University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATION 2012/2013

STAT 40390

Bayesian Analysis

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Time Allowed: 2 hours

Instructions for Candidates

Full marks for *four* complete questions.
Marks are indicated for each question.

Instructions for Invigilators

Calculators are permitted.

1. Data x_i arise from a geometric distribution, $G(\theta)$, for $i = 1, \ldots, n$.

(a) Write down the likelihood for the model, up to the constant of proportionality.

[2 marks]

(b) Define the Jeffreys prior distribution. Why would a Jeffreys prior distribution be chosen for a parameter?

[4 marks]

(c) Show that the Jeffreys prior for θ is proportional to a $Be(0, 0.5)$ distribution.

[6 marks]

(d) Show that the posterior distribution for $\theta | x_1, \ldots, x_n$ is also Beta distributed when the Jeffreys prior is used.

[4 marks]

(e) An alternative model is suggested such that $x_i \sim G(\theta_i)$. What does this new model represent? Suggest a suitable transformed prior distribution for θ_i.

[4 marks]

(f) The Deviance Information Criterion (DIC) is to be used to choose between the model in (a) and that in (e). Define the DIC and explain how it is used to compare between models.

[5 marks]

[Total 25 marks]

2. A possibly faulty air speed indicator taken from a plane crash wreck is subjected to a series of tests to determine its operating state at a fixed speed. The investigator feels that the speed indicator may be over-estimating the true speed. After 8 test runs the resulting bias values x_i (in knots) are:

$$50.5,\ 48.6,\ 41.3,\ 44.5,\ 23.9,\ 47.1,\ 42.6,\ 61.5,$$

with $\bar{x} = 45$. A normal distribution can be assumed for the measurements, with known standard deviation $\sigma = 15$. Of key interest is the mean μ of the distribution.

(a) Show that the likelihood for the above data can be written as:

$$p(\boldsymbol{x}|\mu) \propto e^{-\frac{4}{225}(\mu^2 - 90\mu)}$$

[4 marks]

(b) The air crash investigator believes from previous experience with faulty indicators that the parameter μ follows a gamma distribution with mean 50 and standard deviation 5. Estimate the parameters of a gamma distribution which represent the investigator's beliefs.

[3 marks]

(c) Write down the posterior distribution for the investigator up to a suitable constant.

[4 marks]

(d) Suggest in detail an appropriate procedure for sampling values from the full posterior.

[9 marks]

(e) The investigator would like to know whether the mean bias in the measurements is greater than 20 knots. Explain how you might estimate $p(\theta > 20|\boldsymbol{x})$ from the samples created in part (d). Is there any way you might simplify the problem to produce an answer without resorting to simulation?

[5 marks]

[Total 25 marks]

3. Consider a situation in which data are observed as $x_i|\theta, \phi \sim N(\theta, \phi^{-1})$, where both the mean θ and precision ϕ are to be estimated.

(a) A prior distribution $p(\theta, \phi) \propto \frac{1}{\phi}$ is suggested. Why might this prior be used? Is it appropriate for this situation?

[4 marks]

(b) Show that the posterior distribution can be written as:

$$p(\theta, \phi|\boldsymbol{x}) \propto \phi^{n/2-1} e^{-\frac{\phi}{2}\left[S+n(\bar{x}-\theta)^2\right]}$$

where $S = \sum(x_i - \bar{x})^2$.

[3 marks]

(c) What is a marginal posterior distribution? Explain how a marginal posterior distribution can be computed both algebraically and computationally.

[5 marks]

(d) Show that the marginal posterior distribution of $\phi|\boldsymbol{x}$ is such that:

$$\phi|\boldsymbol{x} \sim Ga\left(\frac{1}{2}(n-1), \frac{S}{2}\right)$$

What is the posterior mean of ϕ?

[6 marks]

(e) Show that the marginal distribution of $\theta|\boldsymbol{x}$ is such that:

$$p(\theta|\boldsymbol{x}) \propto \left[S + n(\bar{x}-\theta)^2\right]^{-\frac{n}{2}}$$

Hint: make the substitution $z = \frac{\phi}{2}(S + n(\bar{x}-\theta)^2)$. To which probability distribution is this related?

[7 marks]

[Total 25 marks]

4. Write short notes on **three** of the following topics.

(a) The use of Bayes factors in comparing models.

[8 marks]

(b) The use of the exponential family in finding prior distributions.

[8 marks]

(c) Sufficiency and its role in Bayesian statistics.

[8 marks]

(d) The EM algorithm.

[8 marks]

[Total 24 marks]

Probability distributions

Geometric distribution

$$p(x|p) = p(1-p)^{x-1}$$

$$\mathbb{E}(x) = \frac{1}{p},\ Var(x) = \frac{1}{1-p^2}$$

Normal distribution

$$p(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

$$\mathbb{E}(x) = \mu,\ Var(x) = \sigma^2$$

Beta distribution

$$p(x|\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

$$\text{where } B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$\mathbb{E}(x) = \frac{\alpha}{\alpha+\beta},\ Var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Gamma distribution

$$p(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$$

$$\text{where } \Gamma(u) = \int_0^\infty t^{u-1} \exp(-t)\,dt$$

$$\mathbb{E}(x) = \frac{\alpha}{\beta},\ Var(x) = \frac{\alpha}{\beta^2}$$