Tessellating the Plane: from periodic tilings to Hat and Spectre

Rory Yarr

May 12, 2025

Abstract

From periodic frieze groups, lattices and wallpaper groups to the aperiodic Hat and Spectre!

Tessellations

A *tessellation* (or tiling) of the plane is a cover of shapes (tiles) that fill the plane with no gaps or overlaps.

Group Operations of Tilings

Types of Symmetries

- Translations
- Reflections
- Glide Reflections
- Rotations.

Translations

Translations are repetitions of a pattern structure. defined mathematically as follows

a mapping $t_a : \mathbb{R}^2 \to \mathbb{R}^2$ where $x \to x + a$ where $x \in X$ [Zhoa(2023)]

$$\text{Pattern} \longrightarrow \text{Pattern}$$

Figure 1: Translations

Reflections and Glide Reflections

Reflections: A reflection over a line through the origin is defined as follows

$$Ref_l(v) = 2\frac{v \cdot l}{l \cdot l}\text{l - v}$$

where v and l are vectors going through the line of origin.

Glide : A glide is a reflection followed by a translation. [Zhoa(2023)]

(a) Reflections (b) Glide Reflection

Figure 2: Reflective Symmetries

Rotations

A rotation is a change of angle around a center point. Or consider it as the linear transformation $R_\theta : T \to T_\theta$ $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where $\theta \in \{0, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\}$. [Zhoa(2023)]

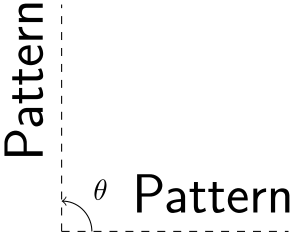


Figure 3: Rotations

Frieze Groups

(a) p1

(b) p11m

(c) p1m1

(d) p11g

(e) p2

(f) p2mm

(g) p2mg

Figure 4: Frieze groups by [Tomruen(2015)]

Lattices

A lattice is the group $(\mathbb{Z}[\vec{a}, \vec{b}], +)$.
i.e., a grid of points where any point $p = n\vec{a} + m\vec{b}$

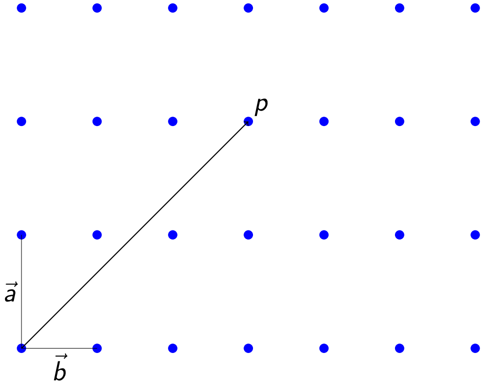
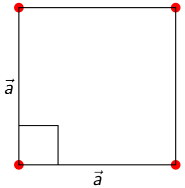


Figure 5: Lattice

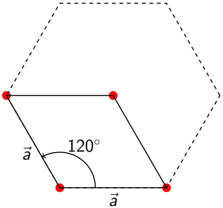
Bravais lattices

(a) Square: $||\vec{a}|| = ||\vec{b}|| < ||\vec{a} - \vec{b}|| = ||\vec{a} + \vec{b}||$
(b) Hexagon: $||\vec{a}|| = ||\vec{b}|| = ||\vec{a} - \vec{b}|| < ||\vec{a} + \vec{b}||$
(c) Rectangle: $||\vec{a}|| < ||\vec{b}|| < ||\vec{a} - \vec{b}|| = ||\vec{a} + \vec{b}||$
(d) Rhombic: $||\vec{a}|| < ||\vec{b}|| = ||\vec{a} - \vec{b}|| < ||\vec{a} + \vec{b}||$
(e) Oblique: $||\vec{a}|| < ||\vec{b}|| < ||\vec{a} - \vec{b}|| < ||\vec{a} + \vec{b}||$

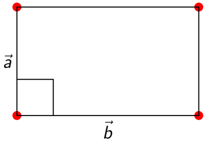
Bravais Lattices



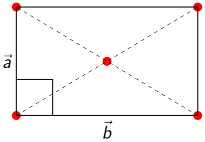
(a) Square Cell



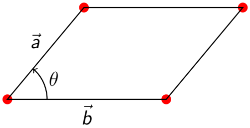
(b) Hexagon Cell



(c) Rectangle Cell



(d) Rhombic Cell



(e) Oblique Cell

Figure 6: All five two-dimensional Bravais lattice cells.

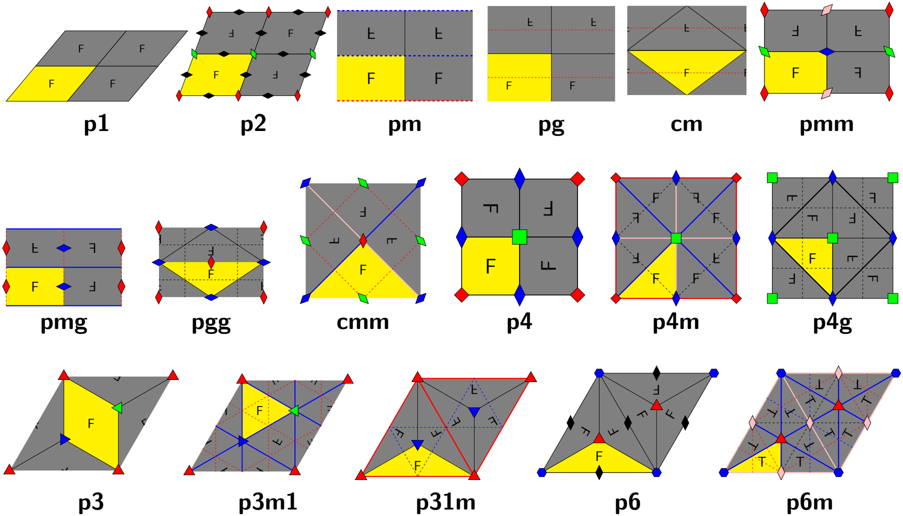


Figure 7: The 17 wallpaper groups, diagrams inspired by [Tomruen(2011)].

Classification of wallpaper groups

- p and c refer to primitive centred cells, respectively.
- The first number refers to the rotational order of the cell.
- m and g refer to mirror(reflections) and glide(reflections), respectively.

Oblique Cells

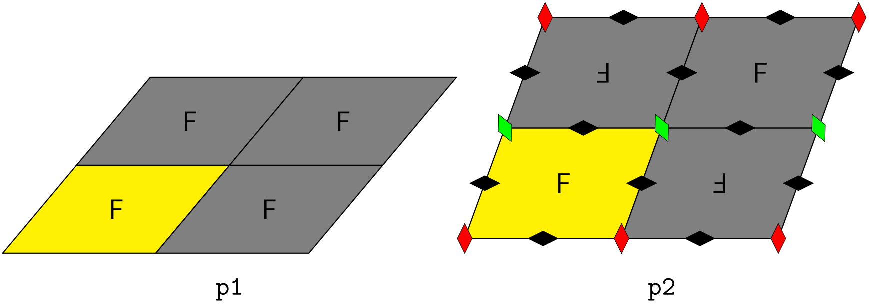


Figure 8: lattice diagrams for oblique cells

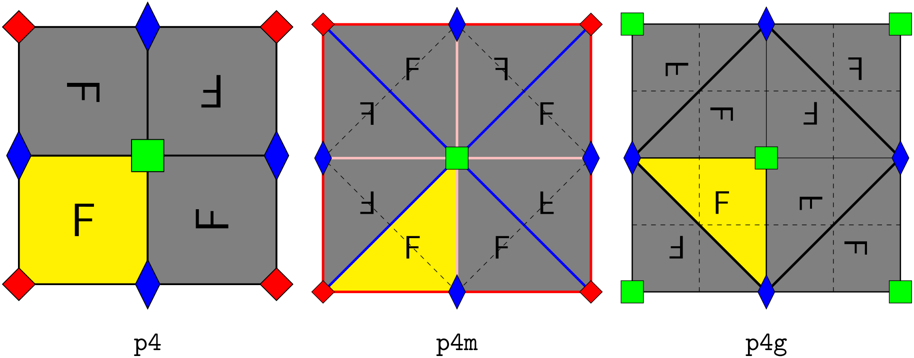


Figure 9: Lattice diagrams for square cells.

Rectangle Cells

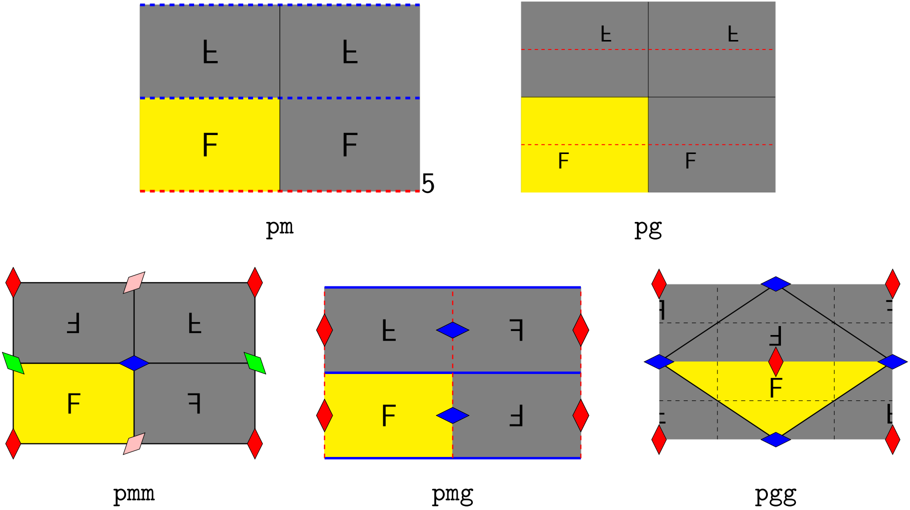
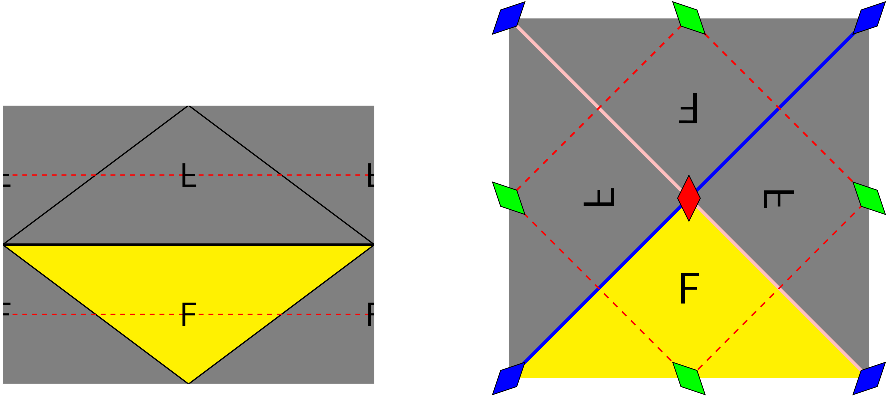


Figure 10: Lattice diagrams for rectangle cells.

Rhombic Cells



cm cmm

Figure 11: Lattice diagrams for rhombic cells.

Hexagon Cells

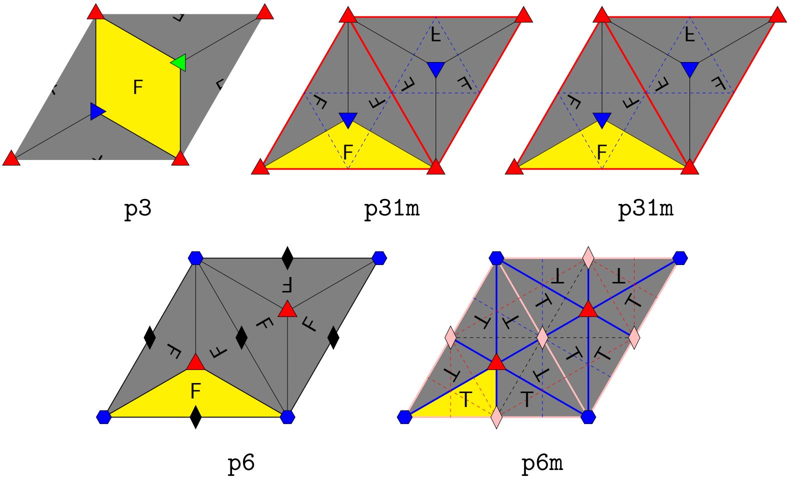


Figure 12: Lattice diagrams for hexagon cells.

Subgroups of Wallpaper Groups

G	H	G	H
p1	trivial	p4	$\mathbb{Z}_4$
p2	$\mathbb{Z}_2$	p4m	D_4
pm	$\mathbb{Z}_2$	p4g	D_4
pg	$\mathbb{Z}_2$	p3	$\mathbb{Z}_3$
pmm	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p3m1	D_3
pmg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p31m	D_3
pgg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p6	$\mathbb{Z}_6$
cm	$\mathbb{Z}_2$	p6m	D_6
cmm	$\mathbb{Z}_2 \times \mathbb{Z}_2$		

Table 1: Wallpaper groups G and their corresponding symmetry subgroups H.

[Sasse(2020)]

Are all tilings periodic?

- In 1902 David Hilbert posed 23 open problems for mathematicians of his time to solve.
- His 18th problem assumed that it was not possible to have a non-periodic. [Hilbert(1902)]
- Hilbert was wrong as I will show you now!

Wang Tiles

- In the 1962 Hao Wang created a way to construct sets of tiles that only tiled the plane aperiodically.
- In 1966 Robert Berger proved that a set of 20426 Wang tiles was aperiodic. [Berger(1966)]
- Which was reduced down to the set of 11 Wang tiles below by[Jeandel and Rao(2021)].

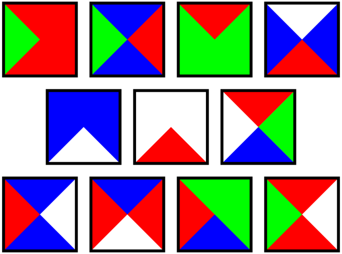


Figure 13: Wang tiles[Taxel(2016)]

Robinson Tiles

In 1971 Raphael Robinson from Berkeley show the following 6 tiles only aperiodically tile the plane.[Robinson(1971)]

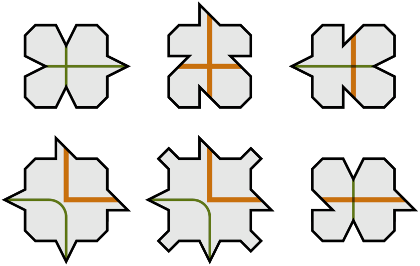


Figure 14: Robinson tiles[Archibald(2005)]

Penrose Tilling P1

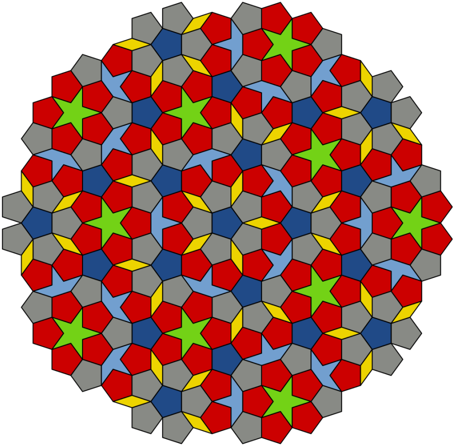


Figure 15: P1 Penrose tilling [Inductiveload(2009a)]

Penrose Tilling P2

Figure 16: Quit of P2 penrose tilling by Matt Zucker.[Zucker(2022)]

Penrose Tilling P3

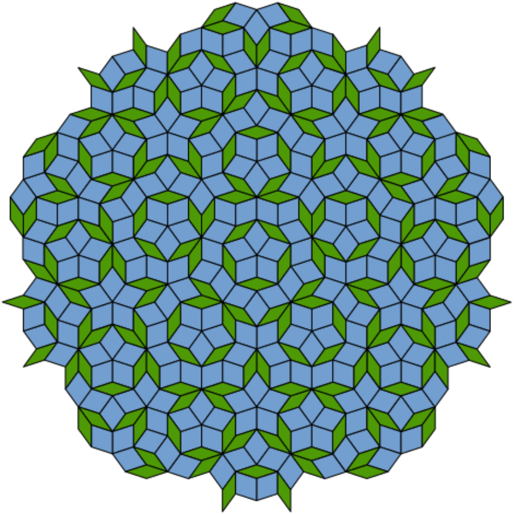


Figure 17: Rhombic Penrose tilling (p1)[Inductiveload(2009b)]

The Einstein(”One Tile”) Problem

- Is it possible to tile a plane using a single tile that only tiles aperiodically.

The Hat

- In 2022 the hobbyist David Smith discovered the "Hat" monotile.
- He then reached out to Craig Kaplan and other mathematicians to prove that it tiles the plane only aperiodically[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss].

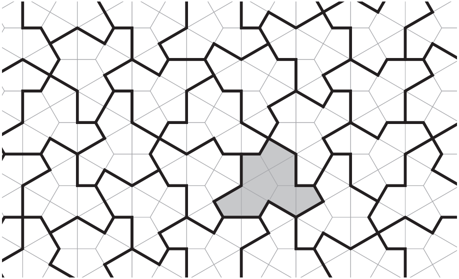


Figure 18: Hat tilling.
[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

Metatiles

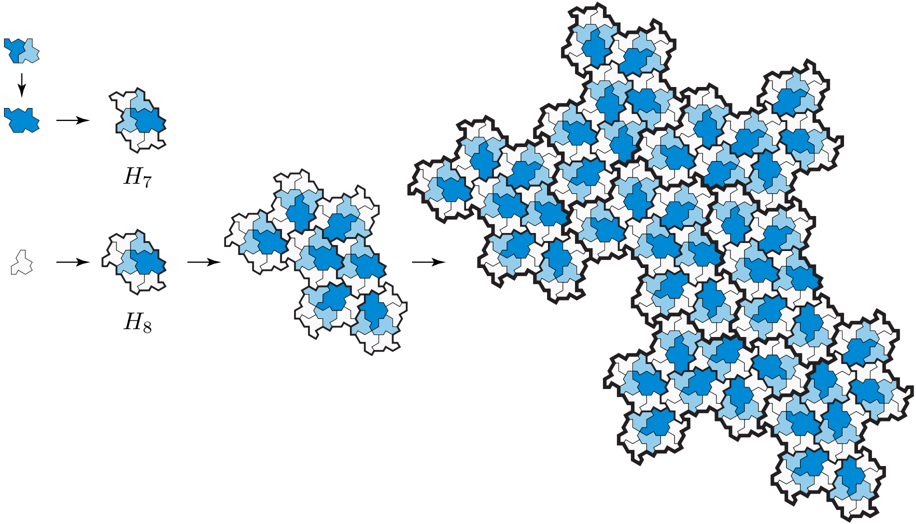


Figure 19: A metatile of the hat.
[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

A Family of Polykites

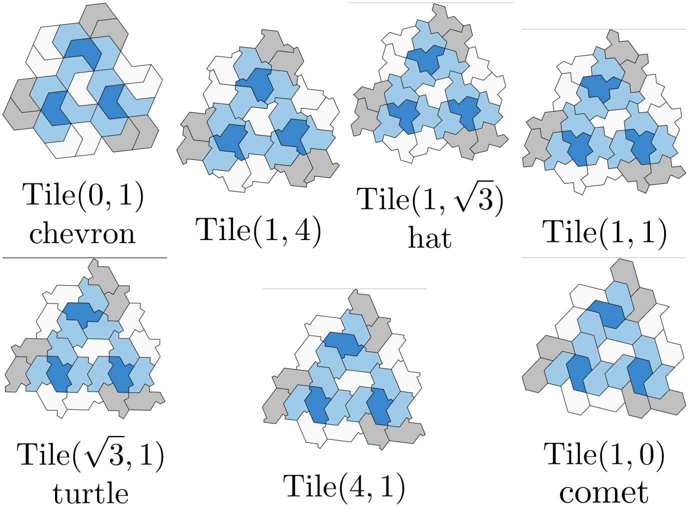


Figure 20: Family of polykites.
[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

The Spectre

Shortly after their groundbreaking paper they were able to add a chirality to the tile and aperiodically tile the plane without mirrors. [Smith et al.(2024b)Smith, Myers, Kaplan, and Goodman-Strauss]

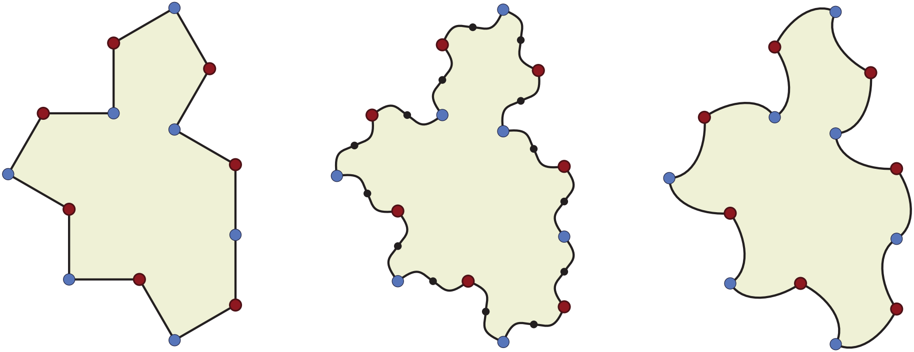


Figure 21: From the turtle to the spectre. [Smith et al.(2024b)Smith, Myers, Kaplan, and Goodman-Strauss]

Family of polykites

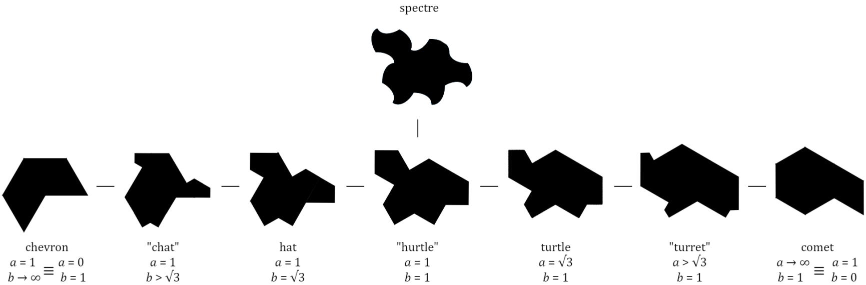


Figure 22: Family of poly-kites and the spectre. [Steckles(2023)]

GitHub

Figure 23: QR code for these slides on my GitHub

Other Resources I

- Numberphile interview with Graig Kaplan.
- Poster from a University of Melbourne student(Jamie Vu).
- App David Smith used to discover the hat monotile.
- Matt Zuckers quilt development webpage.
- Roger Penroses patent for his aperiodic tiling Hat webpage. Spectre webpage.

Bibliography I

Archibald.
Robinson tiles.
Own work. Wikimedia Commons., May 2005.
URL https://commons.wikimedia.org/wiki/File:Robinson_tiles.svg.
Released into the public domain; accessed 2025-05-12.

R. Berger.
Undecidability of the Domino Problem.
Memoirs Series. American Mathematical Society, 1966.
ISBN 9780821812662.
URL https://books.google.com.au/books?id=GxDxwAEACAAJ.

David Hilbert.
Mathematical Problems.
Bulletin of the American Mathematical Society, 8(10):437–479, 1902.
doi: 10.1090/S0002-9904-1902-00923-3.
URL https://doi.org/10.1090/S0002-9904-1902-00923-3.

Bibliography II

Inductiveload.
Penrose Tiling (P1).
Own work. Wikimedia Commons. Public domain (PD-self)., February 2009a.
URL https://commons.wikimedia.org/wiki/File:Penrose_Tiling_(P1).svg.

Accessed: 2025-05-12.

Inductiveload.
Penrose tiling (rhombi), 2009b.
URL https://en.wikipedia.org/wiki/File:Penrose_Tiling_(Rhombi).svg.
Public domain image of a Penrose tiling (P3) using thick and thin rhombi, exhibiting fivefold symmetry.

Emmanuel Jeandel and Michaël Rao.
An aperiodic set of 11 Wang tiles.
Advances in Combinatorics, jan 6 2021.
doi: 10.19086/aic.18614.

Bibliography III

M. Robinson, R.
Undecidability and nonperiodicity for tilings of the plane.
Inventiones Mathematicae, 12:177–209, 1971.
doi: 10.1007/BF01418780.

Vivek Sasse.
Classification of the 17 Wallpaper Groups.
Universiy of Chicago, 2020.
URL https://math.uchicago.edu/~may/REU2020/REUPapers/Sasse.pdf.

David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss.
An aperiodic monotile.
Combinatorial Theory, 4(1), July 2024a.
ISSN 2766-1334.
doi: 10.5070/c64163843.
URL http://dx.doi.org/10.5070/C64163843.

Bibliography IV

David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss.
A chiral aperiodic monotile.
Combinatorial Theory, 4(2), September 2024b.
ISSN 2766-1334.
doi: 10.5070/c64264241.
URL http://dx.doi.org/10.5070/C64264241.

Katie Steckles.
Now that's what I call an aperiodic monotile!
The Aperiodical (News), May 2023.
URL https://aperiodical.com/2023/05/now-thats-what-i-call-an-aperiodic-monotile/.
Accessed: 2025-05-12.

Bibliography V

Parcly Taxel.
Wang 11 tiles.
Own work. Wikimedia Commons., June 2016.
URL https://commons.wikimedia.org/wiki/File:Wang_11_tiles.svg.
Released under the Free Art License; accessed 2025-05-12.

Tomruen.
Wallpaper group diagram.
Own work. Wikimedia Commons, June 2011.
URL
https://commons.wikimedia.org/wiki/File:Wallpaper_group_diagram.svg.
Derived from unrotated version; accessed 2025-05-12.

Tomruen.
Frieze group example with feet under john conway's nicknames, 2015.
URL https://commons.wikimedia.org/wiki/File:Frieze_step.png.
Licensed under Creative Commons Attribution 4.0 International (CC BY 4.0).

Bibliography VI

Angela Zhoa.
A brief survey on wallpaper groups.
2023.
URL https://math.mit.edu/research/highschool/primes/circle/documents/2023/Angela_Zhao.pdf.

Matt Zucker.
Penrose tiling quilt.
Needlessly Complex blog, November 2022.
URL https://mzucker.github.io/2022/11/13/penrose-tiling-quilt.html.
Accessed: 2025-05-12.

P1 Wallpaper group

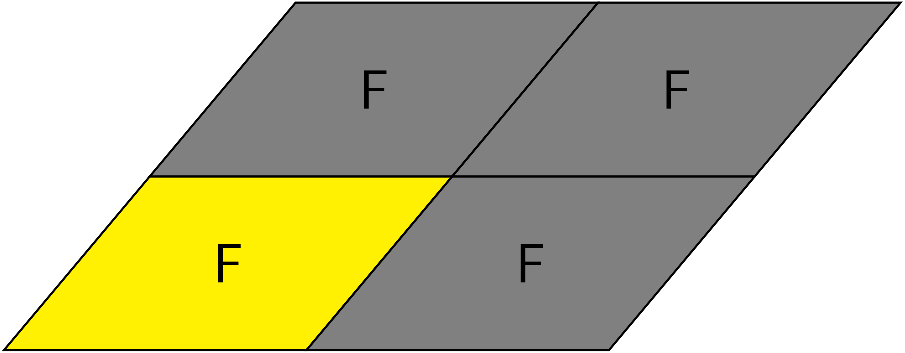


Figure 24: p1

P2 Wallpaper group

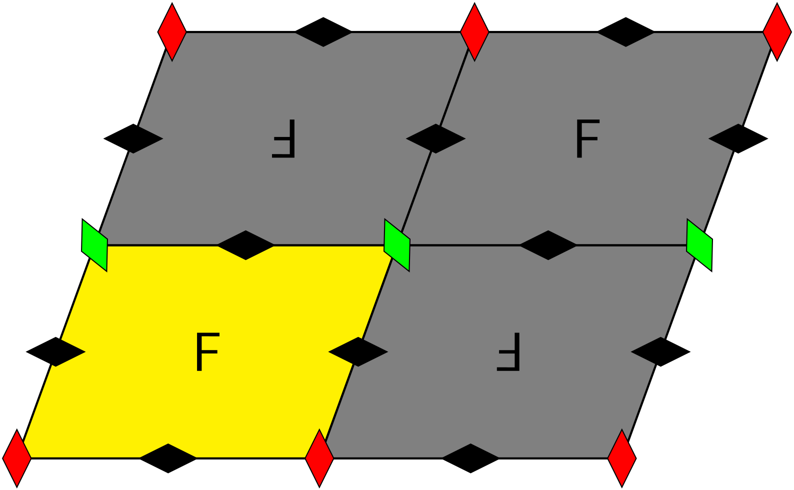


Figure 25: p2

P4 Wallpaper group

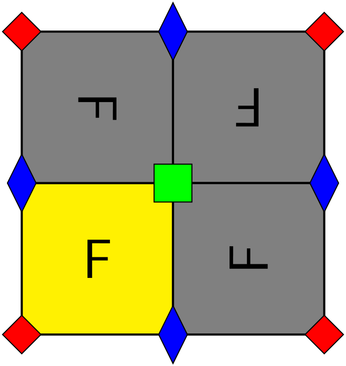


Figure 26: p4

P4M Wallpaper group

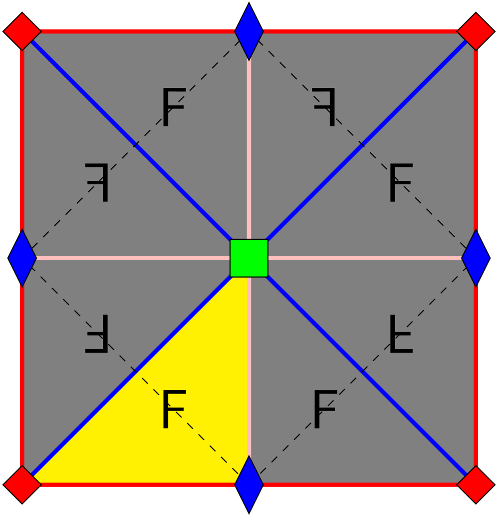


Figure 27: p4m

P4G Wallpaper group

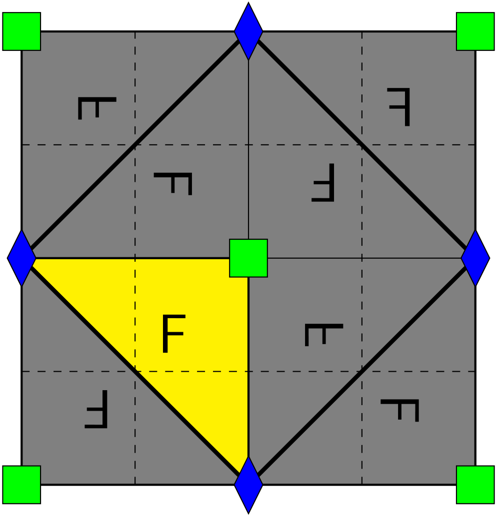


Figure 28: p4g

CM Wallpaper group

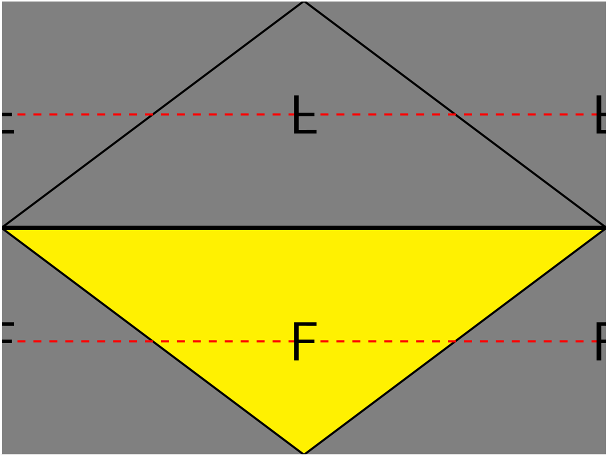


Figure 29: cm

CMM Wallpaper group

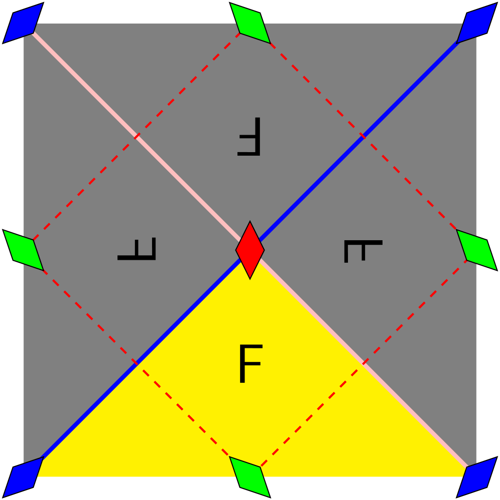


Figure 30: cmm wallpaper group

PG Wallpaper group

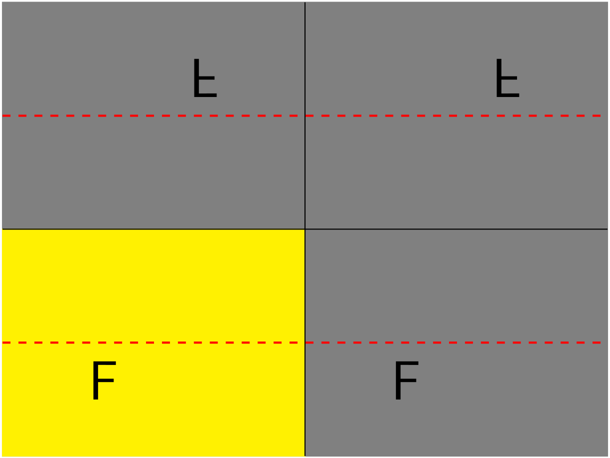


Figure 31: pg wallpaper group

PGG Wallpaper group

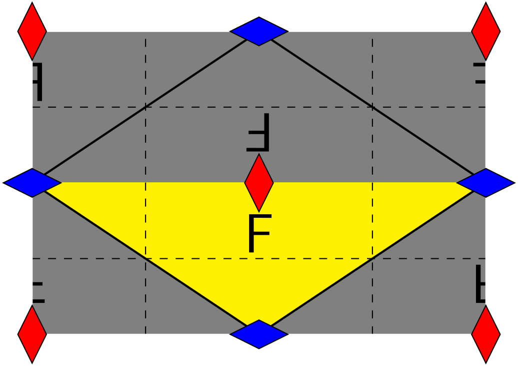


Figure 32: pgg wallpaper group

PMG Wallpaper group

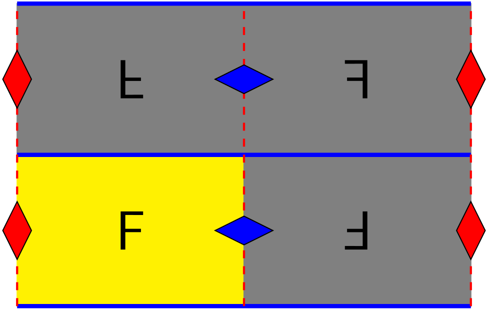


Figure 33: pmg wallpaper group

PM Wallpaper group

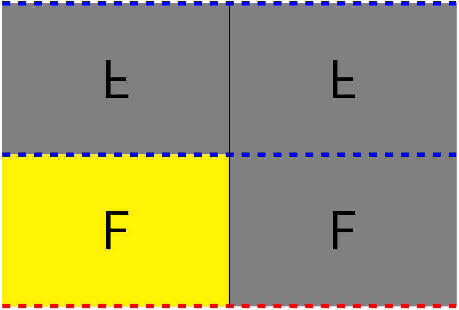


Figure 34: pm wallpaper group

PMM Wallpaper group

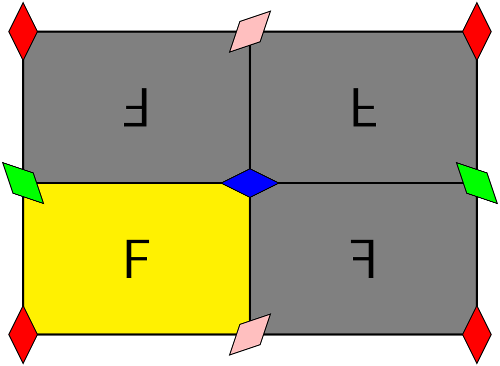


Figure 35: pmm wallpaper group

P3 Wallpaper group

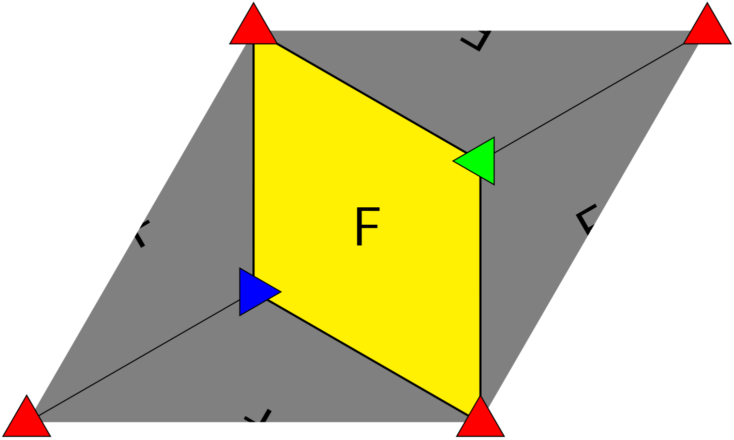


Figure 36: p3

P3M1 Wallpaper group

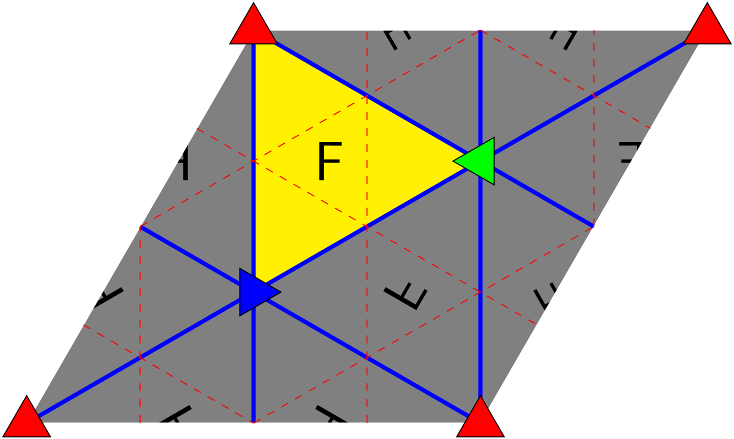


Figure 37: p3m1

P31M Wallpaper group

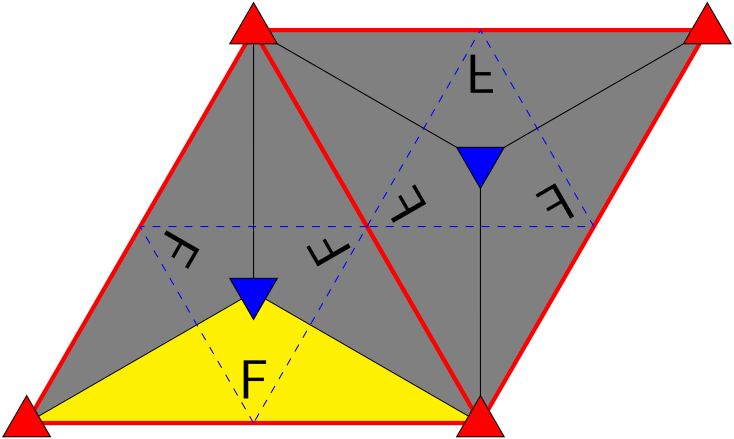


Figure 38: p31m

P6 Wallpaper group

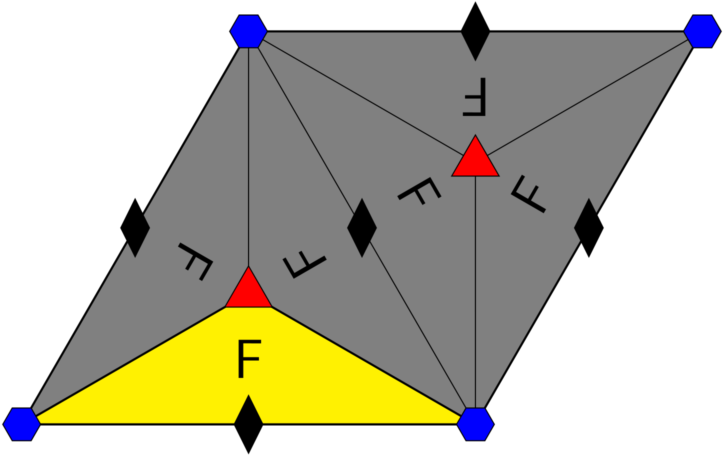


Figure 39: p6

P6M Wallpaper group

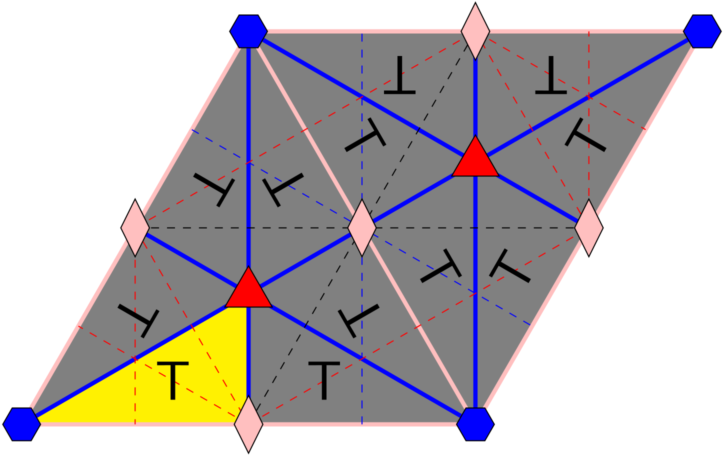


Figure 40: p6m