Frieze Groups, Lattices and Wallpaper groups

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MATH3120 Abstract Algebra
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Wallpaper groups

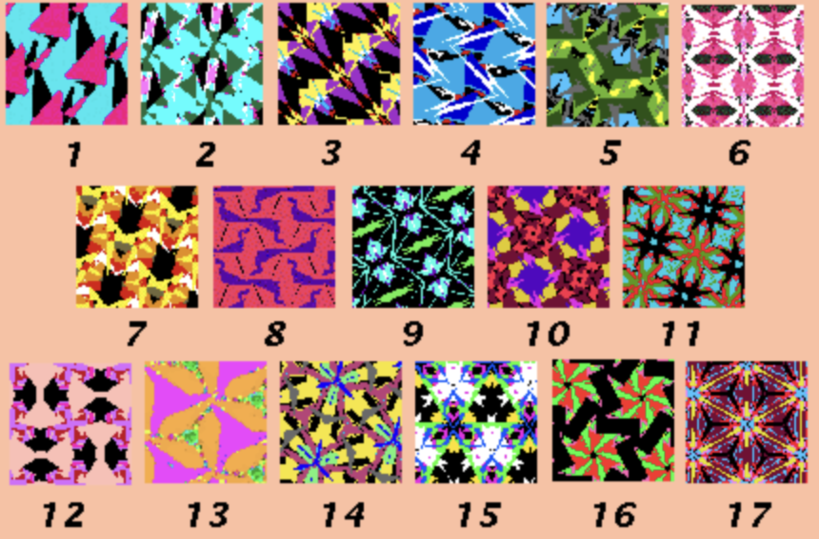


Figure 1: The 17 wallpaper groups [Joyce, 2024]

Introduction, group operations

Definition (Types of Symmetries)

- Translations
- Reflections
- Glide Reflections
- Rotations.

Translations

Definition

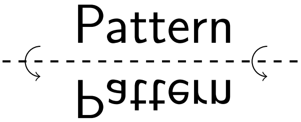
Translations are repetitions of a pattern structure. defined mathematically as follows
a mapping $t_a : \mathbb{R}^2 \to \mathbb{R}^2$ where $x \to x + a$ where $x \in X$ [Zhoa, 2023]

$$\text{Pattern} \longrightarrow \text{Pattern}$$

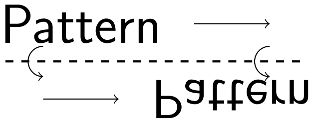
Figure 2: Translations

Definition

Reflections: A reflection over a line through the origin is defined as follows
$Ref_l(v) = 2\frac{v \cdot l}{l \cdot l}$l - v
where v and l are vectors going through the line of origin.
Glide : A glide is a reflection followed by a translation. [Zhoa, 2023]



(a) Reflections



(b) Glide Reflection

Figure 3: Reflective Symmetries

Rotations

Definition

A rotation is a change of angle around a center point.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ Where $\theta \in \{1, 2, 3, 6\}$. [Zhoa, 2023]

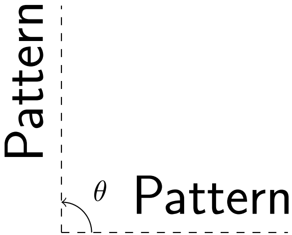


Figure 4: Rotations

Lattices

Definition

A lattice is the group $(\mathbb{Z}[\vec{a}, \vec{b}], +)$.
i.e., a grid of points where any point $p = n\vec{a} + m\vec{b}$

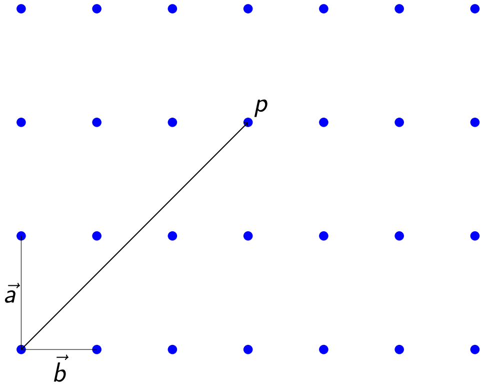


Figure 5: Lattice

Bravais Lattices

Primitive cells are parallelograms, whereas centered cells have lattice points in the center.

Additionally, Square $\subseteq$ Rectangle $\subseteq$ Rhombic, The Rectangle $\subseteq$ Oblique cell and the Hexagonal cell $\subseteq$ Oblique Cell.

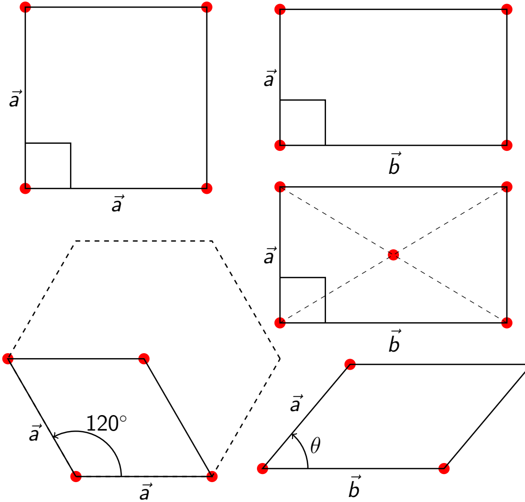


Figure 6: Bravais Lattices

Visual proof that lattices cannot have order 8

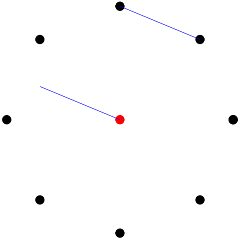


Figure 7: 1

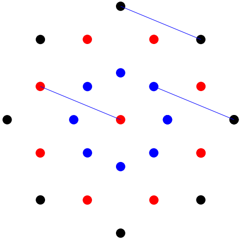


Figure 8: 2

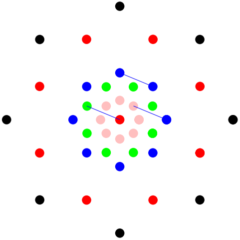


Figure 9: 3

Figure 10: Octagons under translation

Proof sketch

Note that any point can translated by lattice definition. So the translation of the center and vertices gives two smaller octagons. This can further be extended in smaller octagons. This can be repeated infinitely. Creating a dense grid of points on the entire 2-dimensional plane.

Definition

A frieze group is the set of two-dimensional patterns that repeat in one dimension. [Ganapathy, 2021]

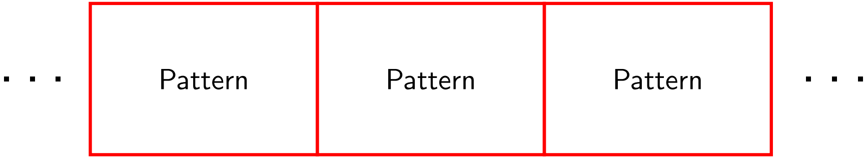


Figure 11: Frieze group example

Frieze Groups

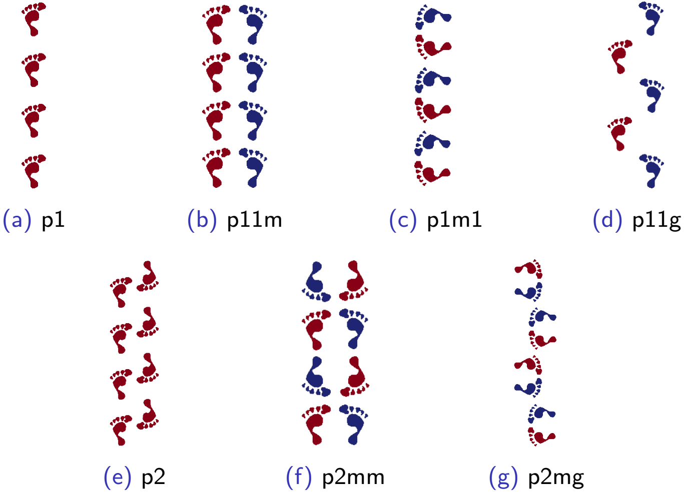


Figure 12: The seven frieze groups [Tomruen, 2015].

Wallpaper Groups

Definition

A wallpaper group is the set of isometries and a fundamental region that fill an entire two-dimensional Euclidean plane. [Ganapathy, 2021]

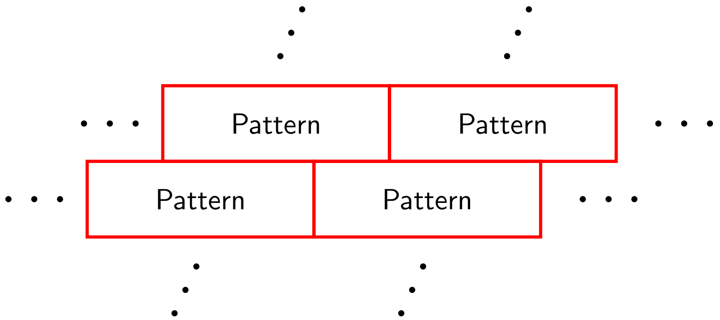


Figure 13: Wallpaper group example

Symmetry group properties, IUC

Wallpaper Group	Lattice	Rotation order	Reflections
p1	Oblique	0	none
p2	Oblique	2	none
pm	Rectangle	0	180°
pg	Rectangle	0	none
cm	Rhombus	0	180°
pmm	Rectangle	2	90°
pmg	Rectangle	2	180°
pgg	Rectangle	2	none
cmm	Rhombus	2	90°

Table 1: Symmetries of the wallpaper groups. [Joyce, 2024]

Symmetry group properties, IUC

Wallpaper Group	Lattice	Rotation order	Reflections
p4	Square	4	none
p4m	Square	$4^{\dagger}$	45°
p4g	Square	4^{*}	90°
p3	Hexagon	3	none
p31m	Hexagon	3^{*}	60°
p3m1	Hexagon	$3^{\dagger}$	30°
p6	Hexagon	6	none
p6m	Hexagon	6	30°

Table 2: Symmetries of the wallpaper groups: † means the rotation centres lie on the reflection axis. * means the group rotation centres are not on the reflection axis. [Joyce, 2024]

Wall Paper Groups

Why are they A group

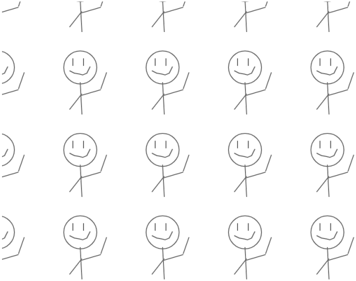
They meet the three group axioms

- Identity: no transformations.
- Inverses: the reversal or opposite of the transformation.
- Associativity: two transformations are still a transformation. [Zhoa, 2023]

Wallpaper groups

Definition

- P1 (primitive of order 1)[Bazett,]
- simplest of the 17 groups
- consists only of translations



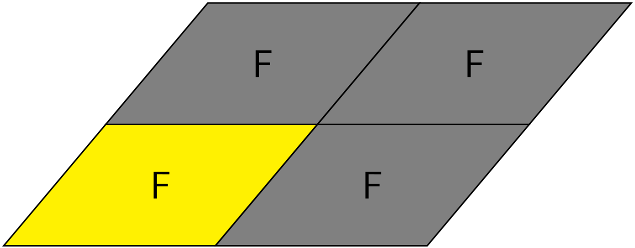
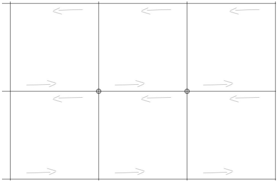
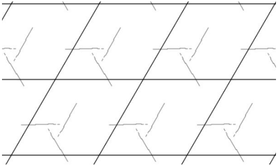


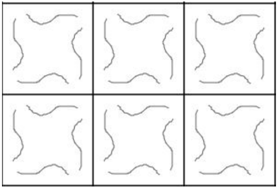
Figure 14: p1 diagram



(a) p2



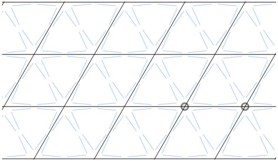
(b) p3



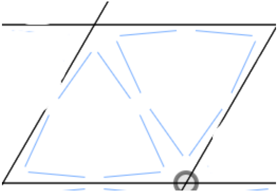
(c) p4

(d) p6

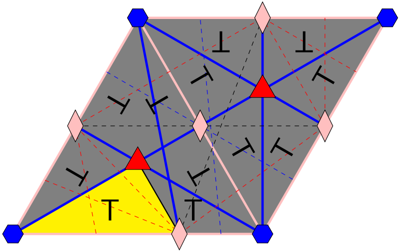
p6m



(a) Image of p6m



(b) p6m zoomed into one cell



(c) p6m Diagram

Figure 16: One of the most complex wallpaper groups

Conclusion

- The 17 Wallpaper groups are a mathematical repetitive 2D pattern.
- There are 4 symmetries within each group: translation, rotation, reflection and glide reflection.
- The 7 Frieze groups are 2D unidirectional patterns.
- There are 5 lattices: square, rectangular, rhombic, oblique, hexagonal.
- Implications or potential applications of the research: art, crystallography, engineering, etc.

Wallpaper generators to play around with if we have time

- Image wallpaper Generator
- Drawing Wallpaper generator

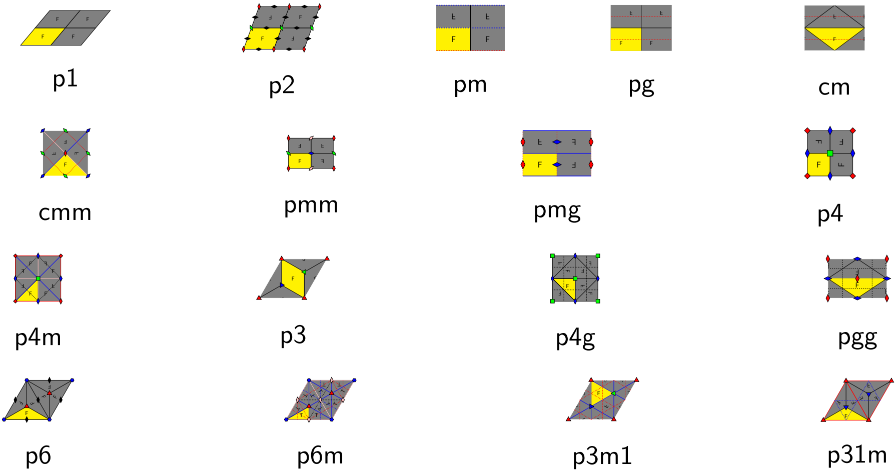


Figure 17: The 17 Wallpaper Groups

References

Bazett, D. T.
The beauty of symmetry: An introduction to the wallpaper group.
Accessed: 2024-05-24.

Ganapathy, T. (2021).
Wallpaper groups: Alhambra, escher and symmetries.
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Joyce, D. E. (2024).
Wallpaper groups:the 17 plane symmetry groups.
Accessed: 2024-05-28.

Tomruen (2015).
Frieze group example with feet under john conway's nicknames.
Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0).

Zhoa, A. (2023).
A brief survey on wallpaper groups.

p1 cell diagram

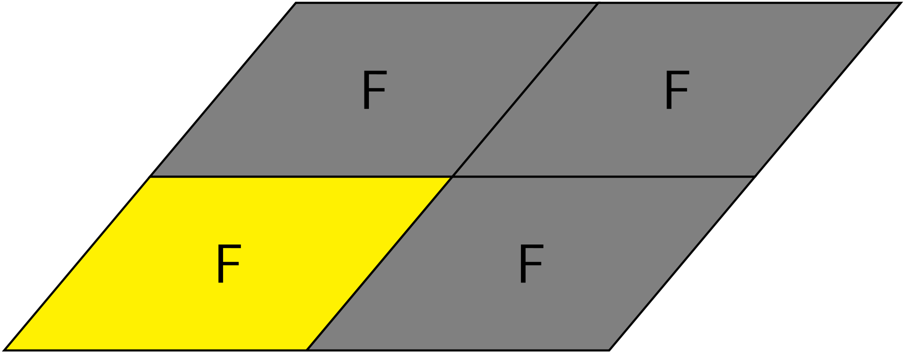

Figure 18: p1 cell diagram.

p2 cell diagram

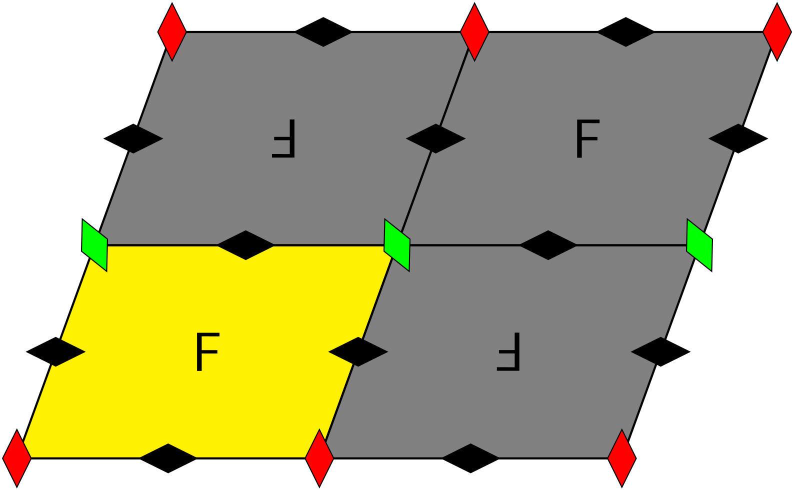

Figure 19: p2 cell diagram.

pm cell diagram

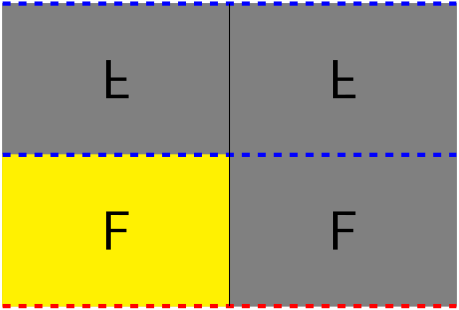

Figure 20: pm cell diagram.

pg cell diagram

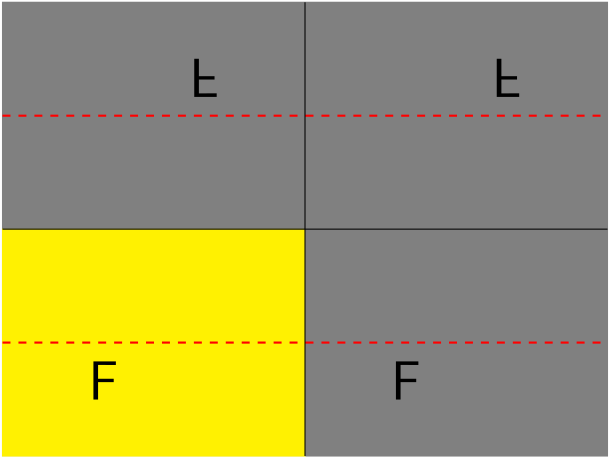

Figure 21: pg cell diagram.

cm cell diagram

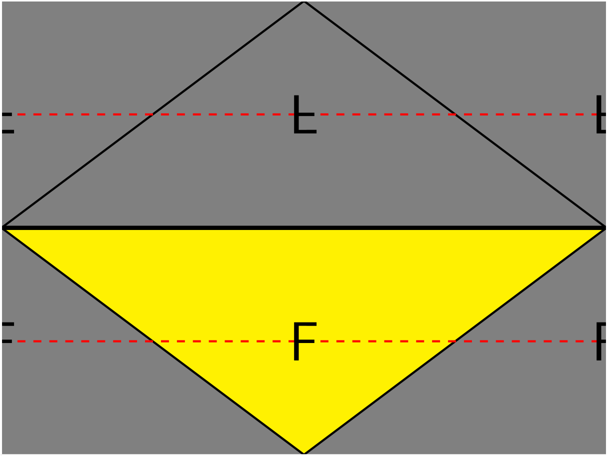

Figure 22: cm cell diagram.

cmm cell diagram

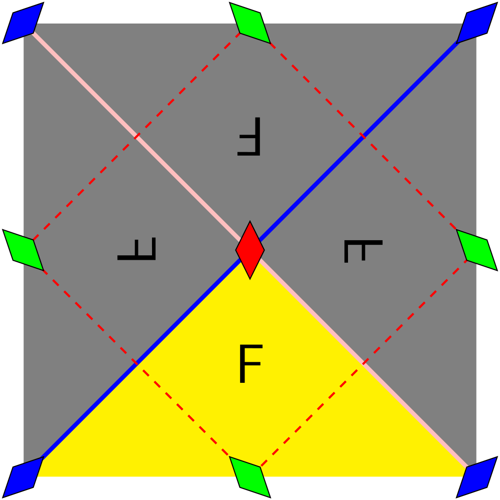

Figure 23: cmm cell diagram.

pmm cell diagram

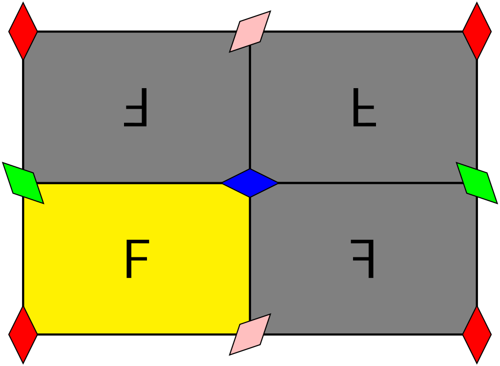

Figure 24: pmm cell diagram.

pmg cell diagram

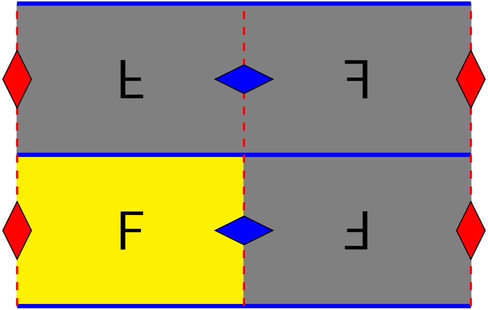

Figure 25: pmg cell diagram.

pgg cell diagram

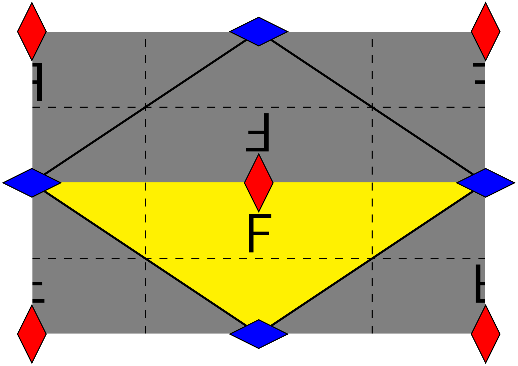

Figure 26: pgg cell diagram.

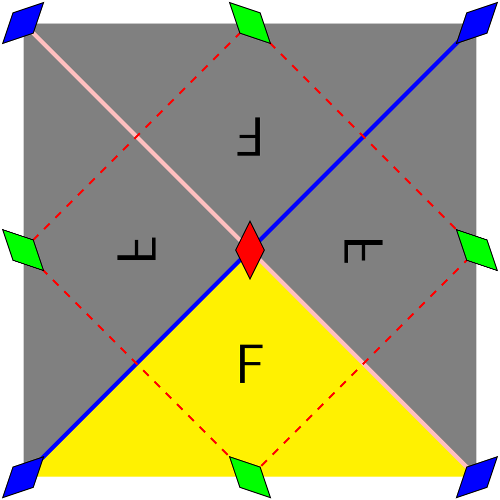

Figure 27: cmm cell diagram.

p4 cell diagram

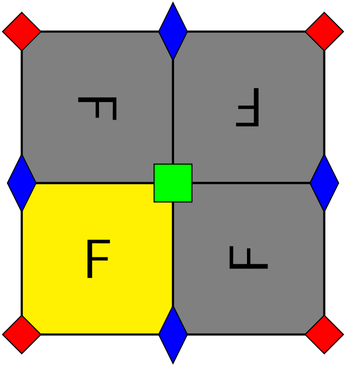

Figure 28: p4 cell diagram.

p4m cell diagram

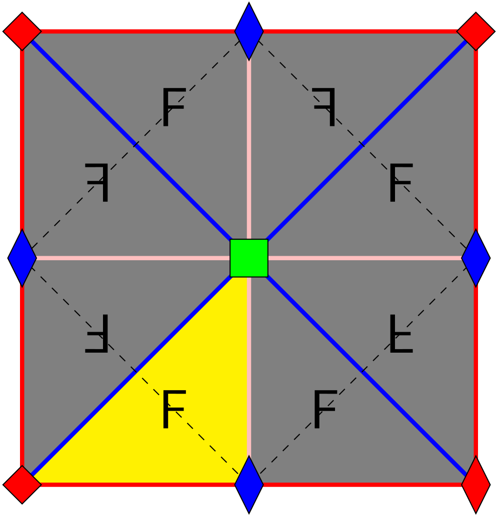

Figure 29: p4m cell diagram.

p3 cell diagram

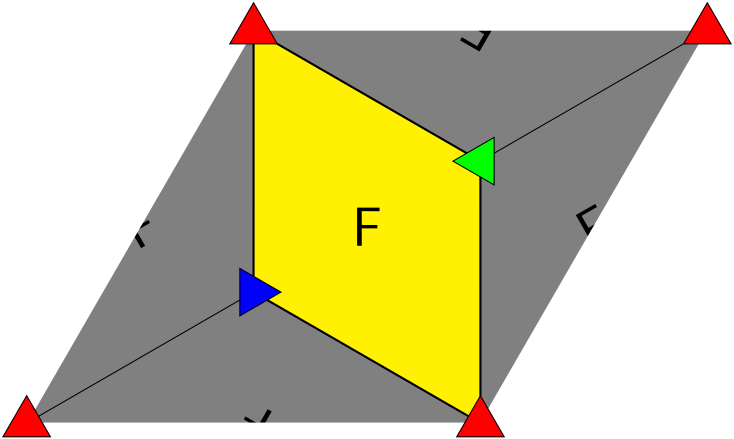

Figure 30: p3 cell diagram.

p31m cell diagram

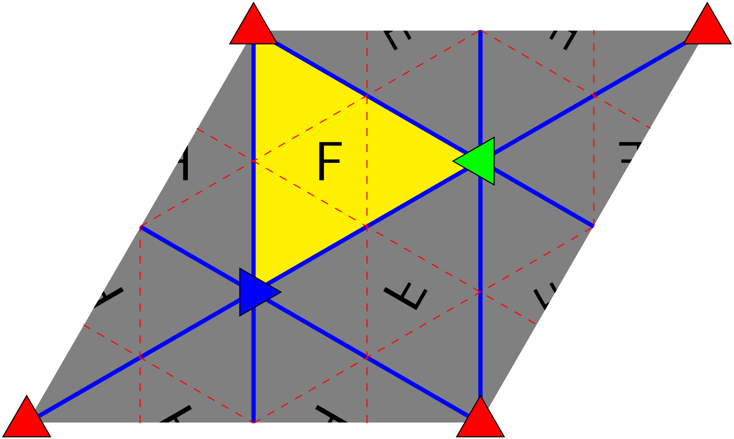

Figure 31: p31m cell diagram.

p3m1 cell diagram

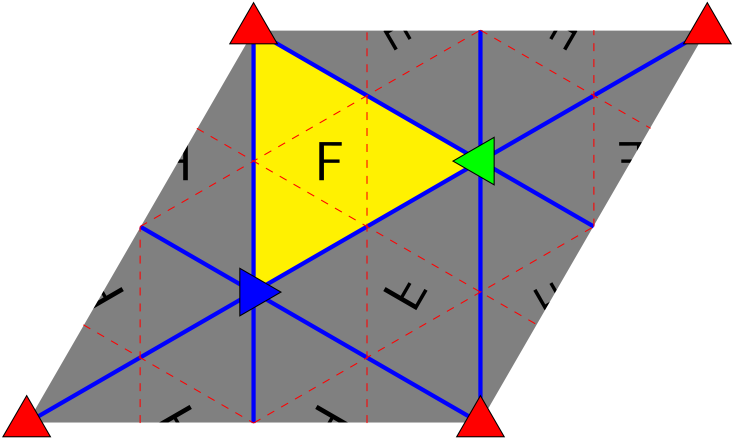

Figure 32: p3m1 cell diagram.

p6 cell diagram

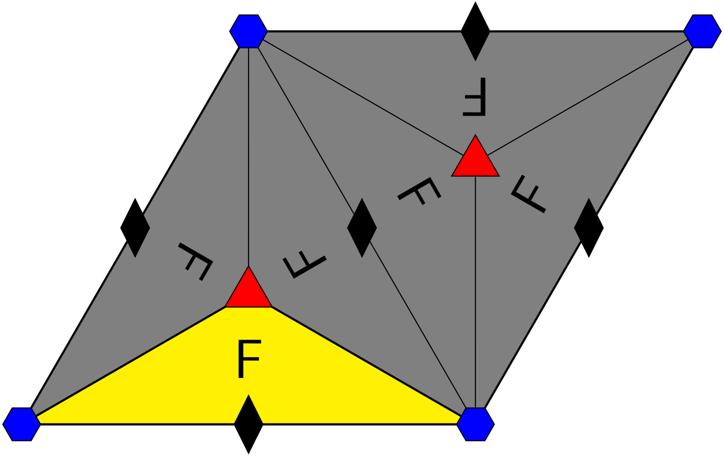

Figure 33: p6 cell diagram.

p6m cell diagram

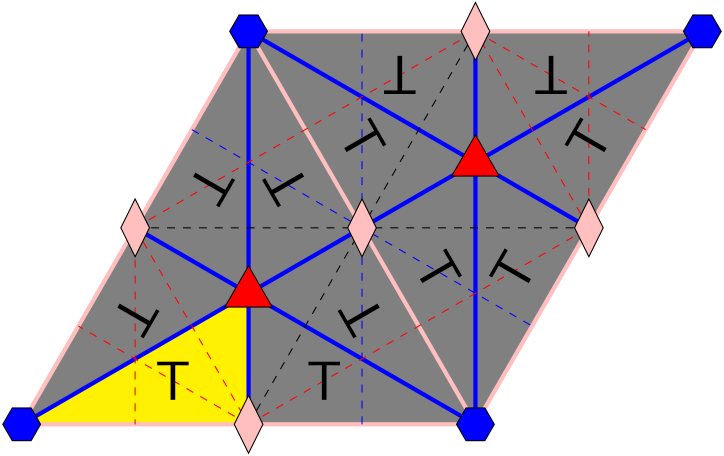

Figure 34: p6m cell diagram.