University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATIONS 2014/2015

STAT 30090 - STAT 40680
Stochastic Models

Dr. George Streftaris

Prof. Gary McGuire

Prof. Nial Friel*

Time Allowed: 2 hours

Instructions for Candidates

Attempt all questions. The total number of marks for each question is given.

Instructions for Invigilators

Candidates will not require
New Cambridge Statistical Tables.
Programmable and non-programmable calculators are permitted.
Graph paper is not required.

1. **(i)** State the definition of a martingale. [2]

   (ii) State the defining properties of a Brownian motion. [3]

   (iii) Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion. We consider the discrete time process $\{W_n\}_{n\in\mathbb{N}}$. Prove that $\{W_n\}_{n\in\mathbb{N}}$ is a martingale. [5]

   Total [10]

2. Suppose that the number of calls arriving to a telesales call centre follows a Poisson process with intensity $\lambda = 7$ calls per hour. Each caller makes a purchase with probability $p = 0.3$.

   (i) Find the probability that exactly 4 calls arrive during the first 90 min. [5]

   (ii) Suppose that 11 calls arrived in the first two hours. What is the probability that 4 of them arrived in the first 30 minutes? [5]

   (iii) What is the expected time until the first purchase? [5]

   (iv) Suppose now that 6 calls arrived in the first hour. What is the probability that 4 of them made a purchase? [5]

   Total [20]

3. A Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with state space $\{1, 2, 3\}$ has the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

   (i) Calculate $\mathbb{P}(X_2 = 3|X_0 = 3)$ and $\mathbb{P}(X_2 = 1|X_0 = 1)$. [3]

   (ii) Prove that $\{X_n\}$ has a unique limit probability distribution π. [4]

   (iii) Following part (ii), calculate π. [5]

   (iv) Define T as the first return time to state 1: $T = \min\{n \geq 1 : X_n = 1\}$. Calculate $\mathbb{E}(T|X_0 = 1)$. [3]

   Total [15]

4. A Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with state space $\{1, 2, 3, 4, 5\}$ has the following transition matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) Plot the corresponding graph of transition probabilities. [3]

(ii) List all the communication classes of this Markov chain. State, for each class, whether it is recurrent or transient. [4]

(iii) Given that $X_0 = 3$, what is the probability that the chain eventually ends up in state 1? [13]

Total [20]

5. A cell produces 0, 1 or 2 offspring with probabilities 0.3, 0.2 and 0.5 respectively. (You may assume that cells produce offspring independently of one another.)

(i) What is the extinction probability for a colony that initially consists of three cells? [5]

(ii) If a colony starts with a single cell, what is the probability that it is extinct in the second generation at the latest? [5]

Total [10]

6. Consider a continuous time process $\{X_t\}_{t\geq 0}$ with state space $\{0, 1\}$, and

$$\begin{cases} \mathbb{P}(X_{t+h} = 1|X_t = 0) = \alpha h + o(h) \\ \mathbb{P}(X_{t+h} = 0|X_t = 1) = \beta h + o(h) \end{cases}$$

for some numbers $\alpha, \beta > 0$. The objective of this question is to calculate the long range probability that $X_t = 0$. For short, we put $p(t) = \mathbb{P}(X_t = 0)$.

(i) Prove that $p(t + h) - p(t) = -h(\beta + \alpha)p(t) + \beta h + o(h)$. [3]

(ii) Prove that $p'(t) = -(\beta + \alpha)p(t) + \beta$. [3]

(iii) In (iii) only, assume that $\{X_t\}$ is stationary. Using (ii), find $p(t)$ in this case. [3]

(iv) Find all the solutions of the differential equation in (ii). Prove that only one solution corresponds to a given initial probability distribution $\pi = (\pi_0, \pi_1) = (\mathbb{P}(X_0 = 0), \mathbb{P}(X_0 = 1))$ and find this solution. [12]

(v) Prove that in any case, the number found in (iii) is the limit of $p(t)$. [2]

(vi) When $(\alpha, \beta) = (1, 2)$, $\pi = (0, 1)$, what is $\lim_{t\to\infty} p(t)$? What is $p(2)$? [2]

Total [25]

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