit plane
5. Project onto x-y plane
6. Project onto circle, calculate angles using atan2(y/x)

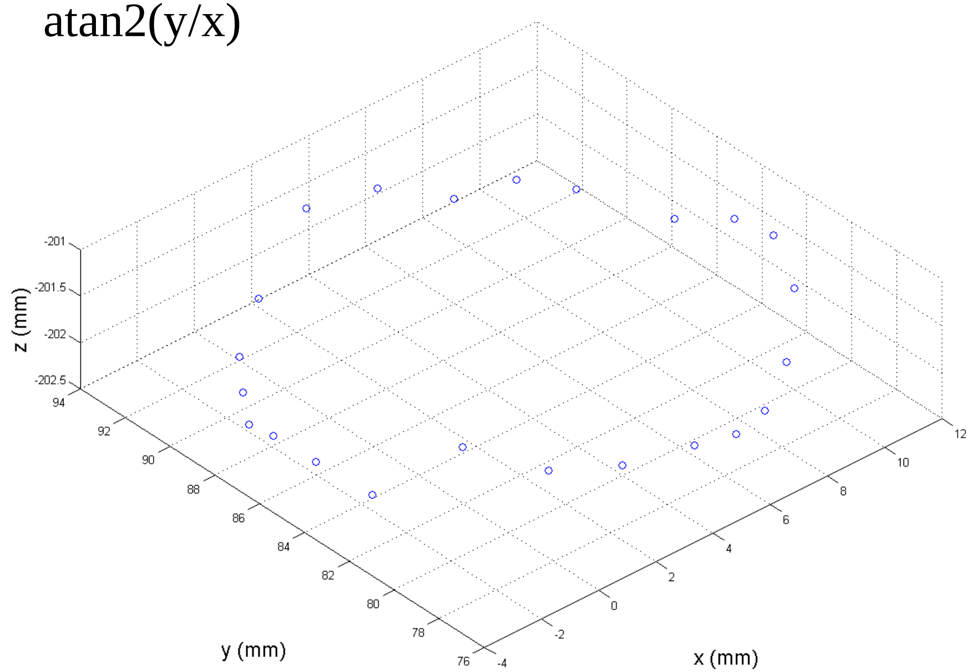


Result of stereo triangulation, in the reference frame of the left camera

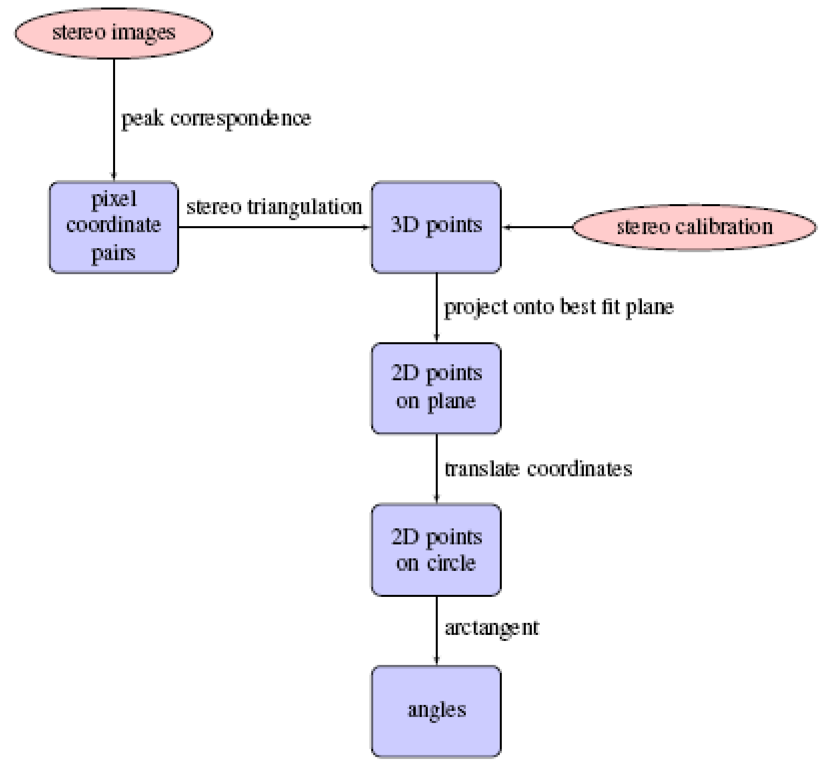


Data analysis procedure

1. Capture images of drop splash
2. Identify corresponding crown peaks by hand
3. Use stereo triangulation to extract 3D points
4. **Rotate into best fit plane**
5. Project onto x-y plane
6. Project onto circle, calculate angles using atan2(y/x)

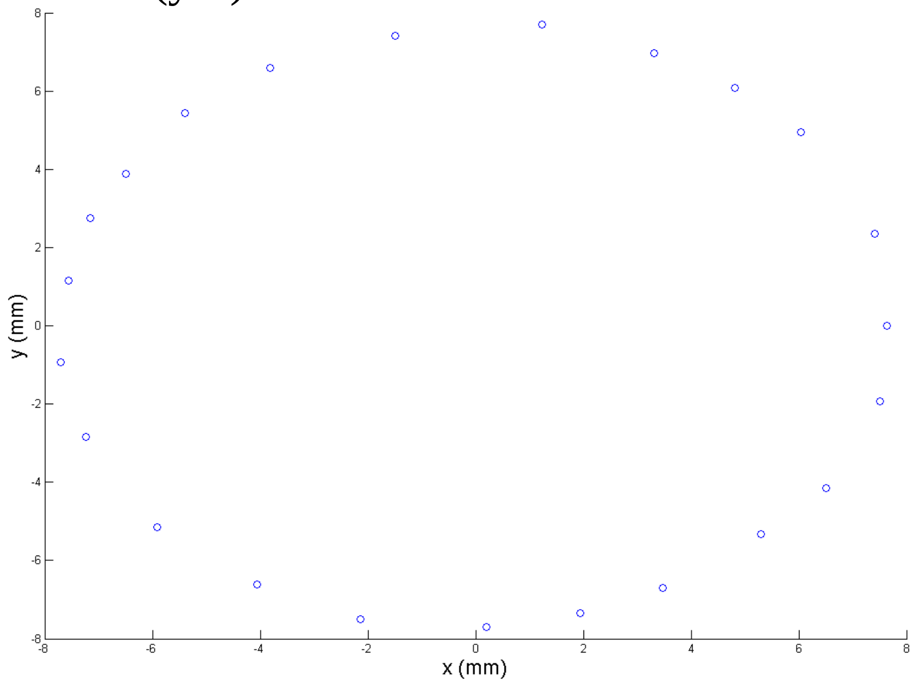


Result of stereo triangulation, in the reference frame of the left camera

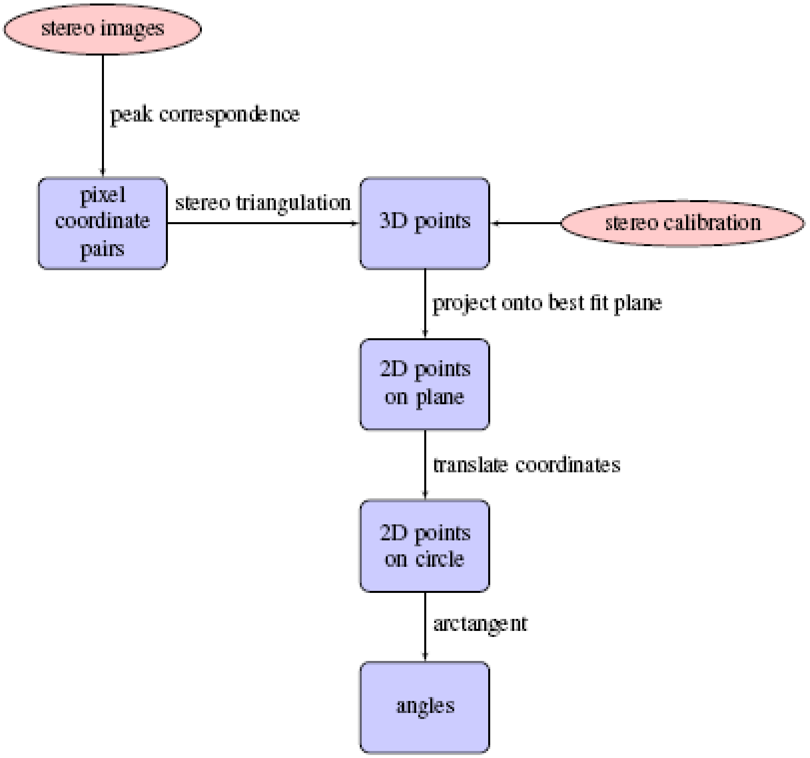


Data analysis procedure

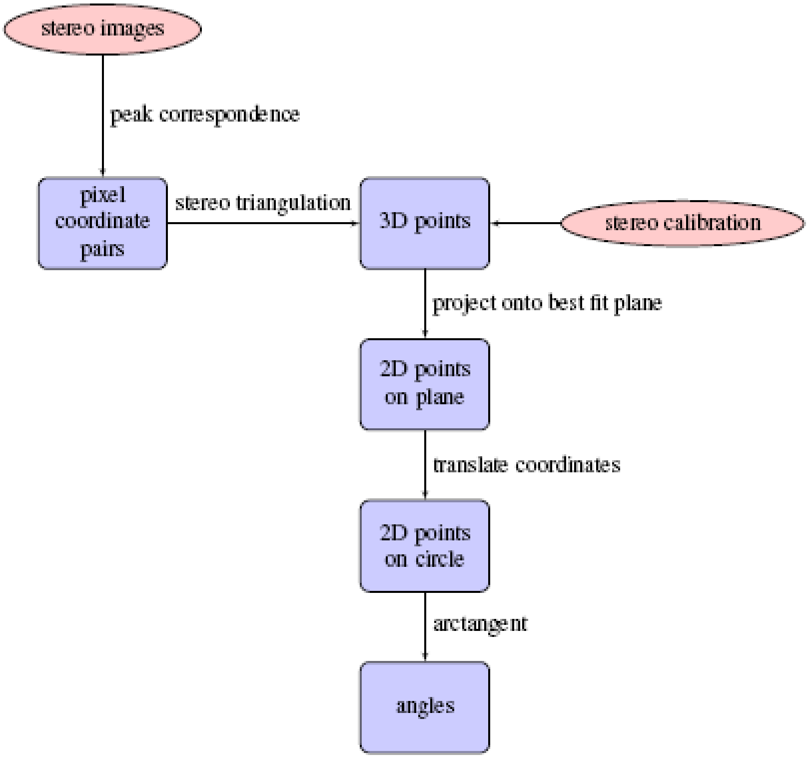
1. Capture images of drop splash
2. Identify corresponding crown peaks by hand
3. Use stereo triangulation to extract 3D points
4. Rotate into best fit plane
5. **Project onto x-y plane**
6. Project onto circle, calculate angles using atan2(y/x)



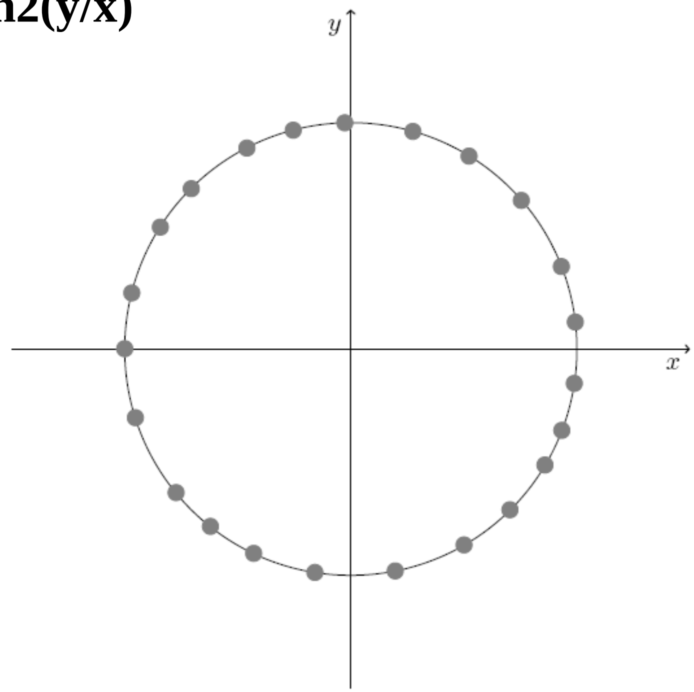
2D points in x-y plane, origin coincident with center of best fit circle



Data analysis procedure



1. Capture images of drop splash
2. Identify corresponding crown peaks by hand
3. Use stereo triangulation to extract 3D points
4. Project onto plane
5. Project onto circle
6. **Project onto circle, calculate angles using atan2(y/x)**

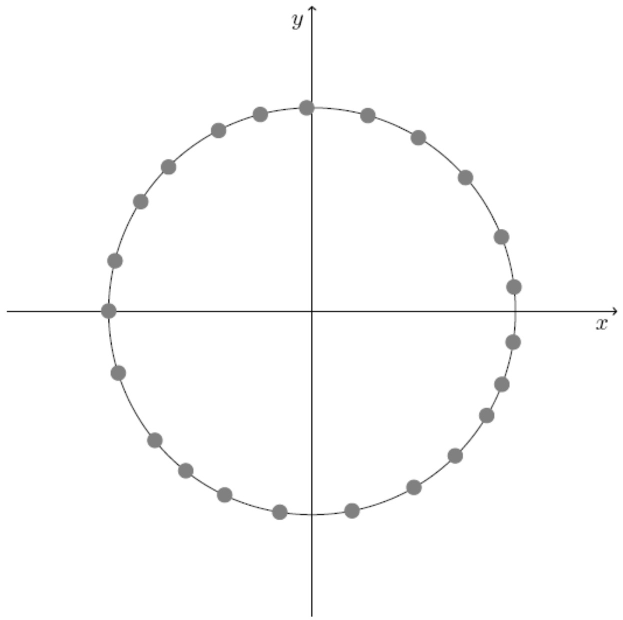


The result of data analysis procedure: angles, displayed here as points on a circle

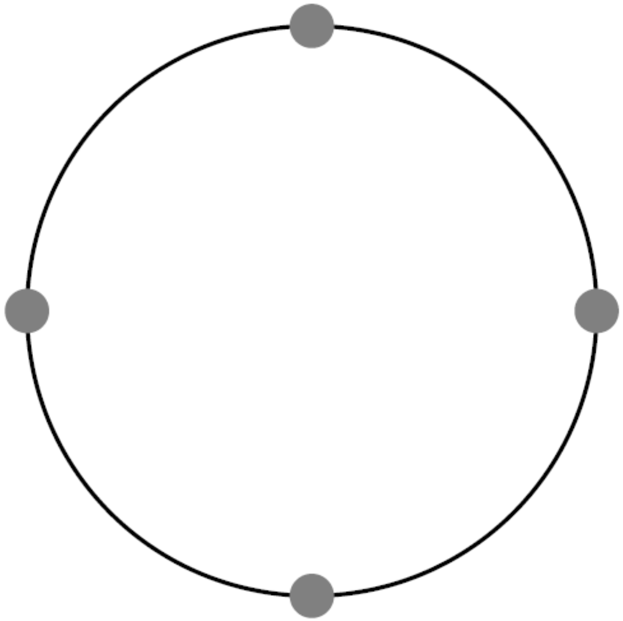
Theory: patterns

Types of patterns
1. Single-wavenumber patterns
2. Regular patterns composed of single-wavenumber patterns (frustration)
3. Irregular patterns

Two key questions
1. Under what conditions is a regular pattern identifiable?
2. What is the “pattern decomposition”?



Experimentally determined pattern



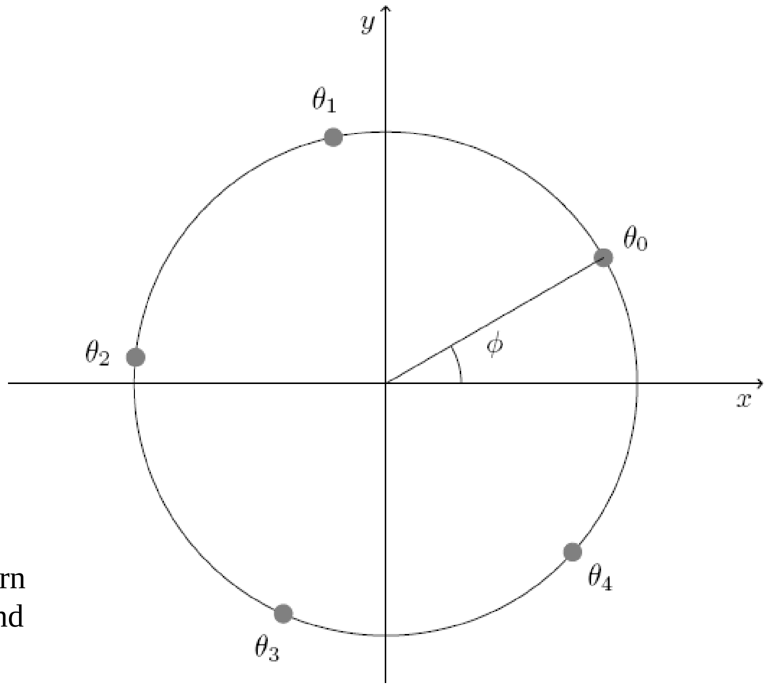
Ideal pattern

Definition: single-wavenumber pattern

1. **Informally: a set of uniformly spaced points**

2. **Ideal single-wavenumber pattern, formally:**

Definition 1 (Ideal single-wavenumber pattern). Let $\Theta = \left\{\theta_1, \ldots, \theta_k\right\}$ be a set of k elements. If $\Theta = \left\{\theta_n \in [0, 2\pi) \,|\, \theta_n = n\lambda + \phi, \text{ for } n = 0, \ldots, k-1\right\}$, where $\lambda = 2\pi/k$, then Θ is an *ideal single-wavenumber pattern.*



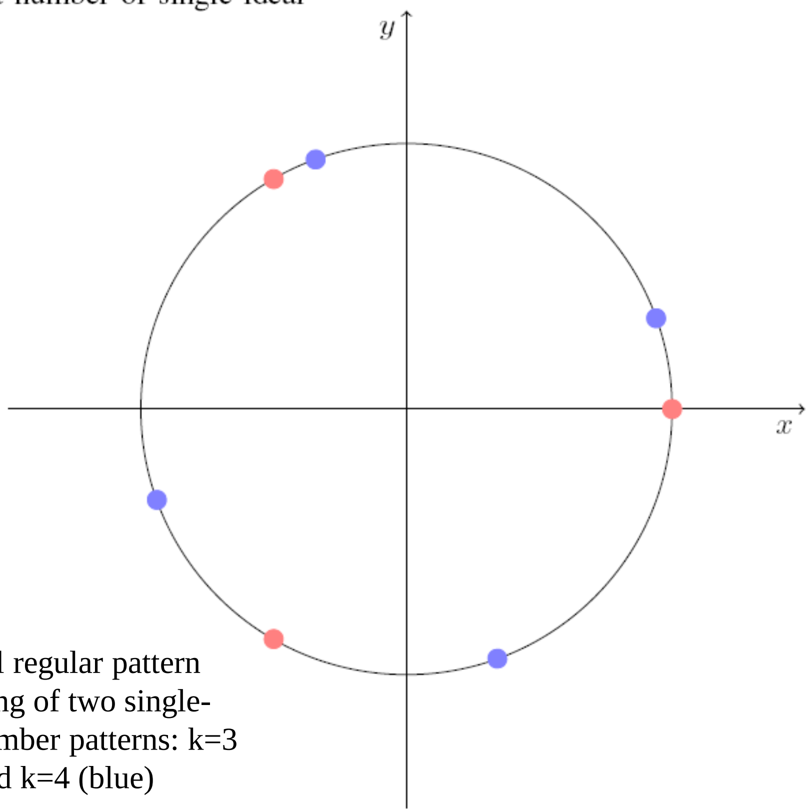
A single-wavenumber pattern with wavenumber k = 5 and phase shift φ

Definition: ideal regular pattern

1. Informally: a superposition of multiple single-wavenumber patterns

2. Ideal regular pattern, formally:

Definition 2 (Ideal regular pattern). If $\mathbf{\Theta} = \bigcup_{i=1}^{m} \Theta^{(i)}$, where m is the least number of single ideal regular patterns required, then $\mathbf{\Theta}$ is an *ideal regular pattern*.



An ideal regular pattern consisting of two single-wavenumber patterns: k=3 (red) and k=4 (blue)

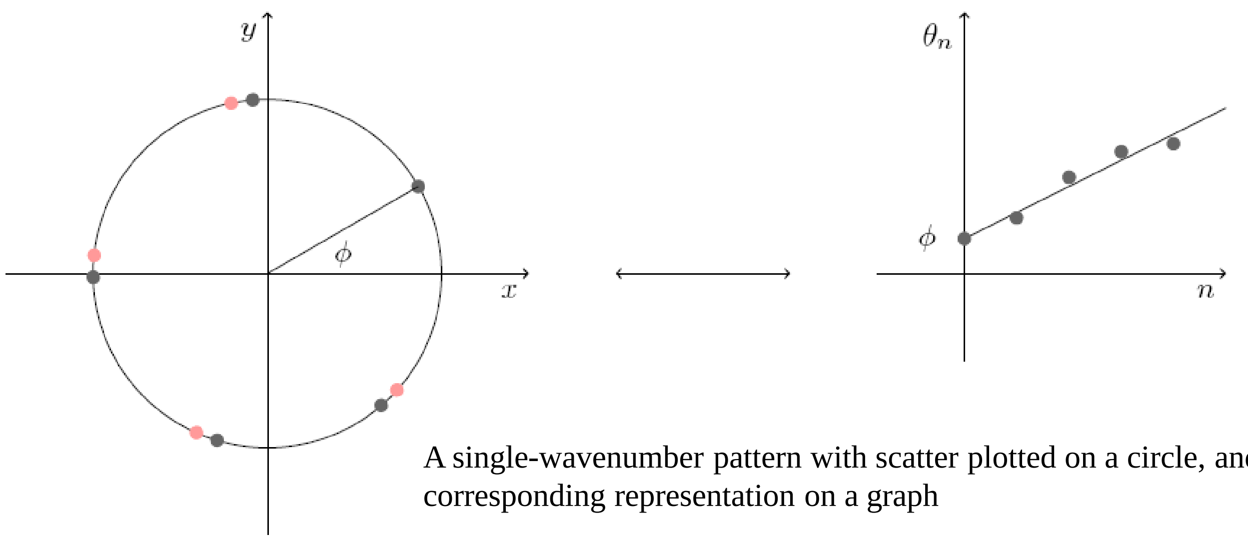
Definition: single-wavenumber pattern with scatter

1. **Informally: an ideal single-wavenumber pattern plus some scatter – i.e. the difference between actual and ideal patterns**

2. **Single-wavenumber patterns with scatter, formally:**

Definition 4 (Single-wavenumber pattern with scatter). Let $\Theta = \{\theta_1, \ldots, \theta_k\}$ be a set with k elements. If $\Theta = \{\theta_n \in [0, 2\pi) | \theta_n = n\lambda + \phi + \epsilon_n, \text{ for } n = 0, \ldots, k-1, |\epsilon_n| \leq \delta\}$, then Θ is a *single-wavenumber pattern with scatter*.

3. **Plotting single-wavenumber patterns**
   a. **Plot coordinates (n, θ_n)**
   b. **fitting of line is straightforward, standard statistical analysis applies**



A single-wavenumber pattern with scatter plotted on a circle, and its corresponding representation on a graph

Pattern identification

Theoretical stumbling blocks

1. Scatter (due to goodness of fit and experimental uncertainty)
2. Overlapping patterns (due to resolution, and merging of spikes due to surface tension)
3. Incomplete patterns (patterns which are missing one or more points)

Motivates three questions

1. How do we identify a given pattern as regular?
2. Under what conditions can we identify regular patterns?
3. How do we quantify how regular it is?

Pattern identification: other approaches

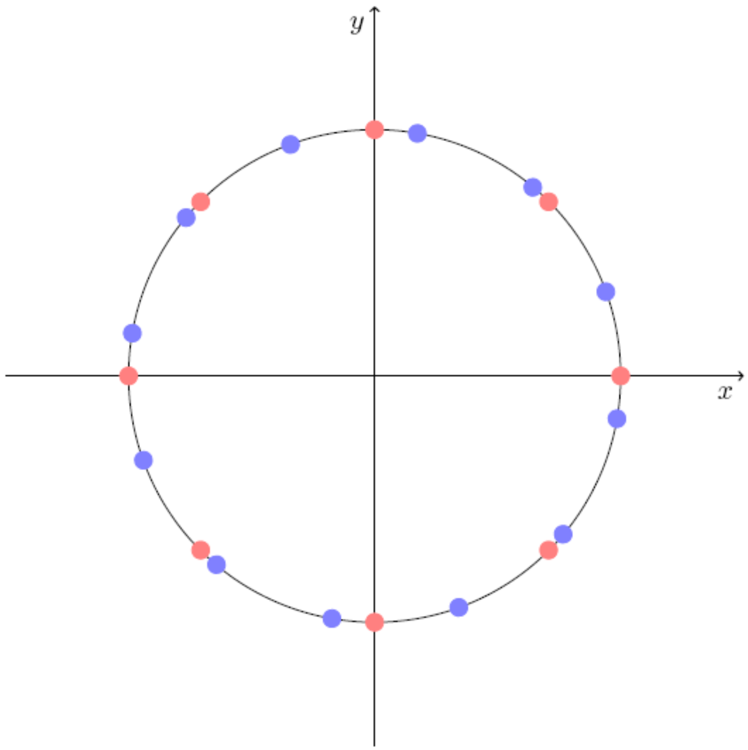
1. "Ad hoc" methods
   a. by inspection
   b. look for regular spacing between elements

2. Fourier transform
   a. not appropriate for data type
   b. sampling, aliasing and Nyquist Theorem

3. Complex order parameter

$$r_k = \frac{1}{N}\left|\sum_{n=1}^{N} e^{i\theta_n k}\right|$$

4. Computation approaches



An ideal regular pattern consisting of two single-wavenumber patterns: k=8 (red) and k=12 (blue)

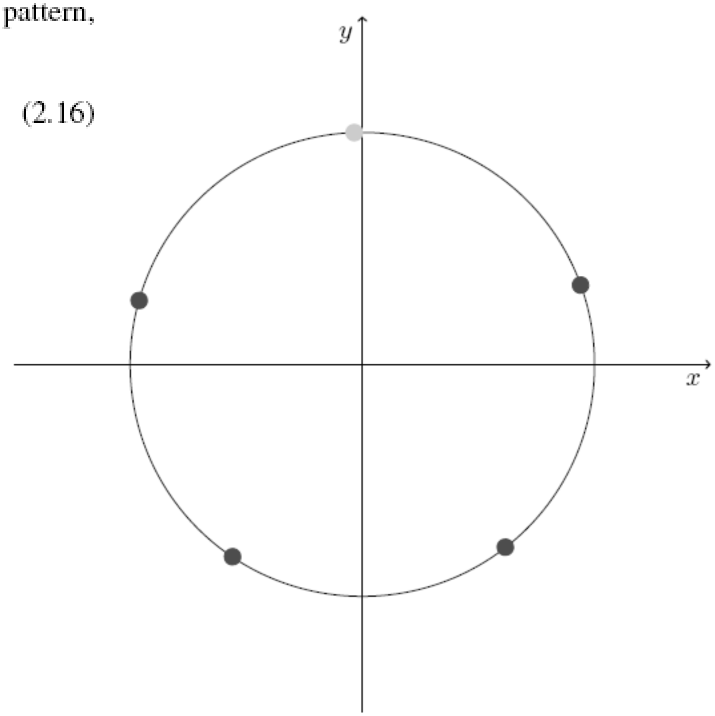
Definition: incomplete patterns

1. Informally: patterns which are regular, but missing one or more points

2. Incomplete patterns, formally:

Definition 3. A *single incomplete-wavenumber pattern* Θ^I is single regular pattern, Θ^R, minus a set of subtracted points Θ^- where $\Theta^- \subset \Theta^R$.

$$\Theta^I(k, \phi) = \Theta^R(k, \phi) \setminus \Theta^- \tag{2.16}$$

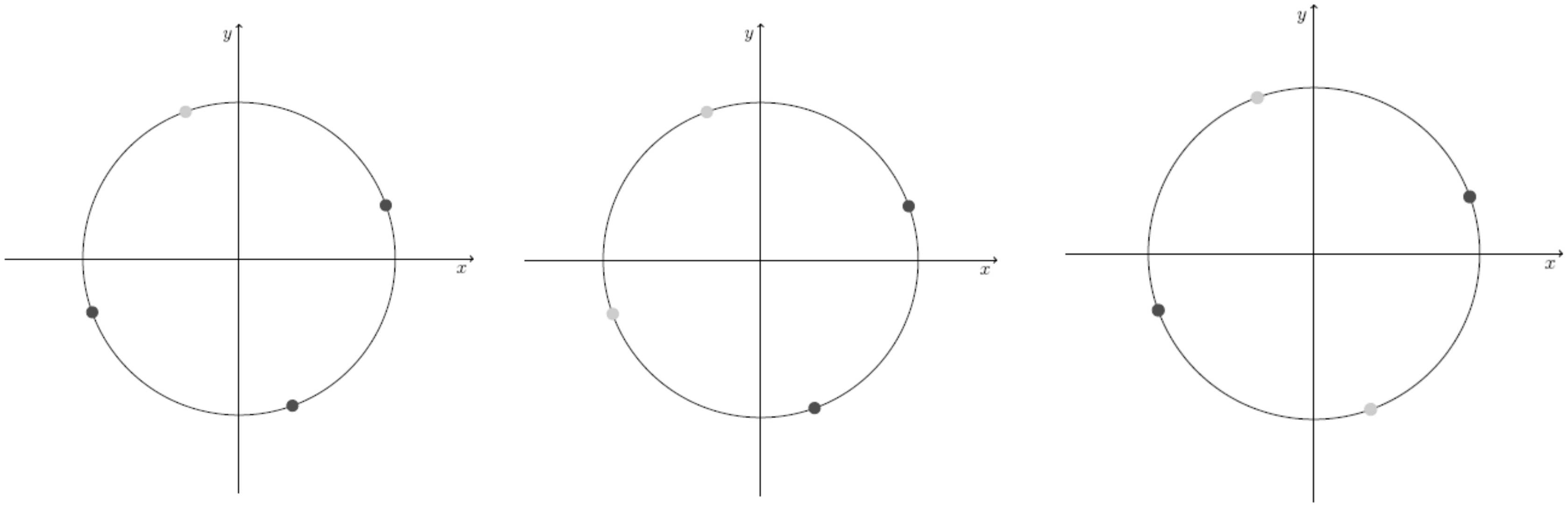


An incomplete pattern consisting of a single-wavenumber pattern with one point missing

Incomplete patterns

1. Want an incomplete pattern definition which retains phase and wavenumber information

2. Consider as an example a complete pattern with k=4
   a. remove a single point
   b. remove two points – leads to trouble



Claim 1. If Θ^- is not regular (does not contain any regular subsets), then $\Theta^I(k,\phi)$ has the same properties as Θ^R and is well-defined among all possible Θ^-.

Summary of theoretical results

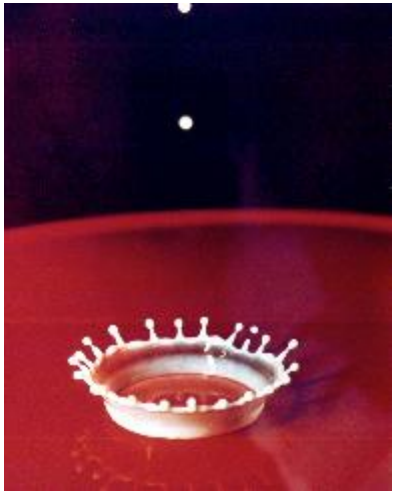
1. A new theoretical framework for patterns

   a. Clear definitions of patterns: single-wavenumber, regular, irregular

   b. Classification of theoretical problems: ideal regular patterns, patterns with overlaps, incomplete patterns

2. Algorithms – implemented in Matlab

   a. can identify non-overlapping ideal regular patterns with and without scatter



Edgerton's "milk coronet," 1936

Drop splash video

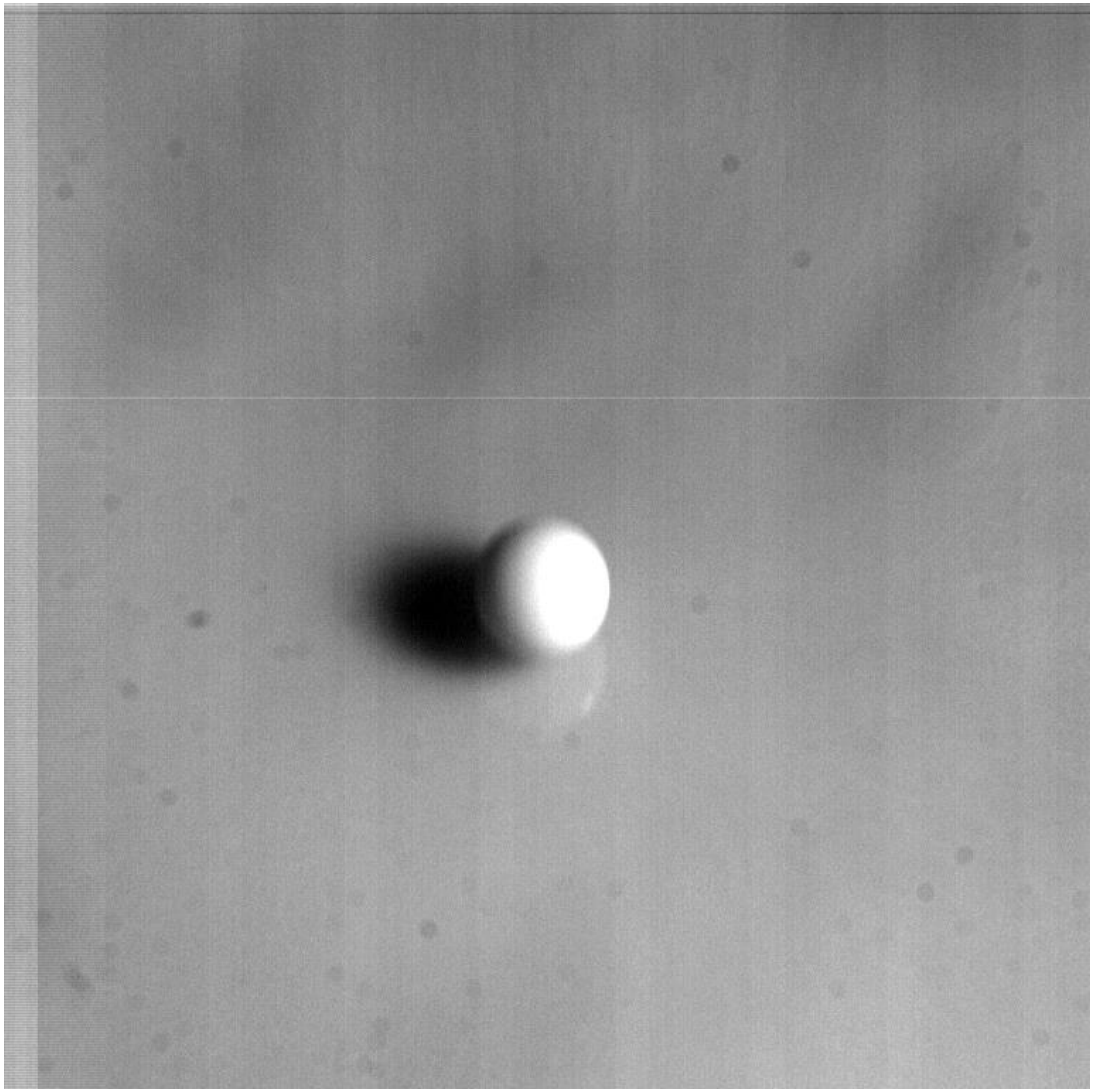
Formation of a regular crown

Film thickness = 0.88mm
Droplet diameter = 4.4mm
Drop height = 28cm

$$We_{drop} = \frac{\rho v^2 d}{\sigma} \approx \frac{inertia}{surface\ tension}$$

$$We_{drop} = 605$$

$$\alpha = \frac{We_{drop}}{We_{film}} \approx \frac{drop\ inertia}{film\ inertia} = 5$$

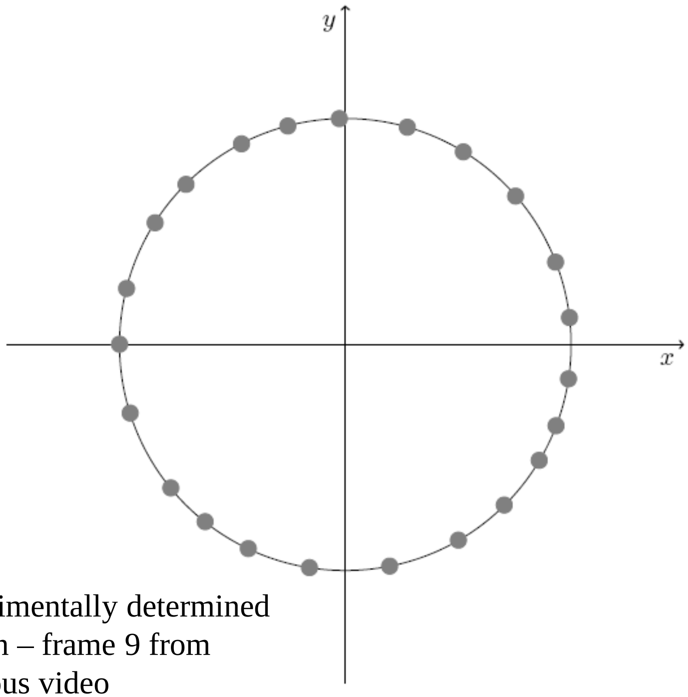


Application to experimental data: regular crown

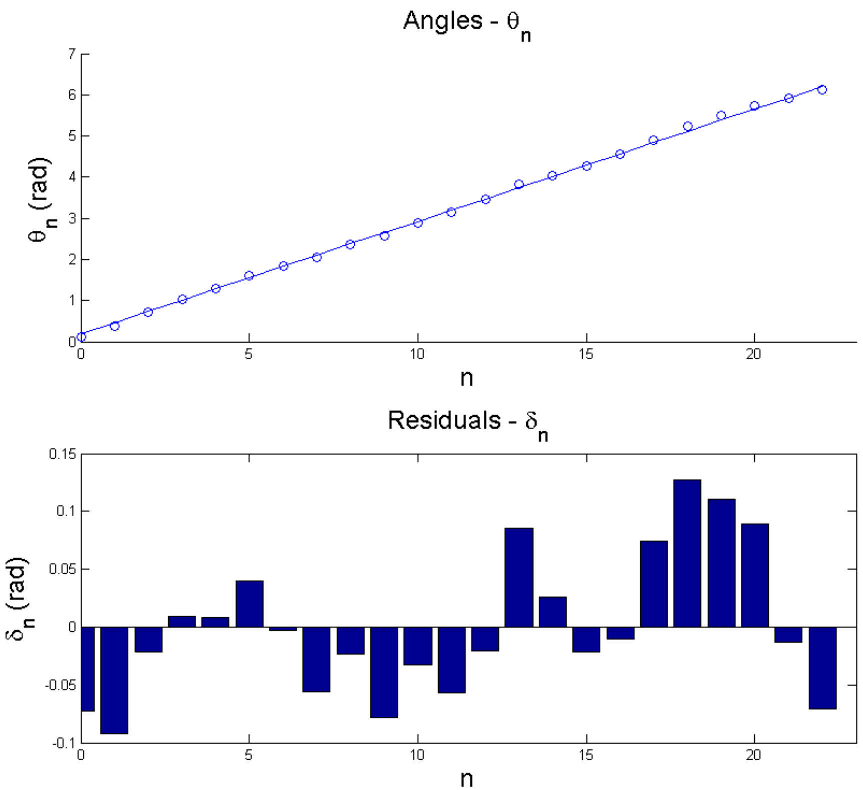
1. Formation of a milk crown – experimental data

2. Pattern identification: single-wavenumber pattern

3. Tolerance or scatter: $\frac{\delta}{\lambda} = 0.47$



Experimentally determined pattern – frame 9 from previous video



The result of fitting the experimental pattern to the single-wavenumber pattern with scatter model

Conclusions and Future Work

Key results

1. New experimental setup with quantitative data on rim structure

2. New theoretical framework for pattern identification

Outstanding issues

1. Further exploration of parameter space
   a. single-wavenumber rims with less scatter
   b. identify a case of “frustration”

2. Theoretical treatment of incomplete patterns with scatter

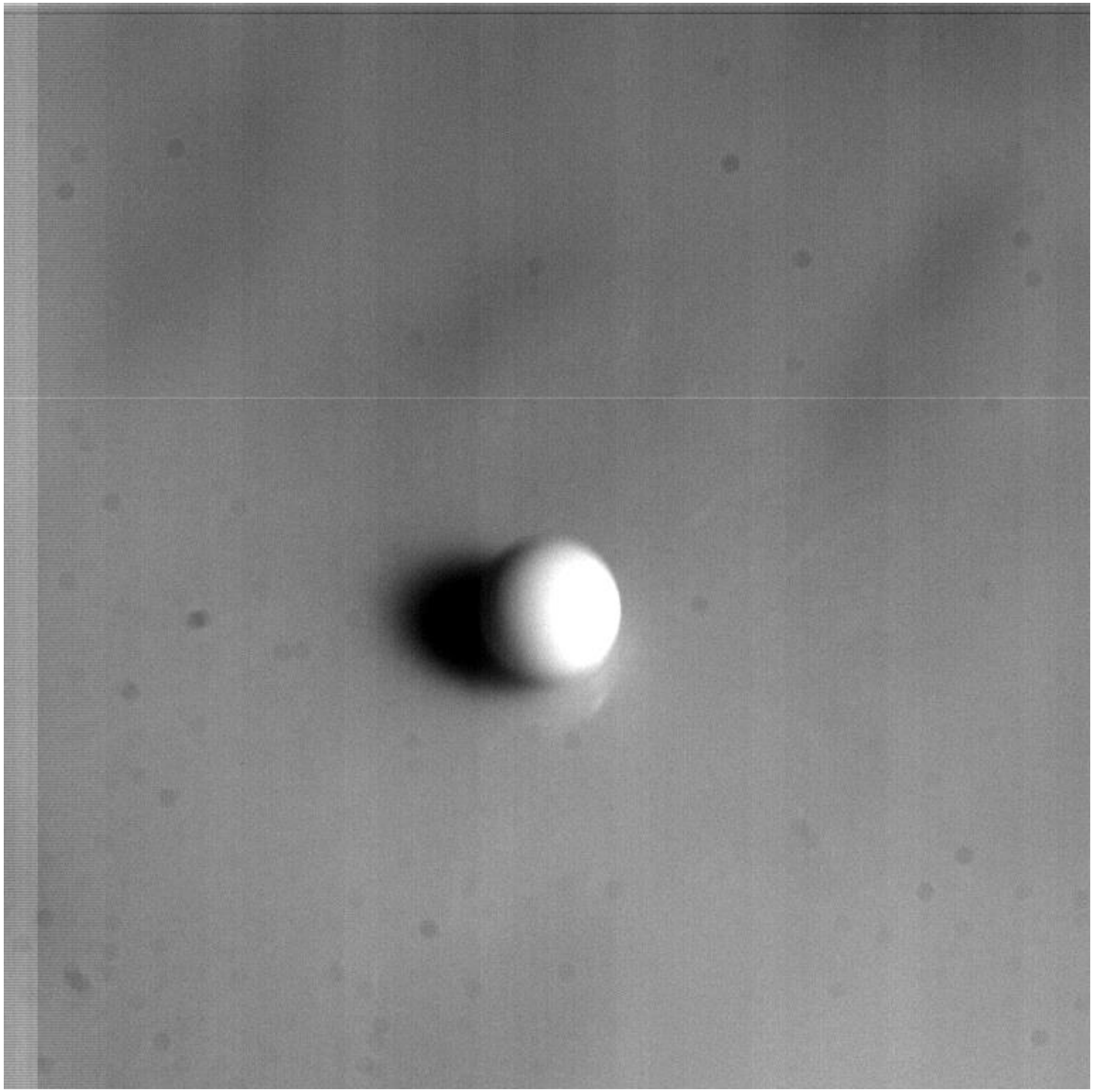
Image taken from Parker Hannifin brochure

Appendix: video of an irregular crown

Formation of an irregular crown

Drop height = 16cm

$We_{drop} = 345$
$\alpha = 5$



Appendix: motivation – frustrated rims

Why are there a certain number of spikes?
1. Along the rim instabilities: Richtmyer-Meshkov instability
2. Superposition of regular patterns = “frustration”

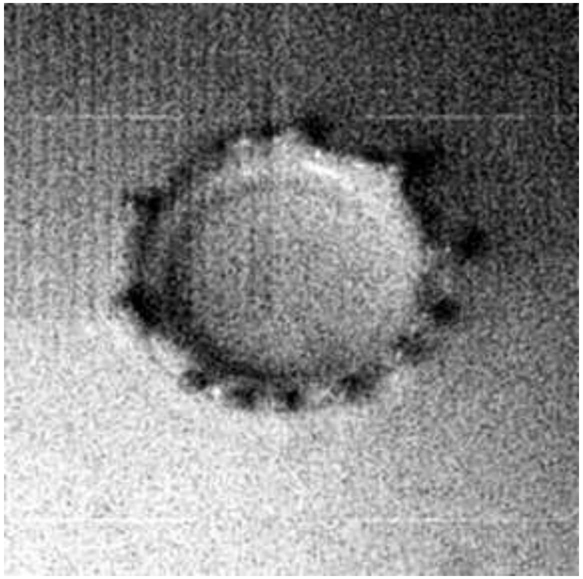
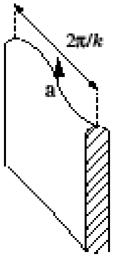
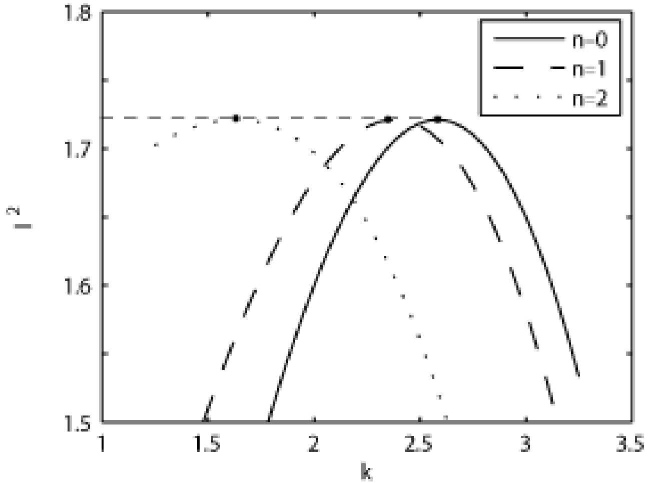


Image taken from G. Homsy, R. Krechetnikov. “Crown-forming instability in the drop splash problem.”
A frustration picture?





Equal growth rates indicates the possible superposition of three wavenumbers

Appendix: single-wavenumber pattern factoring

Example:
A single-wavenumber pattern with k=6 is also:
1. **two k=3 patterns**
2. **three k=2 patterns**

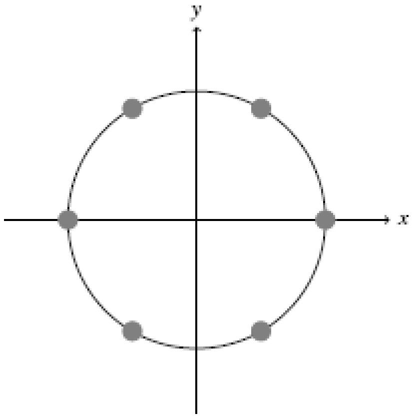


Figure 2.2: A single regular pattern $\Theta(k = 6, \phi = 0)$)

Theorem 1. *Any ideal single-wavenumber pattern $\Theta(n, \phi)$ may be expressed as a union of ideal single-wavenumber patterns of $k = p$ where p divides n.*

$$\Theta(n, \phi_0) = \bigcup_{i=1}^{n/p} \Theta^{(i)}(p, \phi_0 + \lambda_n(i-1)) \tag{2.5}$$

Appendix: pattern identification algorithm

Key idea: identify all patterns (steps 1-6) and then remove subpatterns (step 7)

Finite differences:

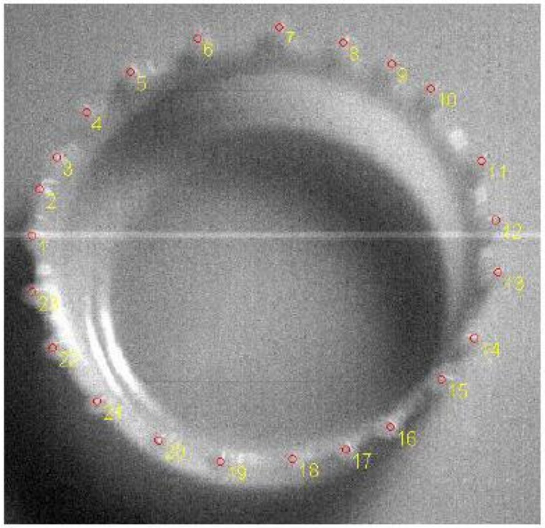
$$\Delta\mathbf{\Theta}_{ij} = \theta_i - \theta_j.$$

For ideal regular patterns with possible overlaps

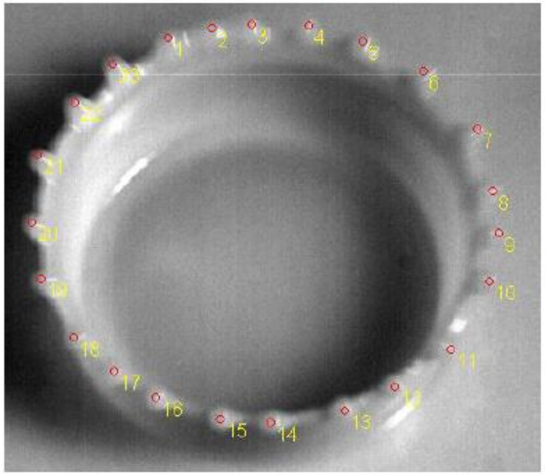
Partition algorithm for ideal patterns with overlaps

1. Given $\Theta = \left\{\theta_1, \ldots, \theta_N\right\}$, compute $\Delta\Theta$
2. Let $k = N$ and compute $\lambda_k = \frac{2\pi}{k}$
3. Find all indicies i, j for which $\Delta\Theta_{ij} = n\lambda_k$ for any possible $n \in \mathbb{N}$
4. From indicies i, j, place the corresponding θ_i into a new set Θ'
5. Partition Θ' into blocks of single ideal regular patterns, with $k = N$, according to phase shift
   (a) Given $\Theta' = \left\{\theta'_1, \ldots, \theta'_L\right\}$ in ascending order
   (b) Let $\theta'_1 = \theta_0^{(1)} \in \Theta^{(1)}$. Therefore $\phi^{(1)} = \theta'_1$
   and $\Theta_k^{(1)} = \left\{\theta_n \in [0, 2\pi) | \theta_n = n\lambda_k + \theta'_1 \text{ for } n = 0, \ldots, k-1\right\}$
   (c) Let $\Theta' \to \Theta' \setminus \Theta^{(1)}$
   (d) Repeat procedure, forming $\Theta_k^{(2)}, \ldots, \Theta_k^{(L/k)}$
6. Repeat procedure for $k = N-1, N-2, \ldots, 2$
7. Considering all $\Theta_i^{(j)}$ and $\Theta_l^{(m)}$, remove any $\Theta_l^{(m)}$ for which $\Theta_l^{(m)} \subset \Theta_i^{(j)}$

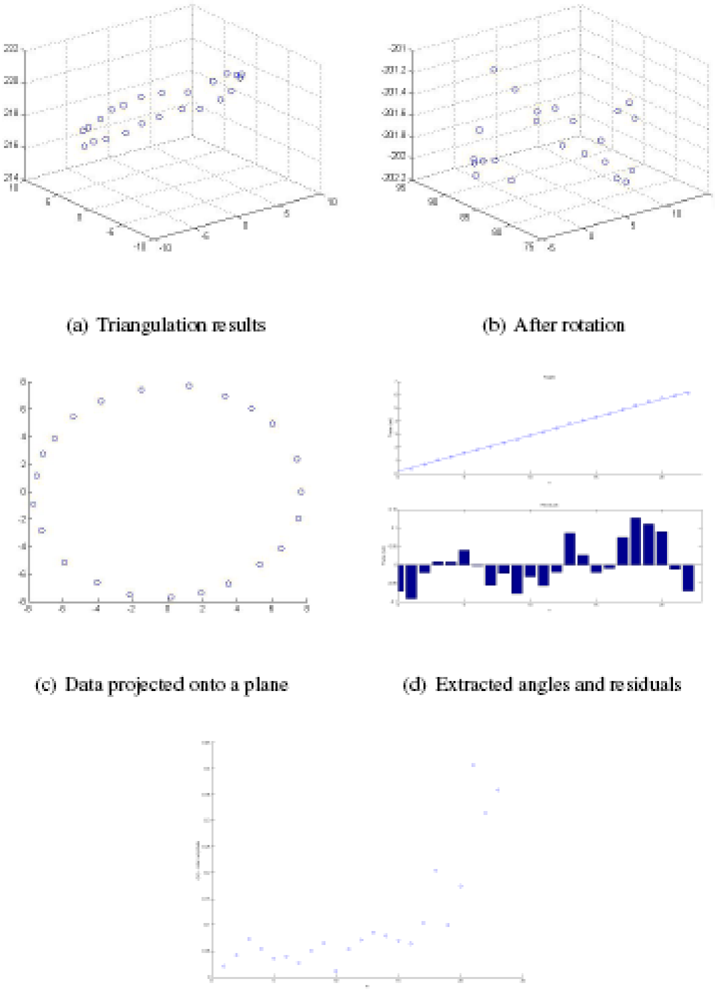
Data analysis example



(a) Left image



(b) Right image



(a) Triangulation results (b) After rotation

(c) Data projected onto a plane (d) Extracted angles and residuals

(e) Complex order amplitudes

Figure 3.4: Example of data reduction and analysis: (a) results of triangulation in the reference frame of the left camera, (b)first coordinate transform, (c) second coordinate transform, (d) the resulting fit with residuals, and (e) the plot of complex order amplitudes for the crown in Fig. 3.3, $We_{\text{drop}} = 605$, $H = 28cm$, $\alpha = 5$, Knudsen 2% milk.

Presentation Outline

1. Introduction

2. Pattern identification theory

3. Experimental study

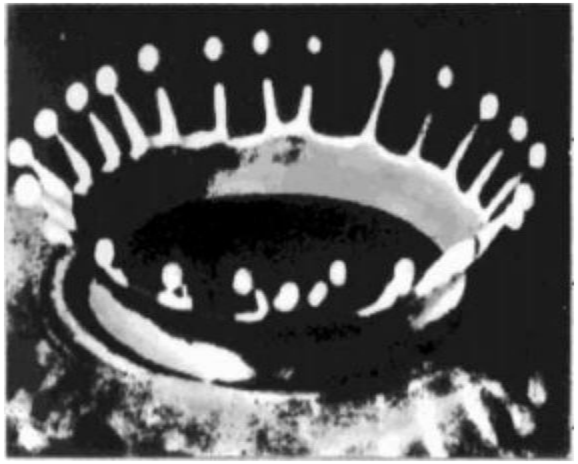
Image taken from JFM Online

Image taken from efluids, credit: Abhishek Joshi

Introduction

1. Systems with O(2) symmetry
   a. domain of a circle
   b. Time domain
   c. Spatial domain

2. A different type of data
   a. 1-Dimensional
   b. Patterns which deal with order of data points



S. P. Betyaev, *Physics Uspekhi*