Adimensionalitzation of the Gross-Pitaievskii equation

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The Gross-Pitaievskii equation describing the evolution of Bose-Einstein condensates in 1D and $V_{ext} = 0$ reads as:

$$i\hbar\frac{\partial\Psi(z,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(z,t)}{\partial z^2} + gn\Psi(z,t) \tag{1}$$

where g is $g = \frac{4\pi\hbar^2 a}{m}$, $n = |\Psi(z,t)|^2$ and a is the s-wave scattering lenght. Using a new set of units:

$$\tilde{z} = \frac{z}{\xi}, \qquad \tilde{v} = \frac{v}{c}, \qquad \tilde{t} = \frac{t(|g|n_i)}{\hbar} \tag{2}$$

with c the sound's speed, $n_i = n_\infty$ for grey solitons and $n_i = n_0$ for bright solitons. ξ is the healing lenght of the soliton. We substitute for the new variables:

$$\frac{\partial\Phi(\tilde{z},\tilde{t})}{\partial t} = \frac{\partial\tilde{t}}{\partial t}\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}} = \frac{|g|n_i}{\hbar}\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial\Phi(\tilde{z},\tilde{t})}{\partial z}\right) = \frac{\partial\tilde{z}}{\partial z}\frac{\partial}{\partial\tilde{z}}\left(\frac{\partial\tilde{z}}{\partial z}\frac{\partial\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}}\right) = \frac{1}{\xi^2}\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2}$$

and we plug this expressions in equation (1):

$$\frac{i|g|n_i}{\hbar}\hbar\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}} = -\frac{\hbar^2}{2m\xi^2}\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} + gn\Psi(\tilde{z},\tilde{t}) \rightarrow$$

If we work in conditions where:

$$\frac{\hbar^2}{m\xi^2} \approx |g|n_i$$

then the equation reads as

$$i|g|n_i\frac{\partial\Psi(\tilde{z},\bar{t})}{\partial\tilde{t}} = -\frac{|g|n_i}{2}\frac{\partial^2\Psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} + gn\Psi(\tilde{z},\tilde{t}) \rightarrow$$

$$i\frac{\partial\Phi(\tilde{z},\bar{t})}{\partial\tilde{t}} = -\frac{1}{2}\frac{\partial^2\Phi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} + \frac{g}{|g|}\frac{n}{n_i}\Phi(\tilde{z},\bar{t}) \tag{3}$$

where $\frac{g}{|g|} = 1$ for grey solitons and -1 for bright solitons.