University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATIONS 2011/2012

STAT 30090
Models – Stochastic Models

Professor Michael Tretyakov

Dr. Patrick Murphy

Professor Nial Friel*

Time Allowed: 2 hours

Instructions for Candidates

Attempt all questions. The number of marks for each question is given. The total number of marks is given.

Instructions for Invigilators

Candidates will require:
New Cambridge Statistical Tables
Calculators are permitted.
Graph paper is not required.

1. **(a)** Define each of the following

   (i) A stationary stochastic process. [3]

   (ii) A martingale. [3]

   (iii) A white noise process. [2]

   (iv) A simple random walk. [2]

   (b) Prove that the expected value of an increment for a martingale is zero. [5]

   Total: [15]

2. Suppose that the number of calls arriving to a telesales call centre follows a Poisson process with intensity $\lambda = 6$ calls per hour. Each caller makes a purchase with probability 0.4.

   (i) Find the probability that exactly 7 calls arrive in the first hour and a half. [2]

   (ii) Suppose that 10 calls arrived in the first two hours. What is the probability that 4 of them arrived in the first 30 minutes? [3]

   (iii) What is the expected time until the first purchase? [2]

   (iv) Suppose now that 5 calls arrived in the first hour. What is the probability that 3 of them made a purchase? [3]

   Total: [10]

3. Let X_n be a Markov chain with states $\{1, 2, 3\}$ and transition matrix:

$$P = \begin{bmatrix} 0 & ? & 0.4 \\ 0.8 & ? & 0.2 \\ ? & 0.5 & 0 \end{bmatrix}$$

(i) Fill in the missing entries of the matrix. [2]

(ii) Which states of the chain are recurrent? What is their period? [3]

(iii) Does this chain have a stationary distribution? Explain the reason and find the stationary distribution (if it exists). [5]

Total: [10]

4. The size of a population can be described as a birth-and-death process, where the death rate is proportional to the size of the population, with proportionality constant $\mu > 0$. The birth rate is a constant value $\lambda > 0$.

Justify the exsistence of a stationary distribution.
(You may find it useful to note that $\sum_{n=1}^{\infty} x^n n! = \infty$ for all $x > 0$.)

Calculate the expected size of the population after a long time. **Total:** [10]

5. It is known that a cell produces 0 or 1 offspring with probabilities 0.4 and 0.2 respectively. In addition, a cell can also produce 2 or 3 offspring. It is also known that a population of cells beginning from a single cell dies out with probability 0.9. What is the probability that a cell produces 3 offspring? (You may assumes that cells produce offspring independently of one another.)
Total: [10]

6. A continuous-time Markov chain X_t with state space $\{1, 2, 3\}$ has the infinitesimal generator:

$$A = \begin{bmatrix} -6 & ? & 2 \\ 5 & ? & 3 \\ 3 & 2 & ? \end{bmatrix}$$

(i) Fill in the missing entries of the matrix. [2]

(ii) Suppose that the chain starts in state 1. What is the expected amount of time that the chain spends in state 1 until the chain moves to state 3 for the first time? [5]

(iii) Find the stationary distribution of the Markov chain. [3]

Total: [10]

7. Let W_t be the standard Brownian motion.

(i) Give the definition of a (standard) Brownian motion W_t. [3]

(ii) Calculate the variance of $4W_3 - 2W_2$. [3]

(iii) Calculate the probabilty that W_t exceeds 1 for some t in the interval $[0, 2]$. [4]

Total: [10]