Solitons

Rosa Flaquer Galmés

1 Adimensionalitzation

The Gross-Pitaievskii equation describing the evolution of Bose-Einstein condensates in 1D reads as:

$$i\hbar\frac{\partial\Psi(z,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(z,t)}{\partial z^2} + gn\Psi(z,t) + V_{ext}(z,t) \tag{1}$$

where g is $g = \frac{4\pi\hbar^2 a}{m}$ and $n = |\Psi(z,t)|^2$. Using a new set of units:

$$\tilde{z} = \frac{z}{\xi}, \qquad \tilde{v} = \frac{v}{c}, \qquad \tilde{t} = \frac{t(|g|n_i)}{\hbar} \tag{2}$$

with c the speed of sound, $n_i = n_\infty$ for grey solitons and $n_i = n_0$ for bright solitons. ξ is the healing lenght of the soliton. We substitute for the new variables:

$$\frac{\partial\Phi(\tilde{z},\tilde{t})}{\partial t} = \frac{\partial\tilde{t}}{\partial t}\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}} = \frac{|g|n_i}{\hbar}\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial\Phi(\tilde{z},\tilde{t})}{\partial z}\right) = \frac{\partial\tilde{z}}{\partial z}\frac{\partial}{\partial\tilde{z}}\left(\frac{\partial\tilde{z}}{\partial z}\frac{\partial\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}}\right) = \frac{1}{\xi^2}\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2}$$

and we plug this expressions in equation 1:

$$\frac{i|g|n_i}{\hbar}\hbar\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}} = -\frac{\hbar^2}{2m\xi^2}\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} + gn\Psi(\tilde{z},\tilde{t}) + V_{ext}(\tilde{z},\tilde{t})$$

If we work in conditions where:

$$\frac{\hbar^2}{m\xi^2} \approx |g|n_i$$

then the equation reads as

$$i|g|n_i\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}} = -\frac{|g|n_i}{2}\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} + gn\Psi(\tilde{z},\tilde{t}) + V_{ext}(\tilde{z},\tilde{t}) \quad \rightarrow$$

$$i\frac{\partial\Phi(\tilde{z},\bar{t})}{\partial\tilde{t}} = -\frac{1}{2}\frac{\partial^2\Phi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} + \frac{g}{|g|}\frac{n}{n_i}\Phi(\tilde{z},\bar{t}) + \frac{V_{ext}(\tilde{z},\tilde{t})}{|g|n_i} \tag{3}$$

where $\frac{g}{|g|} = 1$ for grey solitons and -1 for bright solitons.

2 Solutions for $V_{ext} = 0$

We take the solutions for the time-independent GP equation. Bright and dark solitons must be considered separately.

Grey soliton:

$$\Psi(\tilde{z},\tilde{t}) = \sqrt{n_\infty}(i\tilde{v} + \sqrt{1-\tilde{v}^2})tanh\left[\left(\frac{\tilde{z}}{\sqrt{2}} - \tilde{v}\tilde{t}\right)\sqrt{1-\tilde{v}^2}\right] \tag{4}$$

For $\tilde{v} = 0$ it is called a black soliton.

Bright soliton:

$$\Psi(\tilde{z},\tilde{t}) = \sqrt{n_0}\frac{1}{cosh(\tilde{z}-\tilde{z_0}-\tilde{v}\tilde{t})}\exp^{i\tilde{v}(\tilde{z}-\tilde{z_0})}\exp^{i\tilde{t}(\frac{1}{2}-\frac{\tilde{v}^2}{2})} \tag{5}$$

As we are going to work more with bright solitons it is convenient to check if it is solution of the GP equation.

$$\frac{\partial\Psi(\tilde{z},\tilde{t})}{\partial\tilde{t}} = \sqrt{n_0}\left[\left(-\frac{sinh(A)}{cosh^2(A)}(-\tilde{v})B\right) + \frac{iB}{cosh(A)}\left(\frac{1}{2}-\frac{\tilde{v}^2}{2}\right)\right]$$

with $A \equiv \tilde{z} - \tilde{z_0} - \tilde{v}\tilde{t}$ and $B \equiv \exp^{i\tilde{v}(\tilde{z}-\tilde{z_0})}\exp^{i\tilde{t}(\frac{1}{2}-\frac{\tilde{v}^2}{2})}$.

$$\frac{\partial\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}} = \sqrt{n_0}\left[-\frac{sinh(A)}{cosh^2(A)}B + \frac{iB\tilde{v}}{cosh(A)}\right]$$

$$\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} = \sqrt{n_0}\left[\left(-\frac{cosh^2(A)-2sinh^2(A)}{cosh^3(A)}\right)B - i\tilde{v}B\frac{sinh(A)}{cosh^2(A)} + i\tilde{v}B\left(-\frac{sinh(A)}{cosh^2(A)} + \frac{i\tilde{v}}{cosh(A)}\right)\right]$$

$$\frac{gn}{|g|n_0}\Psi(\tilde{z},\tilde{t}) = -\frac{1}{cosh^2(A)}\frac{\sqrt{n_0}}{cosh(A)}B$$

We substitute this expressions in equation 1 for $V_{ext} = 0$ and consider the immaginary part and the real one separately.

Imaginary:

$$\frac{sinh(A)}{cosh^2(A)}\tilde{v} = -\frac{1}{2}\left[-\frac{sinh(A)}{cosh^2(A)}\tilde{v} - \frac{sinh(A)}{cosh^2(A)}\tilde{v}\right] \quad \rightarrow 1 = 1$$

Real:

$$-\frac{1}{cosh(A)}\left(\frac{1}{2}-\frac{\tilde{v}^2}{2}\right) = -\frac{1}{2}\left[-\frac{cosh^2(A)-2sinh^2(A)}{cosh^3(A)} - \frac{\tilde{v}^2}{cosh(A)}\right] - \frac{1}{cosh^3(A)} \rightarrow$$

$$-\frac{1}{2} = \frac{1}{2}\left(1-2\frac{sinh^2(A)}{cosh^2(A)}\right) - \frac{1}{cosh^2(A)} = +\frac{1}{2}\left[1-2\left(1-\frac{1}{cosh^2(A)}\right)\right] - \frac{1}{cosh^2 A} = -\frac{1}{2}$$

And we ensure that this solution fullfills the GP equation with our adimensionalized variables.

3 Crank Nicolson

We derive the Crank Nicolson method for equation 3. We include the external potential because altough we will set it equal to 0 when computing the time evolution of solitons it will be useful to make some tests of the method with an harmonic potential and g=0.

$$\begin{aligned}
\frac{\partial\Psi(\tilde{z},\tilde{t})}{\partial\tilde{t}} =& \frac{\Psi(\tilde{z},\tilde{t}+\frac{\Delta\tilde{t}}{2})-\Psi(\tilde{z},\tilde{t})}{\frac{\Delta\tilde{t}}{2}} = i\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} \quad \Rightarrow \\
\Psi(\tilde{z},\tilde{t}+\frac{\Delta\tilde{t}}{2}) =& \big[1+i\frac{\Delta\tilde{t}}{2}\big(\frac{1}{2}\frac{\partial^2}{\partial\tilde{z}^2}+\frac{g}{|g|}\frac{|\Psi(\tilde{z},\tilde{t})|^2}{n_j}+V_{ext}(\tilde{z})\big)\big]\Psi(\tilde{z},\tilde{t}) \\
\Psi(\tilde{z},\tilde{t}+\frac{\Delta\tilde{t}}{2}) =& \big[1-i\frac{\Delta\tilde{t}}{2}\big(\frac{1}{2}\frac{\partial^2}{\partial\tilde{z}^2}+\frac{g}{|g|}\frac{|\Psi(\tilde{z},\tilde{t}+\Delta\tilde{t})|^2}{n_j}+V_{ext}(\tilde{z})\big)\big]\Psi(\tilde{z},\tilde{t}+\Delta\tilde{t})
\end{aligned} \tag{6}$$

from where we find:

$$\begin{aligned}
&\big[1+i\frac{\Delta\tilde{t}}{2}\big(\frac{\partial^2}{\frac{1}{2}\partial\tilde{z}^2}+\frac{g}{|g|}\frac{|\Psi(\tilde{z},\tilde{t})|^2}{n_j}+V_{ext}(\tilde{z}\big)\big]\Psi(\tilde{z},\tilde{t}) = \\
&\qquad \big[1-i\frac{\Delta\tilde{t}}{2}\big(\frac{1}{2}\frac{\partial^2}{\partial\tilde{z}^2}+\frac{g}{|g|}\frac{|\Psi(\tilde{z},\tilde{t}+\Delta\tilde{t})|^2}{n_j}+V_{ext}(\tilde{z}\big)\big]\Psi(\tilde{z},\tilde{t}+\Delta\tilde{t})
\end{aligned} \tag{7}$$

We then use a three point formula for the spatial derivative and change the notation to $\Psi(\tilde{z},\tilde{t}) \equiv \Psi_i^t$ so that $\Psi(\tilde{z}+\Delta\tilde{z},\tilde{t}) = \Psi_{i+1}^t$ and $\Psi(\tilde{z},\tilde{t}+\Delta\tilde{t}) = \Psi_i^{t+1}$

$$\begin{aligned}
&\Psi_i^t + i\frac{\Delta\tilde{t}}{2}\left[\frac{1}{2}\frac{\Psi_{i+1}^t-2\Psi_i^t+\Psi_{i-1}^t}{(\Delta\tilde{z})^2}-\frac{g}{|g|}\frac{|\Psi_i^t|^2}{n_j}\Psi_i^t - V_i|\Psi_i^t\right] = \\
&\qquad \Psi_i^{t+1} i\frac{\Delta\tilde{t}}{2}\left[-\frac{1}{2}\frac{\Psi_{i+1}^{t+1}-2\Psi_i^{t+1}+\Psi_{i-1}^{t+1}}{(\Delta\tilde{z})^2}+\frac{g}{|g|}\frac{|\Psi_i^{t+1}|^2}{n_j}\Psi_i^{t+1}-V_i\Psi_i^{t+1}\right]
\end{aligned} \tag{8}$$

Defining $r \equiv i\frac{\Delta\tilde{t}}{4(\Delta\tilde{z})^2}$ it takes the form of

$$\begin{aligned}
&\left(1+2r+\frac{2g\Delta\tilde{z}^2|\Psi_i^{t+1}|^2}{|g|n_j}+2r\Delta\tilde{z}^2V_i\right)\Psi_i^{t+1} - r\Psi_{i+1}^{t+1} - r\Psi_{i-1}^{t+1} = \\
&\qquad \left(1-2r-\frac{2g\Delta\tilde{z}^2|\Psi_i^t|^2}{|g|n_j}-2r\Delta\tilde{z}^2V_i\right)\Psi_i^t + r\Psi_{i+1}^t + r\Psi_{i-1}^t
\end{aligned} \tag{9}$$

Which turns to be a matrix equation if we define a vector u^t that contains all Ψ_i^t values at a given time.

$$Au^{t+1} = Bu^t \tag{10}$$

A is a tridiagonal matrix with $\left(1+2r+\frac{2g\Delta\tilde{z}^2|\Psi_i^{t+1}|^2}{|g|n_j}+2r\Delta\tilde{z}^2V_i\right)$ in its main diagonal and $-r$ on the upper and lower diagonals, all the other elements are 0. B is also a tridiagonal matrix with $1-2r-\frac{2g\Delta\tilde{z}^2|\Psi_i^t|^2}{|g|n_j}-2r\Delta\tilde{z}^2V_i$ in its main diagonal and r on the upper and lower diagonals, all the other elements are 0. We notice that the main diagonal of A depends on the vale of $\Psi_i^{t+1}|^2$ wich is precisely what we are trying to compute. To solve this problem we will approximate it to $\Psi_i^t|^2$.

There are various ways to check if the method is working properly: conservation of the norm and conservation of the energy. The easiest is the first one. First of all we ensure that our $\Psi(\tilde{z},\tilde{t})$ is well normalizet to 1 using any integration method, for instance Simpson:

$$1 = \|N\|^2\int_a^b \Psi^*(\tilde{z},\tilde{t})\Psi(\tilde{z},\tilde{t})d\tilde{z} \approx \|N\|^2\frac{d\tilde{z}}{3}\sum_{k=0}^{M/2-1}(u_{2k}^t+4u_{2k+1}^t+u_{2k+2}^t) \tag{11}$$

Where M is the number of points where the function is avaluated. Once we have the wavefunction propperly normalized we can check at every step of the Crank-Nicolson method if it is conserved by computing the norm of u^{t+1} and storing the maximum difference between the norm at $\tilde{t} = 0$ and the evolved function.

To compute the expextation value of the energy:

$$\langle E \rangle = \int_{-\infty}^{\infty} \Psi^*(\tilde{z},\tilde{t}) \hat{H} \Psi(\tilde{z},\tilde{t}) d\tilde{z} = \int_{-\infty}^{\infty} \Psi^*(\tilde{z},\tilde{t}) \big(-\frac{\partial^2}{\partial \tilde{z}^2} \big) \Psi(\tilde{z},\tilde{t}) d\tilde{z} = \int_{-\infty}^{\infty} \Psi^*(\tilde{z},\tilde{t}) \big(i \frac{\partial \Psi(\tilde{z},\tilde{t})}{\partial \tilde{t}} \big) d\tilde{z} \tag{12}$$

It is useful to write $(\hat{H})$ like this (using equation 3) because we can then write $\frac{\partial \Psi(\tilde{z},\tilde{t})}{\partial \tilde{t}}$ as:

$$\frac{\partial \Psi(\tilde{z},\tilde{t})}{\partial \tilde{t}} \approx \frac{\Psi(\tilde{z},\tilde{t}+\delta\tilde{t}) - \Psi(\tilde{z},\tilde{t})}{\delta\tilde{t}} \tag{13}$$

and both of this terms are known in every iteration of the Crank Nicolson method so we only have to focus on computing the integral. Using the u_1^t notation:

$$\langle E \rangle = \int_{-\infty}^{\infty} (u_i^t)^* \frac{i}{\delta\tilde{t}} (u_i^{t+1} - u_i^t) d\tilde{z} \tag{14}$$

This integral can be solved using, for example, the Simpson's method again.

4 First test, $V_{ext} = 0$ and g=0

To start the resoluiton of the problem we compute with the Crank Nicolson method the time evolution for $g = 0$ and $V_{ext} = 0$ then the equation reads as:

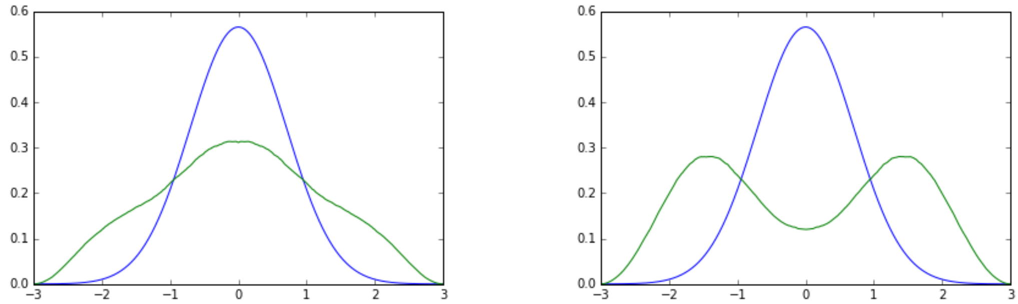
$$i\frac{\partial\Psi(\tilde{z},\tilde{t})}{\partial\tilde{t}} = -\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} \tag{15}$$

Note: this was the first test done and the equations used were slightly different from the ones I used in the rest of the problem. In this case the factor $\frac{1}{2}$ was included in the set of adimensionalized variables, this implies that the bright solution of the soliton reads as: $\Psi(\tilde{z},\tilde{t}) = \sqrt{N_0}\frac{1}{cosh(\frac{\tilde{z}-\tilde{z}_0}{\sqrt{2}}-\tilde{v}\tilde{t})}e^{i\tilde{v}\frac{\tilde{z}-\tilde{z}_0}{\sqrt{2}}}e^{i\tilde{t}(\frac{1}{2}-\frac{\tilde{v}^2}{4})}$ and also implies that the parameter r of the Crank Nicolson method is $r = r \equiv i\frac{\Delta\tilde{t}}{2(\Delta\tilde{z})^2}$.

4.1 Gaussian

To start testing the method we compute the time evolution for a gaussian function centered at zero and with $\sigma = 1$. We expect it to evolve as it is not an eigenfunction of $H = -\frac{\partial^2}{\partial\tilde{z}^2}$.

We compute the method for a spacing of $\Delta\tilde{z} = 0.01$ with $\tilde{z} \in [-3, 3]$ and a time interval $\delta\tilde{t} = 0.001$.



(a) $\tilde{t} = 5$ (b) $\tilde{t} = 10$

Figure 1: The initial $|\Psi|^2$ is represented in blue and the evolved one in green

	Norm difference
$\tilde{t} = 5s$	$1.162x10^{-7}$
$\tilde{t} = 10s$	$1.162x10^{-7}$

Table 1: Computing time and norm diference for the plots in fig 1

4.2 Bright soliton

4.2.1 null velocity

We compute the method for a spacing of $\Delta\tilde{z} = 0.01$ with $\tilde{z} \in [-10, 10]$ and a time interval $\delta\tilde{t} = 0.001$ for a bright soliton solution with $n_0 = 1$

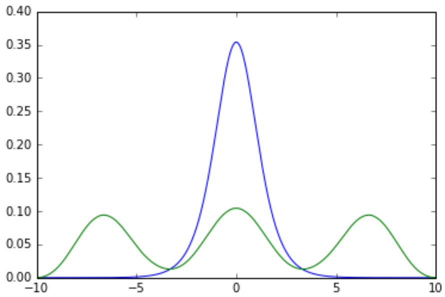


Figure 2: $|\Psi|^2$ *at time* $\tilde{t} = 0$ *in blue and at time* $\tilde{t} = 10$ *in green*

The maximum diference between the norm at $\tilde{t} = 0$ and the norm of the evolving function was: $1.4.59786764218x10^{-9}$

4.2.2 0.5 velocity

We compute with $\tilde{v} = 0.5$

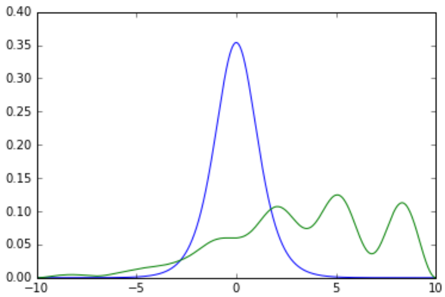
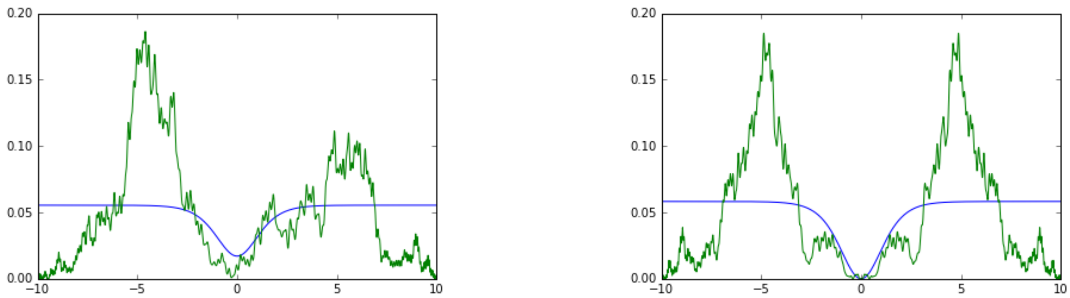


Figure 3: $|\Psi|^2$ *at time* $\tilde{t} = 0$ *in blue and at time* $\tilde{t} = 5$ *in green with* $\delta\tilde{t} = 0.01$

The maximum diference between the norm at $\tilde{t} = 0$ and the norm of the evolving function was: $3.06995195931x10^{-9}$

4.3 Grey soliton

We run the code for the grey solution for a spacing of $\Delta\tilde{z} = 0.01$ with $\tilde{z} \in [-10, 10]$ and a time interval $\delta\tilde{t} = 0.01$ and the results found are:



(a) $\tilde{v} = 0.5$ (b) $\tilde{v} = 0$

Figure 4: The initial $|\Psi|^2$ is represented in blue and the evolved one in green

After discusing this results we see the dificulty on working with grey solitons as the Crank Nicolson method supposes implicitely a vanishing function at the extrems whereas a grey soliton

	Norm difference
$\tilde{v} = 0.5$	$9.441x10^{-5}$
$\tilde{v} = 0$	$9.703x10^{-5}$

Table 2: Computing time and norm diference for the plots in fig 4

has $|\Psi(\tilde{z}, \tilde{t})|^2 = n_\infty$ which is different from 0. We also see that the interval used in the gaussian section is too small as the probability density does not mantain its shape during the time evolution.

5 Second test: Harmonic Oscillator

5.1 Free space

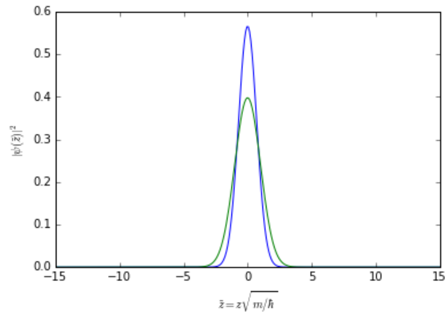
To make some checks to the Crank-Nicolson method developed we focus on a known case, the ground state of an 1-D harmonic oscillator. We know that the eigenfunction for the time independent Shrödinger equation is:

$$\Psi(\bar{z}) = AH_0(\bar{z})e^{-\frac{\bar{z}^2}{2}} = \frac{e^{-\frac{\bar{z}^2}{2}}}{\pi^{\frac{1}{4}}} \tag{16}$$

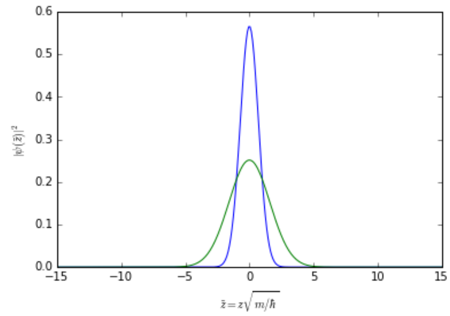
with $\bar{z} = \sqrt{\frac{m\omega}{\hbar}}z$. We start by studiyng the evolution of this wavefunction in free space:

$$i\hbar\frac{\partial\Psi(z,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(z,t)}{\partial z^2} \Rightarrow i\frac{\partial\Psi(\tilde{z},t)}{\partial t} = -\frac{1}{2}\frac{\partial^2\Psi(\tilde{z},t)}{\partial\tilde{z}^2} \tag{17}$$

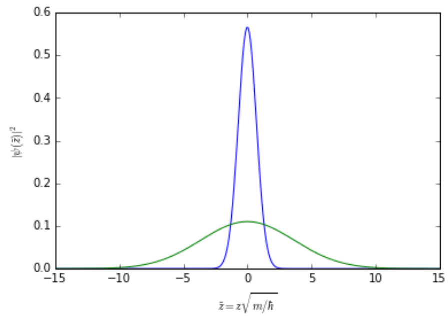
Wiht $\tilde{z} = \sqrt{\frac{m}{\hbar}}$. As the wave function is an eigenfunction for the harmonic potential we expect that when we compute the time evolution for free space it will mantaint the shape but it will widen with time. We run the method for different values of t:



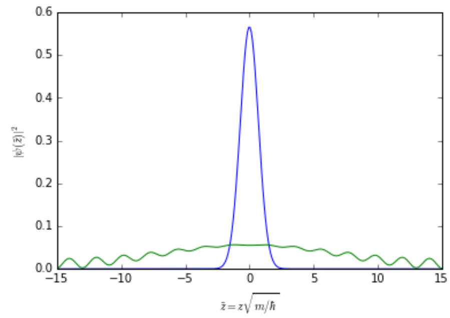
(a) t=1s



(b) t=2s



(c) t=5s



(d) t=10s

Figure 5: Results obteined for a box of $\tilde{z} = [-15, 15]$ with intervals of $d\tilde{z} = 0.1$ and a time interval of $dt = 0.009$ which means that the parameter r for the Crank Nicolson method is $r = 0.225j$. In blue it's represented the value for $|\Psi(\tilde{z})|^2$ at time $t = 0$ and in green the evolved.

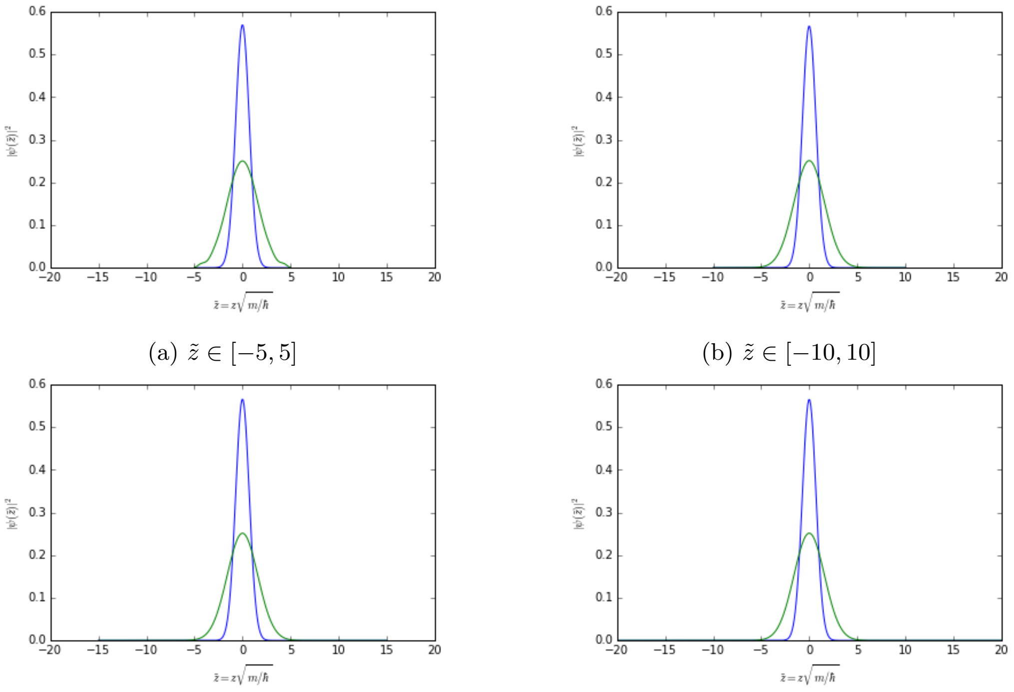
	Norm difference	computing time (s)
t=1s	$2.664x10^{-15}$	0.329
t=2s	$5.33x10^{-15}$	2.234
t=5s	$9.202x10^{-8}$	4.895
t=10s	$8.979x10^{-6}$	10.650

Table 3: Computing time and norm diference for the plots in fig 5

As can be seen the method works properly for times no much bigger than t=5s.

5.1.1 Dependence on the z limits

We make some tests to see how the extension of the system affects the results.



(a) $\tilde{z} \in [-5, 5]$

(b) $\tilde{z} \in [-10, 10]$

(c) $\tilde{z} \in [-15, 15]$

(d) $\tilde{z} \in [-20, 20]$

Figure 6: Results obteined for a fixed value of t=2s. The intervals used are $d\tilde{z} = 0.1$ and $dt = 0.009$ which means that the parameter r for the Crank Nicolson method is $r = 0.225j$. In blue it's represented the value for $|\Psi(\tilde{z})|^2$ at time $t = 0s$ and in green the evolved to t=2s.

We look at the conservation of the norm

	Norm difference
$\tilde{z} \in [-5, 5]$	$5.440x10^{-6}$
$\tilde{z} \in [-10, 10]$	$3.625x10^{-12}$
$\tilde{z} \in [-15, 15]$	$5.330x10^{-15}$
$\tilde{z} \in [-20, 20]$	$5.773x10^{-15}$

Table 4: Computing time and norm diference for the plots in fig 6

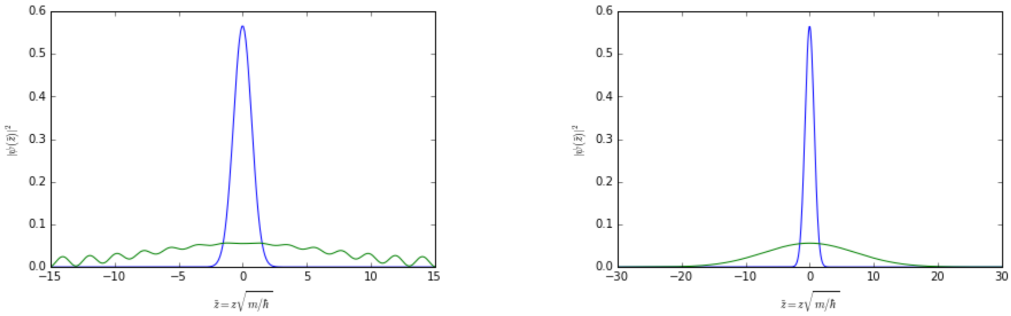
Finally we show that if we want to compute the time evolution for long times we we will need a higher number of points z. As it can be seen in figure 7 for the same time t=10s if we double the points z the function obtained is much reasonable plus there is a factor two in improving of the conservation of the norm.

	Norm difference	computing time (s)
$\tilde{z} \in [-15, 15]$	$8.987x10^{-6}$	2.689
$\tilde{z} \in [-30, 30]$	3.610^{-8}	5.052

Table 5: Computing time and norm diference for the plots in fig 7

5.2 Harmonic potential

Now we compute the method with the harmonic potential. The time-dependent Schrödinger equation reads as:



(a) $\tilde{z} \in [-15, 15]$

(b) $\tilde{z} \in [-30, 30]$

Figure 7: Results obteined for a fixed value of t=10s. The intervals used are $d\tilde{z} = 0.1$ and $dt = 0.009$ which means that the parameter r for the Crank Nicolson method is $r = 0.225j$. In blue it's represented the value for $|\Psi(\tilde{z})|^2$ at time $t = 0s$ and in green the evolved to t=10s.

$$i\hbar\frac{\partial\Psi(z,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(z,t)}{\partial z^2} + \frac{kx^2}{2}\Psi(z,t) \Rightarrow i\frac{\partial\Psi(\tilde{z},t)}{\partial t} = -\frac{1}{2}\big(\frac{\partial^2}{\partial\tilde{z}^2} + \tilde{z}^2\big)\Psi(\tilde{z},t) \qquad (18)$$

with $\tilde{z} = \sqrt{\frac{m\omega}{\hbar}}z$ and $\omega = \sqrt{\frac{k}{m}}$. We expect that the probability $|\Psi(\tilde{z})|^2$ does not evolve as the function we are using is the eigene-function of the ground state of this system. We compute the method for a time t=100s and the resul obtained is:

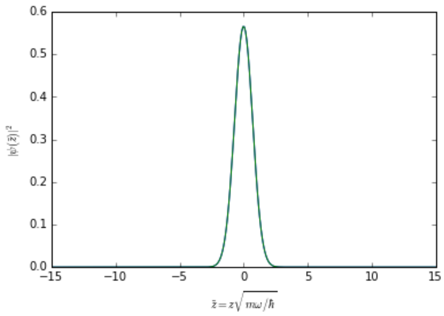


Figure 8: Results obteined for a box of $\tilde{z} = [-15, 15]$ with intervals of $d\tilde{z} = 0.1$ and a time interval of $dt = 0.009$ which means that the parameter r for the Crank Nicolson method is $r = 0.225j$. In blue it's represented the value for $|\Psi(\tilde{z})|^2$ at time $t = 0$ and in green at t=100s. The difference in the norm at t=0s and at t=100s is $1.924x10^{-13}$.

6 Third test: Interaction

Here we compute the time evolution for bright solitons which, in the adimensionalized variables we are using reads as:

$$\Psi(\tilde{z},\tilde{t}) = \sqrt{n_0}\frac{1}{cosh(\tilde{z}-\tilde{z_0}-\tilde{v}\tilde{t})}e^{i\tilde{v}(\tilde{z}-\tilde{z_0})}e^{i(\frac{1}{2}-\frac{\tilde{v}^2}{4})\tilde{t}} \tag{19}$$

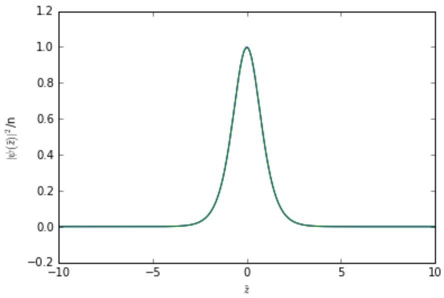
whith $\tilde{z} = \frac{z}{\xi}, \tilde{v} = \frac{v}{c}, \tilde{t} = \frac{t(|g|n_i)}{\hbar}$.

6.1 Null velocity

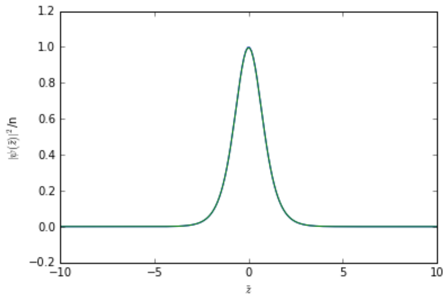
We start by computing the evolution for a bright soliton whith null velocity. We expect that if the method is working properly the density profile will not evolve whith time as it is an eigenestate of the Gross-Pitaievskii equation. Relying on the results obtained for the harmonic oscillator, we use the following spacing and intervals:

1. $\tilde{z} \in [-15, 15]$
2. $d\tilde{z} = 0.1$
3. $d\tilde{t} = 0.009$

Whith this spacing $r = 0.225j$.



(a) $t = 10s$



(b) $t = 100s$

Figure 9: Plot of the density profile normalized to n_0. In both figure 9a and 9b the it is represented the density profile at t=0s in blue and the evolved one in green. The norm is conserved up to 10^{-12} for figure 9a and to 10^{-11} for figure 9b.

6.2 Velocity

When we try to calcule the time evolution of a soliton with velocity different than 0 we see that the probability density does not evolve mantaining its shape, this does not match with what we expect form the theory. I tried to see if the problem is that the function that I am using has some problems in the arguments of the complex exponential $\exp^{i\tilde{v}(\tilde{z}-\tilde{z_0})}\exp^{i\tilde{t}(\frac{1}{2}-\frac{\tilde{v}^2}{2})}$ which will vanish for $\tilde{v} = 0$ and pass the first test without any problem. As far as I have checked it I still do not find the problem.

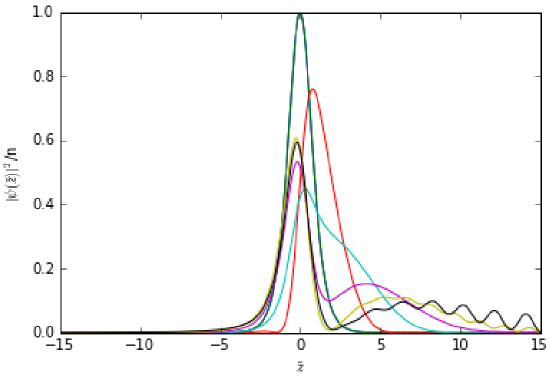


Figure 10: Representations of $|\Phi(\tilde{z},\tilde{t})$ every 200 steps of the time evolution from $\tilde{t}=0s$ to $\tilde{t}=10s$ with a time step of $d\tilde{t}=0.009$ and a velocity $\tilde{v}=0.9$