STAT40180 — Stochastic Models

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Week 2

Multivariate Parameters

Multivariate Parameters

- In this class, we look at maximum likelihood inference for models with multivariate parameters.
- Again, the emphasis will be on using computational methods.

Time-to-Event: Kiama Blowhole

- Let's reconsider the Kiama blowhole data
- A more general model for the data is the Weibull model

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right], \text{ where } \alpha, \beta > 0.$$

- If $(\alpha, \beta) = (1, 1/\theta)$ then the Weibull model is the same as an exponential distribution.
- However, when $\alpha \neq 1$ it is has a different shape.
- Also,

$$\mathbb{E}(X) = \beta\Gamma\left(1 + \frac{1}{\alpha}\right) \text{ and } \mathbb{V}\mathrm{ar}(X) = \beta^2\left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right)^2\right]$$

So, method of moments would be very difficult for this model!

Maximum likelihood

- The likelihood is of the form

$$
\begin{aligned}
L(\alpha, \beta) &= \prod_{i=1}^{n} f(x_i) \\
&= \prod_{i=1}^{n} \frac{\alpha}{\beta} \left(\frac{x_i}{\beta} \right)^{\alpha - 1} \exp \left[- \left(\frac{x_i}{\beta} \right)^{\alpha} \right] \\
&= \frac{\alpha^n}{\beta^n} \left(\frac{\prod_{i=1}^{n} x_i}{\beta^n} \right)^{\alpha - 1} \exp \left[- \sum_{i=1}^{n} \left(\frac{x_i}{\beta} \right)^{\alpha} \right]
\end{aligned}
$$

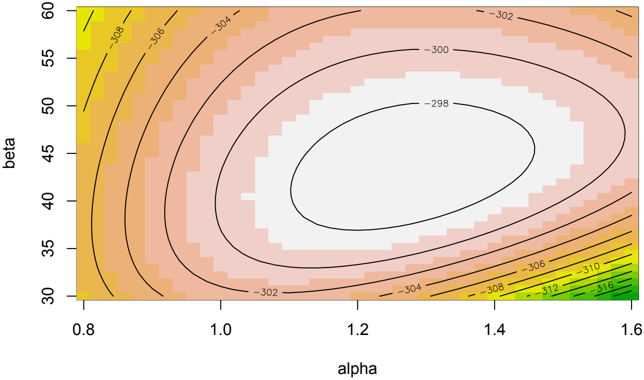
- The log-likelihood is cannot be maximized directly.

Maximum likelihood: Numerical

- We can maximize the likelihood using numerical methods.
- The method for maximization is very similar to the previous example, but we need to optimize with respect to α and β.
- For simplicity, we will let the unknown parameters be written as $\theta = (\theta_1, \theta_2) = (\alpha, \beta)$.
- We use the `optim()` command to optimize the log-likelihood with respect to the parameter θ.

Maximum likelihood: Likelihood

- We can produce a contour plot of the likelihood function to see how it varies with the value of $\theta = (\alpha, \beta)$.



Maximum Likelihood: Code

- The code for doing the maximum likelihood estimation.

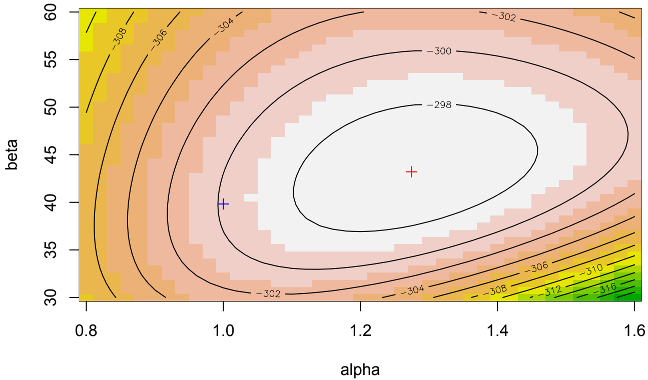
```
x <- scan()
83  51  87  60  28  95   8  27  15  10  18  16  29  54  91   8
17  55  10  35  47  77  36  17  21  36  18  40  10   7  34  27
28  56   8  25  68 146  89  18  73  69   9  37  10  82  29   8
60  61  61  18 169  25   8  26  11  83  11  42  17  14   9  12

loglik <- function(theta,x)
{
alpha <- theta[1]
beta <- theta[2]
sum(dweibull(x,alpha,beta,log=TRUE))
}

alpha0 <- 1
beta0 <- mean(x)
theta0 <- c(alpha0,beta0)
fit <- optim(par=theta0,loglik,method="BFGS",x=x,control=list(fnscale=-1),hessian=TRUE)
```

- We can see that

$$\hat{\theta} = (\hat{\alpha}, \hat{\beta}) = (1.27, 43.2).$$

Maximum likelihood: Likelihood

- The maximum likelihood estimate (red) and the exponential model fit (blue) are shown.



- The exponential model fit has log-likelihood value 2.91 lower than the maximum likelihood value.

Maximum Likelihood: Contour Code

- The code for doing the contour plot (with estimates) is below.

```
alphagrid <- seq(0.8,1.6,length=41)
betagrid <- seq(30,60,length=41)
thetagrid <- expand.grid(alphagrid,betagrid)
thetagrid <- as.matrix(thetagrid)

lvec<-rep(NA,41^2)
for (i in 1:nrow(thetagrid))
{
lvec[i] <- loglik(thetagrid[i,],x)
}

lmat <- matrix(lvec,41,41)

image(alphagrid,betagrid,lmat,col=terrain.colors(12),xlab="alpha",ylab="beta")
contour(alphagrid,betagrid,lmat,add=TRUE)

points(fit$par[1],fit$par[2],pch=3,col="red")
points(theta0[1],theta0[2],pch=3,col="blue")
```

Maximum likelihood: Inference

- The Hessian (second derivative) in this case is a matrix.
- Minus the Hessian is called the information matrix.
- The standard error of each parameters is given by

$$\mathsf{SE}(\hat{\theta}_1) = \sqrt{[I(\hat{\theta})^{-1}]_{11}} \text{ and } \mathsf{SE}(\hat{\theta}_2) = \sqrt{[I(\hat{\theta})^{-1}]_{22}}.$$

- That is, the square root of the diagonal of the inverse Information matrix gives the standard error for each parameter.

Maximum likelihood: Inference

- For the Weibull model we can use the following code to give the standard errors.

```
inf<- -fit$hessian
sqrt(diag(solve(inf)))
```

- We thus get,

$$SE(\hat{\alpha}) = 0.12 \text{ and } SE(\hat{\beta}) = 4.49.$$

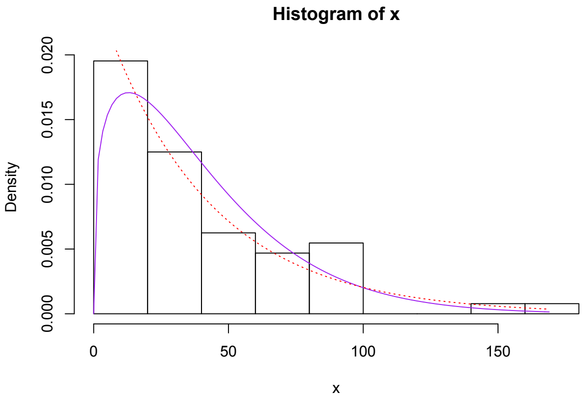
- Thus, approximate 95% confidence intervals for the parameters are:

$$\alpha : 1.274 \pm 0.236 \text{ and } \beta : 43.2 \pm 8.8.$$

- It is worth noting that 1 is not in the confidence interval for α.
- Thus, we have some evidence to support that $\alpha \neq 1$.

Model Fit

- We can informally compare the model fit (purple) to the data histogram.



- The exponential model is included (red) for comparison.