Final Project: Analyzing the climate change problem in an overlapping generation framework

Quantitative Macroeconomics

Author: Rosanna Gomar Lloret

Professor: Raül Santaeulàlia-Llopis

TA: Albert Rodríguez Sala

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"Benefits that we enjoy today are not "partly" result of what people did before but 99.99% of it. It all belongs to humanity."

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1 Abstract

As we all know the climate change will be irreversible in the near future and there will be negative consequences in human's lives, for that reason, we observe that nowadays there are some macroeconomic models including the environment in the analysis. Regarding my study, I have considered introducing this feature in two different ways. Looking at the production side, we have two types of capital, pollutant, and non-pollutant, as using non-pollutant capital has a higher cost, the firms prefer using pollutant capital. On the other hand, the state of the environment has an impact on the utility of the individuals, or in other words, people care about the environment. The environment is affected by both types of capital. If the pollutant capital is high, the environmental quality will be lower, on the other hand, if the capital used is non-pollutant, the environment will be in better conditions. The aim of this project can be summarized into three steps: first, I start to program a basic OLG of three periods in Phyton since we have not worked using these models along the course, then, I solved the environmental model analytically [1], last, I will try to program this in Phyton.

[1] Any mistake along this document is mine since I have tried to develop the model explained (firms and households) without having any paper reference.

2 Introduction

Climate change will be irreversible soon, so if we want to mitigate the impact on human health and the environment, we cannot take any longer to make important decisions about this issue. The United Nations panel on climate change states that the impact of climate change can still be slowed down if countries around the world take unprecedented action to reduce their use of fossil fuels and release less carbon dioxide and other greenhouse gases into the air. As we all know, climate change is a macroeconomic phenomenon since it is global. Climate change presents a grave risk to current and future generations since it will bring extreme-weather events (such as floods, drought, extreme storms, desertification, tsunamis, earthquakes), a rise in sea levels, an impact on ecosystems (mass extinctions and so on), an alteration in human lives and health, influence in food and water security, and so forth. Human activities have produced huge amounts of greenhouse gases in the atmosphere over the last 150 years. In conclusion, we observe that climate change is a negative externality in the economy, maybe the worst one on the planet.

To summarize, we can design some policies (public or private) or regulatory instruments that mitigate these pollution problems and improve air quality or public health since we focus on market failure. As I have said before, the goal to reduce the impact of climate change consists of reducing emissions. In my opinion, when we analyze pollution, we are looking at a distribution topic, so it can be very appealing to ask these three questions:

1. Who benefits from the activity that is creating this problem? If nobody is taking advantage, the activity wouldn't have happened.
2. Who is harmed? If nobody is harmed when we look at the current and future generations, this would not be a problem at least in the topic of human well-being.
3. What is it about the relationship between the beneficiaries and about who's harmed in the topic that we are concern about it and leads this problem to persist?

I will focus on an OLG economy, following Kotlikoff et al. (2019). The reason for doing this is that HH's do not have an infinite planning horizon and new households are born over time. The arrival of new households in the economy is not only a realistic feature, but it also introduces a range of new economic interactions. These economic interactions have no counterpart in the neoclassical growth model. The OLG model is useful for several reasons: first, it captures the potential interaction of different generations of individuals in the marketplace. Second, it provides a tractable alternative to the infinite-horizon representative agent models. Third, some of the key implications are different from those of the neoclassical growth model (dynamic inefficiency). Finally, the OLG model provides a flexible framework to study the effects of macroeconomic policies. In addition to this, an infinitely-lived agent model relies, implicitly, on intergenerational altruism. While using an OLG we assume that private and social discounting are naturally separated, and the distribution and distributional conflict between generations can be modeled explicitly. In other words, an OLG captures two aspects of distribution: intra-generational between households of the same age cohort and inter-generational distribution between young and old cohorts. Then, climate change, and its mitigation, involve inter-generational redistribution.

In my OLG model, agents live for three periods, working in the first two periods and consuming all periods. When they are young, they can invest in two goods: pollutant and non-pollutant capital. I consider that the environment depends on both types of capital. The agents get utility from consumption, and of the environmental quality, I introduce this as a new argument of the utility function. This is related to our previous explanation, if the pollution is high, they will suffer more health problems. I could include taxes into the analysis, but in this case, it is not necessary since we can observe the trade-off without introducing it. I will analyze this in detail in the section Quantitative Model. It will be interesting to extend the analysis into other dimensions, for example, taking into account the relationship between consumption and environment bequeathed to future generations (following John & Pecchenino (1994)), if the consumption is high, this degrades the environment bequeathed to future generations. On the other hand, if consumption is lower, they assign more investment in environmental quality, which improves the environment bequeathed to future generations. Another case could be an upper bound in the resources we can consume since the resources are limited.

3 Relation to the Literature

One important contribution in the literature has been made by Nordhaus (2007), Nobel Prize in 2018, in a few words "for integrating climate change into long-run macroeconomic analysis", the author emphasizes that the NGM (Neoclassical Growth Model) ignores important externalities in the growth process. He focuses on a dynamic integrated climate-economy (DICE) model to study optimal climate policy. This model is a neoclassical growth model that accounts for how carbon emissions today will affect global warming, and economic output, in the future. A representative, forward-looking household maximizes the present-discounted value of utility from consumption subject to an accumulation constraint, and a constraint describing the evolution of carbon damage over time. He focuses on the emissions of CO2 instead of quotas since, according to him, the price of these quotas is not stable as there is a market for the quotas, there is huge volatility. The main innovation is the modeling of climate change and its effect on the economy. World output is a function of technology, capital, and labor, as well as climate change. For Nordhaus: "Climate sceptics are like the people who refused to accept that smoking causes cancer".

Moreover, Kotlikoff et al. (2019) develop a large-scale, dynamic 55-period, OLG model to calculate the carbon tax policy delivering the highest uniform welfare gain to all generations. Their contribution is using the damage function of Nordhaus but introducing technological and demographic change, a clean energy sector, and the extraction of coal, oil, gas, and so on. They state that if carbon tax starts at \$30 tax, rises annually at 1.5% and raises the welfare of all generations (current and future) by 0.73% on a consumption-equivalent basis but to do so, the future generations have to be taxed by as much as 8.1% and the early generations subsidized by as much as 1.2% lifetime consumption. This is a win-win policy if this redistribution is not done, the carbon tax constitutes a win-lose policy.

Another important contribution is made by Hassler & Krusell (2018). For the authors the climate change should be regarded as the first-order issue for macroeconomics, the key parameters are: governing utility discounting, carbon depreciation, and damages, and the impact of these parameters are nonlinear. According to the authors, one important issue is the sustainability of the world's resources, or in other words, meeting the needs of the present without compromising the ability of future generations to meet their needs.

Additionally, Karp & Rezai (2014) focus on the two-sector OLG model and analyze the intergenerational effects of a tax, that protects an environmental stock. The authors argue the majority of analyses related to environmental problems that assume people alive today must make sacrifices to preserve consumption opportunities for those alive in the future. They deal with an OLG model with endogenous asset prices since all generations are better off from an environmental policy if the winner alive compensate those who would be harmed by the policy. Without this transfer, the tax harms the young generation by decreasing their real wage. While future generations benefit from the tax-induced improvement in environmental stock. The principal intergenerational conflict arising from public policy is between generations alive when society imposes the policy, not between generations alive at different times. A Pareto-improving policy can be implemented under various political economy settings.

Obviously, the literature on this topic is growing, and I can find more authors that have been interested in this topic: Golosov (2014) decentralizes the SP solution with carbon taxes and introduces more than one form of dirty energy, Cai (2018) shows that optimal carbon tax rates will be higher if the carbon damage is uncertain and so on.

4 The Quantitative Model

Before I start with this model, I think it is best to consider a simple OLG of three periods (this can be seen in section 8 (Appendix)). First, I write the problems of individuals and firms, then I solve the steady-state and time path iteration. The next step is to learn how to program an OLG in Phyton since I have not had the opportunity of doing this before, the code of this model can be seen in "OLG_3periods_Rosanna.py", and the graphical results are displayed in Figure 1.

In our economy, time is discrete. Each individual lives three periods, for the generation born in period t, they live in period t, t+1, and t+2, the economy always has three generations in each period. I assume no population growth; at each date t, a new generation of N identical agent is born. I normalize the size of each generation to unity.

4.1 Firm's problem

The innovation here consists of introducing two types of capital in the economy: the pollutant and the non-pollutant. I denote by $K_{t,p}$ the aggregate pollutant capital stock, and by $K_{t,np}$ the aggregate non-pollutant capital stock. Therefore, K_t is in turn the total CES (Constant Elasticity of substitution) aggregate of capital. As we know, the elasticity of substitution represents the envelope of all technically feasible combinations of input factors that produce a certain amount of output. There are different studies about the elasticity of substitution between capital and labor, but I am going to analyze this between different types of capital:

$$K_t = \left(\phi K_{t,p}^{\rho} + (1-\phi) K_{t,np}^{\rho}\right)^{\frac{1}{\rho}} \tag{1}$$

where ϕ reflects the intensity of pollutant capital in the production function and (1-ϕ) represents the intensity of non-pollutant capital in the production function. We observe that $\frac{1}{1-\rho}$ is the elasticity of substitution between two types of capital, if $\rho < 1$, the pollutant and non-pollutant capital are imperfect substitutes in production. If $\rho = 1$, we have a linear relation between them, and if ρ goes to zero, in the limit we have the Cobb-Douglas specification. Obviously, this CES aggregate of capital is combined with labor to produce output Y_t, according to Cobb-Douglas production function:

$$Y_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha} = \left(\left(\phi K_{t,p}^{\rho} + (1-\phi) K_{t,np}^{\rho}\right)^{\frac{1}{\rho}}\right)^{\alpha} L_t^{1-\alpha} \tag{2}$$

The production function is twice continuously differentiable, positive, increasing strictly concave, and satisfies the Inada conditions.

The parameter α measures the elasticity of output concerning capital and on the other hand, $1-\alpha$ is the elasticity of output for labor. The production function presents constants returns to scale, and there is perfect competition among the different firms of the industry. For these reasons, the profits are equal to zero, and we can assume the existence of a single representative firm in this economy.

Now let me analyze the problem of the representative firm, the firms do not face a dynamic decision problem as the variable chosen at period t do not affect the constraints nor returns at later periods. Then, the static profit maximization problem for the representative firm can be written as:

$$\max \pi = \max\Bigg(\Big(\phi K_{t,p}^{\rho} + (1-\phi)K_{t,np}^{\rho}\Big)^{\frac{1}{\rho}}\Bigg)^{\alpha} L_t^{1-\alpha} - w_t L_t - r_{t,p}K_{t,p} - (r_{t,np}+x)K_{t,np} \tag{3}$$

where the wage rate per unit of labor is equal to w_t, $r_{t,p}$ is the interest rate for pollutant firms, and $r_{t,np}$ for non-pollutant. In addition to this, I have another parameter, x, that captures the cost of investing in less pollutant capital, for example, imagine that firms need to buy less polluting machinery or using another source of energy in the production function, and so on. I denote by $k_t = \frac{K_t}{L_t}$ the "capital intensity". Then, I divide the entire problem by L_t:

$$\max \pi = \max\Bigg(\Big(\phi K_{t,p}^{\rho} + (1-\phi)K_{t,np}^{\rho}\Big)^{\frac{1}{\rho}}\Bigg)^{\alpha} L_t^{-\alpha} - w_t - r_{t,p}\frac{K_{t,p}}{L_t} - (r_{t,np}+x)\frac{K_{t,np}}{L_t} \tag{4}$$

I factor L_t^{α} into the production function, then the entire firm problem in intensive form can be written as:

$$\max \pi = \max\Bigg(\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}}\Bigg)^{\alpha} - w_t - r_{t,p}k_{t,p} - (r_{t,np}+x)k_{t,np} \tag{5}$$

Then, the FOC of the firms are defined by:

$$\left(\frac{\partial \pi}{\partial k_{t,p}}\right) = \alpha\Bigg(\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}}\Bigg)^{\alpha-1}\frac{1}{\rho}\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}-1}\rho\phi k_{t,p}^{\rho-1} - r_{t,p} = 0 \tag{6}$$

$$\left(\frac{\partial \pi}{\partial k_{t,np}}\right) = \alpha\Bigg(\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}}\Bigg)^{\alpha-1}\frac{1}{\rho}\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}-1}\rho(1-\phi) k_{t,np}^{\rho-1} - (r_{t,np}+x) = 0 \tag{7}$$

Thus:

$$\alpha\Bigg(\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}}\Bigg)^{\alpha-1}\frac{1}{\rho}\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}-1}\rho\phi k_{t,p}^{\rho-1} = r_{t,p}$$

$$\alpha\Bigg(\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}}\Bigg)^{\alpha-1}\frac{1}{\rho}\Big(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\Big)^{\frac{1}{\rho}-1}\rho(1-\phi) k_{t,np}^{\rho-1} = (r_{t,np}+x)$$

And dividing one by the other, I obtain:

$$\frac{\phi}{1-\phi}\left(\frac{k_{t,np}}{k_{t,p}}\right)^{1-\rho} = \frac{r_{t,p}}{(r_{t,np}+x)} \tag{8}$$

We observe that the cost of capital depends on the elasticity of substitution between the two types of capital and the weights assigned to each type unless $\rho = 1$. Since the production function exhibits positive marginal products, $r_{t,np}, r_{t,p}$, and $w_t > 0$ in any competitive equilibrium because, otherwise, factor demands would become unbounded.

As we have competitive firms, the profits of the firms are equal to 0, therefore I can isolate the wage of equation 5:

$$w_t = \left(\left(\phi k_{t,p}^{\rho} + (1-\phi)k_{t,np}^{\rho}\right)^{\frac{1}{\rho}}\right)^{\alpha} - r_{t,p}k_{t,p} - (r_{t,np} + x)k_{t,np} \Longrightarrow$$

$$w_t = k_t^{\alpha} - r_{t,p}k_{t,p} - (r_{t,np} + x)k_{t,np} \tag{9}$$

Another way to see this is that the production function is assumed to have constant returns to scale, which means that it is homogeneous of degree one (if you multiply each input by m then you will get m times the output:

$$F(mk, ml) = mF(k, l)$$

Then, using Euler's Theorem we know that if the function is homogeneous of degree one, we have:

$$\frac{\partial F}{\partial L}L + \frac{\partial F}{\partial K}K = F(L, K)$$

Therefore, this implies that:

$$F(k_t, l_t = F_k(k_t, l_t)k_t + F_l(k_t, l_t)l_t$$

As $l_t = 1$:

$$f(k_t) = f'(k_t)k_t + w_t$$

$$r_t = f'(k_t)$$

Rearranging we get:

$$w_t = f(k_t) - f'(k_t)k_t \tag{10}$$

That this expression is equal to expression 9, taking into account that I have two types of capital.

4.2 Household's problem

I am going to follow the same explanations as in section 8. The age of the individuals is reflected by the subindex s, s={1,2,3}. There is no uncertainty. Moreover, the labor is inelastically supplied by the individuals in the first two periods of life:

$$L_{s,t} = \begin{cases} 1 & \text{if } s = 1, 2 \\ 0 & \text{if } s = 3 \qquad\qquad \forall s, t \end{cases} \tag{11}$$

The individuals born with no savings ($k_{1,t} = 0$) and they do not save income in the last period of their lives ($k_{4,t} = 0$). In this case, the individuals have a logarithm utility function that takes the following form:

$$u(c_{s,t}) = ln(c_{s,t}) + \eta ln(E_t) \tag{12}$$

Regarding the notation, c represents the consumption, and E can be interpreted as a public good that reflects the environmental quality, for example, E_t can include the quality of soil or groundwater, the cleanliness of rivers and oceans, or the inverse of the atmospheric concentration of chlorofluorocarbons. I assume that the utility function is twice continuously differentiable, and satisfies the Inada Conditions. In this case, I have supposed that it is additively separable, and η reflects the impact of the public good in the utility. This public good depends on both types of capital, if the firms produce with the pollutant capital, the environment will be worse off. However, the impact will be positive if the capital used is non-pollutant. Therefore:

$$E_t = \alpha_1 k_{t,np} - \alpha_2 k_{t,p} \tag{13}$$

In this set-up agents choose the capital, hence the agents choose how much to save and how much to consume, while the firms have a static problem as we have seen before. In the budget constraints, I have both types of capital since it gives income, but both types of capital have a cost in the environment, as we have seen in the firm's problem. In other words, we can observe an interesting trade-off the non-pollutant capital is expensive but has a benefit for the whole economy, while the pollutant capital is more profitable privately but hurts the environment. I suppose that the depreciation rate for both types of capital is constant across time, for that reason it does not have the subindex t. The maximization problem of the individuals can be written as:

$$\max u(c_{1,t}) + \beta u(c_{2,t+1}) + \beta^2 u(c_{3,t+2}) \tag{14}$$

where:

$$u(c_{1,t}) = ln(c_{1,t}) + \eta ln(E_t)$$
$$u(c_{2,t+1}) = ln(c_{2,t+1}) + \eta ln(E_{t+1})$$
$$u(c_{3,t+2}) = ln(c_{3,t+2}) + \eta ln(E_{t+2})$$

The three-age specific budget constraints are:

$$c_{1,t} + k_{2,t+1,np} + k_{2,t+1,p} = w_t \tag{15}$$

$$c_{2,t+1} + k_{3,t+2,np} + k_{3,t+2,p} = w_{t+1} + (1 + r_{t+1,np} - \delta_{np})k_{2,t+1,np} + (1 + r_{t+1,p} - \delta_p)k_{2,t+1,p} \tag{16}$$

$$c_{3,t+2} = (1 + r_{t+2,np} - \delta_{np})k_{3,t+2,np} + (1 + r_{t+2,p} - \delta_p)k_{3,t+2,p} \tag{17}$$

Note that the budget constraint already imposed the outcome that households supply all their labor (unit 1 in the first two periods). The last constraint incorporates notion that individuals only spend money on their own end of life consumption (no altruism or bequest motive). Let's set up the Lagrangian:

$$\begin{aligned}
\mathcal{L} =& ln(c_{1,t}) + \eta ln(\alpha_1 k_{t,np} - \alpha_2 k_{t,p}) + \beta\Bigg(ln(c_{2,t+1}) + \eta ln(\alpha_1 k_{t+1,np} - \alpha_2 k_{t+1,p})\Bigg) + \beta^2\Bigg(ln(c_{3,t+2}) + \\
&+ \eta ln(\alpha_1 k_{t+2,np} - \alpha_2 k_{t+2,p})\Bigg) + \lambda_1\Bigg(w_t - c_{1,t} - k_{2,t+1,np} - k_{2,t+1,p}\Bigg) + \\
&+ \beta\lambda_2\Bigg(w_{t+1} + (1 + r_{t+1,np} - \delta_{np})k_{2,t+1,np} + (1 + r_{t+1,p} - \delta_p)k_{2,t+1,p} - c_{2,t+1} - k_{3,t+2,np} - k_{3,t+2,p}\Bigg) + \\
&+ \beta^2\lambda_3\Bigg((1 + r_{t+2,np} - \delta_{np})k_{3,t+2,np} + (1 + r_{t+2,p} - \delta_p)k_{3,t+2,p} - c_{3,t+2}\Bigg)
\end{aligned} \tag{18}$$

In order to derive the Euler Equations, we need to take first order conditions with respect to the control (choice) variables of the household's, these are: $c_{1,t}, c_{2,t+1}, c_{3,t+2}, k_{3,t+2,np}, k_{3,t+2,p}, k_{2,t+1,np}$ and $k_{2,t+1,p}$.

$$\frac{\partial \mathcal{L}}{\partial c_{1,t}} = \frac{1}{c_{1,t}} - \lambda_1 = 0 \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2,t+1}} = \beta\frac{1}{c_{2,t+1}} - \beta\lambda_2 = 0 \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial c_{3,t+2}} = \beta^2\frac{1}{c_{3,t+2}} - \beta^2\lambda_3 = 0 \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial k_{2,t+1,np}} = \beta\eta\frac{1}{E_{t+1}}\alpha_1 - \lambda_1 + \beta\lambda_2(1 + r_{t+1,np} - \delta_{np}) = 0 \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial k_{2,t+1,p}} = -\beta\eta\frac{1}{E_{t+1}}\alpha_2 - \lambda_1 + \beta\lambda_2(1 + r_{t+1,p} - \delta_p) = 0 \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial k_{3,t+2,np}} = \beta^2\eta\frac{1}{E_{t+2}}\alpha_1 - \beta\lambda_2 + \beta^2\lambda_3(1 + r_{t+2,np} - \delta_{np}) = 0 \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial k_{3,t+2,p}} = -\beta^2\eta\frac{1}{E_{t+2}}\alpha_2 - \beta\lambda_2 + \beta^2\lambda_3(1 + r_{t+2,p} - \delta_p) = 0 \tag{25}$$

From expressions 19, 20 and 21 [2], we isolate the values of λ and plug in the others FOC's, then:

$$\beta\eta\frac{1}{E_{t+1}}\alpha_1 - \frac{1}{c_{1,t}} + \beta\frac{1}{c_{2,t+1}}(1 + r_{t+1,np} - \delta_{np}) = 0 \tag{26}$$
$$-\beta\eta\frac{1}{E_{t+1}}\alpha_2 - \frac{1}{c_{1,t}} + \beta\frac{1}{c_{2,t+1}}(1 + r_{t+1,p} - \delta_p) = 0 \tag{27}$$
$$\beta^2\eta\frac{1}{E_{t+2}}\alpha_1 - \beta\frac{1}{c_{2,t+1}} + \beta^2\frac{1}{c_{3,t+2}}(1 + r_{t+2,np} - \delta_{np}) = 0 \tag{28}$$
$$-\beta^2\eta\frac{1}{E_{t+2}}\alpha_2 - \beta\frac{1}{c_{2,t+1}} + \beta^2\frac{1}{c_{3,t+2}}(1 + r_{t+2,p} - \delta_p) = 0 \tag{29}$$

I can isolate the same term $c_{1,t}$ in expressions 26 and 27, or in other words, if I combine the FOC for c_t with $k_{t+1,np}$ and $k_{t+1,p}$, I get:

$$\frac{1}{c_{1,t}} = \beta\frac{1}{c_{2,t+1}}(1 + r_{t+1,np} - \delta_{np}) + \beta\eta\frac{1}{E_{t+1}}\alpha_1 \tag{30}$$
$$\frac{1}{c_{1,t}} = \beta\frac{1}{c_{2,t+1}}(1 + r_{t+1,p} - \delta_p) - \beta\eta\frac{1}{E_{t+1}}\alpha_2 \tag{31}$$

As I have two types of capital, I obtain two Euler equations. But as I am studying an OLG, I need to combine the FOC for $c_{3,t+2}$ with $k_{t+2,np}$ and $k_{t+2,p}$, or in other words, the same for $c_{2,t+1}$ in expressions 28 and 29, I get:

$$\frac{1}{c_{2,t+1}} = \beta\frac{1}{c_{3,t+2}}(1 + r_{t+2,np} - \delta_{np}) + \beta\eta\frac{1}{E_{t+2}}\alpha_1 \tag{32}$$
$$\frac{1}{c_{2,t+1}} = \beta\frac{1}{c_{3,t+2}}(1 + r_{t+2,p} - \delta_p) - \beta\eta\frac{1}{E_{t+2}}\alpha_2 \tag{33}$$

I have obtained two new Euler Equations, then we have four Euler Equations in total. These expressions are similar to the basic NGM model where:

$$u'(c_t) = u'(c_{t+1})(1 + r_{t+1} - \delta)$$

Nevertheless, we observe some differences, in the first place, as in my model there are two types of capital, we have two interest rates and two depreciation rates. In addition to this, there is a second argument in the utility, the environment, then I obtain a second term in the EE. As I have explained the environment depends positively on non-pollutant capital and negatively on pollutant capital, as we can observe in expressions 30 and 31. I cannot solve this problem analytically, but we can observe the interesting trade-off of my analysis in the previous expressions. If I could substitute the interest rate of the Firms problem, we would observe that the cost of using non-pollutant capital is higher due to the

[2] The equations have cross references along the document.

parameter x, since both types of interest rate should be equal in equilibrium. On the other hand, the households obtain higher utility with non-pollutant capital and lower with pollutant capital.

I am going to simplify a little bit the problem, I can equalize equations 30 and 31, and equations 32 and 33:

$$\frac{1}{c_{2,t+1}}(1+r_{t+1,np}-\delta_{np})+\eta\frac{1}{E_{t+1}}\alpha_1=\frac{1}{c_{2,t+1}}(1+r_{t+1,p}-\delta_p)-\eta\frac{1}{E_{t+1}}\alpha_2$$
$$\eta\frac{1}{E_{t+2}}\alpha_1+\frac{1}{c_{3,t+2}}(1+r_{t+2,np}-\delta_{np})=-\eta\frac{1}{E_{t+2}}\alpha_2+\frac{1}{c_{3,t+2}}(1+r_{t+2,p}-\delta_p)$$

Rearranging a little bit:

$$\eta\frac{1}{E_{t+1}}(\alpha_1+\alpha_2)=\frac{1}{c_{2,t+1}}\Big((1+r_{t+1,p}-\delta_p)-(1+r_{t+1,np}-\delta_{np})\Big) \tag{34}$$
$$\eta\frac{1}{E_{t+2}}(\alpha_1+\alpha_2)=\frac{1}{c_{3,t+2}}\Big((1+r_{t+2,p}-\delta_p)-(1+r_{t+2,np}-\delta_{np})\Big) \tag{35}$$

5 Definition of Equilibrium

I have done the following assumption on the ownership structure of the economy: I assume that consumers own all factors of production (i.e. they own the capital stock at all times) and rent it out to the firms. I also assume that households own the firms, i.e. are claimants of the firm's profits. Now I have to specify the equilibrium concept and the market structure. Another assumption that I have dealt with is that there is nothing dynamic about the firm's problem, and it has been separated into an infinite number of static maximization problems. Households instead face a fully dynamic problem in this economy. They own the capital stock and hence have to decide how much labor and capital services to supply, how much to consume, and how much capital to accumulate.

- First of all, we should answer the question: What about the number of markets?

There is a labor market in which households still trade their labor with the firms. In this market, the labor demand by firms, L_t, should be equal to the total endowment of time of the representative agent (who supplies all his/her labor as he/she does not value leisure) which is equal to one. Thus, L_t=1. The other two markets in the economy are the capital market since in the case capital is owned by households and used by firms to produce, there it is traded, and the second one is the goods markets, as in the standard NGM.

The competitive equilibrium in sequence form is an allocation x^h=$\{\{c_{t,s}, l_{t,s}, k_{t+1,s}\}_{t=0}^{\infty}\}_{s=1}^{3}$, x^f=$\{L_t, K_t\}_{t=0}^{\infty}$ and prices x^p=$\{r_{t,np}, r_{t,p}, w_t\}_{t=0}^{\infty}$ such that:

1. Given the prices x^h solves the maximization problem of the HH's.

2. Given the prices x^f solves the maximization problem of the firm.

3. Markets clears, $\forall t$:

   - Goods market: $C_t + K_{t+1} = F(K_t, 1) + (1 - \delta)K_t$
   - Labor market: L_t =$\sum_{s=1}^{3} l_{s,t} = 1 + 1 + 0 = 2$
   - Capital market: $K_t = \sum_{s=2}^{3} k_{s,t} = k_{2,t,np} + k_{3,t,np} + k_{2,t,p} + k_{3,t,p}$

It is not necessary write the environment in the allocation of the household since it is redundant (it depends on the capital). A *stationary competitive equilibrium* of the economy is defined as a competitive equilibrium where all variables grow at a constant rate.

6 Results

To get the solution of the model, we generally seek for a fixed point such that all markets clear. The easiest but not most efficient way to solve these models is accordingly by a **fixed point iteration**. The general solution method does not differ much between steady-state and transition calculations. I have studied the different equations for the firm's and HH's, we are going to solve for the steady-state in this OLG economy.

6.1 Steady-state

In steady-state growth, all variables, such as output, population, capital stock, saving, investment, and technical progress, grow at a constant rate. For this reason, we start to look at the solution in the steady-state of the economy.

I have written the fixed point iteration as an iteration searching for the equilibrium interest rate. The different steps can be summarized as:

1. Start with an initial guess for the interest rate.

2. Then for each iteration:

   - Calculate the wage, modifying the first-order conditions of the firm.
   - The following step consists of solving the household model. I choose an arbitrary initial guess for the steady-state distribution of wealth. Then, find a root of the vector function.
   - Use market clearing condition for capital, and labor, since the good market is redundant (Walras' Law). In other words, I aggregate across all households living in the steady-state to get the aggregate capital stock.
   - Obtain the interest rate for the following period.
   - Look at the distance between the interest rate and the interest rate in the following period. If the distance is smaller than the tolerance error (ϵ), I stop. If the distance is larger than the tolerance error:

$$||r' - r|| < 0.00001$$

I update the interest rate or I state a a new transition path for the interest rate that depends on the convex combination of r and r' such as:

$$r_{path} = x_i * r'_{path} + (1 - x_i) * r_{path} \tag{36}$$

where x_i is some dampening factor. The lower the value of x_i, the more conservative is the update of the guess for the equilibrium value of r to be used in the next iteration step. Then, I go to first step again.

6.2 Time path

To calculate the time path, I need to obtain the initial value for the marginal product of capital, and the value for the final period. Then, I obtain the entire transition, r_{path}, immediately. As before, I need to each iteration:

- Compute the wage path, this is very immediately once I have computed the interest rate since the other elements involved are parameters.
- Solve the problem of the Households, for all households born in $t = 1, \ldots, T$.
- Solve all full lifetimes. Use the market-clearing for labor and capital, recall that the good market is redundant. Once I have the path for labor and capital, I can calculate the income path.
- Find implied r' for all periods of time.
- As before, I need to check the distance in absolute value between r' and r:

  $$||r' - r|| < \epsilon$$

  If the previous condition holds, I stop and print: "The time path SOLVED!!!!". On the other hand, I state a new transition path for the interest rate that depends on the convex combination of r and r' such as:

  $$r_{path} = x_i * r'_{path} + (1 - x_i) * r_{path} \tag{37}$$

  I update the iteration counter each time, and come back to the first step.

Then, I display the results and graphs. I cannot be able to implement this in the model that I have developed in section 4, but the attempts are made in "OLG_3periods_environment.py". On the other hand, in the code "OLG_3periods_RosannaGomar.py" can be seen all the previous steps of the simple model, which has been explained in Appendix section, 8.

7 Conclusion

Throughout the course, we have dealt with value function iteration with discrete and continuous methods, a neoclassical transition, ABHI economies, and so on. As I have explained in the abstract, this work aims to analyze an OLG dealing with the environment in two different ways (in the production function and the utility of the consumers). So first, I have developed and coded the simple OLG of three periods since it is the starting point of our analysis, and we have not analyzed these models along the course, this can be seen in the Appendix of this document. For doing this, it has been very useful the tools of fixed point iteration and transition path provided along the course.

The following step (once I have explained the motivation in the introduction and the related literature on this topic) consists of analyzing the Quantitative Model. To develop this, I have introduced two types of capital in the production function, and both have an impact on the environment, moreover, the households care about this environment. Therefore, there is an interesting trade-off in their maximization problem. I would like to remark that all possible mistakes are mine since I have not seen any specific paper dealing with this issue in this way. Then, any comment about this document would be grateful.

In addition to this, I have tried to solve by fixed-point iteration the steady-state, parallel that, the time path, as in the simple OLG, but I have not succeeded in this complex model. As I have repeated several times, I find this field fascinating. The environmental problems will be a turning point in our lives, so it is interesting taking the aforementioned into account. There are different ways of introducing the environment in the macroeconomic analysis, so this project can be a good starting point to continue working on it.

8 Appendix

In this section I am going to explain a perfect foresight, 3-period-lived agent overlapping generations model, that is the one that has been programmed in Phyton. As I have said before, it is better to start programming the “simple” one and then try to introduce the new modifications.

8.1 Individuals

Focusing on the individuals, each generation lives for three periods, and one new generation is born each period. The age of individuals will be reflected by the subindex s, $s = \{1, 2, 3\}$. In addition to this, I assume that the labor is inelastically supplied by the individuals in the first two periods of life, however as the individuals are retired in the last period, the labor supply is equal to 0.

$$n_{s,t} = \begin{cases} 1 & \text{if } s = 1, 2 \\ 0 & \text{if } s = 3 \qquad\qquad \forall s, t \end{cases} \tag{38}$$

In this model, I have a three age-specific budget constraints defined as:

$$c_{1,t} + b_{2,t+1} = w_t \tag{39}$$

$$c_{2,t+1} + b_{3,t+2} = w_{t+1} + (1 + r_{t+1})b_{2,t+1} \tag{40}$$

$$c_{3,t+2} = (1 + r_{t+2})b_{3,t+2} \tag{41}$$

The individuals are born with no savings and they do not save income in the last period of their lives, therefore, $b_{1,t} = 0$ and $b_{4,t} = 0 \quad \forall t$. Other important assumptions are that: the consumption cannot be negative and $b_{2,t} + b_{3,t}$ has to be strictly positive due to the aggregate capital stock. Furthermore, the individuals have a CRRA (Constant Relative Risk aversion) utility function that takes the following form, where θ represents the coefficient of relative risk aversion:

$$u(c_{s,t}) = \frac{(c_{s,t})^{1-\theta} - 1}{1 - \theta} \tag{42}$$

Then, the maximization problem of the individuals can be written as:

$$\max_{\{c_{s,t+s-1}\}_{s=1}^{3}, \{b_{s+1,t+s}\}_{s=1}^{2}} u(c_{1,t}) + \beta u(c_{2,t+1}) + \beta^2 u(c_{3,t+2}) \tag{43}$$

$$c_{1,t} + b_{2,t+1} = w_t \tag{44}$$

$$c_{2,t+1} + b_{3,t+2} = w_{t+1} + (1 + r_{t+1})b_{2,t+1} \tag{45}$$

$$c_{3,t+2} = (1 + r_{t+2})b_{3,t+2} \tag{46}$$

I can simplify the problem if I use the different BC (equations 43, 45 and 46) to isolate the consumption and plug in in the objective function:

$$\max \mathcal{L} = u(w_t - b_{2,t+1}) + \beta u\Big(w_{t+1} + (1+r_{t+1})b_{2,t+1} - b_{3,t+2}\Big) + \beta^2 u\Big((1+r_{t+2})b_{3,t+2}\Big) \tag{47}$$

$$\frac{\partial \mathcal{L}}{\partial b_{3,t+2}} = -\beta u'\Big(w_{t+1} + (1+r_{t+1})b_{2,t+1} - b_{3,t+2}\Big) + \beta^2(1+r_{t+2})u'\Big((1+r_{t+2})b_{3,t+2}\Big) = 0 \tag{48}$$

I observe that that the optimal savings for age-2 individuals is a function $\psi_{2,t+1}$ of the wage in that period, the interest rate in that period and in the nest one, and how much savings the individual saved in the previous period:

$$b_{3,t+2} = \psi_{2,t+1}(b_{2,t+1}, w_{t+1}, r_{t+1}, r_{t+2}) \tag{49}$$

If I do the derivative of the Lagrangian with respect to $b_{2,t+1}$, I will observe that includes derivatives of $b_{3,t+2}$ with respect to $b_{2,t+1}$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial b_{2,t+1}} =& - u'(w_t - b_{2,t+1}) + \beta(1+r_{t+1})u'\Big(w_{t+1} + (1+r_{t+1})b_{2,t+1} - b_{3,t+2}\Big) - \\ & \beta u'\Big(w_{t+1} + (1+r_{t+1})b_{2,t+1} - b_{3,t+2}\Big)\frac{\partial \psi_{2,t+1}}{\partial b_{2,t+1}} + \beta^2(1+r_{t+2})u'(c_{3,t+2})\frac{\partial \psi_{2,t+1}}{\partial b_{2,t+1}} = 0 \end{aligned}$$

I can rearrange it:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial b_{2,t+1}} =& - u'(w_t - b_{2,t+1}) + \beta(1+r_{t+1})u'\Big(w_{t+1} + (1+r_{t+1})b_{2,t+1} - b_{3,t+2}\Big) - \\ & \beta\frac{\partial \psi_{2,t+1}}{\partial b_{2,t+1}}\Bigg(u'\Big(w_{t+1} + (1+r_{t+1})b_{2,t+1} - b_{3,t+2}\Big) + \beta(1+r_{t+2})u'(c_{3,t+2})\Bigg) = 0 \end{aligned}$$

And by equation 48, we know that the last term is equal to 0, using the envelope condition. The important fact is that as the individuals optimize tomorrow given the current period, we do not need to worry about the effect of the choices today on the choices tomorrow. Then:

$$\frac{\partial \mathcal{L}}{\partial b_{2,t+1}} = -u'(w_t - b_{2,t+1}) + \beta(1+r_{t+1})u'\Big(w_{t+1} + (1+r_{t+1})b_{2,t+1} - b_{3,t+2}\Big) = 0 \tag{50}$$

The optimal savings for age-1 individuals is a function of the wages in that period and the following one, the same for the interest rate.

$$b_{2,t+1} = \psi_{1,t}(w_t, w_{t+1}, r_{t+1}, r_{t+2}, w_{t+2}) \tag{51}$$

I have seen what happens with the age-1 and age-2 savings decisions of a particular individual, but I can study the age-2 savings decisions of the middle-aged in period t ($b_{3,t+1}$) iterating backward the equation 48:

$$u'\big(w_t + (1+r_t)b_{2,t} - b_{3,t+1}\big) = \beta(1+r_{t+1})u'\big((1+r_{t+1})b_{3,t+1}\big) \tag{52}$$

8.2 Firms

I do not want to extent so much in this part since the firms produce according to a Cobb-Douglas production technology:

$$Y_t = F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha} \tag{53}$$

As the price of the output is equal to 1, the maximization problem can be written as:

$$\max_{K_t, L_t} \pi = AK_t^{\alpha} L_t^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{54}$$

And the FOC's are:

$$\frac{\partial \pi}{\partial K_t} = A\alpha K_t^{\alpha-1} L_t^{1-\alpha} - (r_t + \delta) = 0 \tag{55}$$

$$\frac{\partial \pi}{\partial L_t} = A(1-\alpha) K_t^{\alpha} L^{-\alpha} - w = 0 \tag{56}$$

If I isolate the interest rate of expression 55, I get the inverse demand function:

$$r_t = A\alpha \left(\frac{L_t}{K_t} \right)^{1-\alpha} - \delta \tag{57}$$

I do some arrangements that are going to be useful in the FOC for labor:

$$\left(\frac{K_t}{L_t} \right)^{1-\alpha} = \frac{A\alpha}{r_t + \delta}$$

$$\left(\frac{K_t}{L_t} \right) = \left(\frac{A\alpha}{r_t + \delta} \right)^{\frac{1}{1-\alpha}}$$

Then, I can substitute this in the expression 56 and I get:

$$w_t = A(1-\alpha) \left(\frac{K_t}{L_t} \right)^{\alpha}$$

$$w_t = A(1-\alpha) \left(\frac{A\alpha}{r_t + \delta} \right)^{\frac{\alpha}{1-\alpha}} \tag{58}$$

8.3 Market clearing condition

There are three markets in this economy: the capital market, the good market, and the labor market. By Walras' Law, the goods market-clearing condition is redundant. I state supply equals demand to each market, so:

$$L_t = \sum_{s=1}^{3} n_{s,t} = 1 + 1 + 0 = 2 \qquad \forall t \tag{59}$$

$$K_t = \sum_{s=2}^{3} b_{s,t} = b_{2,t} + b_{3,t} \qquad \forall t \tag{60}$$

$$Y_t = C_t + I_t \qquad \forall t \tag{61}$$

8.4 Equilibrium

The steady-state equilibrium in the perfect foresight OLG with 3-period live agents is defined as constant allocations of consumption, capital, and labor, i.e. $c_t = c^*$, $k_t = k^*$ and $N = N^*$ for all t, and the interest rate and wages do not vary over time, $r_t = r^*$ and $w_t = w^*$ for all t, such that:

1. Household optimizes according to equations 52 and 48.
2. Firms optimize according to equations 57 and 58.
3. And markets clearing conditions, equations 60 and 59.

While the non-steady-state functional equilibrium is defined as stationary allocation functions of the state $\psi_1(b_{2,t}b_{3,t})$ and $\psi_2(b_{2,t}b_{3,t})$ and stationary price functions w(b_2, b_3, t) and r(b_2, b_3, t) such that the previous points (1, 2, 3) hold. The solution consists of a fixed point in function space, we choose two functions ψ_1 and ψ_2 and we verify that satisfy the Euler Equation for all points in the state space.

8.5 Time path iteration

This has been explained in detail in the section of Results but summarizing, the time path iteration (TPI) method solves the non-steady-state rational expectations equilibrium transition path of the distribution of savings that I have explained in the previous point. As we can see in Auerbach et al. (1987) I need to find a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. This can be seen clearly in Phyton, but the basic steps can be summarized as assuming a transition path for aggregate capital, then I can compute the transition path for interest rate and wages, and finally, I can solve the optimal savings decisions. The fixed point necessary for the equilibrium transition path can be found when the distance between r and r' is arbitrarily close to zero:

$$||r' - r|| < 0.00001 \tag{62}$$

If not, I state a new transition path for the interest rate that depends on the convex combination of r and r' such as:

$$r_{path} = x_i * r'_{path} + (1 - x_i) * r_{path} \tag{63}$$

8.6 Results of the Phyton code

The following step consists of programming this model in Phyton, you can see it in the folder Final Project and it is named as "OLG_3periods_RosannaGomar". First, I set the parameters of the model: $\beta, \alpha, \delta, \theta$, and so on. After this, I need to create the different functions of the model and later I can calculate the steady-state and the transition. We need 152 iterations to solve for the steady-state and 99 to solve for the time-path. Focusing on the steady-state, the results are:

```
The time path SOLVED!!!!
Coefficient of relative risk aversion= 2.0000
Discount factor= 0.9000
Capital share in production= 0.30
Delta= 0.1000
SS interest rate is  0.822146872512887
Maximum Euler error in the SS is  1.7493855075656484e-07
SS capital= 0.40
SS consumption= 1.20
SS investment= 0.04
SS output= 1.24
capital/output= 0.33
investment/output 0.03
Maximum Euler error along the time path is  1.3423928635347693e-10
```

And the graphical representation is displayed on the following page in figure 1. First of all, the horizontal axe is between a value of 0 and 20 since people live three periods of time, and we suppose that each period longs 20 years, so the older generation at this moment will die in 20 years. If we change our T by 25, the results do not change, so it is not significant if each period is 20 or 25 years.

We observe that the labor is equal to 2 for all periods since we have three generations living in the same moment. Recall that when s is equal to 1 and 2, the individuals supply labor inelastically equal to 1 and 0 when they are retired, or equivalently, when s is equal to 3. The interest rate grows along the path, therefore the capital, the investment, and the production decrease. Due to this fact, as we had expected before plotting, the wages experiment a fall, and the same for consumption.

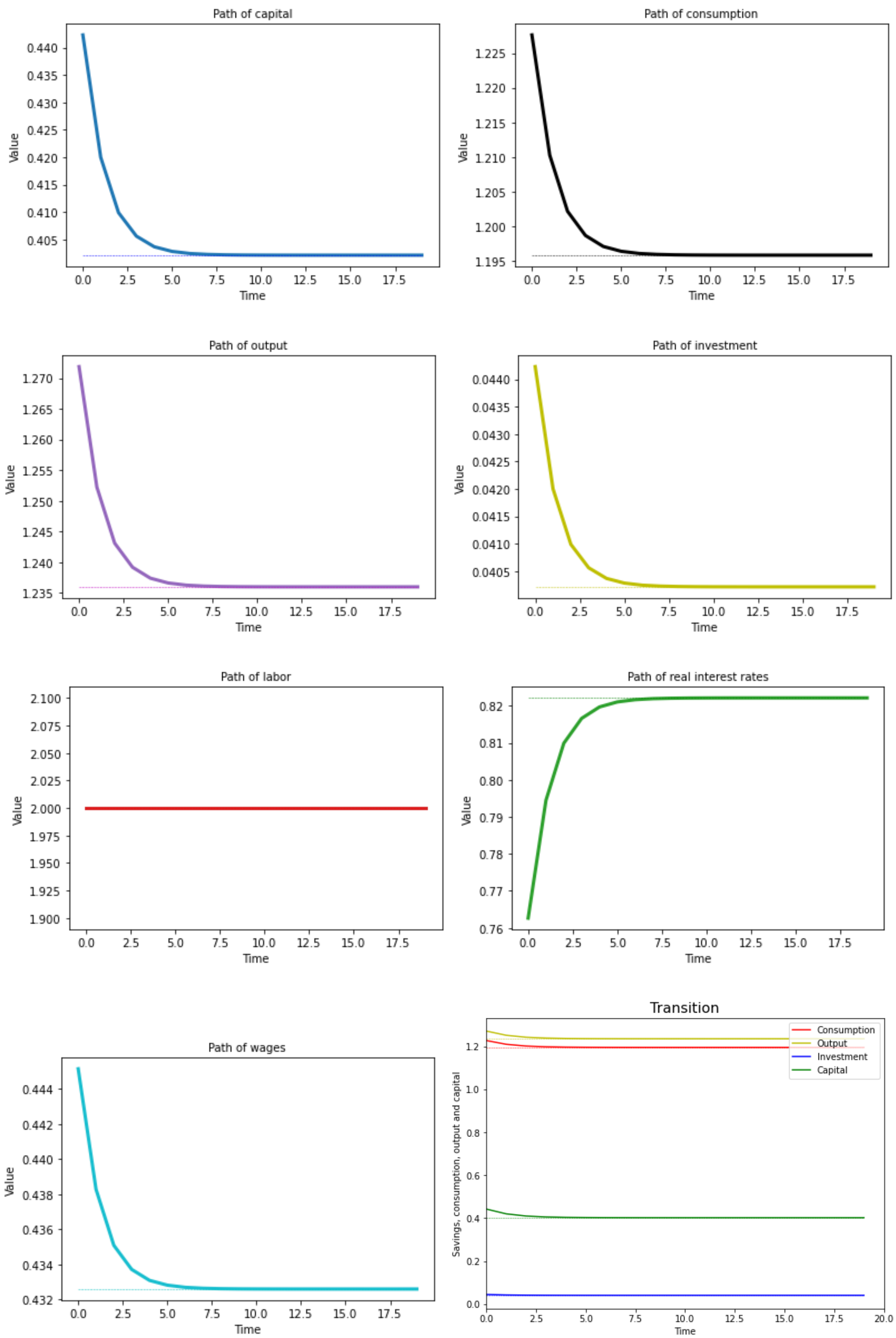

Figure 1: Time path for labor, capital, consumption, investment, output, interest rate and wages.

9 References

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