Please submit your answers to Blackboard as a SINGLE .pdf file by midnight on Friday 29th April.

Q 1. A linear regression model is proposed as:

$$y = \mathbf{X}\boldsymbol{\beta} + \epsilon$$

where y is a vector of n response values, $\mathbf{X}$ an $n \times (p+1)$ design matrix of known values, $\boldsymbol{\beta}$ a vector of $p+1$ unknown regression parameters, and $\epsilon \sim N(0, \phi\mathbf{I})$. A prior distribution $p(\boldsymbol{\beta}, \phi) \propto \frac{1}{\phi}$ is proposed. Show that the posterior distribution of $\boldsymbol{\beta}|\phi, \mathbf{y}$ is:

$$N\left(\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}, \phi\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\right).$$

Similarly, show that the posterior distribution of $\phi|\boldsymbol{\beta}, \mathbf{y}$ is $IG(n/2, S/2)$ where $S = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$

Q 2. The energy of particles being emitted from a radioactive source are modelled using a $N(\mu, \rho\mu^2)$ distribution where ρ is set at 1. A nuclear physicist said he believes that particles will be emitted with a mean energy of 80MeV but he's not certain about that and it could be anywhere between 50MeV and 110MeV.

(a) Specify a gamma prior that reflects the nuclear physicist's opinions on the average energy of the particles. Assume that the range specified is ± 3 standard deviations from the mean.

(b) The energy of eight particles was recorded as: 50, 60, 60, 80, 40, 40, 80 and 70MeV. Show that the likelihood is of the form:

$$p(\mathbf{x}|\mu) \propto \frac{1}{\mu^8}\exp\left[-\frac{15300}{\mu^2} + \frac{480}{\mu} - 4\right].$$

(c) Find the posterior density, up to a constant

(d) Describe, in detail, a suitable procedure for sampling values from the posterior. Comment on the efficiency of the method you propose.

(e) Suppose that the physicist wants to extend the model so that ρ is treated as unknown. Outline the extra steps that would need to be taken to complete a Bayesian inference for this extended model.