Tessellating the Plane: from periodic tilings to Hat and Spectre

Rory Yarr

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Abstract

From periodic frieze groups, lattices and wallpaper groups to the aperiodic Hat and Spectre!

Tessellations

A *tessellation* (or tiling) of the plane is a cover of shapes (tiles) that fill the plane with no gaps or overlaps.

Group Operations of Tilings

Types of Symmetries We can think of any symmetry group as a pair $(\mathbf{v}, M)$ for $\mathbf{v} \in \mathbb{R}^2$ and M a linear transformation. Allowing a symmetry group operations to be defined as:

$$(\mathbf{w}, N)(\mathbf{v}, M) = (\mathbf{w} + N\mathbf{v}, NM)$$

- Translations
- Reflections
- Glide Reflections
- Rotations.

Translations

Translations are repetitions of a pattern structure.

$$T_a(\vec{x}) = \vec{x} + a$$

[Zhoa(2023)]

$$\text{Pattern} \longrightarrow \text{Pattern}$$

Figure 1: Translations

Reflections and Glide Reflections

Reflections on some line $\vec{l}$ can be defined as follows

$$R_l(\vec{x}) = \frac{l \cdot \vec{x}}{l \cdot l} - \vec{x}$$

Glide : A glide is a reflection followed by a translation.

$$G_{l,a}(\vec{x}) = R_l(\vec{x}) + a$$

[Zhoa(2023)]

(a) Reflections (b) Glide Reflection

Figure 2: Reflective Symmetries

A rotation is a change of angle around a centre point.

$$R_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \vec{x}$$

where $\theta \in [0, 2\pi)$ [Zhoa(2023)]

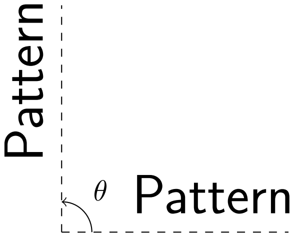


Figure 3: Rotations

Frieze Groups

(a) p1

(b) p11m

(c) p1m1

(d) p11g

(e) p2

(f) p2mm

(g) p2mg

Figure 4: Frieze groups by [Tomruen(2015)]

Lattices

A lattice is the group $(\mathbb{Z}[\vec{a}, \vec{b}], +)$.
i.e., a grid of points where any point $p = n\vec{a} + m\vec{b}$

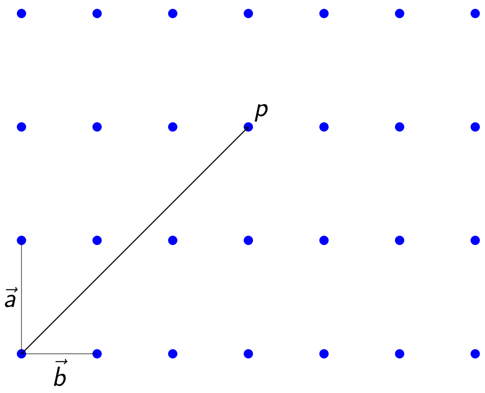
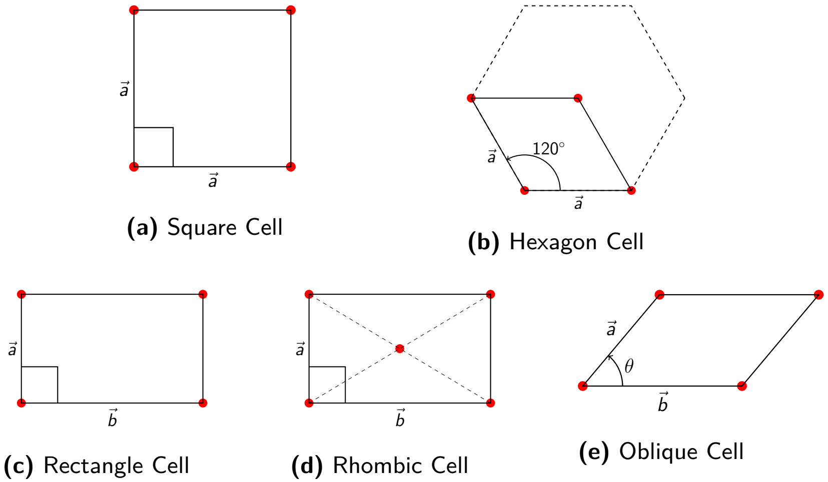


Figure 5: Lattice

Bravais lattices

(a) Square: $||\vec{a}|| = ||\vec{b}|| < ||\vec{a} - \vec{b}|| = ||\vec{a} + \vec{b}||$
(b) Hexagon: $||\vec{a}|| = ||\vec{b}|| = ||\vec{a} - \vec{b}|| < ||\vec{a} + \vec{b}||$
(c) Rectangle: $||\vec{a}|| < ||\vec{b}|| < ||\vec{a} - \vec{b}|| = ||\vec{a} + \vec{b}||$
(d) Rhombic: $||\vec{a}|| < ||\vec{b}|| = ||\vec{a} - \vec{b}|| < ||\vec{a} + \vec{b}||$
(e) Oblique: $||\vec{a}|| < ||\vec{b}|| < ||\vec{a} - \vec{b}|| < ||\vec{a} + \vec{b}||$

Bravais Lattices



(a) Square Cell

(b) Hexagon Cell

(c) Rectangle Cell

(d) Rhombic Cell

(e) Oblique Cell

Figure 6: All five two-dimensional Bravais lattice cells.

Wallpaper groups

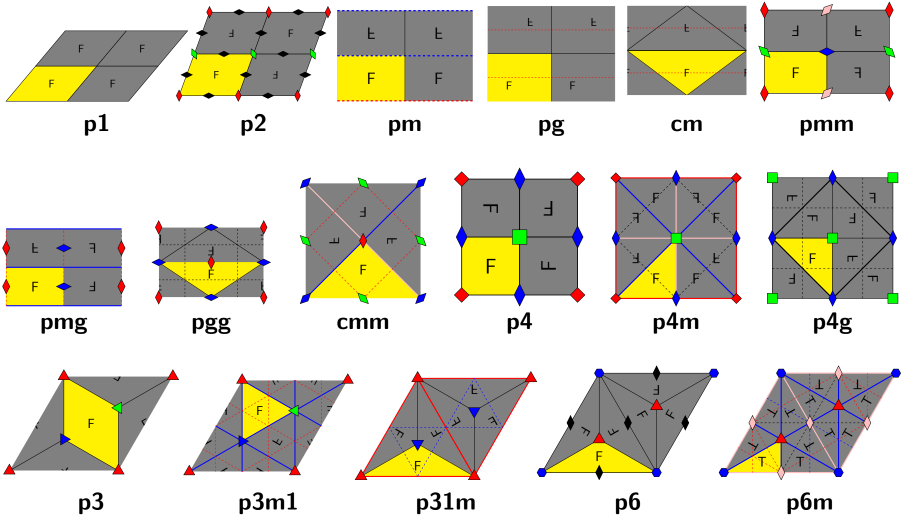
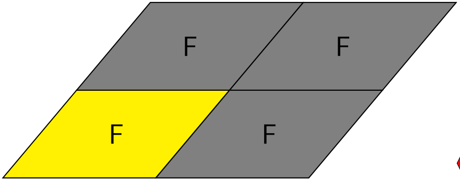


Figure 7: The 17 wallpaper groups, diagrams inspired by [Tomruen(2011)].

Classification of wallpaper groups

- p and c refer to primitive centred cells, respectively.
- The first number refers to the rotational order of the cell.
- m and g refer to mirror(reflections) and glide(reflections), respectively.

Oblique Cells



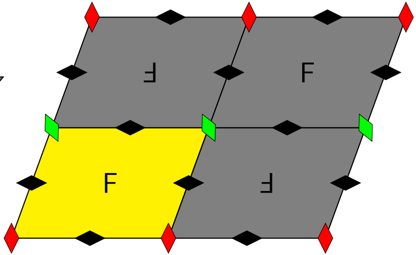


Figure 8: lattice diagrams for oblique cells

Square Cells

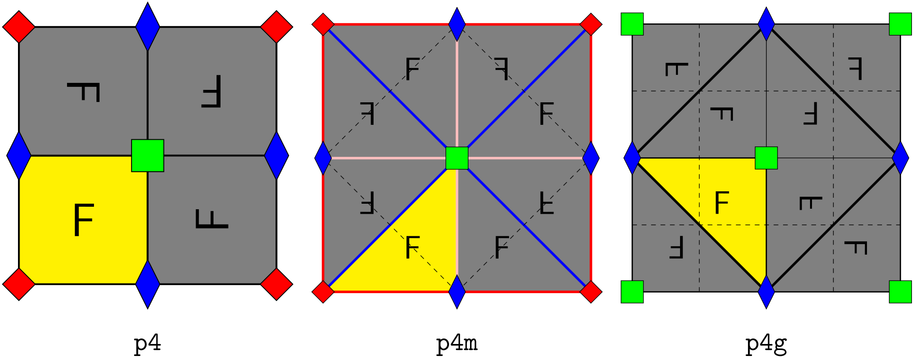


Figure 9: Lattice diagrams for square cells.

Rectangle Cells

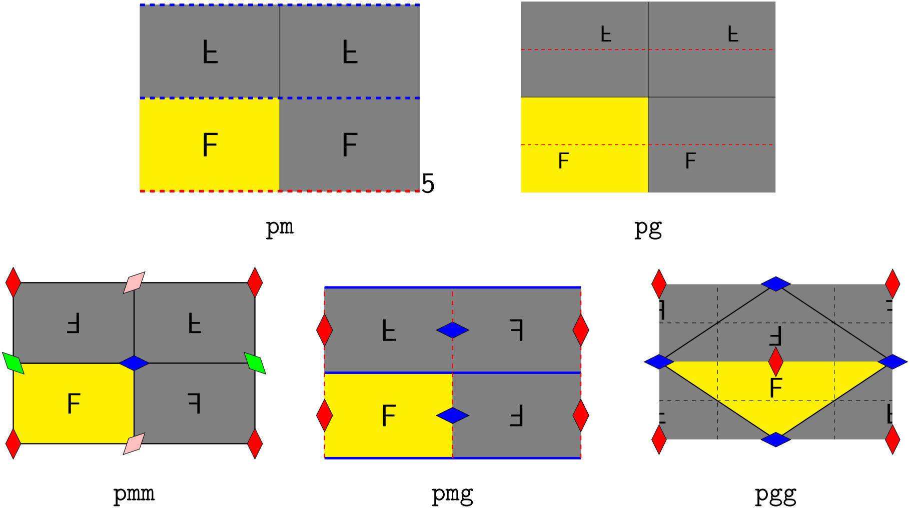


Figure 10: Lattice diagrams for rectangle cells.

Rhombic Cells

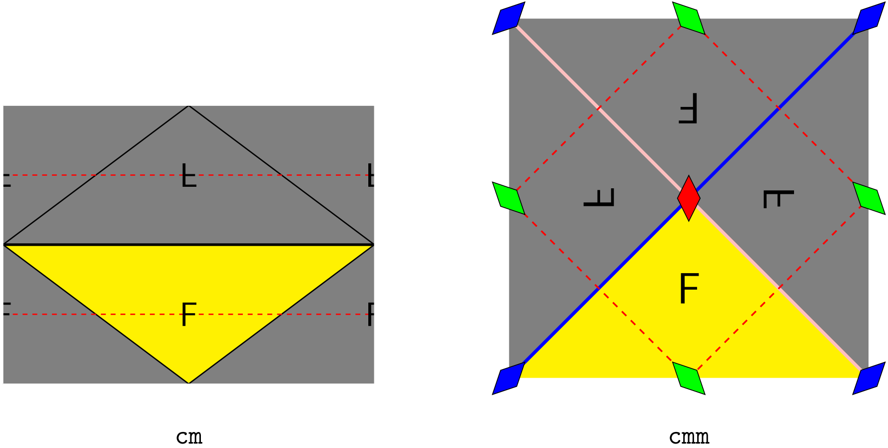


Figure 11: Lattice diagrams for rhombic cells.

Hexagon Cells

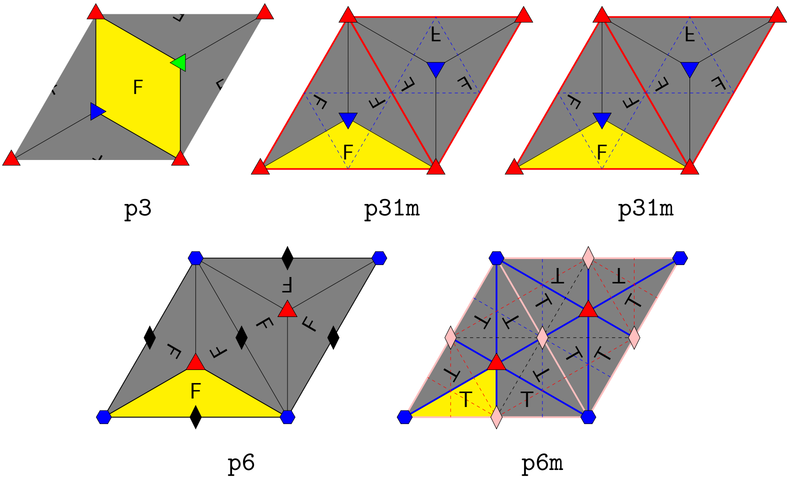


Figure 12: Lattice diagrams for hexagon cells.

Subgroups of Wallpaper Groups

G	**H**	**G**	**H**
p1	trivial	p4	$\mathbb{Z}_4$
p2	$\mathbb{Z}_2$	p4m	D_4
pm	$\mathbb{Z}_2$	p4g	D_4
pg	$\mathbb{Z}_2$	p3	$\mathbb{Z}_3$
pmm	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p3m1	D_3
pmg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p31m	D_3
pgg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	p6	$\mathbb{Z}_6$
cm	$\mathbb{Z}_2$	p6m	D_6
cmm	$\mathbb{Z}_2 \times \mathbb{Z}_2$		

Table 1: Wallpaper groups G and their corresponding symmetry subgroups H.

[Sasse(2020)]

Are all tilings periodic?

- In 1902 David Hilbert posed 23 open problems for mathematicians of his time to solve.
- His 18th problem assumed that it was not possible to have a non-periodic. [Hilbert(1902)]
- Hilbert was wrong as I will show you now!

Wang Tiles

- In the 1962 Hao Wang created a way to construct sets of tiles that only tiled the plane aperiodically.
- In 1966 Robert Berger proved that a set of 20426 Wang tiles was aperiodic. [Berger(1966)]
- Which was reduced down to the set of 11 Wang tiles below by[Jeandel and Rao(2021)].

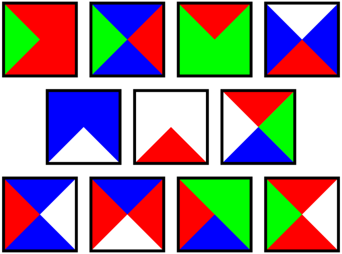


Figure 13: Wang tiles[Taxel(2016)]

Robinson Tiles

In 1971 Raphael Robinson from Berkeley show the following 6 tiles only aperiodically tile the plane.[Robinson(1971)]

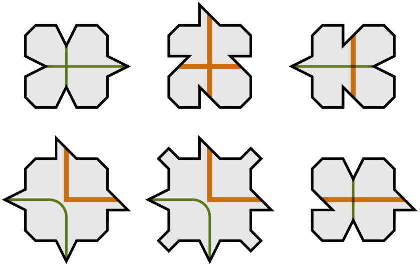


Figure 14: Robinson tiles[Archibald(2005)]

Penrose Tilling P1

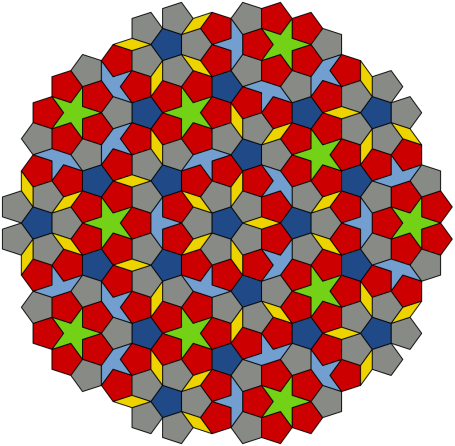


Figure 15: P1 Penrose tilling [Inductiveload(2009a)]

Penrose Tilling P2

Figure 16: Quit of P2 penrose tilling by Matt Zucker.[Zucker(2022)]

Penrose Tilling P3

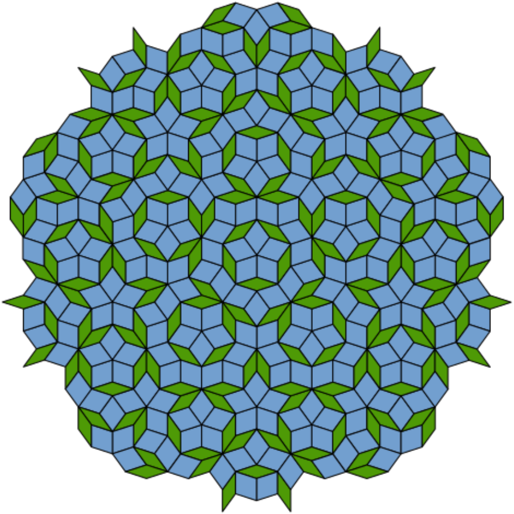


Figure 17: Rhombic Penrose tilling (p1)[Inductiveload(2009b)]

The Einstein("One Tile") Problem

- Is it possible to tile a plane using a single tile that only tiles aperiodically.

The Hat

- In 2022 the hobbyist David Smith discovered the "Hat" monotile.
- He then reached out to Craig Kaplan and other mathematicians to prove that it tiles the plane only aperiodically[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss].

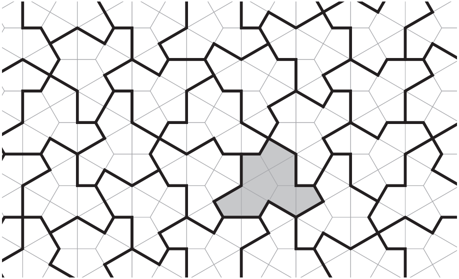


Figure 18: Hat tilling.
[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

Proof Sketch of the Hat Monotile

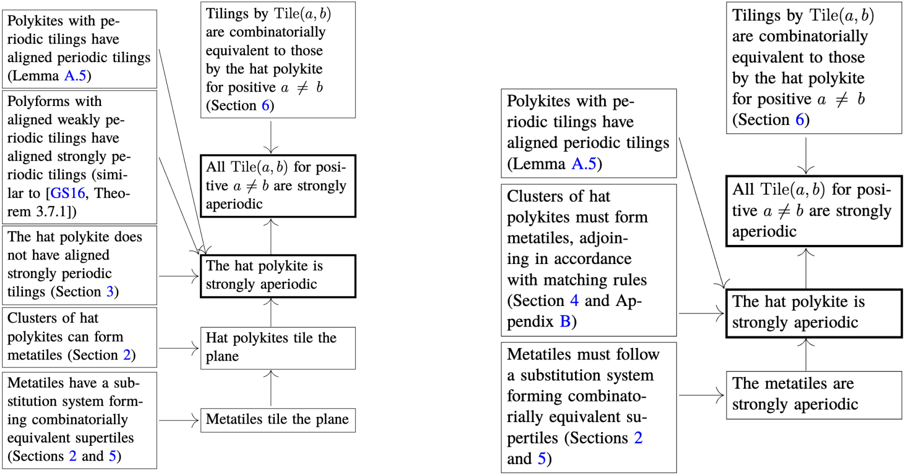


Figure 19: Proof sketches of the aperiodicity Hat Monotile.[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

Super Tiles and Metatiles

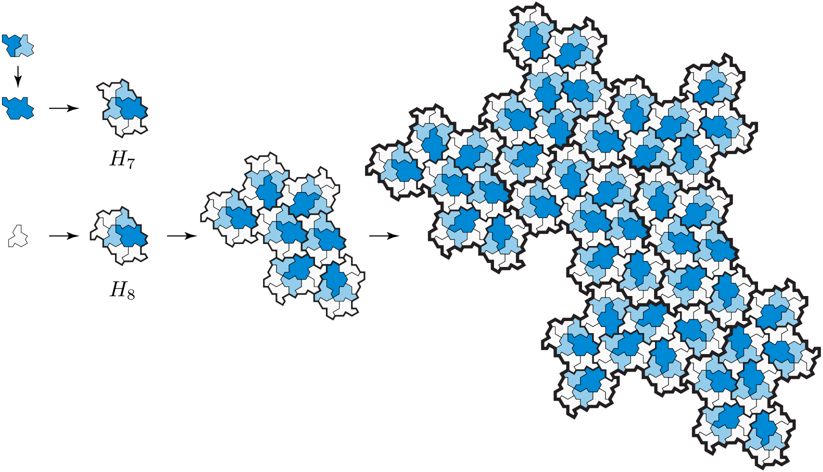


Figure 20: A metatile of the hat.
[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

Super Tiles

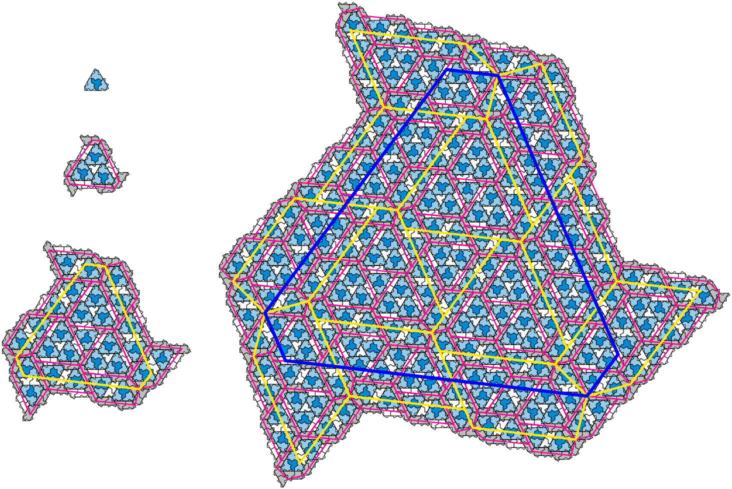


Figure 21: Hexagonal super clusters[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

A Family of Polykites

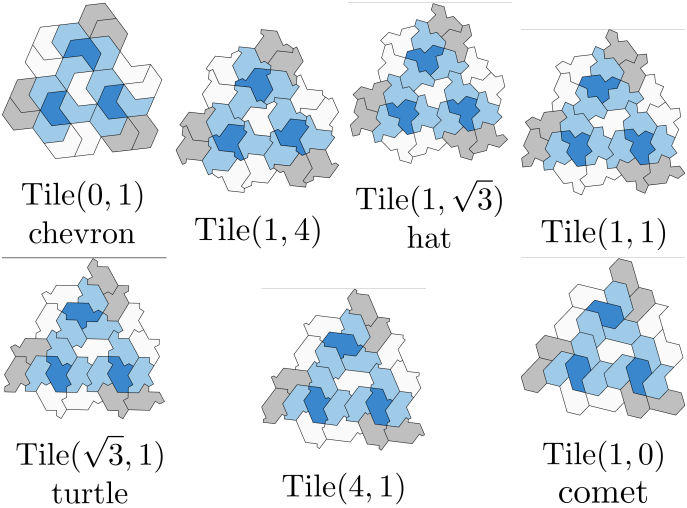


Figure 22: Family of polykites.
[Smith et al.(2024a)Smith, Myers, Kaplan, and Goodman-Strauss]

The Spectre

Shortly after their groundbreaking paper they were able to add a chirality to the tile and aperiodically tile the plane without mirrors. [Smith et al.(2024b)Smith, Myers, Kaplan, and Goodman-Strauss]

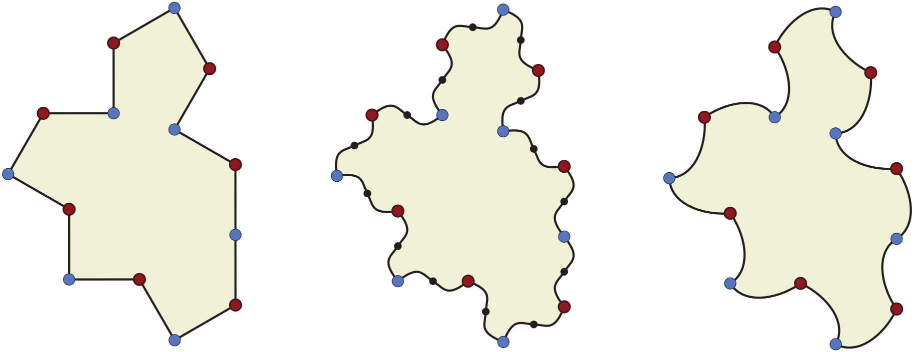


Figure 23: From the hurtle to the spectre. [Smith et al.(2024b)Smith, Myers, Kaplan, and Goodman-Strauss]

From the Polykites to the Spectre.

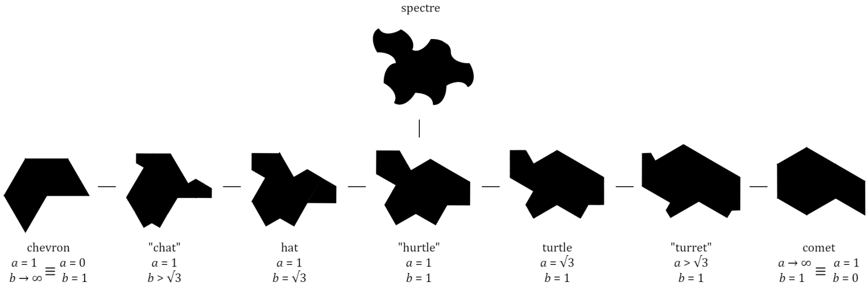


Figure 24: Family of poly-kites and the spectre. [Steckles(2023)]

GitHub

Figure 25: QR code for these slides on my GitHub

Other Resources I

- Numberphile interview with Graig Kaplan.
- Poster from a University of Melbourne student(Jamie Vu).
- App David Smith used to discover the hat monotile.
- Matt Zuckers quilt development webpage.
- Roger Penroses patent for his aperiodic tiling
- Hat webpage.
- Spectre webpage.
- eschermath.org

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P1 Wallpaper Group

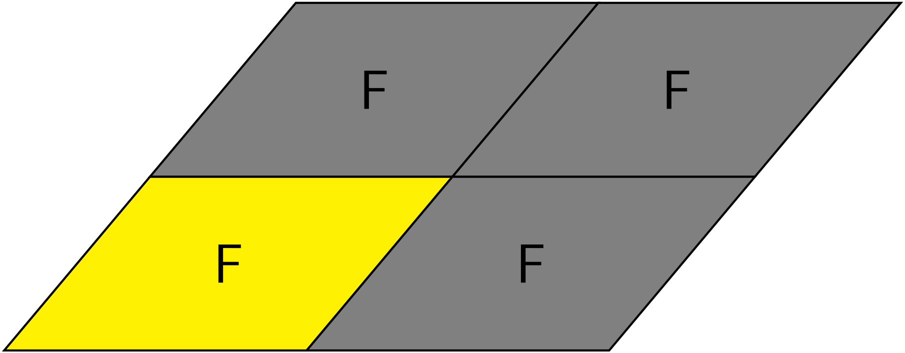

Figure 26: p1

P2 Wallpaper Group

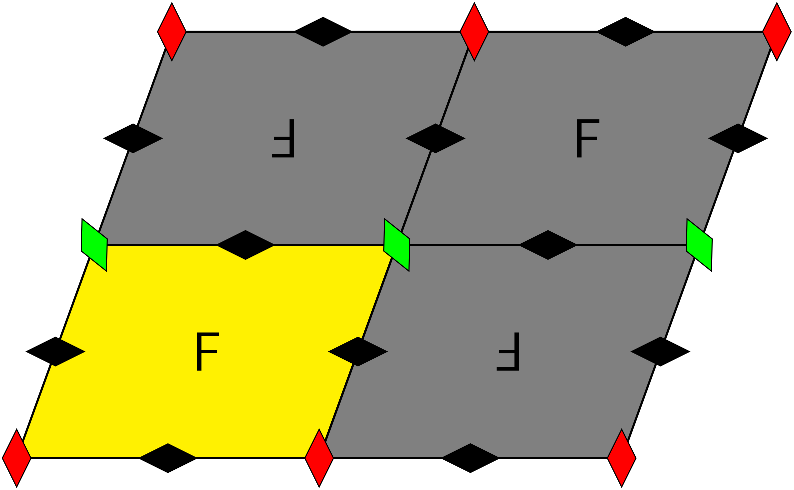

Figure 27: p2

P4 Wallpaper Group

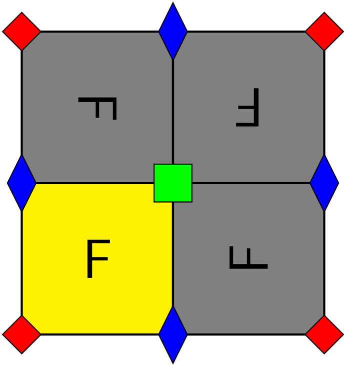

Figure 28: p4

P4M Wallpaper Group

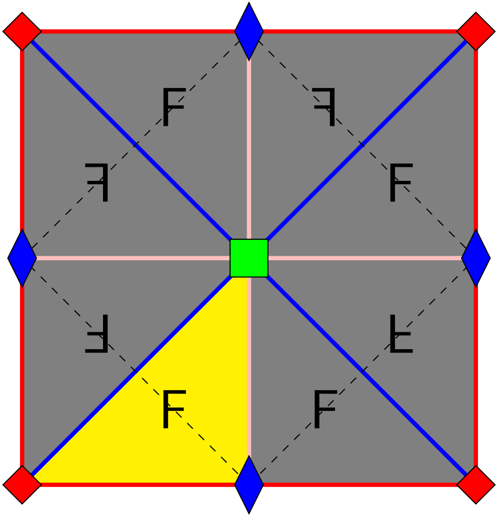

Figure 29: p4m

P4G Wallpaper Group

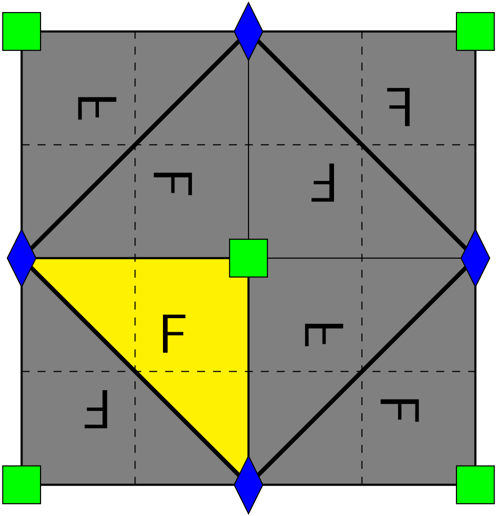

Figure 30: p4g

CM Wallpaper Group

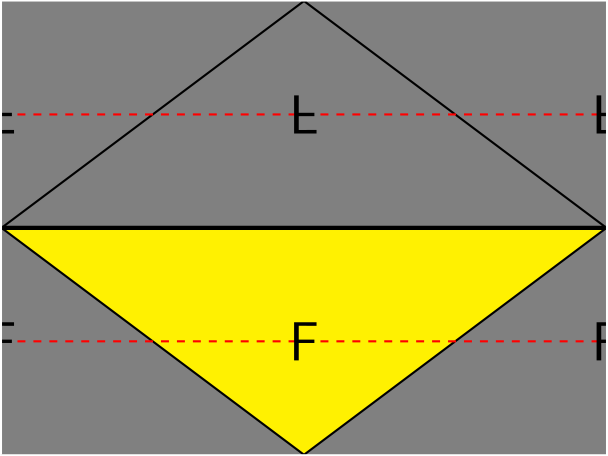

Figure 31: cm

CMM Wallpaper Group

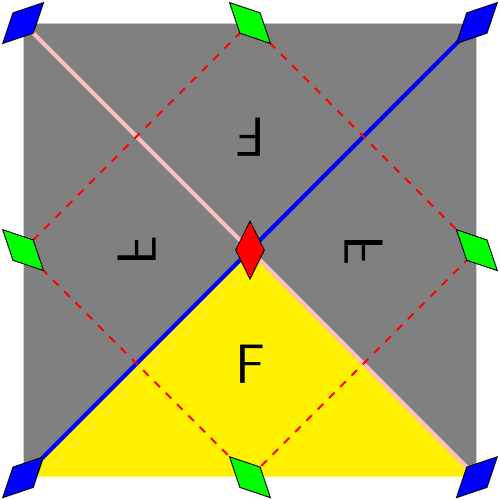

Figure 32: cmm

PG Wallpaper Group

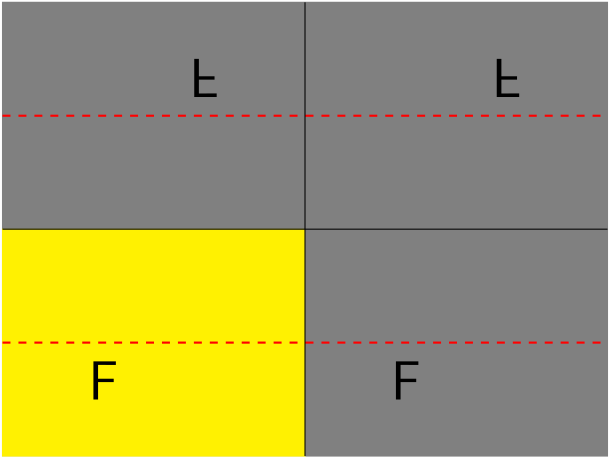

Figure 33: pg

PGG Wallpaper Group

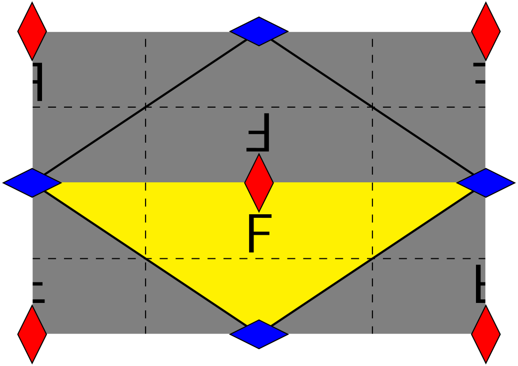

Figure 34: pgg

PMG Wallpaper Group

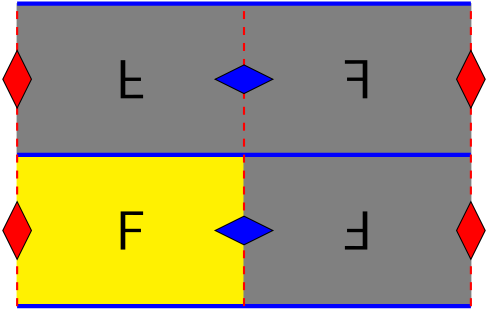

Figure 35: pmg

PM Wallpaper Group

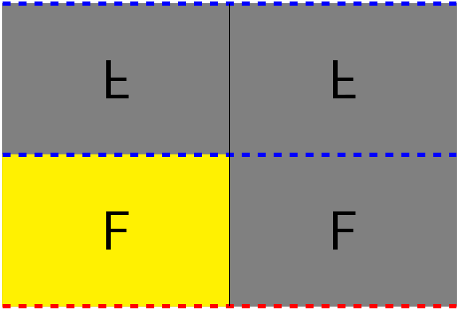

Figure 36: pm

PMM Wallpaper Group

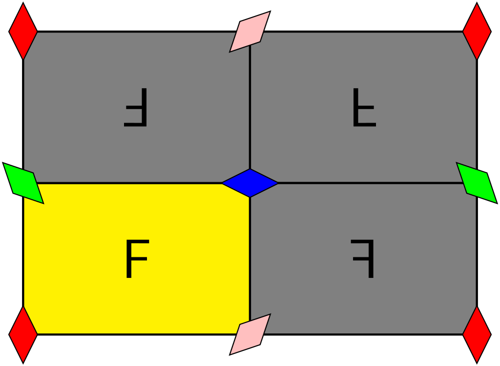

Figure 37: pmm

P3 Wallpaper Group

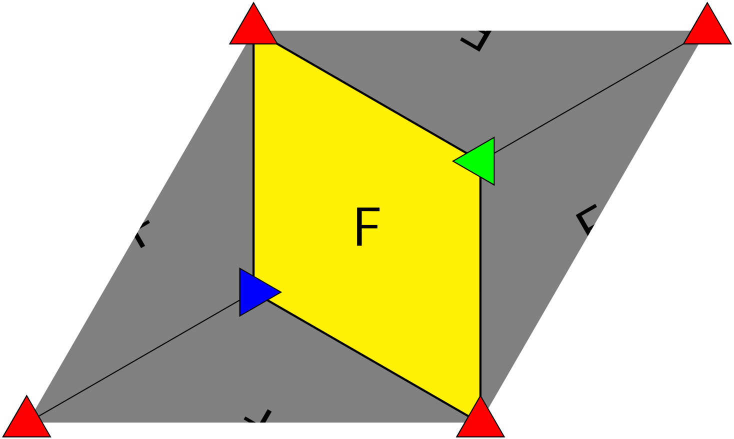

Figure 38: p3

P3M1 Wallpaper Group

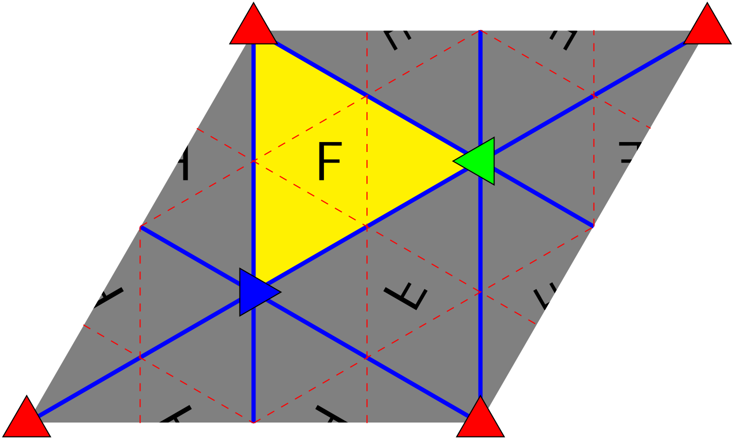

Figure 39: p3m1

P31M Wallpaper Group

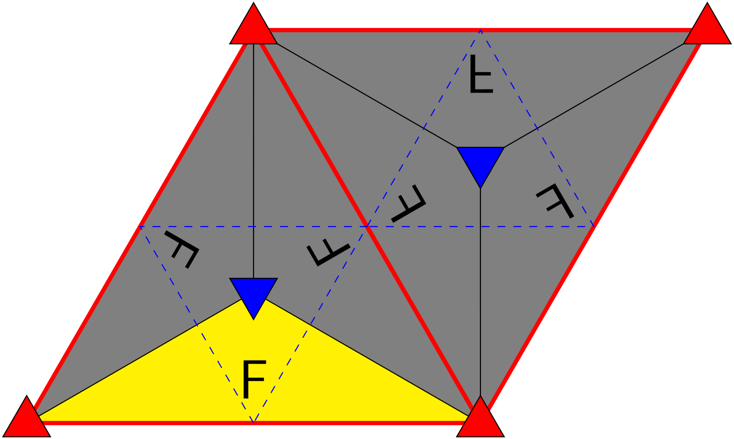

Figure 40: p31m

P6 Wallpaper Group

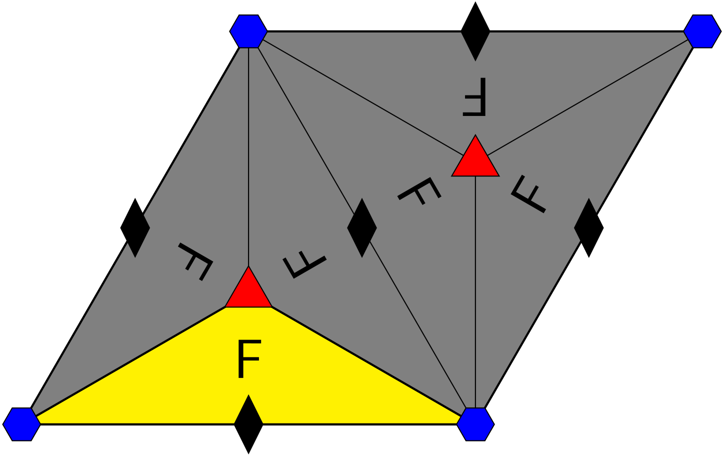

Figure 41: p6

P6M Wallpaper Group

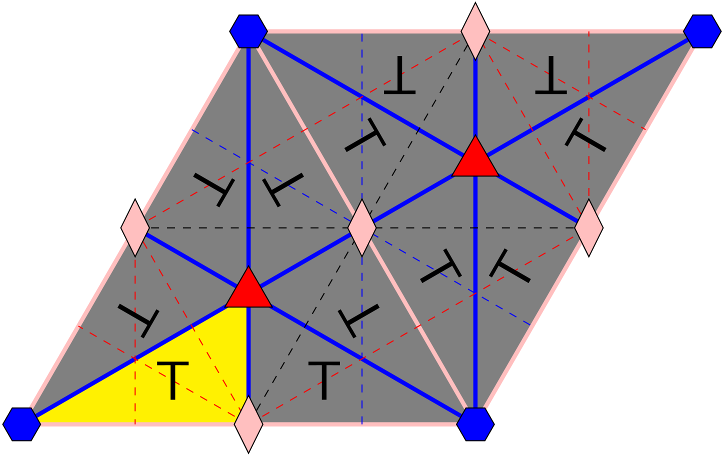

Figure 42: p6m