University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER 1 EXAMINATIONS 2013/2014

STAT 30090 - STAT 40680

Models – Stochastic Models

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Time Allowed: 2 hours

Instructions for Candidates

Attempt all questions. The number of marks for each question is given. The total number of marks is given.

Instructions for Invigilators

Candidates will not require
New Cambridge Statistical Tables.
Calculators are permitted.
Graph paper is not required.

1. Let $\{X_n, n \in \mathbb{N}\}$ be a martingale with independent increments.

   (i) Give the definition of process with independent increments. [2]

   (ii) Give the definition of a martingale. [3]

   (iii) Prove that the expectation of the increments of (X_n) is 0. [5]

   (iv) Assume that the variance of the increments is constant, equal to σ^2. Let $Y_n = X_n^2 - n\sigma^2$. Prove that (Y_n) is a martingale. [10]

   Total [20]

2. A finite Markov chain X_n with state space $\{1, 2, 3, 4, 5, 6, 7\}$ has the transition matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & ? & 0.5 & 0 & 0 \\ 0 & 0 & 0 & ? & 0 & 1 & 0 \\ 0 & 0 & 0 & ? & 0.5 & 0 & 0 \\ 0 & 0 & 0 & ? & 0 & 0.1 & 0 \\ 0.5 & 0 & 0.5 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & ? & 0 & 0.5 & 0 \\ 0 & 0 & 0 & ? & 0 & 1 & 0 \end{bmatrix}.$$

   (i) Complete the matrix. [2]

   (ii) Plot the corresponding graph of transition probabilities. [4]

   (iii) List all the communication classes of this Markov chain. State, for each class, whether it is recurrent or transient. [3]

   (iv) Starting from $X_0 = 1$, what is the long term probability that the chain is in state 5? In state 6? [6]

   Total: [15]

3. A finite Markov chain X_n with state space $\{1, 2, 3, 4\}$ has the transition matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.3 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.7 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

   (i) Plot the corresponding graph. [2]

   (ii) List all the communication classes of this Markov chain, and precise, for each class, whether it is recurrent or transient. [2]

   (iii) When the initial distribution of the chain is given by $\bar{p}_0 = (0.5, 0, 0.2, 0.3)$, what is the probability that $X_2 = 2$? [5]

(iv) Let Y_j be the total number of visits to state j. What is $\mathbb{E}(Y_2|X_0 = 1)$? $\mathbb{E}(Y_1|X_0 = 1)$? $\mathbb{E}(Y_4|X_0 = 4)$? [6]

Total: [15]

4. Claims arrive at a small insurance company according to a Poisson process. On average, 25 claims arrive every week. Claims' sizes are independent of each other and have an exponential distribution with mean 250 euros.

   (i) Find the probability that over a period of four weeks, at most one claim will exceed $1,000$ euros. [5]

   (ii) Given that over a week the company receives 10 claims, find the probability that all but one arrives within the first three days. [5]

   (iii) What is the distribution of the waiting time between claims in excess of 500 euros? [5]

Total [15]

5. A mutant cell produces 0 offspring with probability 0.5 and 3 offsprings with probability 0.5. (You may assume that cells produce offsprings independently of one another.) Let X_n denote the size of the population at generation n.

   (i) What is the extinction probability for a colony when $X_0 = 1$? [5]

   (ii) We define the following sequence of numbers, $f_n = \mathbb{P}(X_n = 0|X_0 = 1)$. Prove that for any $n \geq 1$, $f_n = 0.5f_{n-1}^3 + 0.5$. [8]

   (iii) Calculate f_0, f_1, ..., f_6 and compare to (i). [4]

   Let d_n be the probability that the population dies at time n given that it was still alive at time $n-1$.

   (iv) Prove that d_n can be written as a function of f_n and f_{n-1}. [6]

   (v) Give a numerical approximation for d_7. [2]

Total [25]

6. Let $(W_t, t \in \mathbb{R}_+)$ be the standard Brownian motion. We are interested in the random variable

$$X = \int_0^{\frac{\pi}{2}} W_u \sqrt{\sin(u)} \mathrm{d}W_u.$$

   (i) What are the defining properties of (W_u)? [3]

   (ii) Calculate $\mathbb{E}(X)$. [2]

   (iii) Calculate $\mathrm{Var}(X)$. [5]

Total [10]

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