STAT40380/40390 - Bayesian Analysis

Tutorial Sheet 2

Question 1 - 2 marks

Show that $\sum x_i$ is sufficient for θ if $x_i \sim Bin(k, \theta)$ for $i = 1, \ldots, n$.

$$
\begin{aligned}
p(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \binom{k}{x_i} \theta^{x_i} (1-\theta)^{k-x_i} \\
&= \left[\prod_{i=1}^{n} \binom{k}{x_i}\right] \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{k-\sum_{i=1}^{n} x_i} \\
&= p(x|t)p(t|\theta)
\end{aligned}
$$

ie $p(x|t)$ only depends on x and t, and $p(t|\theta)$ only depends on θ and t.

Question 2 - 3 marks

Show that the Poisson distribution with rate parameter λ is a member of the 1-parameter exponential family. What will be the general form of the conjugate prior distribution for λ? How is this form related to the gamma distribution?

A 1-parameter exponential family is defined as:

$$p(x|\theta) = g(x)h(\theta)\exp\left[t(x)\psi(\theta)\right]$$

We have:

$$
\begin{aligned}
p(x|\lambda) &= \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \frac{1}{x!} e^{-\lambda} \exp(x \log \lambda)
\end{aligned}
$$

so

$$g(x) = 1/x!,\ h(\theta) = e^{-\lambda},\ t(x) = x,\ \psi(\theta) = \log \lambda$$

Thus the general form for the prior distribution is:

$$p(\theta) \propto h(\theta)^{\nu} \exp[\tau\psi(\theta)]$$

so we have:

$$p(\lambda) \propto e^{-\nu\lambda} \exp(\tau \log(\lambda) = e^{-\nu\lambda}\lambda^{\tau}$$

When $\nu = \beta$ and $\tau = \alpha - 1$ we have a gamma distribution.

Question 3 - 5 marks

A chocolate manufacturer is trying to assess the market for chocolate bar. Weekly sales of chocolate bars are measured in thousands of bars, from long experience, known to be normally distributed and in this case the standard deviation in sales per week is assumed to be 1 (thousand bars). The company therefore wants to know what the mean weekly sale, denoted μ, will be. The sales manager believes that the mean weekly sale should be about 8, but admits that she is rather uncertain about this and thus estimates a standard deviation of 2.

1. Specify a gamma prior distribution on μ that reflects the sales manager's opinion on average weekly sales. Write down the pdf of your prior distribution.

2. An 8 week trial of the bar is conducted. The 8 weekly sales figures are 5, 6, 6, 8, 4, 4, 8 and 7. The sum of these 8 observations is 48.

   (a) Show that the likelihood, as a function of μ, can be written

   $$p(x_1, \ldots, x_8 \,|\, \mu) \propto \exp(48\mu - 4\mu^2),$$

   where x_i is the ith week's sales.

   (b) Show that, up to a constant, the posterior distribution of μ is:

   $$p(\mu \,|\, x_1, \ldots, x_8) \propto \mu^a \exp(-4\mu^2 + b\mu)$$

   What are the values of a and b?

1) We would like a suitable prior such that $\mu \sim Ga(\alpha, \beta)$. Note that $M = \frac{\alpha}{\beta}$ and $V = \frac{\alpha}{\beta^2}$. If $M = 8$ and $V = 4$ then $\alpha = 16$ and $\beta = 2$. The prior pdf is now:

$$p(\mu) \propto \mu^{15} e^{-2\mu}$$

2)
(a)

$$\begin{aligned} p(\boldsymbol{x}|\mu) &\propto \prod_{i=1}^{6} p(x_i|\mu) \\ &\propto \exp\left[-\frac{1}{2}\left(n\mu^2 - 2\mu \sum x_i\right)\right] \\ &= \exp\left[48\mu - 4\mu^2\right] \end{aligned}$$

(b)

$$\begin{aligned} p(\mu|\boldsymbol{x}) &\propto p(\mu)p(\boldsymbol{x}|\mu) \\ &= \mu^{15} e^{-2\mu} e^{48\mu - 4\mu^2} \\ &= \mu^{15} e^{46\mu - 4\mu^2} \end{aligned}$$

so $a = 15$ and $b = 46$.