Final Project:
The Three Body System

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Intro to Math Modeling

December 19, 2022

Contents

1 Abstract

The three-body problem is a basic mechanical model in celestial mechanics which is an interesting topic in Physics, Astronomy, Mathematics. In this report, we first analyze the stability of the binary system, then we add a third mass which is small into that system, which is the well-known restricted three-body problem, and in the end we generalize the case of the three body problem and analyze the escape time of one of the three stars. We have set up two ways to define the escape time. Thus, I analyze the distance from each star to the center of mass, and my partner, Yi, analyze the energy of the escaping star.

2 Introduction

In the science fiction named *The Three Body Problem* written by Liu Cixin, there is an alien civilization living on a planet which is surrounded by three stars. The life of the people who lives in this chaotic world is divided into the "Chaotic Eras" and the "Stable Eras".

The chaos in such a world is due to the fact that the three-body system is not stable because the three-body problem cannot be solved exactly, which means all mathematical scenarios of the three-body problem cannot be predicted, and only a few special cases can be solved.

Also, the three-body problem refers to the law of movement of three celestial bodies whose masses, initial positions and initial velocities are arbitrary, which can be regarded as mass points, under the force of the mutual gravitation [3], with the fact that with a very small difference in initial conditions will result in a significant difference in the result [6].

3 Analysis and Results

3.1 Binary Star

The simpliest model for the n-body problem is the two-star system. For example, the sun and the earth form a binary system where the smaller star orbits around the bigger one. But actually, the sun does not remain fixed, it actually is both of the two stars circle around the center of mass of the two-body system (shown in 2 in Figure 1). But the mass of the sun extremely dominates that of the earth, so the moving of the sun is not obvious. Besides this kind of situation, there is another more usual situation where two stars orbiting in two ellipses around the center of mass (shown in 3 in Figure 1). [4]

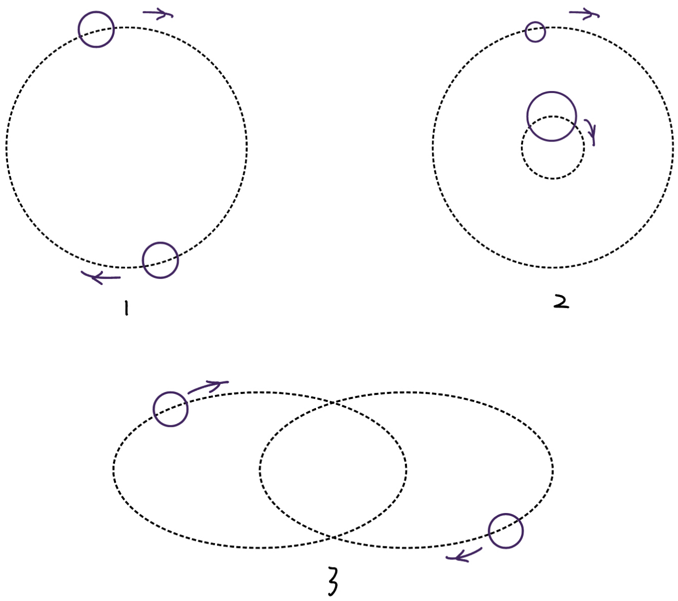

Figure 1: Two-Star System

We assume that the position of two stars is denoted by vector X_1,X_2, the mass is denoted by m_1, m_2, and G is gravitational constant. Then the position vector is $r_1 = X_1 - X_2$ and $r_2 = X_2 - X_1$ respectively. Let F_{12} be the force on m_1 due to interaction with m_2 and F_{21} is the force on m_2 due to interaction with m_1.

Applying Newton's second law, we have

$$F_{12} = m_1\ddot{r}_1 = m_1(\ddot{X}_1 - \ddot{X}_2)$$

$$F_{21} = m_2\ddot{r}_2 = m_2(\ddot{X}_2 - \ddot{X}_1)$$

Let R be the position of center of mass [5], and thus we have

$$(m_1 + m_2)R = m_1X_1 + m_2X_2$$

$$R = \frac{m_1X_1 + m_2X_2}{m_1 + m_2}$$

$$r_1 = R + \frac{m_2}{m_1 + m_2}(X_1 - X_2)$$

$$r_2 = R + \frac{m_1}{m_1 + m_2}(X_2 - X_1)$$

3.2 Restricted Three Star

The restricted three-body problem is a simplified version of the three-body problem where the third mass m_3 is much smaller than m_1 and m_2, which means m_1 and m_2

move in Keplerian orbits and will not be affected by m_3 [7]. Assume that the three stars move in the same plane and $m_1 > m_2$, then we have [7]

$$\frac{m_1 m_2}{m_1 + m_2} R\Omega^2 = \frac{G m_1 m_2}{R^2}$$

which is

$$\Omega^2 = \frac{G(m_1 + m_2)}{R^3}$$

where R is the distance between m_1 and m_2 and Ω is the angular frequency of m_1 and m_2 which are in circular orbit around each other.

By putting the three stars in a rotating frame and write the Lagrangian of m_3, we get the five Lagrangian points which is calculated as followed [1].

$$L_1: \quad r \approx R\sqrt[3]{\frac{m_2}{3m_1}} \tag{1}$$

$$L_2: \quad r \approx R\sqrt[3]{\frac{m_2}{3m_1}} \tag{2}$$

$$L_3: \quad r \approx R\frac{7m_2}{12m_1} \tag{3}$$

where r is the distance from the point to m_2 and $L_{1,2,3}$ are all on the same line with m_1 and m_2 (shown in Figure 2 [8]).

For L_4 and L_5, they lie at the third vertices of the two equilateral triangles whose base is the line formed by connecting m_1 and m_2 (shown in Figure 2 [8]).

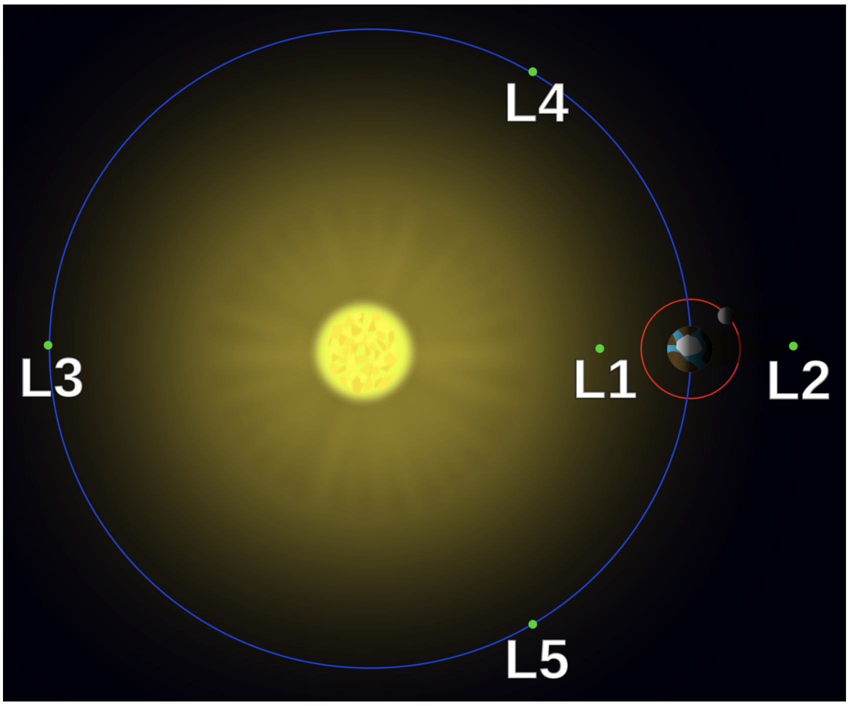

Figure 2: The Five Lagrangian Points

At the Lagrangian point, the gravitational and centrifugal forces of m_1 and m_2 balance each other, which means they are the great choice for the position of the satellites.

For the stability analysis, $L_{1,2,3}$ are always unstable while $L_{4,5}$ are stable if and only if $(m_1 + m_2)^2 \geq 27 m_1 m_2$, and more generally, for the unrestricted three-body problem, $L_{4,5}$ are stable if and only if $(m_1 + m_2 + m_3)^2 \geq 27(m_1 m_2 + m_2 m_3 + m_3 m_1)$[7].

Therefore, we can reasonably extrapolate that for the three-body problem, it is only predictable (has exact solutions) if the third star is at the Lagrangian point.

3.3 Three Star

Let X_1, X_2, and X_3 be the position vector of star 1, 2 and 3, then according to Newton's second law and Newton's law of universal gravitation, we get the following three equations: [3]

$$\ddot{X}_1 = G\left(\frac{m_2(X_2 - X_1)}{||X_2 - X_1||^3} + \frac{m_3(X_3 - X_1)}{||X_3 - X_1||^3}\right) \tag{4}$$

$$\ddot{X}_2 = G\left(\frac{m_3(X_3 - X_2)}{||X_3 - X_2||^3} + \frac{m_1(X_1 - X_2)}{||X_1 - X_2||^3}\right) \tag{5}$$

$$\ddot{X}_3 = G\left(\frac{m_1(X_1 - X_3)}{||X_1 - X_3||^3} + \frac{m_2(X_2 - X_3)}{||X_2 - X_3||^3}\right) \tag{6}$$

We set the position of the center of mass be the origin point, which means

$$m_1 X_1 + m_2 X_2 + m_3 X_3 = 0 \tag{7}$$

Due to the conservation of momentum, we have

$$m_1 \dot{X}_1 + m_2 \dot{X}_2 + m_3 \dot{X}_3 = 0 \tag{8}$$

3.3.1 Distance Model

We solve the above ODE system numerically by using the ODE solver named "ode45" in MATLAB (the detailed code is shown here [2]). We choose the initial position and the initial velocity for star 1 and star 2 randomly and according to the Eq. 7 and Eq. 8, we get the initial position and the initial velocity for star 3.

We use $\mathbf{u_1}, \mathbf{u_3}, \mathbf{u_5}$ to represent the positions of the three stars and $\mathbf{u_2}, \mathbf{u_4}, \mathbf{u_6}$ to represent the velocity of the three stars.

$$
\begin{aligned}
\dot{\mathbf{u}}_1 &= \mathbf{u}_2 \\
\dot{\mathbf{u}}_2 &= G\left(\frac{m_2\left(\mathbf{u}_3-\mathbf{u}_1\right)}{\left\|\mathbf{u}_3-\mathbf{u}_1\right\|^3}+\frac{m_3\left(\mathbf{u}_5-\mathbf{u}_1\right)}{\left\|\mathbf{u}_5-\mathbf{u}_1\right\|^3}\right) \\
\dot{\mathbf{u}}_3 &= \mathbf{u}_4 \\
\dot{\mathbf{u}}_4 &= G\left(\frac{m_3\left(\mathbf{u}_5-\mathbf{u}_3\right)}{\left\|\mathbf{u}_5-\mathbf{u}_3\right\|^3}+\frac{m_1\left(\mathbf{u}_1-\mathbf{u}_3\right)}{\left\|\mathbf{u}_1-\mathbf{u}_3\right\|^3}\right) \\
\dot{\mathbf{u}}_5 &= \mathbf{u}_6 \\
\dot{\mathbf{u}}_6 &= G\left(\frac{m_1\left(\mathbf{u}_1-\mathbf{u}_5\right)}{\left\|\mathbf{u}_1-\mathbf{u}_5\right\|^3}+\frac{m_2\left(\mathbf{u}_3-\mathbf{u}_5\right)}{\left\|\mathbf{u}_3-\mathbf{u}_5\right\|^3}\right)
\end{aligned}
$$

As shown above, for each vector, it contains the x,y,z-coordinates. Therefore, Eq. 4,5,6 become a system of 18 ODEs.

Since if there is an escape for one of these three stars, the escaping star will be the one that goes farther and farther away from the other two stars, which means it will also go farther and farther away from the mass center of the three-star system. Additionally, since the distance between the escaping star and the other two stars is too large, the gravitational force exerted on it by the other two stars is very small and can be neglected, which can roughly be viewed as the escaping star does not receive any external force. Then according to Newton's first law of motion, the escaping star will move at a constant velocity.

Therefore, we define the escape time to be the time at which one of the stars almost starts to move at a constant velocity.

We have tried some situations where the three stars lie on the three vertices of an equilateral triangle (just like the Lagrange point) with the same mass, and we see that the trajectories is regular. To see something chaotic, we change the special cases of the initial conditions. We give certain initial velocities and random initial positions, and also make the mass of the three stars be different.

The initial position and the initial velocity of the three stars are shown in Table 1.

star 1 position	(0.1, -0.06, 0.02)
star 1 velocity	(1, 0, 4)
star 2 position	(-0.1, -0.06, 0.01)
star 2 velocity	(0, 2, 0)
star 3 position	(0.1, 0.12, -0.025)
star 3 velocity	(-0.5, -3, -2)

Table 1: The Initialization of the Three Stars

Then we have done 6 simulations and the results are shown in Table 2 (G=6.6743×10^{-11}).

	star 1 mass	star 2 mass	star 3 mass	escape star	escape time
condition 1	1/G	3/G	1/G	star 3	0.05
condition 2	1/G	3/G	2/G	star 3	4.5
condition 3	1/G	3/G	3/G	star 1	1.9
condition 4	1/G	3/G	4/G	star 1	1.5
condition 5	1/G	3/G	5/G	star 1	0.61
condition 6	3/G	3/G	3/G	star 2	2.25

Table 2: The Results of Escaping for the 6 Conditions

For example, for condition 2, the plot of the distance from the stars to the mass center Vs. time is shown in Figure 3 and the animation picture is shown in Figure 4 and Figure 5.

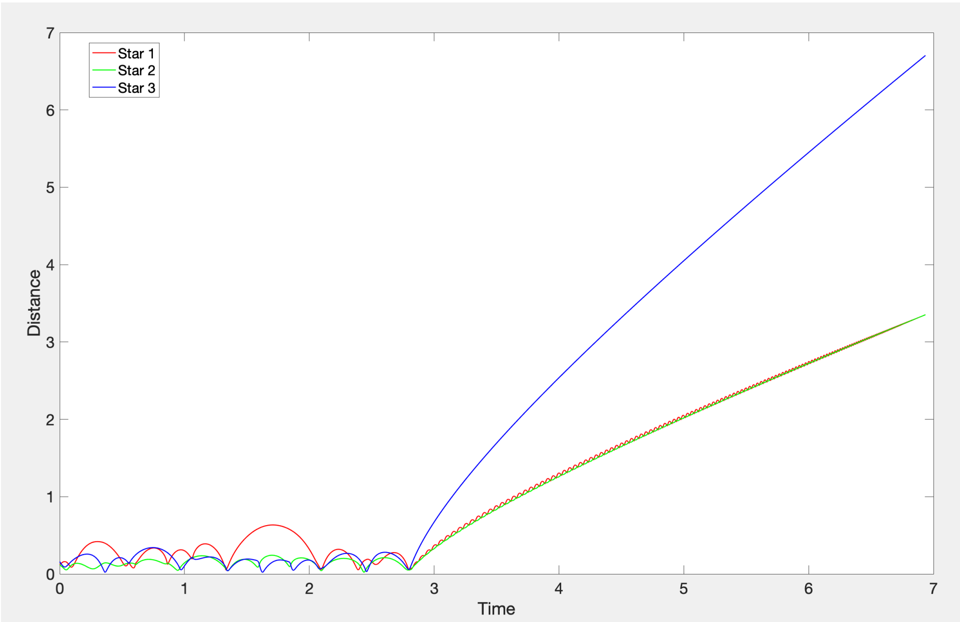

Figure 3: The Distance from the Stars to the Mass Center

We see that at about $t = 2.8$, the third star starts to go into a direction that will never come back as before, and star 1 and star 2 start to orbit around each other. At about $t = 4.5$, the blue line starts to have a constant slope, which is the escaping time we want to find. When time gets larger, star 3 will go out of the three star system while star 1 and star 2 will form a binary-body system.

We can also analyze Table 2. We see set the mass of star 1 and star 2 be not changing and change the mass of star 3 from one times the mass of star 1 to 5 times the mass of star 1 (condition 1-5). The escaping star does not always be the star with the smallest mass. That is because for condition 1 and 2, these three stars' mass are close and do not have a clear dominance, and also there exists the initial velocity for the three stars.

Observing condition 3-5, the escaping star is always the star 1 and with the growing of the mass of star 3, the escaping time becomes smaller and smaller.

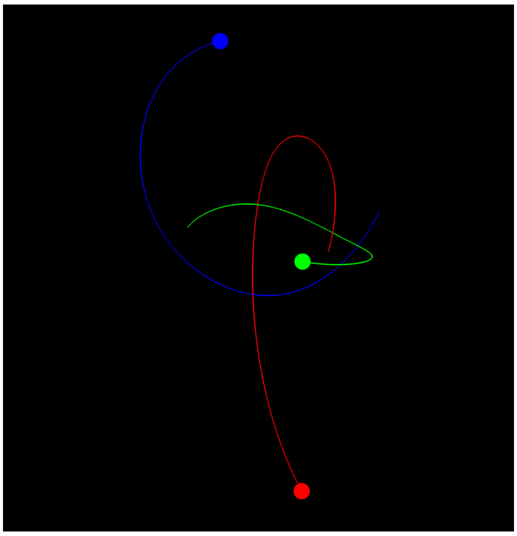

Figure 4: Short Simulation Time

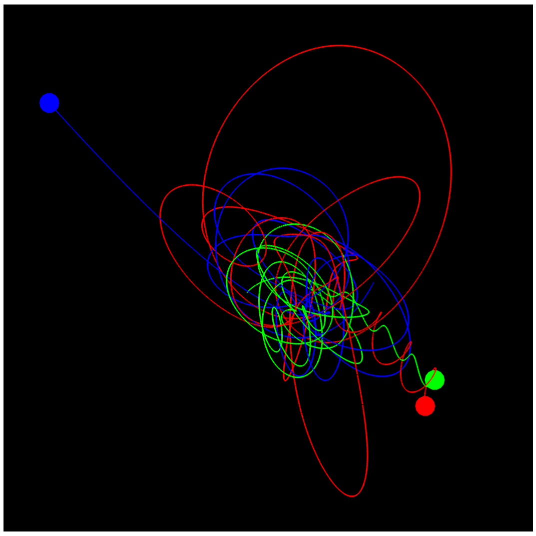

Figure 5: Long Simulation Time

3.3.2 Energy Model

My partner Yi tried another way to evaluate the escaping time, which is based on the change of the energy of the escaping star.

Let KE and PE denote the kinetic energy and potential energy respectively and the expressions are shown below:

$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2$$

$$\begin{aligned} PE &= -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_1m_3}{r_{13}} \\ &= -\frac{Gm_1m_2}{||X_1 - X_2||} - \frac{Gm_2m_3}{||X_2 - X_3||} - \frac{Gm_1m_3}{||X_1 - X_3||} \end{aligned}$$

Since the total energy E contains the kinetic energy and potential energy, then we have

$$\begin{aligned} E &= KE + PE \\ &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 - \frac{Gm_1m_2}{||X_1 - X_2||} - \frac{Gm_2m_3}{||X_2 - X_3||} - \frac{Gm_1m_3}{||X_1 - X_3||} \end{aligned}$$

We assume that the escaping star is m_3 which has the smallest mass. From the equations above, we can find that the initial total energy is negative. As time goes on, the distance between m_3 and the other two stars becomes larger, so the absolute value of the potential energy of m_3 which is $\frac{Gm_2m_3}{||X_2-X_3||} + \frac{Gm_1m_3}{||X_1-X_3||}$ becomes smaller. When the escaping happens, m_3 will never come back. Therefore, the potential energy of the escaping star m_3 which is $(-\frac{Gm_2m_3}{||X_2-X_3||} - \frac{Gm_1m_3}{||X_1-X_3||})$ is approximately 0.

Since after escaping, m_3 will move at a constant velocity (which has been discussed in the previous section), which means the kinetic energy of m_3 will be positive. Then the total energy of m_3 is positive. Therefore, we define the time at which the total energy of m_3 becomes positive be the escaping time.

The code for the energy analysis is shown here. The animation of the real-time energy plot with the motion of three stars is shown here

As shown in Figure 6 where the blue line represents the kinetic energy, the yellow line represents the potential energy, and the green line represents the total energy, we find that the energy is periodic. The large jump occurs when m_1 and m_2 orbit close

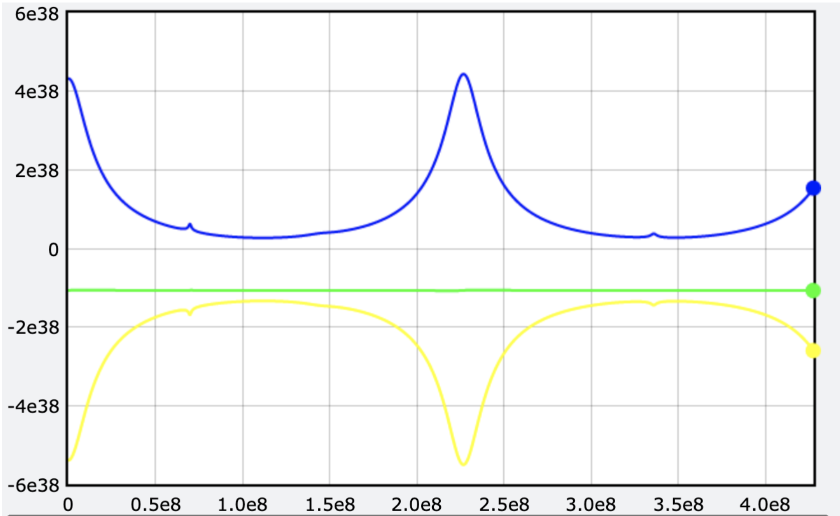

Figure 6: The Energy of the Three Stars

to each other, accelerating and causing a sudden jump in kinetic energy. Also, we noticed that there exists a small jump in the middle, which is the point where m_3 orbits relatively close to m_1 or m_2.

The time will be printed when the energy of m_3 changes from negative to positive, which is the escaping time as defined. By fixing the mass of the other two stars and other initial conditions be the same, we change the mass of the m_3, and the result is shown in Table 3.

mass of star 3	**escape time**
10^{29}	3.672×10^8
10^{28}	4.275×10^8
10^{27}	4.358×10^8
10^{26}	4.366×10^8
10^{25}	4.367×10^8

Table 3: The Result of Escaping Time for Different m_3

From this table, we find that the escaping time will be shorter when the mass of star 3 becomes larger.

4 Discussions and Limitations

As shown in the theory introduction in our Result part, these two kinds of ways defining the escaping time are correct. However, we find some interesting things that the results under these two ways are not consistent. For the ”Distance Model”, when the mass of the other two stars are more dominant, the escaping time will be smaller. However, for the ”Energy Model”, with m_3 getting larger which means the mass of the other two stars are less dominant, the escaping time will be smaller. I think this may because the difference of the mass of three stars is not on the same order of magnitude for the two models. For the ”Distance Model”, there is only a few times of the difference between the escaping star and the other two, but for the ”Energy Model”, there is about 10^x times of the difference. More experiments and analysis will be done on this.

For the limitation of the ”Distance Model”, we cannot set the mass of the three stars be too big because it will cause the time step error (the time step is not small enough to do the integral when calling the ODE solver) and the simulation will break. And for the time step, it should be very small and the whole simulation time will be very short due to the time step error. For the limitation of the ”Energy Model”, we cannot set the mass of the three stars be too small because it will cause the escaping time to be 0. And for the time step, it should be large, or we cannot see the trend of the change of the energy clearly. And that is the reason why the numbers shown in Table 2 are much smaller than the ones shown in Table 3.

5 Conclusions

In conclusion, unlike the binary system and the restricted three-body system at the Lagrange points, the three-body system is almost always be chaotic and unpredictable. One of the chaotic situations is one of the stars escapes from the three-body system. Our project successfully simulate the escaping time by defining the escaping time in two ways, which is observing the distance and observing the energy. We conclude that in most cases, the escaping star is the one with the smallest mass. But some more study should be done to see the relationship between the escaping time and the difference of the mass of three stars.

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