Math 271.1: Exercise 2 (#2)

INSTRUCTION: Polynomial Regression (Quadratic) via Normal Equations and Cholesky Factorization

(a) Derive the normal equations for the least squares solution.
(b) Implement your own Cholesky factorization routine to solve the resulting equations.
(c) Plot the observed data points (xi,yi) and the fitted quadratic curve on the same graph.

Importing Libraries needed

```
import numpy as np #for array operations and numeric computations
import matplotlib.pyplot as plt # for plotting the data and results

from IPython.display import display # for displaying arrays and matrices nicely
```

(a) Deriving the normal equations

Initializing the dataset (observed data points)

- creates the input data (x) snd the output data (y)
- will be used to fit the quadratic regression model

```
x_data = np.array([0, 1, 2, 3, 4, 5, 6, 7, 8, 9])
y_data = np.array([2.03, 4.02, 7.02, 10.45, 15.03,
                   20.60, 27.40, 35.35, 44.42, 54.38])

display(x_data, y_data)
```

```
array([0, 1, 2, 3, 4, 5, 6, 7, 8, 9])
array([ 2.03,  4.02,  7.02, 10.45, 15.03, 20.6 , 27.4 , 35.35, 44.42,
       54.38])
```

Build design matrix for quadratic model

- transforms data into the format needed for the quadratic regression
- we use each column to represent a term in the quadratic equation: $\beta_0 + \beta_1 x + \beta_2 x^2$

```
n = len(x_data)
X = np.zeros((n, 3))
X[:, 0] = 1      # Column of ones
X[:, 1] = x_data      # x values
X[:, 2] = x_data**2   # x^2 values

display(X)
```

```
array([[ 1.,  0.,  0.],
       [ 1.,  1.,  1.],
       [ 1.,  2.,  4.],
       [ 1.,  3.,  9.],
       [ 1.,  4., 16.],
       [ 1.,  5., 25.],
       [ 1.,  6., 36.],
       [ 1.,  7., 49.],
       [ 1.,  8., 64.],
       [ 1.,  9., 81.]])
```

```
# Symmetric positive definite matrix required for Cholesky
XTX = X.T @ X
XTy = X.T @ y_data

display(XTX, XTy)
```

```
array([[1.0000e+01, 4.5000e+01, 2.8500e+02],
       [4.5000e+01, 2.8500e+02, 2.0250e+03],
       [2.8500e+02, 2.0250e+03, 1.5333e+04]])
array([  220.7 ,  1469.16, 10847.84])
```

(b) Implementing Cholesky Factorization

Compute Cholesky factorization A = LL^T

```
A = XTX

n = A.shape[0]
L = np.zeros((n, n))

for i in range(n):
    for j in range(i+1):
        if i == j:
            # Diagonal element
            sum_val = np.sum(L[i, :i]**2)
            L[i, i] = np.sqrt(A[i, i] - sum_val)
        else:
            # Off-diagonal element
            sum_val = np.sum(L[i, :j] * L[j, :j])
            L[i, j] = (A[i, j] - sum_val) / L[j, j]

display(L)
```

```
array([[ 3.16227766,  0.        ,  0.        ],
       [14.23024947,  9.08295106,  0.        ],
       [90.12491331, 81.74655956, 22.97825059]])
```

Solving for coefficients

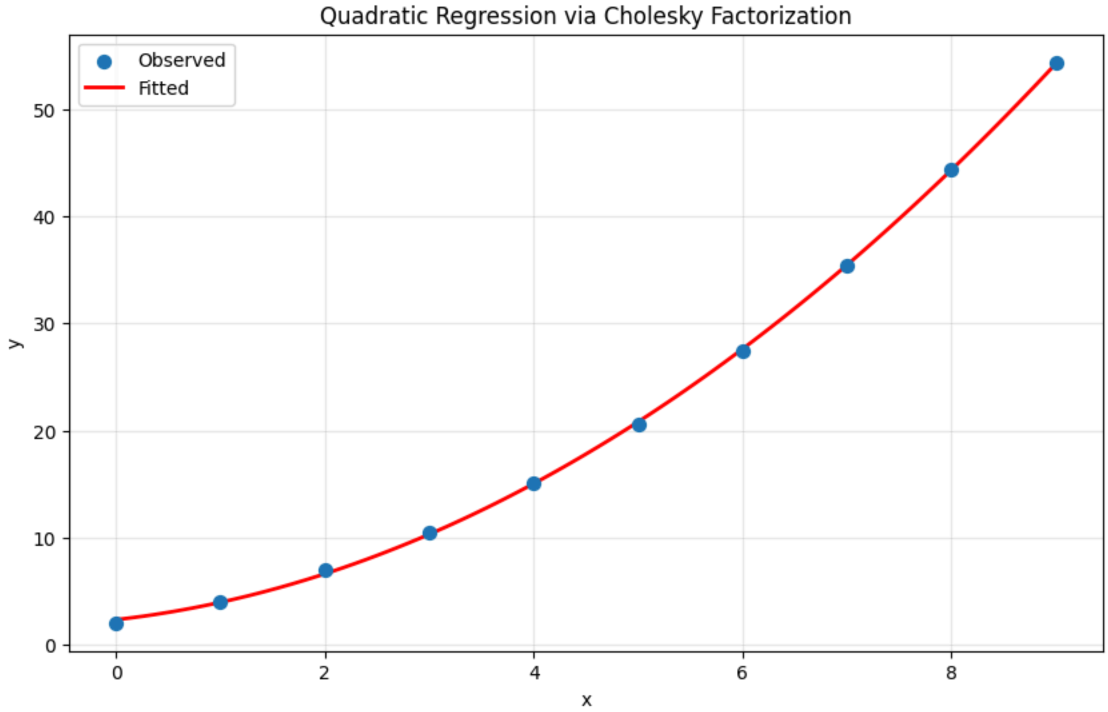
- Uses forward substitution to solve Lz = X^T y
- Uses backward substitution to solve L^T β = z
- The resulting coefficients define the quadratic model

```
# Solve for beta
z = np.linalg.solve(L, XTy)
beta = np.linalg.solve(L.T, z)

print(f"β₀ = {beta[0]:.4f} (true: 2.0)")
print(f"β₁ = {beta[1]:.4f} (true: 1.5)")
print(f"β₂ = {beta[2]:.4f} (true: 0.5)")
```

```
β₀ = 2.3285 (true: 2.0)
β₁ = 1.1028 (true: 1.5)
β₂ = 0.5186 (true: 0.5)
```

(c) Visualization: Plotting the coefficients and the fitted curve

```
x_plot = np.linspace(0, 9, 100)
y_plot = beta[0] + beta[1]*x_plot + beta[2]*x_plot**2

plt.figure(figsize=(10, 6))
plt.scatter(x_data, y_data, s=50, label='Observed', zorder=3)
plt.plot(x_plot, y_plot, 'r-', linewidth=2, label='Fitted')
plt.xlabel('x')
plt.ylabel('y')
plt.title('Quadratic Regression via Cholesky Factorization')
plt.legend()
plt.grid(True, alpha=0.3)
plt.show()
```



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