Q 1. You have just taken a test for a rare but serious disease, and your test returns a positive result. If you truly have the disease, the test gives the correct answer every time. However if you do not have the disease, the test returns the correct answer only 99.8% of the time. If it is known that only 1 in 12,000 people have the disease, use Bayes' theorem to determine the probability you have the disease given a positive test result. Discuss your answer.

Let D be the event that you have the disease.
Let $\bar{D}$ be the event that you do not have the disease.
Let T be the event that you get a positive test result.
Let $\bar{T}$ be the event that you get a negative test result.

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\bar{D})P(\bar{D})}$$

$$P(D) = \frac{1}{12000}, \quad P(T|D) = 1, \quad P(\bar{T}|\bar{D}) = 0.998 \text{ so } P(T|\bar{D}) = 0.002.$$

$$\text{Therefore, } P(D|T) = \frac{1 \times \frac{1}{12000}}{1 \times \frac{1}{12000} + 0.002 \times \frac{11999}{12000}} \simeq 0.04.$$

So theres only a 4% chance that you have the disease even though you had a positive test!

Q 2. Suppose that $x_i \sim N(\theta, \phi)$ for $i = 1, \ldots, n$, with ϕ, the variance known. If prior information shows that $\theta \sim N(\theta_0, \phi_0)$, find the posterior distribution of $\theta|\mathbf{x}$. Show your work.

We have $p(\theta) \propto e^{-\frac{1}{2\phi_0}(\theta-\theta_0)^2}$, and $p(x_i|\theta) \propto e^{-\frac{1}{2\phi}(x_i-\theta)^2}$. Now

$$\begin{aligned}
p(\theta|\mathbf{x}) &\propto e^{-\frac{1}{2\phi_0}(\theta-\theta_0)^2} \times \prod_{i=1}^{n} e^{-\frac{1}{2\phi}(x_i-\theta)^2} \\
&= e^{-\frac{1}{2\phi_0}(\theta-\theta_0)^2} \times e^{-\frac{1}{2\phi}\sum(x_i-\theta)^2} \\
&\propto e^{-\frac{1}{2\phi_0}(\theta^2-2\theta\theta_0)} \times e^{-\frac{1}{2\phi}\sum(-2\theta x_i+\theta^2)} \\
&= e^{-\frac{1}{2}\left[\frac{\theta^2}{\phi_0}-\frac{2\theta\theta_0}{\phi_0}+\frac{n\theta^2}{\phi}-\frac{2\theta n\bar{x}}{\phi}\right]} \\
&= e^{-\frac{1}{2}\left[\theta^2\left[\frac{1}{\phi_0}+\frac{n}{\phi}\right]-2\theta\left[\frac{\theta_0}{\phi_0}+\frac{n\bar{x}}{\phi}\right]\right]}.
\end{aligned}$$

$$\text{Let } \phi_1 = \left[\frac{1}{\phi_0}+\frac{n}{\phi}\right]^{-1}, \text{ and } \theta_1 = \phi_1\left[\frac{\theta_0}{\phi_0}+\frac{n\bar{x}}{\phi}\right] \ldots$$

$$\implies p(\theta|\mathbf{x}) = e^{-\frac{1}{2\phi_1}\theta^2+\frac{\theta\theta_1}{\phi_1}}$$

Multiply across by constant $e^{-\frac{\theta_1^2}{2\phi_1}}$... You get a perfect square...

$$\begin{aligned}
\implies p(\theta|\mathbf{x}) &\propto e^{-\frac{1}{2\phi_1}\theta^2+\frac{\theta\theta_1}{\phi_1}--\frac{\theta_1^2}{2\phi_1}} \\
&= e^{-\frac{1}{2\phi_1}(\theta-\theta_1)^2}
\end{aligned}$$

Multiply across again by constant $\dfrac{1}{\sqrt{2\pi\phi_1}}$

$$\implies p(\theta|\mathbf{x}) \propto \frac{1}{\sqrt{2\pi\phi_1}} e^{-\frac{1}{2\phi_1}(\theta-\theta_1)^2}$$

$$\implies \theta|\mathbf{x} \sim N(\theta_1, \phi_1) \text{ with } \phi_1 = \left[\frac{1}{\phi_0}+\frac{n}{\phi}\right]^{-1}, \text{ and } \theta_1 = \phi_1\left[\frac{\theta_0}{\phi_0}+\frac{n\bar{x}}{\phi}\right].$$

Q 3. Suppose that observations arise from a Binomial distribution such that $x_i \sim Bin(k, \theta)$ with k known. Show that the Beta distrubution is a conjugate prior for the Binomial and thus derive the form of the posterior.

Let θ $Be(\alpha, \beta)$. We have $p(x_i|\theta) \propto \theta^{x_i}(1-\theta)^{k-x_i}$ and $p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$. Thus:

$$
\begin{aligned}
p(\theta|\mathbf{x}) &\propto p(\theta) \prod_{i=1}^{n} p(x_i|\theta) \\
&= \theta^{\alpha-1}(1-\theta)^{\beta-1} \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{k-x_i} \\
&= \theta^{\alpha+\sum x_i - 1}(1-\theta)^{\beta+nk-\sum x_i - 1} \\
\text{so } \theta|\mathbf{x} &\sim Be(\alpha + \sum x_i, \beta + nk - \sum x_i).
\end{aligned}
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