University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATIONS 2015/2016

STAT 30090 - STAT 40680
Stochastic Models

Dr. George Streftaris

Kathryn Boult, BSc, FIA.

Prof. Gary McGuire

Prof. Nial Friel*

Time Allowed: 2 hours

Instructions for Candidates

Attempt all questions. The total number of marks for each question is given.

Instructions for Invigilators

Candidates will not require
New Cambridge Statistical Tables.
Programmable and non-programmable calculators are permitted.
Graph paper is not required.

1. Let W_t be the standard Brownian motion.

   (i) Write down the defining properties of W_t. [3]

   (ii) Calculate the probability that W_t will exceed 3 for some t in the interval $[0, 5]$. [4]

   (iii) Let $Y_t = W_t^n$ (the n^{th} power of W_t, for $n \in \{1, 2, \dots\}$). Show that Y_t satisfies the stochastic differential equation

$$dY_t = \frac{n(n-1)}{2} Y_t^{1-2/n} dt + n Y_t^{1-1/n} dB_t$$

[8]

Total [15]

2. The numbers of claims to an insurance company from smokers and non-smokers follow independent Poisson processes. On average 4 claims from non-smokers and 6 claims from smokers arrive every day independently of each other. The sizes of claims are independent, with an exponential distribution with mean €100.

   (i) Given that 8 claims arrived in a day, what is the probability that 5 of them were from smokers? [4]

   (ii) What is the probability that over 3 days at most one claim will exceed €500? [5]

   (iii) What is the expected waiting time between claims in excess of €500? [6]

Total [15]

3. Let X_n be a Markov chain with the following transition probability matrix:

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0.1 & 0.4 \end{bmatrix}.$$

(i) The initial distribution of the chain is given by $\bar{p}_0 = (0, 0.4, 0.6)$. What is the distribution of X_2? [5]

(ii) Explain how the distribution of X_n is related to the initial distribution and the transition probability matrix. Hence or otherwise obtain the equation for the stationary distribution. [6]

(iii) Explain clearly why the stationary distribution of the Markov chain exists for P above. Find the stationary distribution. [9]

Total [20]

4. It is known that a cell produces 0 or 1 offspring with probabilities 0.4 and 0.2 respectively. In addition, a cell can also produce 2 or 3 offspring. It is also known that a population of cells beginning from a single cell dies out with probability 0.9. What is the probability that a cell produces 3 offspring? (You may assumes that cells produce offspring independently of one another.)

Total [15]

5. The size of a population can be described as a birth-and-death process, where the death rate is proportional to the size of the population, with proportionality constant $\mu > 0$. The birth rate is a constant value $\lambda > 0$.

   Justify the exsistence of a stationary distribution.
   (You may find it useful to note that $\sum_{n=1}^{\infty} x^n n! = \infty$ for all $x > 0$.)

   Calculate the expected size of the population after a long time.

Total [15]

6. Earthquakes occur in a seismic region following a Poisson process with intensity $\lambda = 3$ per year.

   (i) Given that 4 earthquakes occurred in the past 2 years, find the probability that 2 of them occurred in the first year. [7]

   (ii) The ith earthquake gives rise to a random amount of damage D_i, which decreases exponentially in time. It is assumed that the D_i's are independent and identically distributed, each with expected value μ. The total amount of damage after time t is given by

$$D(t) = \sum_{i=1}^{X_t} D_i \exp\{-2(t - S_i)\},$$

   where X_t is the number of earthquakes after time t and S_i is the time of the ith earthquake. Using Campbell's theorem or otherwise, calculate, in terms of μ, the expected total damage at exactly 10 years. [13]

Total [20]

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