University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATIONS 2010/2011

STAT 30090

Models – Stochastic Models

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Time Allowed: 2 hours

Instructions for Candidates

Attempt all questions. The number of marks for each question is given. The total number of marks is given.

Instructions for Invigilators

Candidates may use their own copy of "*Formulae and Tables for Actuarial Examinations*".
Calculators are permitted.

1. Claims arrive at a small insurance company according to a Poisson process. On average, 15 claims arrive every week. Claims' sizes are independent and have an exponential distribution with mean 150 euro.

   (i) Find the probability that over a period of four weeks, at most one claim will exceed 1000 euro. [4 **MARKS**]

   (ii) Given that over a week the company receives 10 claims, find the probability that all but one arrive within the first three days. [3 **MARKS**]

   (iii) What is the distribution of the waiting time between claims in excess of 500 euro? [3 **MARKS**]

   Total: [10 **MARKS**]

2. Let X_n be a Markov chain with transition probability matrix given by

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0 & 0.3 & 0.7 \\ 0.2 & 0 & 0.8 \end{bmatrix}.$$

   (a) Find the distribution of X_2 if the initial distribution of the chain is given by

$$\bar{p}_0 = (0.5, 0.5, 0).$$

   [3 **MARKS**]

   (b) Write down a formula that gives the distribution of X_n in terms of the initial distribution and the tranisition probabability matrix. Explain how it is possible to obtain from this formula the equation for the stationary distribution. [3 **MARKS**]

   (c) Explain why the stationary distribution of the Markov chain X_n exists and find the stationary distribution $\bar{\pi}$. [4 **MARKS**]

   Total: [10 **MARKS**]

3. The Jones family receive the newspaper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3, someone takes all the newspapers in the pile and puts them in the recycling bin. Also, if there ever are at least 5 papers in the pile, Mr. Jones (with probabability 1) takes the papers to the bin. Consider the number of papers in the pile in the evening.

(i) Is it reasonable to model this by a Markov chain? If so, what is the state space and the transition probability matrix? Is this Markov chain aperiodic, recurrent? [5 **MARKS**]

(ii) Suppose there are no papers in the pile, how many days should one expect until the next day that there are no papers in the pile? [5 **MARKS**]

Total: [10 **MARKS**]

4. What should be the birth and death rates of a birth-and-death process in order that it is exactly a (homogeneous) Poisson process? Is it a recurrent or transient process? **Total:** [5 **MARKS**]

5. A cell produces 0, 1 or 2 offspring with probabilities 0.1, 0.4 and 0.5, respectively. (You may assume that cells produce offspring independently of one another.)

   What is the extinction probability for a colony that initially consists of two cells? **Total:** [5 **MARKS**]

6. A continuous time Markov chain X_t with state space $\{1, 2, 3\}$ has the infinitesimal generator

$$A = \begin{bmatrix} -6 & 2 & ? \\ 2 & ? & 3 \\ ? & 3 & -5 \end{bmatrix}.$$

(i) Complete the matrix. [2 **MARKS**]

(ii) The chain is currently in state 2. Find the probability that the next state will be 1. [2 **MARKS**]

(iii) If the chain starts in state 2, what is the expected total time spent in state 2 before the chain first enters state 3. [6 **MARKS**]

Total: [10 **MARKS**]

7. Let W_t be the standard Brownian motion.

(i) Write down the defining properties of W_t. [3 **MARKS**]

(ii) Calculate the probability that W_t will exceed 4 for some t in the interval $[0, 3]$? [3 **MARKS**]

(iii) Calculate the variance of $4W_3 - 3W_2$. [4 **MARKS**]

Total: [10 **MARKS**]