THE SPATIAL Λ-FLEMING-VIOT PROCESS IN A RANDOM ENVIRONMENT

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Abstract. We study large scale behaviour of a population consisting of two types which evolve in dimension $d = 1, 2$ according to a spatial Lambda-Fleming-Viot model subject to random time-independent selection.

If one of the two types is rare compared to the other, we prove that its evolution can be approximated by a superBrownian motion in a random time-independent environment. Without the sparsity assumption, a diffusion approximation leads to a Fisher-KPP equation in a random potential. We discuss the longtime behaviour of the limiting processes addressing Wright's claim that the variation in spatial conditions contributes positively to genetic variety in the populations.

The crucial technical components of the proofs are two-scale Schauder estimates for semidiscrete approximations of the Laplacian and of the Anderson Hamiltonian.

Key words: Spatial Lambda Fleming-Viot model, superprocesses, SuperBrownian motion, Anderson Hamiltonian, scaling limits, Schauder estimates

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Introduction

A fundamental challenge in population genetics is to understand the interplay between different evolutionary and ecological factors and their overall contribution to genetic variety, i.e. the distribution of different types within a population. A prominent example of such a force is a random neutral process of 'genetic drift', which occurs due to random reproduction of organisms. Another one is the adaptive process of selection. Both genetic drift and selection work, in different ways, to reduce the genetic variability of populations. However, other ecological and evolutionary forces may counterbalance those factors and explain durable heterogeneity within the populations.

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Starting with the pioneering works by Wright [54], spatial structure has played a key role in understanding genetic diversity. Since individuals inhabit different, possibly distant geographical regions and do not move too far from their place of birth, the likelihood of mating between geographically distant populations is very small. This leads to a greater differentiation between subpopulations, as distant individuals evolve essentially independently of each other. In extreme cases, this mechanism, which is usually referred to as isolation by distance may even lead to creation of different species. Even though, in principle, selection acts to reduce the genetic variety, Wright argued in the same article that if the selection is spatially heterogeneous, that is, if selection favors different types of individuals in different regions in space, it may further enhance the differentiation coming from isolation by distance. A large body of empirical evidence suggests that this may indeed be the case. Studies on plants [43], bacteria [46], animals [34] seem to all confirm that the spatial environmental inhomogeneity enhance the diversity. For more in-depth description of biological literature, including less favorable viewpoints of the phenomena we are concerned with, we refer to [50], [30], [49].

Our work is similarly motivated by the question: does spatially heterogeneous selection enhance the genetic diversity?

There are many approaches one could take to model a spatially structured population. The stepping stone models (see i.e. [35]), where the population evolves in separated islands distributed on a lattice and interacts only with neighboring islands, lead to an artificial subdivision of population. Approaches based around Wright–Malécot formula [4, 38, 54] (which was introduced to study the isolation by distance phenomena) suffer from either inconsistencies in their assumptions or lead to unnatural 'clumping' of the population. We refer to [6] for an overview of difficulties associated with modelling spatially distributed populations. The spatial Lambda-Fleming-Viot (SLFV) class of models, introduced in [21] and formally constructed in [5], has been proposed specifically to overcome those difficulties, and is at the basis of our work. Here the population in distributed over continuous space, whereas the reproductive events involve macroscopic regions of space (in this work balls of a fixed radius $\varepsilon \in (0,1)$) and are driven by a space-time Poisson point process.

In the neutral SLFV there is no bias in the relative fitness of the populations at hand. Our work considers instead the case in which the population consists of just two types ($\mathfrak{a}$ and $\mathfrak{A}$) and their relative fitness is modelled by a sign changing selection coefficient $s_\varepsilon(x), x \in \mathbb{T}^d$ (the latter being the d-dimensional torus), so that $\mathfrak{a}$ is favored in the location x if $s_\varepsilon(x) > 0$ and $\mathfrak{A}$ is favored in the opposite case. Instead of choosing a specific selection coefficient, we sample it from a probability distribution $\mathbb{P}$. We will consider the proportion $X_\varepsilon(\omega, t, x)$, evaluated at time $t \geq 0$ and position $x \in \mathbb{T}^d$, of particles of type $\mathfrak{a}$ with respect to the total population, given the realization $s_\varepsilon(\omega)$ of the selection coefficient. The parameter $\varepsilon > 0$ indicates the size of the impact area of reproductive events: we are interested in the limit $\varepsilon \to 0$ and will scale the magnitude of the reproductive events and the strength of the selection coefficient s_ε according to ε as well. All our scaling limits are diffusive and the effect of selection is *weak* with respect to neutral events.

We study two different scenarios. In the first one, we assume that type $\mathfrak{a}$ is rare compared to $\mathfrak{A}$. The rarity is described by considering an initial condition $X_\varepsilon(\omega, 0, x)$ of order ε^ϱ for certain values of $\varrho > 0$. In this scenario $\mathfrak{a}$ represents a mutation which tries to establish itself among the wild type $\mathfrak{A}$. Just as a small sub-population in the Wright-Fisher model is described by a branching process, we expect the limit to be a superBrownian motion (see [20] for an introduction to superprocesses) in a random time-independent environment. A similar scaling result without selection was first obtained by [12] (see also [17] for an analogous result regarding the voter model) and recently extended in [16] to critical values of the parameter ϱ. A scaling limit for a model with a selection coefficient which is white in time and correlated in space, was obtained by [13] using a lookdown representation.

We will assume, instead, that s_ε scales to a spatial white noise ξ on the torus $\mathbb{T}^d$ and consider only dimension $d = 1, 2$. In this setting, the limit (cf. Theorem 1) is the rough superBrownian motion introduced in [45], which formally solves the following stochastic partial differential equation (SPDE) for some $\nu_0 > 0$ (in $d = 2$ the SPDE has to be replaced by the associated martingale problem):

$$\partial_t Y = \nu_0 \Delta Y + (\xi - \infty 1_{\{d=2\}})Y + \sqrt{Y}\widetilde{\xi}, \qquad Y(0) = Y_0. \tag{1}$$

Here $\widetilde{\xi}$ is a space-time white noise independent of ξ. The ∞ appearing in $d = 2$ is the consequence of the renormalization required to make sense of the Anderson model, which is described by the SPDE

$$\partial_t Y = \nu_0 \Delta Y + (\xi - \infty 1_{\{d=2\}})Y, \qquad Y(0) = Y_0. \tag{2}$$

The latter equation is *singular* in $d = 2$ because the expected regularity of the solution Y is not sufficient to make sense of the product $\xi \cdot Y$ and requires theories such as regularity structures or paracontrolled distributions (cf. Section 6 or see [29, 28] for complete works on singular SPDEs). In particular, there is no understanding of the Anderson model in dimension $d \geq 4$. We restrict to $d \leq 2$ as these are the biologically interesting cases and in $d = 3$ renormalization is more involved. The quoted solution theories for singular SPDEs work pathwise, conditional on the realization of the noise ξ and some functionals thereof. As a consequence, solutions to (1) are defined as martingale solutions conditional on the realization of ξ and uniqueness in distribution of solutions to (1) is then proven through a conditional duality argument. This is in contrast with cases where the environment is white in time [41], where the martingale term can contain also the environment.

A crucial step in the proof of the scaling limit is to show that the continuous Anderson Hamiltonian $\mathcal{H} = \nu_0 \Delta + \xi - \infty 1_{\{d=2\}}$ is the limit of approximations $\mathcal{H}_\varepsilon = \mathcal{A}_\varepsilon + \xi_\varepsilon - c_\varepsilon 1_{\{d=2\}}$ (cf. Theorem 4). In the latter operator, the approximate Laplacian $\mathcal{A}_\varepsilon$ acts on $L^2(\mathbb{T}^d)$ and is expressed in terms of local averages of functions: we call this setting semidiscrete, as opposed to the fully discrete setting, where the underlying space is for example a lattice. Fully discrete approximations of singular SPDEs have been the object of many studies (see [40, 19, 14, 39] for a partial literature). Instead, approximations in the present semidiscrete case appear new. In the study of such SPDEs the smoothing effect of the Laplacian is crucial: the first step towards understanding the convergence of the operators is to establish the regularization properties of the approximate Laplacian $\mathcal{A}_\varepsilon$, commonly known as Schauder estimates. Through a two-scale argument, we separate macroscopic scales in frequency space, at which $\mathcal{A}_\varepsilon$ regularizes analogously to the Laplacian, and microscopic scales, which are small but see no regularization (see Theorem 3). Once we are provided with the Schauder estimates and the convergence of $\mathcal{H}_\varepsilon$, the scaling limit is proven through an application of the Krein-Rutman theorem. At this point it is particularly important that the space is compact, while all other results in this work seem to extend from $\mathbb{T}^d$ to $\mathbb{R}^d$.

In the second scenario, s_ε is chosen to scale to a smooth random function $\bar{\xi}$, and we do not take the sparsity assumption. This regime corresponds studying the long time behaviour of a large population. In this case under diffusive scaling one obtains (cf. Theorem 2) convergence to a solution of the (in $d = 1$ stochastic) Fisher-KPP equation

$$\partial_t X = \nu_0 \Delta X + \bar{\xi} X(1 - X) + \sqrt{X(1 - X)}\widetilde{\xi} 1_{\{d=1\}}, \qquad X(0) = X_0. \tag{3}$$

As before $\widetilde{\xi}$ is a space-time white noise independent of $\bar{\xi}$. In a nutshell, the intensity of the martingale term is governed by a parameter $\eta \geq 0$ and there exists a critical value $\eta_c(d) \geq 0$ such that the martingale term is of order $\varepsilon^{\eta - \eta_c}$. In dimension $d = 1$ we consider $\eta = \eta_c$, while in dimension $d = 2$ we take $\eta > \eta_c$. In some models, by taking into account dual processes, cf. [21, 24] , one can prove that in $d = 2$ the deterministic limit holds also *at* the critical value. To the best of our knowledge the process we consider does not have a dual: hence although a similar result is expected, it remains open as the quoted methods do not apply. Due to the same lack of duality, in $d = 1$ uniqueness of the solutions seems out of reach. Similar results have been obtained in [22] where the selection coefficient is constant in space and time (in this case the process admits a dual) and in [7], where the selection coefficient is fluctuating in time and space and correlated in the latter, giving rise to an additional martingale term.

The treatment of this second regime is apparently much simpler, as the solution is bounded between 0 and 1. The only difficulty is to prove convergence in a topology, in which one can pass to the limit inside the nonlinearity. Unlike the previous works [22, 7] we can make good use of the Schauder estimates and directly prove tightness for a smoothed version of X_ε in a Sobolev space of positive regularity (see Theorem 2).

Eventually, we study the longtime behavior of the limiting processes. As already observed in [45], the time-independent random environment is beneficial for the survival of the rough superBrownian motion, which then depends just on the positivity of the largest eigenvalue of the Anderson Hamiltonian $\mathcal{H}$. We will show that with positive probability, with respect to the environment, the process will survive (cf. Proposition 1.10). As a comparison, observe that if the environment is fluctuating also in time, the process can die out quite dramatically, for example see [42]. As for the second setting, in $d = 1$ the presence of genetic drift implies that the solution to (3) becomes trivial in finite time. If $d = 2$, if the initial condition is not identically 1 or 0, and if the noise is sufficiently strong (see Assumption 1.16 and Remark 1.17), the solution converges to a unique non-trivial $\overline{X}(\omega)$, which amounts to longtime coexistence of both types (cf. Proposition 1.18).

In conclusion, this work extends previous scaling limits to incorporate a sign changing, possibly rough, selection. Choosing the selection at random provides a natural setting which exhibits interesting longtime behavior. We believe this could be the starting point for some ulterior studies: for example Equation (3) in $d = 1$ with $x \in \mathbb{R}$ (so globally in space) can be recovered with the same methods and could have interesting longtime properties, as the selection could balance out the genetic drift. The methods we used are based on two-scale Schauder estimates and do not rely on duality. They allow us to establish a connection to singular SPDEs, but appear to be a fairly simple, powerful tool to treat nonlinearities appearing in the SLFV.

Structure of the paper. In Section 1, we describe the notations, define the models and state main results. Section 2 is devoted to relation between the Spatial Lambda-Fleming-Viot process with selection in rough potential and the rough superBrownian motion, whereas in Section 3 the similar relationship with Fisher-KPP equation in rough potential is established. The long time behaviour of the limiting processes is discussed in Section 4. The rest of the paper is devoted to analytical backbone of our results. Section 5 covers Schauder estimates. Finally, Section 6 discusses the analytical and probabilistic aspects of the Anderson model.

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1. Models and statement of main results

We begin with stating the notation used throughout the paper in Subsection 1.1. In Subsection 1.2 we describe the Spatial-Lambda-Fleming-Viot. In Subsection 1.3 describe the small families limit which is the first of our main results. Subsection 1.4 is devoted to diffusive scaling which leads to Fisher-KPP equation. In Subsection 1.5 we describe the main analytical components of the proofs, which we believe may be of separate, purely mathematical interest.

1.1. **Notations.** We write $\mathbb{N} = \{0, 1, 2, \dots\}$, and $\mathbb{R}_+ = [0, \infty)$. Fix $d \in \mathbb{N}$. The $d-$dimensional $\mathbb{T}^d$ torus is defined as $\mathbb{T}^d := [-1/2, 1/2]^d /\sim$, where $\sim$ is the equivalence relation which glues opposite edges. For notational convenience we use the following convention.

$$\text{We write} \quad \varepsilon \in (0, 1/2) \quad \text{for} \quad \varepsilon = 1/n, \quad \text{for some} \quad n \in \mathbb{N}, n \geq 2.$$

Then we introduce the following scaling for balls and cubes. Indicate with $|A|$ the Lebesgue measure of a Borel set $A \subseteq \mathbb{T}^d$. Let then $B_\varepsilon(x) \subseteq \mathbb{T}^d$ be the ball (w.r.t. the Euclidian norm) of volume ε^d about x. Similarly, let $Q_\varepsilon(x) \subset \mathbb{T}^d$ be the d-dimensional cube

$$y \in Q_\varepsilon(x) \iff (y-x)_i \in \big[-\varepsilon/2, \varepsilon/2\big), \ \forall i \in \{1, \dots, d\}.$$

In particular, in our notation

$$|B_\varepsilon(x)| = |Q_\varepsilon(x)| = \varepsilon^d.$$

Now, for integrable $w\colon \mathbb{T}^d \to \mathbb{R}$ define $\Pi_\varepsilon w(x)$ as an average integral of w over $B_\varepsilon(x)$, that is

$$\Pi_\varepsilon w(x) := \fint_{B_\varepsilon(x)} w(y)\,\mathrm{d}y := \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} w(y)\,\mathrm{d}y.$$

Furthermore, consider the lattice

$$\mathbb{Z}^d_\varepsilon = \Big(\varepsilon^{-1}\mathbb{Z}^d\Big) \cap \mathbb{T}^d.$$

Since $\varepsilon = 1/n$, cubes Q_ε, centred at the points of lattice $\mathbb{Z}^d_\varepsilon$ are disjoint and satisfy

$$\mathbb{T}^d = \bigcup_{x \in \mathbb{Z}_\varepsilon} Q_\varepsilon(x).$$

We make use of the Fourier transform both on the torus and in the full space. For $\varphi \in \mathcal{S}'(\mathbb{T}^d)$, that is for an element of the space of tempered distributions on $\mathbb{T}^d$, we define

$$\widehat{\varphi}(k) = \mathcal{F}_{\mathbb{T}^d}\varphi(k) = \int_{\mathbb{T}^d} e^{-2\pi\iota k\cdot x} f(x)\,\mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

Analogously, for $\psi \in \mathcal{S}'(\mathbb{R}^d)$

$$\mathcal{F}_{\mathbb{R}^d}\psi(k) = \int_{\mathbb{R}^d} e^{-2\pi\iota k\cdot x}\psi(x)\,\mathrm{d}x, \quad k \in \mathbb{R}^d.$$

These Fourier transforms admit inverses, which we denote with $\mathcal{F}^{-1}_{\mathbb{T}^d}, \mathcal{F}^{-1}_{\mathbb{R}^d}$ respectively.

For $a\colon \mathbb{Z}^d \to \mathbb{R}$ with at most polynomial growth we define the Fourier multiplier as an operator of the form

$$a(D)\varphi := \mathcal{F}^{-1}_{\mathbb{T}^d}\big(a(\cdot)\mathcal{F}_{\mathbb{T}^d}\varphi(\cdot)\big), \qquad \forall \varphi \in \mathcal{S}'(\mathbb{T}^d).$$

Since characteristic functions, normalized to integrate to 1 over the entire domain, enter the calculations repeatedly, for a set A we write:

$$\chi_A(x) = \frac{1}{|A|}1_A(x).$$

In the special case of balls and cubes we additionally define

$$\chi_\varepsilon(x) := \varepsilon^{-d}1_{B_\varepsilon(0)}(x), \qquad \widehat{\chi}_\varepsilon(k) = \widehat{\chi}(\varepsilon k) := \mathcal{F}_{\mathbb{T}^d}\chi_\varepsilon(k) = \mathcal{F}_{\mathbb{R}^d}\chi_\varepsilon(k),$$
$$\chi_{Q_\varepsilon}(x) := \varepsilon^{-d}1_{Q_\varepsilon(0)}(x), \qquad \widehat{\chi}_{Q_\varepsilon}(k) = \widehat{\chi}_Q(\varepsilon k) := \mathcal{F}_{\mathbb{T}^d}\chi_{Q_\varepsilon}(k) = \mathcal{F}_{\mathbb{R}^d}\chi_{Q_\varepsilon}(k).$$

Observe that in order to obtain the identity between the Fourier transform on the torus and in the full space, the ε should satisfy $\varepsilon \leq 1/2 < \sqrt{\pi}/2$, as otherwise the ball of radius ε about 0 intersects the boundary of the torus.

A special role in the paper is played by an operator $\mathcal{A}_\varepsilon$ defined as

$$\mathcal{A}_\varepsilon(\varphi)(x) = \varepsilon^{-2}\fint_{B_\varepsilon(x)}\fint_{B_\varepsilon(y)} \varphi(y) - \varphi(x)\,\mathrm{d}z\,\mathrm{d}y = \varepsilon^{-2}\big(\Pi^2_\varepsilon\varphi - \varphi\big)(x). \tag{4}$$

Such an operator is a Fourier multiplier with

$$\mathcal{A}_\varepsilon = \vartheta_\varepsilon(D), \qquad \vartheta_\varepsilon(k) = \varepsilon^{-2}\frac{1}{\widehat{\chi}^2(\varepsilon k) - 1}.$$

We proceed with a definition of Besov spaces. Following [3, Proposition 2.10], fix a dyadic partition of the unity $\{\varrho_j\}_{j\geq -1}$. We assume that $j \geq 0$, $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ is a radial, smooth compactly supported function. For a distribution $\varphi \in \mathcal{S}'(\mathbb{T}^d)$ define $\Delta_j \varphi = \varrho_j(D)\varphi$ and hence define the spaces $B^\alpha_{p,q}$ for $\alpha \in \mathbb{T}, p, q \in [1,\infty]$ via the norms

$$\|\varphi\|_{B^\alpha_{p,q}} = \|(2^{\alpha j}\|\Delta_j \varphi\|_{L^p(\mathbb{T}^d)})_{j\geq -1}\|_{\ell^q(j\geq -1)}.$$

Since the partition of unity was chosen to be smooth, we define the Besov spaces on full space via the same formula. It is convenient to introduce notation

$$K^x_j(y) = \mathcal{F}_{\mathbb{T}^d}\rho_j(x-y).$$

For $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$ and $p, q = \infty$ the above definition coincides with that of classical Hölder spaces. We therefore write

$$\mathcal{C}^\alpha := B^\alpha_{\infty,\infty}, \qquad \mathcal{C}^\alpha_p := B^\alpha_{p,\infty}.$$

We shall denote the norm of the Hölder space $\mathcal{C}^\alpha$ by $\|\cdot\|_\alpha$.

Let $\mathcal{M}(\mathbb{T}^d)$ denote the space of finite positive measures over $\mathbb{T}^d$. For metric spaces X, Y let $C(X;Y)$ and $C_b(X;Y)$ the space of continuous, and bounded and continuous, respectively functions from X to Y. If $Y = \mathbb{R}$, we may drop the second argument. In addition for a metric space X we define $\mathbb{D}([0,\infty);X)$ to be the space of cadlag functions with values in X, endowed with the Skorohod topology as in [23, Section 3.5] (similarly for finite time horizon $T > 0$ we write $\mathbb{D}([0,T];X)$). If X is a Banach space we write $L^2([0,T];X)$ for the space of measurable functions φ on $[0,T]$ taking values in X and satisfying $\|\varphi\|_{L^2([0,T];X)} = \big(\int_0^T \|\varphi(s)\|^2_X \,\mathrm{d}s\big)^{1/2} < \infty$. The local variant of the space for $T = \infty$ is then defined as $L^2_{\mathrm{loc}}([0,\infty);X) = \bigcap_{T>0} L^2([0,T];X)$.

1.2. **Spatial Λ-Fleming-Viot process in a random environment.** We now turn our attention to the description of the underlying popuation model, the Spatial Lambda-Fleming-Viot model in a random environment. We are interested in a population with two genetic types, $\mathfrak{a}$ and $\mathfrak{A}$. At each time $t \geq 0$, X^ε_t is a random function such that

$$X^\varepsilon_t = \text{ proportion of individuals of type } \mathfrak{a} \text{ at time } t \text{ at position } x.$$

The dynamics of the Spatial Lambda-Fleming-Viot model is determined by reproduction events, driven by independent Poisson point processes. In order to incorporate selection, we follow the usual strategy and distinguish two types of reproduction events, neutral and selective. In simple terms

Neutral: Both types have the same chance of reproducing,
Selective: One of the two types is more likely to reproduce than the other.

The strength, as well as the direction of the selection are encoded by the magnitude and sign of the random function $s_\varepsilon(\omega)$. The function s_ε should satisfy

$$\Omega \ni \omega \mapsto s_\varepsilon(\omega) \in L^\infty(\mathbb{T}^d;\mathbb{R}), \quad |s_\varepsilon(\omega,x)| < 1, \quad \text{with } (\Omega,\mathcal{F},\mathbb{P}) \text{ a probability space.} \tag{5}$$

Conditional on the realization $s_\varepsilon(\omega)$ of the environment, the process $X^\varepsilon(\omega)$ will be a Markov process. Its dynamics are defined below, deferring some technical steps regarding existence and construction of the probability space $(\Omega\times\Omega', \mathcal{F}\otimes\mathcal{F}', \mathbb{P}\ltimes\mathbb{P}^\omega)$ on which the process is defined until Appendix A. We write:

$$M = \big\{w \colon \mathbb{T}^d \to [0,1], \quad w \text{ measurable}\big\}.$$

Definition 1.1 (Spatial Λ-Fleming-Viot process with random selection)**.** *Fix $\varepsilon \in (0,\frac{1}{2})$, $\mathfrak{u} \in (0,1)$ and consider s_ε and Ω as in (5). Let $X^{\varepsilon,0} \colon \mathbb{T}^d \to \mathbb{R}$ be such that $0 \leq X^{\varepsilon,0} \leq 1$. Define the process X^ε on the probability space $(\Omega\times\Omega', \mathcal{F}\otimes\mathcal{F}', \mathbb{P}\ltimes\mathbb{P}^\omega)$, so that for every $\omega\in\Omega$ it holds that*

i) *The space $(\Omega',\mathbb{P}^\omega)$ supports a pair of independent Poisson point processes $\Pi^{\mathrm{neu}}_\omega$ and $\Pi^{\mathrm{sel}}_\omega$ on $\mathbb{R}_+\times\mathbb{T}^d$ with intensity measures $\mathrm{d}t\otimes(1-|s_\varepsilon(\omega,x)|)\mathrm{d}x$ and $\mathrm{d}t\otimes|s_\varepsilon(\omega,x)|\mathrm{d}x$ respectively.*

ii) *The random process (defined on Ω') $\mathbb{R}_+ \ni t \mapsto X^\varepsilon_t(\omega)$ is the Markov process started in $X^{\varepsilon,0}$ with values in M associated to the bounded generator*

$$\mathcal{L}(\varepsilon, s_\varepsilon(\omega), \mathfrak{u}) \colon C_b(M;\mathbb{R}) \to C_b(M;\mathbb{R})$$

(see again Lemma A.2 for a rigorous construction), that can be described by the following dynamics.

(1) *If $(t,x) \in \Pi^{\mathrm{neu}}_\omega$, a neutral event occurs at time t in the ball $B_\varepsilon(x)$, namely:*
   (a) *Choose a parental location y uniformly in $B_\varepsilon(x)$.*
   (b) *Choose the parental type $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}$ according to the distribution*

$$\mathbb{P}[\mathfrak{p} = \mathfrak{a}] = X^\varepsilon_{t-}(\omega, y), \quad \mathbb{P}[\mathfrak{p} = \mathfrak{A}] = 1 - X^\varepsilon_{t-}(\omega, y).$$

   (c) *A proportion $\mathfrak{u}$ of the population within $B_\varepsilon(x)$ dies and is replaced by offspring with type $\mathfrak{p}$. Therefore, for each point $z \in B(x,r)$,*

$$X^\varepsilon_t(\omega, z) = X^\varepsilon_{t-}(\omega, z)(1-\mathfrak{u}) + \mathfrak{u}\chi_{\{\mathfrak{p}=\mathfrak{a}\}}.$$

(2) *If $(t,x) \in \Pi^{\mathrm{sel}}_\omega$, a selective event occurs in the ball $B_\varepsilon(x)$, namely:*
   (a) *Choose two parental locations y_0, y_1 independently, uniformly in $B_\varepsilon(x)$.*
   (b) *Choose the two parental types, $\mathfrak{p}_0, \mathfrak{p}_1$, independently, according to*

$$\mathbb{P}[\mathfrak{p}_i = \mathfrak{a}] = X^\varepsilon_{t-}(\omega, y_i), \quad \mathbb{P}[\mathfrak{p}_i = \mathfrak{A}] = 1 - X^\varepsilon_{t-}(\omega, y_i).$$

   (c) *A proportion $\mathfrak{u}$ of the population within $B_\varepsilon(x)$ dies and is replaced by offspring with type chosen as follows:*
      (i) *If $s(x) > 0$, their type is set to be a if $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{a}$, and $\mathfrak{A}$ otherwise. Thus for each $z \in B_\varepsilon(x)$*

$$X^\varepsilon_t(\omega, x) = (1-\mathfrak{u})X^\varepsilon_{t-}(\omega, z) + \mathfrak{u}\chi_{\{\mathfrak{p}_0=\mathfrak{p}_1=\mathfrak{a}\}}.$$

      (ii) *If $s(x) < 0$, their type is set to be $\mathfrak{a}$ if $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{a}$ or $\mathfrak{p}_0 \neq \mathfrak{p}_1$ and $\mathfrak{A}$ otherwise, so that for each $z \in B_\varepsilon(x)$,*

$$X^\varepsilon_t(\omega, z) = (1-\mathfrak{u})X^\varepsilon_{t-}(\omega, z) + \mathfrak{u}(\chi_{\{\mathfrak{p}_0=\mathfrak{p}_1=\mathfrak{a}\}} + \chi_{\{\mathfrak{p}_0\neq\mathfrak{p}_1\}}).$$

Remark 1.2. *Strictly speaking, the process constructed in Appendix A is a Markov jump process $X^\varepsilon(\omega)$. The Poisson point processes mentioned in Definition 1.1 are not constructed explicitly, but can be reconstructed from the jump times and jump locations.*

Most of our arguments take advantage of the martingale representation of the process. We record this representation as a Lemma. The proof can be found in Appendix A. For a function f on $[0,\infty)$ we write

$$f_{t,s} = f_t - f_s.$$

Lemma 1.3. *Fix $\omega \in \Omega$ and X^ε an SLFV as in the previous definition. For every $\varphi \in L^1(\mathbb{T}^d)$ the process $t \mapsto \langle X^\varepsilon_t(\omega), \varphi\rangle$ satisfies the following martingale problem, for every $t \geq s \geq 0$*

$$\langle X^\varepsilon_{t,s}(\omega), \varphi\rangle = \mathfrak{u}\varepsilon^d \int_s^t \langle (\Pi^2_\varepsilon - \mathrm{Id})(X^\varepsilon_r(\omega)), \varphi\rangle + \langle \Pi_\varepsilon\big[s_\varepsilon(\omega)\big(\Pi_\varepsilon X^\varepsilon_r(\omega) - (\Pi_\varepsilon X^\varepsilon_r(\omega))^2\big)\big], \varphi\rangle \,\mathrm{d}r + M^\varepsilon_{t,s}(\varphi)$$

where $M^\varepsilon_{t,s}(\varphi)$ is the increment of a square-integrable martingale with predictable quadratic variation given by

$$\begin{aligned}\langle M^\varepsilon(\varphi)\rangle_t = \mathfrak{u}^2\varepsilon^{2d} \int_0^t &\langle (1+s_\varepsilon(\omega))\Pi_\varepsilon X^\varepsilon_r(\omega), (\Pi_\varepsilon\varphi)^2 - 2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\big)\rangle \\ &+ \langle \big(\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\big)^2, 1\rangle - \langle s_\varepsilon(\omega)(\Pi_\varepsilon X^\varepsilon_r(\omega))^2, (\Pi_\varepsilon\varphi)^2 - 2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\big)\rangle \,\mathrm{d}r.\end{aligned}$$

1.3. **Sparse regime.** First, we consider a scaling regime in which the type $\mathfrak{a}$ is rare, which means that X^ε_t is very close to 0. Heuristically, if ε^ρ is the magnitude of the local population density, then $Y_t = \varepsilon^{-\rho} X^\varepsilon_t$ models the density of the number of individuals of type $\mathfrak{a}$. This justifies the following 'smallness' assumption on the sequence of initial conditions of the process.

Assumption 1.4 (Sparsity). *Fix a $\varrho > 5d/2$ and a sequence $X^{\varepsilon,0} \in L^\infty(\mathbb{T}^d)$ such that for some $Y^0 \in L^\infty(\mathbb{T}^d)$*

$$0 \le X^{\varepsilon,0} \le 1, \qquad \lim_{\varepsilon\to 0} \varepsilon^{-\varrho} X^{\varepsilon,0} = Y^0 \ \textit{in}\ L^\infty(\mathbb{T}^d).$$

We would like to describe a scaling limit in which even though none of the types is overall favourable, the selection can be locally strong. In particular, our selection coefficient will converge to space white noise. However, to obtain a non-trivial scaling limit in dimension $d = 2$, renormalization has to be taken into account. The sequence of renormalization constants c_ε is determined by the sequence which allows constructing solutions to the Anderson model, namely

$$c_\varepsilon = \sum_{k\in\mathbb{Z}^2} \frac{\hat{\chi}^2(\varepsilon k)\hat{\chi}_Q(\varepsilon k)}{-\vartheta_\varepsilon(k)+1}. \tag{6}$$

The assumptions on the noise are summarized in what follows. We emphasize that ξ_ε is an approximation of space white noise.

Assumption 1.5 (White noise scaling). *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined an i.i.d. sequence of random variables $\{Z_\varepsilon(x)\}_{x\in\mathbb{Z}^d_\varepsilon}$ with all moments finite and satisfying:*

$$\mathbb{E}\big[Z^2_\varepsilon(x)\big] = 1, \qquad Z_\varepsilon(\omega,x) \in (-2,2), \qquad \textit{for all}\ \ x\in\mathbb{Z}^d_\varepsilon,\ \ \varepsilon\in(0,1/2),\ \ \omega\in\Omega.$$

Then define

$$s_\varepsilon(\omega,y) = Z_\varepsilon(\omega,x) - \varepsilon^{\frac{d}{2}} c_\varepsilon 1_{\{d=2\}}, \quad \textit{if}\ y\in Q_\varepsilon(x), \qquad \forall\omega\in\Omega, x\in\mathbb{T}^d$$

and write:

$$\xi^e_\varepsilon(\omega,x) = \varepsilon^{-\frac{d}{2}} s_\varepsilon(\omega,x), \qquad \xi_\varepsilon(\omega,x) = \xi^e_\varepsilon(\omega,x) + c_\varepsilon 1_{\{d=2\}}.$$

Under appropriate scaling, we will prove that the process X^ε converges to a rough super-Brownian motion. This process has been introduced and studied by [45] as an approximation of lattice branching process in a static environment. First, recall the construction of the Anderson Hamiltonian, and its relationship to our setting.

Lemma 1.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a white noise $\xi\colon \Omega \to \mathcal{S}'(\mathbb{T}^d)$, that is a process such that for all $f \in \mathcal{S}(\mathbb{T}^d)$ the projection $\langle \xi, f\rangle =: \int_{\mathbb{T}^d} f(x)\xi(\,\mathrm{d}x)$ are Gaussian random variables with covariance*

$$\mathbb{E}\Big[\langle\xi,f\rangle\langle\xi,g\rangle\Big] = \langle f,g\rangle, \qquad \forall f,g\in\mathcal{S}(\mathbb{T}^d).$$

For almost all $\omega\in\Omega$ there exists an operator

$$\mathcal{H}(\omega)\colon \mathcal{D}_\omega \subseteq C(\mathbb{T}^d) \to C(\mathbb{T}^d),$$

with a dense domain $\mathcal{D}_\omega \subseteq C(\mathbb{T}^d)$, such that

$$\mathcal{H}(\omega) = \lim_{\varepsilon\to 0}\Big[\mathcal{A}_\varepsilon + \xi_\varepsilon(\omega) - c_\varepsilon 1_{\{d=2\}}\Big] =: \nu_0\Delta + \xi(\omega) - \infty 1_{\{d=2\}}.$$

The limit is taken in distribution with respect to the probability measure $\mathbb{P}$, with the precise meaning of the procedure provided in Theorem 4. The last notation is just a convenient formalism obtained by exchanging the limit with the sum.

This lemma is a consequence of Proposition 1.19 and Theorem 4 below. The rough super-Brownian motion is then a Markov process conditional on the realization of the spatial white noise and thus on the realization of the Anderson Hamiltonian.

Definition 1.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a white noise ξ and $Y^0 \in \mathcal{M}(\mathbb{T}^d)$. A rough superBrownian motion is a couple composed of an enlarged probability space $(\Omega \times \overline{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \ltimes \overline{\mathbb{P}}^{\omega})$ (where $\mathcal{F} \otimes \overline{\mathcal{F}}$ is the product sigma-field and $\mathbb{P}^{\omega}$ is the conditional law of the process given the realization of the noise, see Appendix A for a definition of semidirect products) and a map*

$$Y \colon \Omega \times \overline{\Omega} \to C([0, \infty); \mathcal{M}(\mathbb{T}^d)).$$

Moreover, for $\omega \in \Omega$ let $\{\mathcal{F}_t^{\omega}\}_{t \geq 0}$ be the filtration generated by $t \mapsto Y_t(\omega)$, right-continuous and enlarged with all null-sets. Then, for all $\omega \in \Omega$ such that the operator $\mathcal{H}(\omega)$ is defined, and for all $\varphi \in \mathcal{D}_{\omega}$ and $T > 0$, the process

$$M_t^{\varphi} := \langle Y_t(\omega), \varphi \rangle - \langle Y^0, \varphi \rangle - \int_0^t \langle Y_s(\omega), \mathcal{H}(\omega)\varphi \rangle \mathrm{d}s$$

is a centered continuous, square-integrable $\mathcal{F}_t^{\omega}$-martingale on $[0, T]$ with quadratic variation

$$\langle M^{\varphi} \rangle_t = \int_0^t \langle Y_s(\omega), \varphi^2 \rangle \; \mathrm{d}s.$$

We are now in position to state the first of the main results.

Theorem 1. *For any $\varrho > \frac{5}{2}d$ consider a random environment s_{ε} as in Assumption 1.5, and initial conditions $X^{\varepsilon,0}$ as in Assumption 1.4. Consider the process X^{ε} as in Definition 1.1, associated to the generator*

$$\varepsilon^{-d-2-\eta} \mathcal{L}(\varepsilon, \varepsilon^{2-\frac{d}{2}} s_{\varepsilon}(\omega), \varepsilon^{\eta}),$$

with η defined by

$$\eta := \varrho + 2 - d. \tag{7}$$

Then the process $t \mapsto Y_t^{\varepsilon} = \varepsilon^{-\varrho} X_t^{\varepsilon}$ converges in distribution, as a stochastic process on the probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \ltimes \mathbb{P}^{\omega})$ (cf. Definition 1.1):

$$\lim_{\varepsilon \to 0} Y^{\varepsilon} = Y \quad \textit{in} \quad \mathbb{D}([0, \infty); \mathcal{M}(\mathbb{T}^d)),$$

where Y is the unique in distribution rough superBrownian motion as in Definition 1.7, started in Y^0.

Remark 1.8. *The scaling in Theorem 1 corresponds to scaling down the impact of each of the events by ε^{η}, the rate of selective events by $\varepsilon^{2-d/2}$, setting the volume of the ball in which of each of the events occurs to ε and speeding up the time by $\varepsilon^{-2-d-\eta}$. A fraction of order ε^{ρ} of the individuals present initially in the population is of type $\mathfrak{a}$.*

The coefficients appearing in the rescaling could be interpreted as follows. The term $\varepsilon^{-d-2-\eta}$ guarantees diffusive scaling, namely that a Laplacian appears in the limit. The scaling of s_{ε} guarantees convergence to space white noise. Finally, the parameter η guarantees that the limit is non-trivial. limit; it's only impact is determination of the only first non-trivial term in the quadratic variation.

An interesting feature of the rough superBrownian motion on a torus is persistence.

Definition 1.9. *A random process $t \mapsto Y(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ is persistent if for any $\varphi \in \mathcal{S}(\mathbb{T}^d)$ and $\varepsilon > 0$*

$$\mathbb{Q}\big(\lim_{t \to \infty} \langle Y(t), \varphi \rangle > \varepsilon \big) > 0.$$

Proposition 1.10. *Let $Y(t)$ be a rough superBrownian motion in a static environment as in Definition 1.7. Then:*

$$\mathbb{P}\big(\textit{The process } t \mapsto Y_t(\omega) \textit{ is persistent w.r.t. the law } \mathbb{P}^{\omega} \big) > 0.$$

The proof of Proposition 1.10 can be found in Subsection 4.1.

1.4. **Diffusive regime.** The second scaling regime we consider is the diffusive one. As before, the impact parameter $\mathfrak{u}$ is scaled as $\varepsilon^{-\eta}$. The restrictions on the value of η follows

Assumption 1.11. *Choose η such that*

$$\eta = 1 \quad \textit{if} \quad d = 1, \qquad \eta > 0 \quad \textit{if} \quad d = 2.$$

In the diffusive regime we still assume that the selection coefficient is random, yet it is not described by a an object converging to space white noise.

Assumption 1.12. *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\overline{\xi}$ be a measurable map:*

$$\overline{\xi} \colon \Omega \to \mathcal{S}(\mathbb{T}^d).$$

Then define:

$$s_\varepsilon(\omega, x) = \varepsilon^2 \overline{\xi}(\omega, x).$$

Then we define the (stochastic if $d = 1$) FKPP equation in a random potential as follows.

Definition 1.13. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a white noise ξ and $X^0 \in B^\alpha_{2,2}$. A (stochastic if $d = 1$) FKPP process in random potential is a couple given by a probability space $(\Omega \times \overline{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \ltimes \overline{\mathbb{P}}^\omega)$ (cf. Definition 1.7) and a map*

$$X \colon \Omega \times \overline{\Omega} \to L^2_{\mathrm{loc}}([0, \infty); B^\alpha_{2,2}),$$

for some $\alpha > 0$. Moreover, for $\omega \in \Omega$ let $\{\mathcal{F}^\omega_t\}_{t \geq 0}$ be the filtration generated by $t \mapsto X_t(\omega)$, right-continuous and enlarged with all null-sets. Then for all $\omega \in \Omega$ it is required that, depending on the dimension:

(1) *In dimension $d = 1$ for all $\varphi \in C^\infty(\mathbb{T}^d)$:*

$$N^\varphi_t := \langle X_t(\omega), \varphi \rangle - \langle X^0, \varphi \rangle - \int_0^t \langle X_s(\omega), \nu_0 \Delta \varphi \rangle - \langle \overline{\xi}(\omega) X_s(\omega)(1 - X_s(\omega)), \varphi \rangle \mathrm{d}s$$

is a continuous in time, square integrable martingale with quadratic variation

$$\langle N^\varphi \rangle_t = \int_0^t \langle X_s(\omega)(1 - X_s(\omega)), \varphi^2 \rangle \mathrm{d}s.$$

(2) *In dimension $d = 2$, X is a solution to*

$$\begin{aligned} \partial_t X_t(\omega) &= \nu_0 \Delta X_t(\omega) + \overline{\xi}(\omega) X_t(\omega)(1 - X_t(\omega)), \\ X_0(\omega, x) &= X^0(\omega, x), \qquad \forall x \in \mathbb{T}^d. \end{aligned}$$

It is interpreted in the sense that for all $\varphi \in C^\infty(\mathbb{T}^2)$

$$\langle X_t(\omega), \varphi \rangle = \langle X^0, \varphi \rangle + \int_0^t \langle X_s(\omega), \nu_0 \Delta \varphi \rangle + \langle \overline{\xi}(\omega) X_s(\omega)(1 - X_s(\omega)), \varphi \rangle \, \mathrm{d}s.$$

Remark 1.14. *Note that in the previous definition, since $X \in L^2_{\mathrm{loc}}([0, \infty); B^\alpha_{2,2})$, the quadratic non-linearity:*

$$\int_0^t \langle X_s^2, \varphi \rangle \, \mathrm{d}s$$

is well-defined. Moreover, up to enlarging the probability space, the process can be represented in $d = 1$ as a solution to an SPDE of the form

$$\partial_t X = \nu_0 \Delta X + \overline{\xi} X(1 - X) + \sqrt{X(1 - X)} \widetilde{\xi},$$

where the spatial noise $\overline{\xi}$ is independent of the space-time white noise $\widetilde{\xi}$, following a classical construction by Konno and Shiga [36] *(see also* [45, Theorem 2.18] *for a similar case in a random environment).*

In this setting, we can prove the following scaling limit.

Theorem 2. *Let η satisfy Assumption 1.11 and s_ε be as in Assumption 1.12. Consider $X_0 \in \mathcal{S}(\mathbb{T}^d)$, and let $X^\varepsilon(\omega)$ be the Markov process associated to the generator*

$$\varepsilon^{-\eta-d-2\cdot}\mathcal{L}(\varepsilon, s_\varepsilon(\omega), \varepsilon^\eta)$$

and started in X_0, as Definition 1.1. There exists an $\alpha > 0$ such that for every $\omega \in \Omega$

$$\{t \mapsto X^\varepsilon_t(\omega)\}_{\varepsilon\in(0,1/2)}$$

is tight in the space $L^2_{\mathrm{loc}}([0,\infty); B^\alpha_{2,2})$. In particular:

1. *In dimension $d = 1$ if $\eta = 1$ any subsequential limit is a stochastic FKPP process in a random potential as in Definition 1.13.*
2. *In dimension $d = 2$ the entire sequence converges in distribution to an FKPP process in a random potential as in Definition 1.13.*

Remark 1.15. *The scaling in Theorem 2 is very similar to that of Theorem 1. Once again the impact of each of the events is scaled by ε^η (with a different value of η), the rate of selective events by ε^2, the volume of the ball in which of each of the events occurs is set to ε time is sped up by $\varepsilon^{-2-d-\eta}$. We do not place any restrictions on the relative sizes of the initial population.*

As before, we can now study the longtime behavior of the limiting process. In dimension $d = 1$ the presence of the genetic drift implies triviality in finite time. In dimension $d = 2$ we have, conditional on the realisation of the noise, a deterministic equation. In order to prove longtime coexistence of the two types we require that the noise is sufficiently strong

Assumption 1.16. *Fix $(\Omega, \mathcal{F}, \mathbb{P})$ as in Definition 1.13. Let $\lambda_1(\Delta - \overline{\xi}), \lambda_1(\Delta + \overline{\xi})$ be the largest eigenvalues of the operators $\Delta + \overline{\xi}$ and $\Delta - \overline{\xi}$ respectively. Then assume that:*

$$\mathbb{P}(\lambda_1(\Delta + \overline{\xi}) > 0, \lambda_1(\Delta - \overline{\xi}) > 0) > 0.$$

These assumptions are not too far-fetched we two cases in which they are naturally satisfied.

Remark 1.17. *If the noise is symmetric and sufficiently "strong" , Assumption 1.16 hold.*

1. *Consider for example $d = 2$ and $\overline{\xi}_\varepsilon = \varepsilon^{-d}\varrho(\varepsilon^{-d}\cdot) * \xi$, where ξ is space white noise on $\mathbb{T}^d$, ϱ is a smooth symmetric positive function with compact support and $\int_{\mathbb{T}^d} \varrho(x)\,\mathrm{d}x = 1$ and $\varepsilon \in (0,1)$. Then there exists an $\varepsilon_0 > 0$ such that for $\varepsilon > \varepsilon_0$*
$$\mathbb{P}(\lambda_1(\Delta + \overline{\xi}_\varepsilon) > 0) > \frac{1}{2},$$
which by symmetry implies
$$\mathbb{P}(\lambda_1(\Delta + \overline{\xi}_\varepsilon), \lambda_1(\Delta - \overline{\xi}_\varepsilon) > 0) > 0.$$
In fact in [2], *for the construction of the continuous Anderson Hamiltonian $\mathcal{H}$ in $d = 2$, the authors prove that there exists a deterministic sequence $c_\varepsilon \simeq \log\frac{1}{\varepsilon}$ such that $\lambda_1(\Delta + \overline{\xi}_\varepsilon(\omega)) - c_\varepsilon \to \lambda_1(\mathcal{H}(\omega)) \in \mathbb{R}$ for almost all $\omega \in \Omega$.*
2. *Instead of taking a sufficiently close approximation of white noise, one can take a sufficiently large domain. Consider ϱ as above and ξ space white noise on $\mathbb{R}^d$, $\overline{\xi} = \varrho * \xi$. Then instead of considering the Laplacian on the unit torus, consider Δ_L the Laplacian on $L^2(L\mathbb{T}^d)$, the latter being the torus of size $L > 0$ (i.e. $[-\frac{L}{2}, \frac{L}{2}]^d$ with periodic boundary conditions). Then there exists an $L_0 > 0$ such that for all $L \geq L_0$*
$$\mathbb{P}(\lambda_1(\Delta_L + \overline{\xi}) > 0) > \frac{1}{2},$$
which as before implies the requirements. An indirect proof of this fact can be found in [11, Theorem 5.1], *see also* [15] *for a similar result with $\overline{\xi}$ being space white noise.*

We can then prove the following result (here $X \equiv c \in \mathbb{R}$ means that $X(x) = c, \forall x \in \mathbb{T}^d$).

Proposition 1.18. *Consider X a process as in Definition 1.13.*

(1) *If $d=1$, then $\mathbb{P}\ltimes\mathbb{P}^\omega$-almost surely there exists a finite random time τ, such that*
$$X_\tau \equiv 1, \quad or \quad X_\tau \equiv 0.$$
(2) *If $d=2$ and Assumption 1.16 is satisfied, and if $X_0 \not\equiv 1,\ X_0 \not\equiv 0$, for every $\omega\in\Omega$ there exists an $\overline{X}\in B^\alpha_{2,2,}$ for some $\alpha>0$, with $0\le \overline{X}\le 1$, such that*
$$\overline{X}(\omega) = \lim_{t\to\infty} X_t(\omega) \quad in \quad B^\alpha_{2,2},$$
and
$$\mathbb{P}(\overline{X}\not\equiv 1, \overline{X}\not\equiv 0) \ge \mathbb{P}(\lambda_1(\Delta+\overline{\xi})>0, \lambda_1(\Delta-\overline{\xi})>0) > 0.$$

The proof of this proposition can be found at the end of Section 4, in Lemmata 4.2 and 4.3.

1.5. **Proof methods.** The main ingredient of the proofs is a careful study of the operator $\mathcal{A}_\varepsilon$. Intuitively, one expects that this operator approximates the Laplacian with periodic boundary conditions and therefore has similar regularizing properties. To quantify this intuition we introduce a division of scales. On large scales, namely for Fourier modes k of order $k\lesssim 1/\varepsilon$ we show that $\mathcal{A}_\varepsilon$ has the regularizing properties of the Laplace operator. On small scales, that is for modes of order $k\gtrsim 1/\varepsilon$ we do not expect any regularization. Instead we prove that small scales are negligible. To divide small and large scales we use 'projection' operators $\mathcal{P}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ on large and small scales respectively. Here we state a slimmed version of the results we require. The proof of the following theorem, as well as additional side results, is the content of Section 5.

Theorem 3. *There exists a smooth radial function with compact support $\daleth\colon \mathbb{R}^d\to\mathbb{R}$ such that for some $0<r<R$*
$$\daleth(k) = 1,\ \forall |k|\le r, \qquad \daleth(k)=0,\ \forall|k|\ge R.$$
Define
$$\mathcal{P}_\varepsilon = \daleth(\varepsilon D), \qquad \mathcal{Q}_\varepsilon = (1-\daleth)(\varepsilon D).$$
For any $\alpha\in\mathbb{R}, p\in[1,\infty]$ the following holds"

i) *For any $\zeta>0$ and $\varphi\in\mathcal{C}^\alpha_p$*
$$\mathcal{A}_\varepsilon\varphi \to \nu_0\Delta\varphi \quad in \quad \mathcal{C}^{\alpha-2-\zeta}_p, \qquad as \quad \varepsilon\to 0,$$
where
$$\nu_0 = \frac{1}{12} \quad for \quad d=1, \qquad \nu_0=\frac{1}{4\pi} \quad for \quad d=2. \tag{8}$$
ii) *Uniformly over $\lambda>1, \varepsilon\in(0,1/2)$ and $\varphi\in\mathcal{C}^\alpha_p$ the following estimates hold:*
$$\|\mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon+\lambda)^{-1}\varphi\|_{\mathcal{C}^{\alpha+2}_p} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon(-\mathcal{A}_\varepsilon+\lambda)^{-1}\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

In particular, a precise control of the regularization effects of the semidiscrete Laplacian $\mathcal{A}_\varepsilon$ allows us to treat semidiscrete approximations of the Anderson model that appear in the study of the rough superBrownian motion. In the next proposition we recall some salient features of the continuous Anderson Hamiltonian.

Proposition 1.19. *Fix $\kappa>0$ and ξ_ε satisfying Assumption 1.5. Up to changing probability space there exists a space white noise $\xi\colon\Omega\to\mathcal{S}'(\mathbb{T}^d)$ for which the following hold true for almost all $\omega\in\Omega$. The Anderson Hamiltonian*
$$\mathcal{H}(\omega) = \nu_0\Delta + \xi(\omega) - \infty 1_{\{d=2\}}$$
associated to $\xi(\omega)$ is defined, as constructed[1] in [25] in $d=1$ and [2] in $d=2$. The Hamiltonian, as an unbounded self-adjoint operator on $L^2(\mathbb{T}^d)$, has a discrete spectrum given by pairs of

[1]To be precise, [25] constructs the operator in dimension $d=1$ with Dirichlet boundary conditions, but their construction can be extended to periodic boundary conditions. Alternatively, the operator can be constructed with arguments similar to the ones presented in Section 6.

eigenvalues and eigenfunctions $\{(\lambda_k(\omega), e_k(\omega))\}_{k\in\mathbb{N}}$ *such that:*

$$\lambda_0(\omega) > \lambda_1(\omega) \geq \lambda_2(\omega) \geq \dots, \qquad \lim_{k\to\infty} \lambda_k(\omega) = -\infty, \qquad e_0(\omega, x) > 0, \forall x \in \mathbb{T}^d.$$

In addition, for every $k \in \mathbb{N}$, $e_k(\omega) \in \mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d)$, *and the set*

$$\mathcal{D}_\omega = \{\textit{Finite linear combination of } \{e_k(\omega)\}_{k\in\mathbb{N}}\}$$

is dense in $C(\mathbb{T}^d)$.

The operator $\mathcal{A}_\varepsilon$ is a tool which provides us with a semidiscrete approximation of the continuous Anderson Hamiltonian.

Theorem 4. *Fix* $\kappa > 0$ *and* ξ_ε *satisfying Assumption 1.5. Up to changing probability space the assertions of Proposition 1.19 hold true. For every* $k \in \mathbb{N}$ *there exists an* $\varepsilon_0(\omega, k) \in (0, 1/2)$ *such that for every* $\varepsilon \leq \varepsilon_0(\omega, k)$ *there exists a pair of eigenvalue and associated eigenfunction* $(\lambda^\varepsilon_k(\omega), e^\varepsilon_k(\omega))$ *for the operator*

$$\mathcal{H}_\varepsilon(\omega) := \mathcal{A}_\varepsilon + (\xi_\varepsilon(\omega) - c_\varepsilon)\Pi^2_\varepsilon, \qquad \mathcal{H}_\varepsilon(\omega)\colon L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d),$$

with c_ε *as in* (6), *such that*

$$\lim_{\varepsilon\to 0} \lambda^\varepsilon_k(\omega) = \lambda_k(\omega), \qquad \lim_{\varepsilon\to 0} \Pi_\varepsilon e^\varepsilon_k(\omega) = e_k(\omega) \quad \textit{in} \quad \mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d).$$

The proof of this result can be found in Section 6.

2. Scaling to the rough super-Brownian motion

This section is devoted to the proof of Theorem 1. We leverage the analytic results of Theorem 4 to obtain tightness of the sequence Y^ε. Uniqueness of the limit points the follows by a conditional duality argument.

2.1. Scaling limit.

The core of the tightness proof is conditioning on the realization of the environment. Since we want to prove convergence in distribution for the sequence Y^ε, the exact choice of the probability space Ω of Definition 1.1 is not important. For this reason we adopt the following standing assumption that allows us to work with a suitably chosen probability space.

Assumption 2.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$, *the probability space appearing in Definition 1.1 and Assumption 1.5 be such that the results of Proposition 1.19 and Theorem 4 hold true for almost all* $\omega \in \Omega$.

The first step towards establishing tightness is to restate the martingale problem of Lemma 1.3 to take into account the scaling assumed in Theorem 1.

Lemma 2.2. *In the setting of Theorem 1 and under Assumption 2.1, for every* $\omega \in \Omega$ *and* $\varepsilon \in (0, 1/2)$, *under the law* $\mathbb{P}^\omega$, *and for every* $\varphi \in L^1(\mathbb{T}^d)$ *the process* $t \mapsto \langle Y^\varepsilon_t(\omega), \varphi\rangle$ *satisfies the following martingale problem:*

$$\begin{aligned} \text{(9)} \quad & \langle Y^\varepsilon_{t,s}(\omega), \varphi\rangle - M^\varepsilon_{t,s}(\varphi) \\ &= \int_s^t \langle \mathcal{A}_\varepsilon(Y^\varepsilon_r(\omega)) + \Pi_\varepsilon[\xi_\varepsilon(\omega)\Pi_\varepsilon Y^\varepsilon_r(\omega)], \varphi\rangle - \varepsilon^\varrho \langle \big(\Pi_\varepsilon Y^\varepsilon_r(\omega)\big)^2, \xi_\varepsilon(\omega)\Pi_\varepsilon(\varphi)\rangle \, \mathrm{d}r, \end{aligned}$$

where $M^\varepsilon_\cdot(\varphi)$ *is a square integrable martingale with predictable quadratic variation given by:*

$$\begin{aligned} \text{(10)} \quad \langle M^\varepsilon(\varphi)\rangle_t = & \int_0^t \langle (1 + \varepsilon^{2-\frac{d}{2}} s_\varepsilon(\omega))\Pi_\varepsilon Y^\varepsilon_r(\omega), (\Pi_\varepsilon\varphi)^2 - 2\varepsilon^\varrho \Pi_\varepsilon(\varphi)\Pi_\varepsilon(Y^\varepsilon_r(\omega)\varphi)\rangle \\ & + \varepsilon^\varrho \langle \big(\Pi_\varepsilon(Y^\varepsilon_r(\omega)\varphi)\big)^2, 1\rangle - \varepsilon^\varrho \langle \varepsilon^{2-\frac{d}{2}} s_\varepsilon(\omega)(\Pi_\varepsilon Y^\varepsilon_r(\omega))^2, (\Pi_\varepsilon\varphi)^2 - 2\varepsilon^\varrho \Pi_\varepsilon(\varphi)\Pi_\varepsilon(Y^\varepsilon_r(\omega)\varphi)\rangle \, \mathrm{d}r, \end{aligned}$$

Remark 2.3. *Note that the only term which is not of lower order in the quadratic variation is*

$$\langle \Pi_\varepsilon Y_r^\varepsilon, \big(\Pi_\varepsilon \varphi\big)^2 \rangle,$$

which, combined with the form of the drift term, provides an algebraic heuristic for obtaining the super-Brownian motion in a static random environment as the scaling limit.

In order to obtain the convergence, the first step is to prove a tightness result.

Proposition 2.4. *In the setting of Theorem 1 and under Assumption 2.1 fix any $\omega \in \Omega$. For any $T > 0$ the sequence $\{Y^\varepsilon(\omega)\}_{\varepsilon \in (0,1/2)}$ is tight in $\mathbb{D}([0,T]; \mathcal{M}(\mathbb{T}^d))$. Moreover any limit point is continuous, i.e. lies in $C([0,T]; \mathcal{M}(\mathbb{T}^d))$.*

The proof will be based on an application of Jakubowski's tightness criterion, which we recall for convenience.

Proposition 2.5. [32, Theorem 3.1] *Let X be a separable metric space. Let F be a family of real, continuous functions on X which separates points and is closed under addition. Then a sequence of probability measures $\{\mathbb{P}_n\}_{n \in \mathbb{N}}$ on $\mathbb{D}([0,T]; X)$ is tight if the following two conditions are satisfied:*

(1) *For each $\varepsilon > 0$ there exists a compact set $K \subset X$ such that*

$$\inf_{n \in \mathbb{N}} \mathbb{P}_n\big(X_t \in K, \ \forall t \in [0,T]\big) \geq 1 - \varepsilon,$$

where X_t is the canonical process on $\mathbb{D}([0,T]; X)$.

(2) *For each $f \in F$ sequence $\mathbb{P}_n \circ f^{-1}$ is tight as a measure on $\mathbb{D}([0,T]; \mathbb{R})$.*

Proof of Proposition 2.4. Since $\omega \in \Omega$ is fixed, we omit the dependence on it. The proof is divided into three steps. In the first two, we check the conditions of Proposition 2.5.

In the first step, we establish the compact containment condition. Since for $R > 0$ sets of the form $K_R = \{\mu \colon \langle \mu, 1 \rangle \leq R\} \subseteq \mathcal{M}(\mathbb{T}^d)$ are compact in the weak topology, it is sufficient to show that

$$\forall \delta > 0, \quad \exists R(\delta) > 0, \ \frac{1}{2} > \varepsilon(\delta) > 0 \text{ such that } \inf_{\varepsilon \in (0, \varepsilon(\delta)]} \mathbb{P}\Big(\sup_{t \in [0,T]} \langle Y_t^\varepsilon, 1 \rangle \leq R(\delta) \Big) \geq 1 - \delta. \tag{11}$$

In the second step, we establish the one-dimensional tightness. By Proposition 4, it is sufficient to show that for every $k \in \mathbb{N}$ process $\langle Y_t^\varepsilon, e_k \rangle$ is tight in $\mathbb{D}([0,T]; \mathbb{R})$. By Aldous' tightness criterion [1, Theorem 1] this reduces to proving that for any sequence of stopping times τ_ε, taking finitely many values and adapted to the filtration of Y^ε, and any sequence δ_ε of constants such that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$

$$\forall \delta > 0, \qquad \lim_{\varepsilon \to 0} \mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon + \delta_\varepsilon}, e_k \rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, e_k \rangle| \geq \delta \Big) = 0. \tag{12}$$

In the third step we address the continuity of the limiting process.

Step 1. By Theorem 4, for any $k \in \mathbb{N}$ and $\varepsilon \leq \varepsilon_0(k)$ there exists an eigenfunction e_k^ε of $\mathcal{H}_\varepsilon$ such that $\Pi_\varepsilon e_k^\varepsilon \to e_k$ in $\mathcal{C}^{2 - \frac{d}{2} - \kappa}(\mathbb{T}^d)$. In particular, since $e_0 > 0$, we may assume that $\Pi_\varepsilon e_0^\varepsilon > 0, \forall \varepsilon \leq \varepsilon_0(0)$ and hence for any positive measure μ there exists a $C > 0$ such that

$$\langle \mu, 1 \rangle \leq C \langle \mu, \Pi_\varepsilon e_0^\varepsilon \rangle, \qquad \forall \varepsilon \leq \varepsilon_0(0).$$

Therefore (11) follows if one can show that

$$\forall \delta > 0, \quad \exists R(\delta) > 0, \ \varepsilon_0(0) \geq \varepsilon(\delta) > 0 \text{ such that } \inf_{\varepsilon \in (0, \varepsilon(\delta)]} \mathbb{P}\Big(\sup_{t \in [0,T]} \langle Y_t^\varepsilon, \Pi_\varepsilon e_0^\varepsilon \rangle \leq R(\delta) \Big) \geq 1 - \delta.$$

We focus our attention on $\langle Y_t^\varepsilon, \Pi_\varepsilon e_0^\varepsilon \rangle$. By the martingale representation (9) one obtains

$$\langle Y_t^\varepsilon, \Pi_\varepsilon e_0^\varepsilon \rangle = \langle Y_0^\varepsilon, \Pi_\varepsilon e_0^\varepsilon \rangle + \int_0^t \lambda_0^\varepsilon \langle Y_r^\varepsilon, \Pi_\varepsilon e_0^\varepsilon \rangle - \varepsilon^\varrho \langle \big(\Pi_\varepsilon Y_r^\varepsilon\big)^2, \xi_\varepsilon \Pi_\varepsilon^2 e_0^\varepsilon \rangle \, \mathrm{d}r + M_t^\varepsilon(\Pi_\varepsilon e_0^\varepsilon).$$

To treat the nonlinear quadratic term, we shall consider a stopped process. For that purpose fix $R > 0$ and consider a stopping time τ_R and a parameter ϱ_0, defined as

$$\tau_R := \inf\{t \geq 0 \;:\; \langle Y^\varepsilon_t, \Pi_\varepsilon e^\varepsilon_0\rangle \geq R\}, \qquad \varrho_0 = \varrho - \frac{d}{2} - 2d.$$

Since $|\xi_\varepsilon(x)| \lesssim \varepsilon^{-\frac{d}{2}}$ one can bound

$$\varepsilon^\varrho |\langle (\Pi_\varepsilon Y^\varepsilon_{r\wedge\tau_R})^2, \xi_\varepsilon \Pi^2_\varepsilon e^\varepsilon_0\rangle| \lesssim \varepsilon^{\varrho-\frac{d}{2}-2d}\langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle^2 \lesssim R\varepsilon^{\varrho_0}\langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle,$$

and therefore

$$\mathbb{E}|\langle Y^\varepsilon_{t\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle|^2 \lesssim \|Y^\varepsilon_0\|_{L^\infty} + (1+R\varepsilon^{\varrho_0})\int_0^t \mathbb{E}|\langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle|^2 \,\mathrm{d}r + \mathbb{E}\langle M^\varepsilon(\Pi_\varepsilon e^\varepsilon_0)\rangle_{t\wedge\tau_R}.$$

Furthermore, using the formula for the predictable quadratic variation from Lemma 2.2 one obtains

$$\mathbb{E}\langle M^\varepsilon(\Pi_\varepsilon e^\varepsilon_0)\rangle_{t\wedge\tau_R} \lesssim \mathbb{E}\int_0^t \langle \Pi_\varepsilon Y^\varepsilon_{r\wedge\tau_R}, (\Pi^2_\varepsilon e^\varepsilon_0)^2\rangle + \langle \Pi_\varepsilon(Y^\varepsilon_{r\wedge\tau_R}\Pi_\varepsilon e^\varepsilon_0), \Pi^2_\varepsilon e^\varepsilon_0\rangle \mathrm{d}r \lesssim \mathbb{E}\int_0^t \langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle \,\mathrm{d}r.$$

Therefore, by Gronwall's inequality, there exists a $C > 0$ such that

$$\sup_{0\leq t\leq T} \mathbb{E}|\langle Y^\varepsilon_{t\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle|^2 \lesssim e^{CR\varepsilon^{\varrho_0}}. \tag{13}$$

It follows that if $\varepsilon \leq R^{-\varrho_0}$

$$\mathbb{P}\Big(\sup_{0\leq t\leq T} |\langle Y^\varepsilon_t, \Pi_\varepsilon e^\varepsilon_0\rangle| \geq R\Big) = \mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_R\wedge T}, \Pi_\varepsilon e^\varepsilon_0\rangle| = R\Big) \lesssim R^{-2}.$$

This concludes the proof of compact containment condition (11).

Step 2. Fix $k \in \mathbb{N}$ and $\gamma > 0$. In view of calculations from Step 1 there exist $R(\gamma), \varepsilon(\gamma)$ for which (11) holds. Up to choosing a smaller $\varepsilon(\gamma)$ we may also assume that

$$\forall \varepsilon \text{ such that } \varepsilon(\gamma) \geq \varepsilon > 0 : \qquad \|e_k - \Pi_\varepsilon e^\varepsilon_k\|_{L^\infty} \leq \frac{\delta}{2R(\gamma)}.$$

Hence for every $\varepsilon \leq \varepsilon(\gamma)$

$$\mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, e_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, e_k\rangle| \geq \delta\Big) \leq \gamma + \mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle| \geq \delta\Big).$$

Using representation of Lemma 2.2

$$\begin{aligned}\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle = \int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \lambda^\varepsilon_k \langle Y^\varepsilon_r, \Pi_\varepsilon e^\varepsilon_k\rangle - \varepsilon^\varrho\langle \big(\Pi_\varepsilon Y^\varepsilon_r\big)^2, \xi_\varepsilon \Pi^2_\varepsilon e^\varepsilon_k\rangle \,\mathrm{d}r \\ + M^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}(\Pi_\varepsilon e^\varepsilon_k) - M^\varepsilon_{\tau_\varepsilon}(\Pi_\varepsilon e^\varepsilon_k).\end{aligned}$$

Hence one obtains (writing for simplicity R instead of $R(\gamma)$):

$$\mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle| \geq \delta\Big) \leq \gamma + \mathbb{P}\Big(|\langle Y^\varepsilon_{(\tau_\varepsilon+\delta_\varepsilon)\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle| \geq \delta\Big).$$

Computations analogous to those in Step 1. guarantee that

$$\begin{aligned}&\mathbb{P}\Big(|\langle Y^\varepsilon_{(\tau_\varepsilon+\delta_\varepsilon)\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle| \geq \delta\Big) \\ &\qquad \leq \delta^{-2}\mathbb{E}\Big[\big|\langle Y^\varepsilon_{(\tau_\varepsilon+\delta_\varepsilon)\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle\big|^2\Big] \lesssim \delta_\varepsilon.\end{aligned}$$

Since γ is arbitrary, this proves (12).

Step 3. So far any limit point Y of the sequence Y^ε lies in the Skorohod space $\mathbb{D}([0,T];\mathcal{M}(\mathbb{T}^d))$. Since $\mathcal{M}(\mathbb{T}^d)$ is endowed with the weak topology, to prove that actually $Y \in C([0,T];\mathcal{M}(\mathbb{T}^d))$, it is sufficient to show that for any continuous function φ, $\langle Y_t, \varphi\rangle$ is continuous in time. Here one can apply a criterion [23, Theorem 3.10.2] according to which it is sufficient to prove that the maximum size of a jump converges weakly to zero. In our case such convergence is even almost sure, since:

$$|\langle Y_t^\varepsilon, \varphi\rangle - \langle Y_{t-}^\varepsilon, \varphi\rangle| \lesssim \varepsilon^d \|\varphi\|_{C(\mathbb{T}^d)}.$$

This follows from the definition of the generator, as well as the exact definition of η (cf. Equation (7)) jumps are bounded as follows:

$$\|Y_t^\varepsilon - Y_{t-}^\varepsilon\|_{L^\infty} \lesssim \varepsilon^{2-d} \lesssim 1.$$

Since a jump has an impact only in a ball $B_\varepsilon(x)$ for some $x \in \mathbb{T}^d$, integrating φ over such ball guarantees the previous bound.

□

Finally we are in position to deduce Theorem 1.

Proof of Theorem 1. By Proposition 2.4 the sequence $Y_\varepsilon(\omega)$ is tight, for every $\omega \in \Omega$, under Assumption 2.1 (recall that we can always put ourselves in the setting of this assumption by changing probability space, which does not affect the convergence in distribution). It remains to show that, for a fixed realization $\omega \in \Omega$, every limit point satisfies the martingale problem for the rough superBrownian motion as in Definition 1.7, which is covered by Step 1, and that solutions to such martingale problems are unique, which is covered by Step 2.

Step 1. As in the proof of Proposition 2.4, since $\omega \in \Omega$ is fixed we omit writing it. Moreover it is sufficient to fix a finite but arbitrary time horizon $T > 0$ and check the martingale property until that time. Assume that (up to taking a subsequence and applying the Skorohod representation theorem) $Y^\varepsilon \to Y$ almost surely in $\mathbb{D}([0,T]; \mathcal{M}(\mathbb{T}^d))$. Since $\mathcal{D}_\omega$ is composed of finite linear combinations of eigenfunctions, it is sufficient to prove the martingale property of Definition 1.7 for $\varphi = e_k$ for some $k \in \mathbb{N}$. In this setting, one has that almost surely

$$\begin{aligned}
M_t^{e_k} &= \langle Y_{t,0}, e_k\rangle - \int_0^t \langle Y_s, \mathcal{H} e_k\rangle \,\mathrm{d}s = \langle Y_{t,0}, e_k\rangle - \lambda_k \int_0^t \langle Y_s, e_k\rangle \,\mathrm{d}s \\
&= \lim_{\varepsilon\to 0} \left[\langle Y_{t,0}^\varepsilon, \Pi_\varepsilon e_k^\varepsilon\rangle - \int_0^t \langle \mathcal{A}_\varepsilon(Y_r^\varepsilon) + \Pi_\varepsilon[\xi_\varepsilon \Pi_\varepsilon Y_r^\varepsilon], \Pi_\varepsilon e_k^\varepsilon\rangle - \varepsilon^\varrho \langle \big(\Pi_\varepsilon Y_r^\varepsilon\big)^2, \xi_\varepsilon \Pi_\varepsilon^2 e_k^\varepsilon\rangle \,\mathrm{d}r \right] \\
&= \lim_{\varepsilon \to 0} M_t^\varepsilon(\Pi_\varepsilon e_k^\varepsilon).
\end{aligned}$$

The convergence of the linear terms in the second line is a consequence of the convergence

$$\Pi_\varepsilon e_k^\varepsilon \to e_k \ \text{ in } \ \mathcal{C}^{2-\frac{d}{2}-\kappa}, \qquad \lambda_k^\varepsilon \to \lambda_k$$

as proved in Theorem 4 (where also the eigenpairs $e_k^\varepsilon, \lambda_k^\varepsilon$ are defined). As for the non-linear term, one has, as in the proof of Proposition 2.4,

$$\langle \big(\Pi_\varepsilon Y_r^\varepsilon\big)^2, \xi_\varepsilon \Pi_\varepsilon^2 e_k^\varepsilon\rangle \lesssim \varepsilon^{\varrho - 2d - \frac{d}{2}} \langle Y_r^\varepsilon, 1\rangle^2 \to 0,$$

by the assumption on ϱ. To prove that M^{e_k} is a martingale, one has to show that such property is conserved when passing to the limit. For $R > 0$ consider the stopping times

$$\tau_R(Y) = \inf\{t \geq 0 \mid \langle Y_t, 1\rangle \geq R\} = \lim_{\varepsilon \to 0} \inf\{t \geq 0 \mid \langle Y_t^\varepsilon, 1\rangle \geq R\} =: \lim_{\varepsilon\to 0} \tau_R(Y^\varepsilon).$$

This sequence is localizing, in the sense that it makes M^{e_k} a local martingale with quadratic variation

$$\langle M^{e_k}_{\cdot\wedge\tau_R(Y)}\rangle_t = \lim_{\varepsilon\to 0}\Bigg[\int_0^{t\wedge\tau_R(Y^\varepsilon)} \langle (1+\varepsilon^{2-\frac{d}{2}}s_\varepsilon)\Pi_\varepsilon Y^\varepsilon_r, (\Pi^2_\varepsilon e^\varepsilon_k)^2 - 2\varepsilon^\varrho \Pi^\varepsilon_\varepsilon(e^\varepsilon_k)\Pi_\varepsilon(Y^\varepsilon_r\Pi_\varepsilon e^\varepsilon_k)\rangle$$
$$+\varepsilon^\varrho\langle\big(\Pi_\varepsilon(Y^\varepsilon_r\Pi_\varepsilon e^\varepsilon_k)\big)^2, 1\rangle - \varepsilon^\varrho\langle\varepsilon^{2-\frac{d}{2}}s_\varepsilon(\Pi_\varepsilon Y^\varepsilon_r)^2, (\Pi^2_\varepsilon e^\varepsilon_k)^2 - 2\varepsilon^\varrho\Pi^2_\varepsilon(e^\varepsilon_k)\Pi_\varepsilon(Y^\varepsilon_r\Pi_\varepsilon e^\varepsilon_k)\rangle\,\mathrm{d}r\Bigg]$$
$$= \int_0^{t\wedge\tau_R}\langle Y_s, e^2_k\rangle\,\mathrm{d}s.$$

To conclude that M^{e_k} is itself a square integrable martingale it suffices to observe that:

$$\sup_{0\le t\le T}\mathbb{E}|\langle Y_t, 1\rangle|^2 < \infty,$$

which follows by applying Fatou's lemma, first over ε and then over R, to Equation (13) in the proof of Propostion 2.4.

Step 2. We conclude by explaining the uniqueness in law of a process Y satisfying the martingale problem of the rough superBrownian motion (in the following as always $\omega\in\Omega$ is fixed, and we omit from writing it. In particular, all averages are still conditional on the realization of the environment). The uniqueness is the consequence of a duality argument. For any $\varphi\ge 0, \varphi\in C^\infty$ we find a process $t\mapsto U_t\varphi$ such that

$$\mathbb{E}\Big[e^{-\langle Y_t,\varphi\rangle}\Big] = e^{-\langle Y^0, U_t\varphi\rangle}. \tag{14}$$

Hence the distribution of $\langle Y_t,\varphi\rangle$ is uniquely characterized by its Laplace transform. This also characterizes the law of the entire process $\langle Y_t,\varphi\rangle$ through a Dynkin-type argument (see [18, Lemma 3.2.5]), proving the required result.

We are left with the task of describing the process $U_t\varphi$. This is the solution, evaluated at time $t\ge 0$, of the nonlinearly damped parabolic equation

$$\partial_t(U.\varphi) = \mathcal{H}(U.\varphi) - \frac{1}{2}(U.\varphi)^2, \qquad U_0\varphi = \varphi,$$

where we consider the solutions in the mild sense, namely

$$U_t\varphi = e^{t\mathcal{H}}\varphi - \frac{1}{2}\int_0^t e^{(t-s)\mathcal{H}}(U_s\varphi)^2\,\mathrm{d}s,$$

as constructed in Lemma 2.6. To obtain Equation (14) consider some $\zeta>0$ and a process $\psi\in C([0,T];\mathcal{C}^\zeta)$ of the form

$$\psi_t = e^{t\mathcal{H}}\psi_0 + \int_0^t e^{(t-s)\mathcal{H}}f_s\,\mathrm{d}s,$$

with $f\in C([0,T];\mathcal{C}^\zeta), \psi_0\in\mathcal{C}^\zeta$. Approximating f through a piece-wise constant function in time $\widetilde{f}$ and approximating both $\widetilde{f}$ and φ via a finite number of eigenvalues in view of Lemma 6.4, and using the continuity of the semigroup as in Equation (33), it follows from the definition of the rough superBrownian motion that for $0\le s\le t$:

$$\langle Y_s, \psi_{t-s}\rangle - \langle Y_0,\psi_t\rangle - \int_0^s\langle Y_r, f_r\rangle\,\mathrm{d}r =: \widetilde{M}_s(\psi)$$

is a continuous martingale with quadratic variation

$$\langle\widetilde{M}(\psi)\rangle_s = \int_0^s\langle Y_r, \psi^2_{t-r}\rangle\,\mathrm{d}r.$$

Now we apply this observation together with Itô's formula to deduce that

$$[0,t]\ni s\mapsto e^{-\langle Y_s, U_{t-s}\varphi\rangle}$$

is a martingale on $[0,t]$. In particular, this implies Equation (14) and conclude the proof. $\square$

The following result states the well-posedness of the dual PDE to the rough superBrownian motion. We will not provide a proof, since it is identical to [45, Proposition 4.5]. The proof is essentially based on an a-priori L^∞ estimate and the regularization properties of the Anderson Hamiltonian (cf. also the proof of Lemma 6.4).

Lemma 2.6. *Under Assumption 2.1, fix $\omega \in \Omega$. For any $\varphi \geq 0, \varphi \in C^\infty$, time horizon $T>0$ and $\zeta < 2-\frac{d}{2}$, there exists a process $(t,x) \mapsto (U^\omega_t \varphi)(x)$ such that $U^\omega\varphi \in C([0,T];\mathcal{C}^\zeta)$, where*

$$U^\omega_t \varphi = e^{t\mathcal{H}(\omega)}\varphi - \frac{1}{2}\int_0^t e^{(t-s)\mathcal{H}(\omega)}(U^\omega_s\varphi)^2 \,\mathrm{d}s.$$

3. Scaling to FKPP

As in the Section 2, throughout this section we fix one realization ω of the environment and work conditional on that realization. Unlike Section 2 the probability space remains fixed.

The first step towards the scaling limit is to restate the martingale problem of Lemma 1.3 in the current setting. The proof is an immediate consequence of the aforementioned lemma.

Lemma 3.1. *Under the assumptions of Theorem 2 fix any $\omega \in \Omega$. For all $\varphi \in L^1(\mathbb{T}^d)$, the process $t \mapsto \langle X^\varepsilon_t, \varphi\rangle$ satisfies*

$$\langle X^\varepsilon_{t,s}(\omega),\varphi\rangle = \int_s^t \langle \mathcal{A}_\varepsilon(X^\varepsilon_r(\omega)),\varphi\rangle + \langle \Pi_\varepsilon\big[\bar\xi(\omega)(\Pi_\varepsilon X^\varepsilon_r(\omega) - (\Pi_\varepsilon X^\varepsilon_r(\omega))^2)\big],\varphi\rangle \,\mathrm{d}r + M^\varepsilon_{t,s}(\varphi), \tag{15}$$

where $M^\varepsilon_\cdot(\varphi)$ is a centered square integrable martingale with predictable quadratic variation

$$\begin{aligned}\langle M^\varepsilon(\varphi)\rangle_t = \varepsilon^{\eta+d-2}\int_0^t &\langle (1+s_\varepsilon(\omega))\Pi_\varepsilon X^\varepsilon_r(\omega), (\Pi_\varepsilon\varphi)^2 - 2\Pi_\varepsilon(\varphi)\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\rangle \\ &+ \langle \big(\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\big)^2, 1\rangle - \langle s_\varepsilon(\omega)(\Pi_\varepsilon X^\varepsilon_r(\omega))^2, (\Pi_\varepsilon\varphi)^2 - 2\Pi_\varepsilon(\varphi)\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\rangle \,\mathrm{d}r.\end{aligned} \tag{16}$$

Now we are able to show tightness for the process.

Proposition 3.2. *Under the assumptions of Theorem 2 fix any $\omega \in \Omega$. Fix $T>0$ and α such that*

$$\begin{cases} \alpha \in (0,1/2) & \text{if } d=1, \\ \alpha \in (0,\eta) & \text{if } d=2. \end{cases}$$

The sequence $\{s \mapsto \Pi_\varepsilon X^\varepsilon_s(\omega)\}_{\varepsilon\in(0,1/2)}$ is tight in the space

$$L^2([0,T];B^\alpha_{2,2}).$$

In addition, the sequence $\{s \mapsto X^\varepsilon_s(\omega)\}_{\varepsilon\in(1,1/2)}$ is tight in $\mathbb{D}([0,T];\mathcal{M}(\mathbb{T}^d))$, and any limit point lies in $C([0,T],\mathcal{M}(\mathbb{T}^d))$.

A crucial step in the proof of Proposition 3.2 is a compactness criterion due to Simon, which we recall for convenience. Here the space $W^{2,\zeta}([0,T];Y) \subset L^2([0,T];Y)$ is defined by the Sobolev-Slobodeckij norm

$$\|f\|_{W^{2,\zeta}([0,T];Y)} = \|f\|_{L^2([0,T];Y)} + \left(\int_0^T\int_0^T \frac{\|f(t)-f(r)\|^2_Y}{|t-r|^{2\zeta+1}}\,\mathrm{d}t\,\mathrm{d}r\right)^{1/2}.$$

Proposition 3.3 (Corollary 5, [48])**.** *Let X,Y,Z be three Banach spaces such that $X\subset Y\subset Z$ with the embedding $X \subset Y$ being compact. Then also the following embedding is compact, for any $s>0$:*

$$L^p([0,T];X)\cap W^{s,p}([0,T];Z) \subseteq L^p([0,T];Y).$$

Now, we pass to the proof of tightness.

Proof of Proposition 3.2. Since $\omega \in \Omega$ is fixed throughout the proof, we omit writing it, to lighten the notation. Tightness of the sequence X^ε in $\mathbb{D}([0,T];\mathcal{M}(\mathbb{T}^d))$ is an immediate consequence of the bound $0 \le X^\varepsilon_t \le 1$. To show that moreover any limit point lies in $C([0,T];\mathcal{M}(\mathbb{T}^d))$ notice that for any $\varphi \in C(\mathbb{T}^d)$

$$|\langle X^\varepsilon_t, \varphi\rangle - \langle X^\varepsilon_{t-}, \varphi\rangle| \lesssim \varepsilon^{\eta+d}\|\varphi\|_{L^\infty},$$

so that the maximal jump size is vanishing as $\varepsilon \to 0$. The continuity of the limit points follows then through [23, Theorem 3.10.2].

Therefore we now concentrate on proving the tightness of the sequence $\Pi_\varepsilon X^\varepsilon_s$. For simplicity, let us define the parameter λ as follows:

$$\begin{cases} \text{If } d=1, & \eta = 1 \;\Rightarrow\; \text{set } \lambda = 0, \\ \text{If } d=2, & \eta = 0 \;\Rightarrow\; \text{set } \lambda = \eta. \end{cases} \tag{17}$$

Our aim is to apply Proposition 3.3 with

$$X = B^{\alpha'}_{2,2}, \quad Y = B^{\alpha}_{2,2}, \quad Z = B^{\alpha''}_{2,2},$$

for appropriate $\alpha' > \alpha > \alpha''$.

Step 1. First, we derive a uniform bound for the second moment of the $B^\alpha_{2,2}$ norm (this in particular implies boundedness of the sequence $\Pi_\varepsilon X^\varepsilon$ in $L^2([0,T];B^\alpha_{2,2})$):

$$\sup_{\varepsilon\in(0,1/2)} \sup_{0\le t\le T} \mathbb{E}\|\Pi_\varepsilon X^\varepsilon_t\|^2_{B^\alpha_{2,2}} < \infty. \tag{18}$$

To obtain this bound it is convenient to prove the following, stronger bound. Uniformly over $s \in [0,T]$

$$\sup_{s\le t\le T} \mathbb{E}\big[\|\Pi_\varepsilon X^\varepsilon_t\|^2_{B^\alpha_{2,2}} \big| \mathcal{F}_s\big] \lesssim_T 1 + \|\Pi_\varepsilon X^\varepsilon_s\|^2_{B^\alpha_{2,2}}, \tag{19}$$

where $\{\mathcal{F}_t\}_{t\ge 0}$ is the natural filtration generated by X^ε (we omit the dependence on ε). We state the bound with the conditional expectation, since in this form it is simpler to derive, via a Gronwall-type argument. For brevity, fix the notation

$$\overline{X}^\varepsilon = \Pi_\varepsilon X^\varepsilon.$$

By the martingale representation of Lemma 3.1 and a change of variables formula

$$\overline{X}^\varepsilon_t = e^{(t-s)\mathcal{A}_\varepsilon}\overline{X}^\varepsilon_s + \int_s^t e^{(t-r)\mathcal{A}_\varepsilon}\Pi^2_\varepsilon\big[\xi\big(\overline{X}^\varepsilon_r - (\overline{X}^\varepsilon_r)^2\big)\big]\,\mathrm{d}r + \int_{s+}^t \Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}\,\mathrm{d}M^\varepsilon_r,$$

where the last integral is understood as a martingale measure (cf. [52]). However, for the purpose of the proof it is sufficient to consider its one dimensional projections, that is for $\varphi \in C(\mathbb{T}^d)$

$$\langle \overline{X}^\varepsilon_t, \varphi\rangle = \langle \overline{X}^\varepsilon_s, e^{(t-s)\mathcal{A}_\varepsilon}\varphi\rangle + \int_s^t \langle \Pi^2_\varepsilon\big[\xi\big(\overline{X}^\varepsilon_r - (\overline{X}^\varepsilon_r)^2\big)\big], e^{(t-r)\mathcal{A}_\varepsilon}\varphi\rangle\,\mathrm{d}r + \int_{s+}^t \mathrm{d}M^\varepsilon_r(\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}\varphi).$$

The $B^\alpha_{2,2}$ norm is estimated by

$$\begin{aligned}\mathbb{E}\big[\|\overline{X}^\varepsilon_t\|^2_{B^\alpha_{2,2}}\big|\mathcal{F}_s\big] \lesssim \|\overline{X}^\varepsilon_s\|^2_{B^\alpha_{2,2}} &+ \mathbb{E}\bigg[\bigg\|\int_s^t e^{(t-r)\mathcal{A}_\varepsilon}\Pi^2_\varepsilon\big[\xi(\overline{X}^\varepsilon_r - (\overline{X}^\varepsilon_r)^2)\big]\,\mathrm{d}r\bigg\|^2_{B^\alpha_{2,2}}\bigg|\mathcal{F}_s\bigg] \\ &+ \mathbb{E}\bigg[\bigg\|\int_{s+}^t \Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}\,\mathrm{d}M^\varepsilon_r\bigg\|^2_{B^\alpha_{2,2}}\bigg|\mathcal{F}_s\bigg].\end{aligned}$$

An extension of the paraproduct estimates of Lemma B.8 to the $B^\alpha_{p,q}$ scale (see [3, Theorems 2.82, 2.85]) guarantees that

$$\|f^2\|_{B^\alpha_{2,2}} \le 2\|f \varolessthan f\|_{B^\alpha_{2,2}} + \|f \odot f\|_{B^\alpha_{2,2}} \lesssim \|f\|_{L^\infty}\|f\|_{B^\alpha_{2,2}},$$

and through the Schauder estimates of Proposition 5.7, the L^∞ bound on $\overline{X}^\varepsilon$ and the fact that $\bar{\xi}$ is smooth one obtains

$$\mathbb{E}\Big[\Big\|\int_s^t e^{(t-r)\mathcal{A}_\varepsilon}\Pi_\varepsilon^2\big[\bar{\xi}(\overline{X}_r^\varepsilon-(\overline{X}_r^\varepsilon)^2)\big]\,\mathrm{d}r\Big\|_{B_{2,2}^\alpha}^2\Big|\mathcal{F}_s\Big]\lesssim|t-s|\sup_{s\le t\le T}\mathbb{E}\big[\|\overline{X}_t^\varepsilon\|_{B_{2,2}^\alpha}^2\big|\mathcal{F}_s\big].$$

As for the martingale term, by the definition of the space $B_{2,2}^\alpha$ one has

$$\varepsilon^{2\lambda}\mathbb{E}\Big[\Big\|\int_{s+}^t\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}\,\mathrm{d}M_r^\varepsilon\Big\|_{B_{2,2}^\alpha}^2\Big|\mathcal{F}_s\Big]=\sum_{j\ge-1}2^{2\alpha}\int_{\mathbb{T}^d}\varepsilon^{2\lambda}\mathbb{E}\Big[\Big|\int_{s+}^t\mathrm{d}M_r^\varepsilon(e^{(t-r)\mathcal{A}_\varepsilon}\Pi_\varepsilon K_j^x)\Big|^2\ \Big|\ \mathcal{F}_s\Big]\,\mathrm{d}x.$$

Using the predictable quadratic variation computed in Lemma 3.1 one obtains, uniformly over x

$$\begin{aligned}(20)\quad &\varepsilon^{2\lambda}\mathbb{E}\Big[\Big|\int_{s+}^t\mathrm{d}M_r^\varepsilon(e^{(t-r)\mathcal{A}_\varepsilon}\Pi_\varepsilon K_j^x)\Big|^2\ \Big|\ \mathcal{F}_s\Big]\\
&=\varepsilon^{2\lambda}\mathbb{E}\Big[\int_s^t\langle\overline{X}_r^\varepsilon,(1+s_\varepsilon)\big[(\Pi_\varepsilon^2e^{(t-r)\mathcal{A}_\varepsilon}K_j^x)^2-2\Pi_\varepsilon^2(e^{(t-r)\mathcal{A}_\varepsilon}K_j^x)\Pi_\varepsilon(X_r^\varepsilon\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x)\big]\rangle\\
&\qquad+\langle\big(\Pi_\varepsilon(X_r^\varepsilon\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x)\big)^2,1\rangle\\
&\qquad-\langle(\overline{X}_r^\varepsilon)^2,s_\varepsilon\big[(\Pi_\varepsilon^2e^{(t-r)\mathcal{A}_\varepsilon}K_j^x)^2-2\Pi_\varepsilon^2(e^{(t-r)\mathcal{A}_\varepsilon}K_j^x)\Pi_\varepsilon(X_r^\varepsilon\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x)\big]\rangle\,\mathrm{d}r\ \Big|\ \mathcal{F}_s\Big]\\
&\lesssim\varepsilon^{2\lambda}\int_s^t\|\Pi_\varepsilon|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x|\|_{L^2}^2\,\mathrm{d}r,\end{aligned}$$

since $|s_\varepsilon|,|X^\varepsilon|\le1$. Now, for $\zeta\in\mathbb{R}$, for example via the Poisson summation formula in Lemma B.1 and a scaling argument on $\mathbb{R}^d$

$$\|K_j^x\|_{\mathcal{C}_1^\zeta}\lesssim2^{j\zeta}$$

and therefore by the Schauder estimates of Proposition 5.7 and Lemma B.6, for $\gamma\in(0,1)$

$$\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x\|_{\mathcal{C}_1^{\zeta+\gamma}}\lesssim(t-r)^{-\frac{\gamma}{2}}2^\zeta.$$

For clarity, dimension $d=1$ and dimension $d=2$ are treated separately. In dimension $d=1$ choose $-\frac{1}{2}<\zeta<-\alpha$ and fix $\gamma\in(0,1)$ such that $\zeta+\gamma>\frac{1}{2}$. Then, by Besov embeddings, one has

$$\|\Pi_\varepsilon|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x|\|_{L^2}^2\le\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x\|_{L^2}^2\lesssim\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x\|_{\mathcal{C}_1^{\zeta+\gamma}}^2\lesssim(t-r)^{-\gamma}2^{2\zeta}.$$

In dimension $d=2$, where $\eta=\lambda$, choose $\kappa>0$ such that $\alpha<\eta-5\kappa$ and set $\gamma=1-\kappa$. Then Lemma B.6 and Besov embeddings B.2 guarantee that

$$\begin{aligned}\|\Pi_\varepsilon|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x|\|_{L^2}&\lesssim\varepsilon^{\eta-\kappa}\||\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x|\|_{\mathcal{C}_2^{-\eta+2\kappa}}\lesssim\varepsilon^{\eta-\kappa}\||\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x|\|_{\mathcal{C}^{-\kappa}_{\frac{2}{1+\eta-3\kappa}}}\\
&\lesssim\varepsilon^{\eta-\kappa}\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x\|_{L^{\frac{2}{1+\eta-3\kappa}}}\lesssim\varepsilon^{\eta-\kappa}\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K_j^x\|_{\mathcal{C}_1^{1-\eta+4\kappa}}\\
&\lesssim\varepsilon^{\eta-\kappa}(t-r)^{\frac{1-\kappa}{2}}\|K_j^x\|_{\mathcal{C}_1^{-\eta+5\kappa}}\lesssim\varepsilon^{\eta-\kappa}(t-r)^{\frac{\gamma}{2}}2^{-j(\eta-5\kappa)}.\end{aligned}$$

In both dimensions, substituting the estimate into (20) one obtains

$$\mathbb{E}\Big[\Big\|\int_s^t\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}\,\mathrm{d}M_r^\varepsilon\Big\|_{B_{2,2}^\alpha}^2\Big|\mathcal{F}_s\Big]\lesssim|t-s|^{1-\gamma}.$$

For sufficiently small, deterministic T^*, chosen uniform over all parameters, inequality (19) is shown for all $(t-s)\le T^*$. Due to the presence of the conditional expectation, one can exploit this argument for general t, s via a Gronwall-type argument. Indeed, to extend the estimate to $2T^*$, observe there exists a $C(T^*)$ such that

$$\begin{aligned}\sup_{t\in[s,s+2T^*]} \mathbb{E}\big[\|\Pi_\varepsilon X_t^\varepsilon\|^2_{B^\alpha_{2,2}}\big|\mathcal{F}_s\big] &\le C(T^*)\Big(1+\sup_{t\in[s,s+T^*]}\mathbb{E}\big[\|\Pi_\varepsilon X_t^\varepsilon\|^2_{B^\alpha_{2,2}}|\mathcal{F}_s\big]\Big)\\ &\le C(T^*)\Big(1+C(T^*)\Big(1+\mathbb{E}\big[\|\Pi_\varepsilon X_s^\varepsilon\|^2_{B^\alpha_{2,2}}\big]\Big)\Big).\end{aligned}$$

Iterating this argument yields the bound for arbitrary T.

Step 2. The next goal is a bound for the expectation of an increment. For this reason fix

$$0<\beta<\alpha,$$

with α as in Step 1. We shall prove that there exists a $\zeta>0$ satisfying:

$$\mathbb{E}\Big[\|\overline{X}_t^\varepsilon-\overline{X}_s^\varepsilon\|^2_{B^\beta_{2,2}}\Big]\lesssim|t-s|^{4\zeta}. \tag{21}$$

Indeed, arguments similar to those in Step 1. show that

$$\begin{aligned}\mathbb{E}\Big[\|\overline{X}_t^\varepsilon-\overline{X}_s^\varepsilon\|^2_{B^\beta_{2,2}}\Big] &\le \mathbb{E}\Big[\|\overline{X}_t^\varepsilon-e^{(t-s)\mathcal{A}_\varepsilon}\overline{X}_s^\varepsilon\|^2_{B^\beta_{2,2}}\Big]+\mathbb{E}\Big[\|e^{(t-s)\mathcal{A}_\varepsilon}\overline{X}_s^\varepsilon-\overline{X}_s^\varepsilon\|^2_{B^\beta_{2,2}}\Big]\\ &\lesssim \mathbb{E}\Big[\|\overline{X}_t^\varepsilon-e^{(t-s)\mathcal{A}_\varepsilon}\overline{X}_s^\varepsilon\|^2_{B^\beta_{2,2}}\Big]+|t-s|^{\alpha-\beta}\mathbb{E}\|\overline{X}_s^\varepsilon\|^2_{B^\alpha_{2,2}}\\ &\lesssim |t-s|^{1-\gamma}(1+\mathbb{E}\|\overline{X}_s^\varepsilon\|^2_{B^\beta_{2,2}})+|t-s|^{\alpha-\beta}\mathbb{E}\|\overline{X}_s^\varepsilon\|^2_{B^\alpha_{2,2}},\end{aligned}$$

where the penultimate step follows from Lemma 5.8. This is enough to establish (21).

Step 3. Notice that (18) and (21) together guarantee that

$$\sup_{\varepsilon\in(0,1/2)}\mathbb{E}\Big[\|\overline{X}^\varepsilon\|^2_{L^2([0,T];B^\alpha_{2,2})}+\|\overline{X}^\varepsilon\|_{W^{2,\zeta}([0,T];B^\beta_{2,2})}\Big]<\infty,$$

with ζ as in (21). Note that this implies tightness in $L^2([0,T];B^{\alpha'}_{2,2})$ for any $\alpha'<\alpha$, which is still sufficient for the result, since α varies in an open set.

□

At this point, the last step is to prove that any limit point satisfies the required martingale problem (in $d=1$) or solves the required PDE (in $d=2$).

Proof of Theorem Theorem 2. As in all previous cases, we fix $\omega\in\Omega$ and do not state explicitly the dependence on it. We treat the drift and the martingale part differently.

Step 1. We start with the drift, which is the same in both dimensions. Since Let X be any limit point of X^ε in $C([0,T];\mathcal{M}(\mathbb{T}^d))$. The previous proposition guarantees that any such X lies almost surely in $L^2([0,T];B^\alpha_{2,2})$ for some $\alpha>0$. In addition, through Skorohod representation, we can assume that $\Pi_\varepsilon X^\varepsilon\to X$ in $L^2([0,T];B^\alpha_{2,2})$ almost surely. In particular, for $\varphi\in C^\infty(\mathbb{T}^d)$, defining

$$N_t^\varphi=\langle X_{t,0},\varphi\rangle-\int_0^t\langle X_s,\nu_0\Delta\varphi\rangle+\langle\bar{\xi}(X_s-X_s^2),\varphi\rangle\,\mathrm{d}s,$$

and since regarding the nonlinear term one can estimate:

$$\int_0^t\int_{\mathbb{T}^d}|X_s^2-(\Pi_\varepsilon X^\varepsilon)^2|\,\mathrm{d}x\,\mathrm{d}s\le\int_0^t\int_{\mathbb{T}^d}2|X_s-\Pi_\varepsilon X^\varepsilon|\,\mathrm{d}x\,\mathrm{d}s\lesssim\|X_s-\Pi_\varepsilon X^\varepsilon\|_{L^2([0,T];B^\alpha_{2,2})}$$

and applying Lemma 5.5, one has almost surely:

$$\begin{aligned}N_t^\varphi&=\lim_{\varepsilon\to0}\Big[\langle\Pi_\varepsilon X_{t,0}^\varepsilon,\varphi\rangle-\int_0^t\langle\mathcal{A}_\varepsilon X_s^\varepsilon,\varphi\rangle+\langle\bar{\xi}\big[\Pi_\varepsilon X_s^\varepsilon-(\Pi_\varepsilon X_s^\varepsilon)^2\big],\Pi_\varepsilon^2\varphi\rangle\,\mathrm{d}s\Big]\\ &=:\lim_{\varepsilon\to0}N_t^{\varphi,\varepsilon}.\end{aligned}$$

Step 2. Now we prove that N_t^φ is a centered continuous martingale, with quadratic variation depending on the dimension. In $d = 2$ the quadratic variation will be zero and hence $N^\varphi \equiv 0$, proving that the limit is deterministic (conditional on the environment). Since $N_t^{\varepsilon,\varphi}$ is a sequence of martingales, by Lemma 3.1, the fact that also N_t^φ is a martingale follows from the uniform bound of Equation (18) (the continuity of N^φ is as well a consequence of that proposition). The quadratic variation of $N^{\varepsilon,\varphi}$ is given by:

$$\langle N^{\varepsilon,\varphi}\rangle_t = \varepsilon^\lambda \int_0^t \langle (1+s_\varepsilon)\Pi_\varepsilon X_r^\varepsilon, (\Pi_\varepsilon^2\varphi)^2 - 2\Pi_\varepsilon^2(\varphi)\Pi_\varepsilon(X_r^\varepsilon\varphi)\rangle \\ + \langle \big(\Pi_\varepsilon(X_r^\varepsilon\Pi_\varepsilon\varphi)\big)^2, 1\rangle - \langle s_\varepsilon(\Pi_\varepsilon X_r^\varepsilon)^2, (\Pi_\varepsilon^2\varphi)^2 - 2\Pi_\varepsilon^2(\varphi)\Pi_\varepsilon(X_r^\varepsilon\Pi_\varepsilon\varphi)\rangle \,\mathrm{d}s,$$

with λ as in Equation (17). Passing to the limit one has:

$$\lim_{\varepsilon\to 0}\langle N^{\varepsilon,\varphi}\rangle_t = 1_{\{\lambda=0\}}\int_0^t \langle X_s, \varphi^2 - 2X_s\varphi^2\rangle + \langle X_s^2, \varphi^2\rangle \,\mathrm{d}s = 1_{\{\lambda=0\}}\int_0^t \langle X_s(1-X_s), \varphi^2\rangle \,\mathrm{d}s.$$

This is of the required form for Theorem 2. Moreover, a localization argument guarantees that that $\langle N^\varphi\rangle_t = \lim_{\varepsilon\to 0}\langle N^{\varepsilon,\varphi}\rangle_t$, thus completing the proof.

□

4. Long-time behaviour of limiting processes

This section is dedicated to proofs of statements on the long time behaviour of the limiting processes. In subsection 4.1, the persistence of rough superBrownian motion is discussed. Subsection 4.2 covers the long time behaviour of Fisher-KPP equation in rough potential.

4.1. **Rough superBrownian motion.**

In this subsection we briefly discuss the persistence of the rough superBrownian motion on a torus, providing a sketch of the proof of Proposition 1.10. The proof is very similar to the proof of [45, Theorem 2.20], but we sketch it here as this result is crucial for our biological motivation. We begin by recalling a result on the behaviour of the eignevalues of Anderson Hamiltonian. For the proof in dimension $d = 2$ we refer to [2, Theorem 1.7], whereas the bound in dimension $d = 1$ can be deduced from results in [8], see also [37, Theorem 2].

Proposition 4.1. *Fix $(\Omega, \mathcal{F}, \mathbb{P})$ as in Assumption 2.1. For $\omega \in \Omega$ let $\lambda_1(\omega)$ be the first eigenvalue of Anderson Hamiltonian $\mathcal{H}(\omega)$ as defined in Proposition 1.19. Then there exists a pair of constants $C_1 > C_2 > 0$ such that for $x > 0$ large enough*

$$e^{-C_2 x^{2-d/2}} \leq \mathbb{P}\big(\lambda_1 > x\big) \leq e^{-C_1 x^{2-d/2}}.$$

Proof of Proposition 1.10. Let Y_t be a rough superBrownian motion on a torus. For $\omega \in \Omega$ Let $(\lambda_1(\omega), e_1(\omega))$ denote the first eigenvalue-eigenfunction pair of the Anderson Hamiltonian $\mathcal{H}(\omega)$, with $\|e_1(\omega)\|_{L^2(\mathbb{T}^d)} = 1$. According to Theorem 1.19 $e_1(\omega)$ is strictly positive. By Proposition 4.1, $\mathbb{P}(\lambda_1 > 0) > 0$. Hence it is sufficient to prove the claim for all realizations ω such that $\lambda_1(\omega) > 0$. Under this assumption, by the martingale representation of $Y_t(\omega)$ observe that

$$\mathbb{E}_\omega[\langle Y_t(\omega), e_1(\omega)\rangle | \mathcal{F}_s] = \langle Y_t(\omega), e^{t\mathcal{H}(\omega)}e_1(\omega)\rangle = \langle Y_t(\omega), e^{(t-s)\lambda_1(\omega)}e_1(\omega)\rangle,$$

which shows that $E(t,\omega) = \langle Y_t(\omega), e_1(\omega)\rangle$ is a martingale. To see that the variance of $E(t,\omega)$ is uniformly bounded in t notice that

$$\mathbb{E}_\omega[|E(t,\omega) - E(0,\omega)|^2] \lesssim \int_0^t e^{t\mathcal{H}(\omega)}\big((e^{-\lambda_1(\omega)s}e_1(\omega))^2\big)(0)\mathrm{d}s \\ \leq \|e_1(\omega)\|_{L^\infty}\int_0^t e^{-\lambda_1(\omega)s}e^{t\mathcal{H}(\omega)}\big(e^{-\lambda_1(\omega)s}e_1(\omega)\big)(0)\mathrm{d}s = \|e_1(\omega)\|_{L^\infty}\int_0^t e^{-\lambda_1(\omega)s}e_1(\omega,0)\mathrm{d}s \lesssim 1.$$

By the martingale convergence Theorem $E(t,\omega)$ converges as t tends to infinity to a random variable $E(\omega)$. Since $\mathbb{E}_\omega[E(\omega)] = E(0,\omega) = e(\omega,0) > 0$, the random variable $E(\omega)$ is positive with a positive probability. The conclusion for general functions φ now follows by an application of Lemma 6.4. ☐

4.2. **Fisher-KPP.** Here we prove Proposition 1.18, dividing it in two steps, according to the dimension.

Lemma 4.2. *Consider a solution X to the stochastic FKPP equation as in Definition 1.13 in $d = 1$. $\mathbb{P} \ltimes \overline{\mathbb{P}}^{\omega}$-almost surely, there exists a finite random time τ such that $X_\tau \equiv 1$ or $X_\tau \equiv 0$.*

Proof. Consider $\omega \in \Omega$ fixed. For convenience, we omit the dependence on ω in the following. For $n \in \mathbb{N}$ define $X^n_t = X_{t+n}$. Through Schauder estimates similar to the ones in Proposition 3.2 (but here we do not have a jump process, so one can apply the classical Kolmogorov-Chentsov criterion), one can show that the sequence X^n is tight in $C^\kappa([0,T]; B^\kappa_{2,2})$ for any $T > 0$ and some $\kappa > 0$. It is easy to see that any limit point $\overline{X}$ is constant in time. Moreover, up to changing probability space, one can assume that almost surely $X^n \to \overline{X} \in C^\kappa([0,2]; B^\kappa_{2,2})$, with $\overline{X} \in B^\kappa_{2,2}$. First, we prove that either $\overline{X} \equiv 1$ or $\overline{X} \equiv 0$. Assume by contradiction that $0 \leq \overline{X} \leq 1$ is not trivial. Up to changing probability space once more, one can additionally find a Brownian motion B such that for $z_t = \langle X_t, 1\rangle$ and $\sigma_t = \int_0^t \langle X_s(1-X_s), 1\rangle \,\mathrm{d}s$:

$$z_t = z_0 + \int_0^t \langle \overline{\xi}, X_s(1-X_s)\rangle \,\mathrm{d}s + B_{\sigma_t}.$$

Note that $z_t \in [0,1]$. Hence if $\langle \overline{\xi}, \overline{X}(1-\overline{X})\rangle \neq 0$ (and since $\sigma_t \simeq t$), the law of the iterated logarithm brings us to a contradiction, since we would have $\lim_{t\to\infty} \frac{z_t}{t} \neq 0$. Assume then that $\langle \overline{\xi}, \overline{X}(1-\overline{X})\rangle \neq 0$ and consider the process $y_t = \sqrt{z_t}$. By Itô's formula:

$$\begin{aligned} y_t = y_0 &+ \int_0^t \frac{1}{2}\frac{\langle \overline{\xi}, X_s(1-X_s)\rangle}{y_s} - \frac{1}{4}\frac{\langle 1, X_s(1-X_s)\rangle}{y_s^3} \,\mathrm{d}s \\ &+ \int_0^t \frac{1}{2}\frac{\langle \overline{\xi}, X_s(1-X_s)\rangle}{y_s} \sqrt{\langle 1, X_s(1-X_s)\rangle} \,\mathrm{d}B_s. \end{aligned}$$

The last term has quadratic variation:

$$\int_0^t \frac{1}{2}\frac{\langle \overline{\xi}, X_s(1-X_s)\rangle^2}{y_s^2} \langle 1, X_s(1-X_s)\rangle \,\mathrm{d}s \lesssim t.$$

Then again the law of the iterated logarithm would imply that $\lim_{t\to\infty} y_t = -\infty$, which again contradicts $y_t \in [0,1]$. Hence, almost surely $\overline{X} \equiv 0$ or $\overline{X} \equiv 1$. The same argument proves that if $\overline{X} \equiv 0$, then this point is reached in finite time. A symmetric argument proves the result if $\overline{X} \equiv 1$. ☐

Lemma 4.3. *Consider the solution X to the random FKPP equation as in Definition 1.13 in $d = 2$. For every $\omega \in \Omega$ such that $\lambda_1(\Delta - \overline{\xi}(\omega)), \lambda_1(\Delta + \overline{\xi}(\omega)) > 0$ and assuming that $X_0 \not\equiv 1$, $X_0 \not\equiv 0$, $\lim_{t\to\infty} X(\omega) = \overline{X}(\omega) \in \mathcal{C}^\alpha$ for any $\alpha > 0$, where the latter is the unique nontrivial (i.e. $\overline{X} \not\equiv 1, \overline{X} \not\equiv 0$) solution to the equation*

$$\Delta\overline{X}(\omega) + \overline{\xi}(\omega)\overline{X}(\omega)(1-\overline{X}(\omega)) = 0.$$

Proof. We fix ω as required and omit writing the dependence on it. The statement is then proven in [31, Theorem 10.1.5] (that the convergence holds in $\mathcal{C}^\alpha$ for any $\alpha > 0$ is a consequence of the smoothness of the noise and the a-priori estimates in L^∞ for the solution), but to be clear we add some comments. First, the author proves the result only for Neumann boundary conditions, but the extension to the periodic case follows with exactly the same argument, mutatis mutandis. Second, the author proves non-triviality of the limit for the equation

$$\Delta\overline{X} + \lambda\overline{\xi}\overline{X}(1-\overline{X}) = 0,$$

for large λ, assuming that $\int_{\mathbb{T}^d} \overline{\xi}(x)\,\mathrm{d}x < 0$. The latter condition can be replaced by the fact that $\lambda_1(\Delta - \overline{\xi}) > 0$, while the first one is equivalent to $\lambda_1(\Delta + \overline{\xi}) > 0$. Indeed, the precise condition is that $\lambda > \lambda_0$ and the latter, as in [31, Lemma 10.1.2], is the bifurcation point at which the eigenvalue $\lambda_1(\Delta + \lambda_0\overline{\xi}) = 0$, so $\lambda_0 < 1$. □

5. Schauder Estimates

This section is devoted to the proof of Theorem 3 and other similar results. Since the central object in this section, the semidiscrete Laplace operator $\mathcal{A}_\varepsilon$ is defined through convolutions with characteristic functions, the first result collects some information that will be useful in the upcoming discussion.

Lemma 5.1. *Let $(D\varphi)_i = \frac{d\varphi}{dx_i}$ and $(D^2\varphi)_{i,j} = \frac{d^2\varphi}{dx_i dx_j}$ indicate the gradient and the Hessian matrix of a smooth function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ respectively. Recall that $\hat{\chi}_\varepsilon(k) = \hat{\chi}(\varepsilon k) = \mathcal{F}_{\mathbb{R}^d}(\varepsilon^{-d}1_{\{B_\varepsilon(0)\}})(k)$. Then:*

$$D\widehat{\chi}(0) = 0, \qquad D^2\widehat{\chi}(0) = -(2\pi)^2\nu_0 \mathrm{Id},$$

with

$$\nu_0 = \frac{1}{12} \quad \text{in} \quad d = 1, \qquad \nu_0 = \frac{1}{4\pi} \quad \text{in} \quad d = 2.$$

In particular, for any choice of constants $c < 1 < C$, there exists a $\kappa(c, C)$ such that

$$c \le \frac{\vartheta_\varepsilon(k)}{-(2\pi)\nu_0|k|^2} \le C, \qquad \forall k\colon \ |k|\varepsilon \le \kappa(c, C).$$

Finally, the decay of $\hat{\chi}$ can be controlled as follows for any $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \{1, \dots, d\}$:

$$\Big|\frac{d^n\hat{\chi}(k)}{dx_{i_1}\cdots dx_{i_n}}\Big| \lesssim_n (1+|k|)^{-\frac{d+1}{2}}.$$

The proof of this result is deferred to Section B.1. Instead, we pass to the central result of this section, from which all other will follow. Recall that $\mathcal{A}_\varepsilon$ is a Fourier multiplier, therefore also the exponential $e^{t\mathcal{A}_\varepsilon}$ and the resolvent $(-\mathcal{A}_\varepsilon + \lambda)^{-1}$ (for $\lambda > 1$) are naturally defined as Fourier multipliers. As explained already in other points, the action of $\mathcal{A}_\varepsilon$ is different on large and small Fourier modes. The next result provides the correct choice for this division.

Proposition 5.2. *There exists a constant $\kappa_0 > 0$ such that the following holds. For any $p \in [1, \infty], \alpha \in \mathbb{R}$ and $j \ge -1$ there exists a $c > 0$ such that uniformly over $\varepsilon \in (0, 1/2), t \ge 0, j \ge -1$ and $\varphi \in \mathcal{C}^\alpha_p$ one can bound:*

$$\begin{aligned} &\|\Delta_j\mathcal{A}_\varepsilon\varphi\|_{L^p(\mathbb{T}^d)} \lesssim 2^{-(\alpha-2)j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{if} \ \ 2^j\varepsilon \le \kappa_0, \\ &\|\Delta_j\mathcal{A}_\varepsilon\varphi\|_{L^p(\mathbb{T}^d)} \lesssim \varepsilon^{-2}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{if} \ 2^j\varepsilon > \kappa_0. \end{aligned} \tag{22}$$

And similarly for the exponential:

$$\begin{aligned} &\|\Delta_j e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{T}^d)} \lesssim e^{-ct2^{2j}}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{for} \ \ 2^j\varepsilon \le \kappa_0, \\ &\|\Delta_j e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{T}^d)} \lesssim e^{-ct\varepsilon^{-2}}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{for} \ \ 2^j\varepsilon > \kappa_0, \end{aligned} \tag{23}$$

and for the resolvent (uniformly over $\lambda > 1$):

$$\begin{aligned} &\|\Delta_j(-\mathcal{A}_\varepsilon + \lambda)^{-1}\varphi\|_{L^p(\mathbb{T}^d)} \lesssim \frac{1}{2^{2j} + \lambda}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{for} \ \ 2^j\varepsilon \le \kappa_0, \\ &\|\Delta_j(-\mathcal{A}_\varepsilon + \lambda)^{-1}\varphi\|_{L^p(\mathbb{T}^d)} \lesssim \frac{1}{\varepsilon^{-2} + \lambda}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{for} \ \ 2^j\varepsilon > \kappa_0, \end{aligned} \tag{24}$$

Proof. Since all of the estimates follow the same pattern and the first one is particularly simple, we will mainly discuss the proof of Inequality (23), pointing out how to adapt the calculations to the other cases. We also restrict to the case

$$j \ge 0,$$

since the case $j = -1$ is similar but simpler. We begin by restating the inequalities for distributions on $\mathbb{R}^d$. This is useful because on the entire space we can use scaling arguments. Then we examine the behaviour on large and small scales separately. The precise separation of modes is chosen based on Lemma 5.1.

Step 1. To restate the problem on $\mathbb{R}^d$ we extend distributions on the torus periodically. Let $\pi\colon \mathcal{S}'(\mathbb{T}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denote the such periodic extension operator of distribution on $\mathbb{T}^d$ to the full space. Its adjoint is the operator $\pi^*\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{T}^d)$, given by

$$\pi^*\varphi(\cdot) = \sum_{k\in\mathbb{Z}^d} \varphi(\cdot + k).$$

We observe that $\pi(\mathcal{A}_\varepsilon\varphi) = \mathcal{A}_\varepsilon\pi(\varphi)$, where with a slight abuse of notation we have extended $\mathcal{A}_\varepsilon$ to act on distributions on the whole space (simply through Equation (4) - and note that it is still a Fourier multiplier, since for $\varphi\colon \mathbb{R}^d \to \mathbb{R}$, $\mathcal{A}_\varepsilon\varphi = \mathcal{F}^{-1}_{\mathbb{R}^d}\vartheta_\varepsilon\mathcal{F}_{\mathbb{R}^d}\varphi$). Similarly, by the Poisson summation formula (Lemma B.1), $\pi(\Delta_j\varphi) = \Delta_j\pi(\varphi)$, the latter defined as the Fourier multiplier $\Delta_j\pi(\varphi) = \mathcal{F}^{-1}_{\mathbb{R}^d}\varrho_j\mathcal{F}_{\mathbb{R}^d}\pi(\varphi)$. As a consequence of this last observation, for any $a > \frac{d}{p}$ (or $a \geq 0$ if $p = \infty$):

$$\|\Delta_j\pi(\varphi)\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} \simeq_{a,p} \|\Delta_j\varphi\|_{L^p(\mathbb{T}^d)},$$

where $\|f\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} = \|f(\cdot)/(1+|\cdot|^2)^{\frac{a}{2}}\|_{L^p(\mathbb{R}^d)}$. Therefore in order to show (23) it is sufficient to show that for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ and setting $a = d+1$:

$$\begin{aligned}
\|\Delta_j e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^{d+1})} &\lesssim e^{-ct2^{2j}}\|\Delta_j\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^{d+1})}, \quad \text{for} \quad 2^j\varepsilon \leq \kappa_0\\
\|\Delta_j e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^{d+1})} &\lesssim e^{-ct\varepsilon^{-2}}\|\Delta_j\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^{d+1})}, \quad \text{for} \quad 2^j\varepsilon > \kappa_0
\end{aligned}$$

The same holds for (22) and (24), with the natural changes. Hence, from now on let us consider all functions and operators are defined on $\mathbb{R}^d$. Let ψ be a smooth radial function with compact support in an annulus (i.e. $\psi(k) = 0$ if $|k| \leq c_1$ or $|k| \geq c_2$ for some $0 < c_1 < c_2$) such that $\rho\psi = \rho$. By Young's inequality for convolutions and by estimating uniformly over $x, y \in \mathbb{R}^d$

$$(1+|x|^2)^{-\frac{(d+1)}{2}} \lesssim (1+|y|^2)^{-\frac{(d+1)}{2}}(1+|x-y|^2)^{\frac{d+1}{2}},$$

one obtains:

$$\|\Delta_j e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^{d+1})} \lesssim \|\mathcal{F}^{-1}_{\mathbb{R}^d}(e^{t\vartheta_\varepsilon(\cdot)}\psi(2^{-j}\cdot))\|_{L^1(\mathbb{R}^d,\langle\cdot\rangle^{-(d+1)})}\|\Delta_j\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^{d+1})}.$$

In this way, through a change of variables, we reduced the problem to a bound for

$$\int_{\mathbb{R}^d}(1+2^{-2j}|x|^2)^{\frac{d+1}{2}}\left|\mathcal{F}^{-1}_{\mathbb{R}^d}\left[e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\right](x)\right|\mathrm{d}x \tag{25}$$

(and similarly for (22) and (24), with $e^{t\vartheta_\varepsilon}$ replaced by ϑ_ε and $(-\vartheta_\varepsilon + \lambda)^{-1}$ respectively). Before we move on, we finally observe that by Lemma 5.1, there exists a $\kappa_0 > 0$ such that for $2^j\varepsilon \leq \kappa_0$:

$$\frac{1}{2} \leq \frac{\vartheta_\varepsilon(2^jk)}{-(2\pi)^2\nu_0 2^{2j}|k|^2} \leq \frac{3}{2}, \qquad \forall k \in \operatorname{supp}(\psi).$$

Step 2. We now estimate (25) on large scales, i.e. $2^j\varepsilon \leq \kappa_0$. In this case the term can be bounded by:

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}_{\mathbb{R}^d}[e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)] + \sum_{i=1}^d\left|\mathcal{F}^{-1}_{\mathbb{R}^d}[\partial_{k_i}^{2(d+1)}e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)]\right|\right\|_{L^\infty(\mathbb{R}^d)}\\
&\qquad\qquad \lesssim \sup_{k\in\operatorname{supp}(\psi)}\left[\left|e^{t\vartheta_\varepsilon(2^jk)}\psi(k)\right| + \sum_{i=1}^d\left|\partial_{k_i}^{2(d+1)}e^{t\vartheta_\vartheta(2^jk)}\psi(k)\right|\right].
\end{aligned}$$

To bound the term involving derivatives we observe that:

$$D[t\vartheta_\varepsilon(2^j\cdot)](k) = f(2^j\varepsilon k)t2^{2j}|k|, \qquad f(k) = 2\widehat{\chi}(k)\frac{D\widehat{\chi}(k)}{|k|}.$$

where f is smooth on $\mathbb{R}^d$, again by Lemma 5.1. In particular, since $2^j\varepsilon \lesssim 1$, taking higher order derivatives one has for any $n \in \mathbb{N}$: $\big|\partial^n_{k_i}[t\vartheta_\varepsilon(2^j\cdot)]\big|(k) \lesssim t2^{2j}$ for $k \in \mathrm{supp}(\psi)$. Now recall Faá di Bruno's formula:

$$\partial^n_x f(g(x)) = \sum_{\{m\}} C(\{m\},n)(\partial_x^{m_1+\dots+m_n} f)(g(x)) \prod_{j=1}^n \Big(\partial^j_x g(x)\Big)^{m_j},$$

where the sum runs over all $\{m\} := (m_1,\dots,m_n)$ such that $m_1+2m_2+\dots+nm_n = n$. Applying this formula and by our choice of κ_0, there exists a constant $c > 0$ such that:

$$\sup_{k\in\mathrm{supp}(\psi)} \Big[|e^{t\vartheta_\varepsilon(2^jk)\psi(k)}| + \sum_{i=1}^d \big|\partial^{2(d+1)}_{k_i} e^{t\vartheta_\varepsilon(2^jk)}\psi(k)\big|\Big] \lesssim e^{-\frac{1}{2}(2\pi)^2\nu_0 t2^{2j}}(1+t2^{2j})^{2(d+1)} \lesssim e^{-c(t2^{2j})}.$$

This concludes the proof of the large-scale bound in (23). For the resolvent equation one similarly has to bound:

$$\sup_{k\in\mathrm{supp}(\psi)} \Big[\Big|\frac{\psi(k)}{-\vartheta_\varepsilon(2^jk)+\lambda}\Big| + \sum_{i=1}^d \Big|\partial^{2(d+1)}_{k_i}\frac{\psi(k)}{-\vartheta_\varepsilon(k)+\lambda}\Big|\Big].$$

Here as before, for the derivative term one has, through the choice of κ_0:

$$\begin{aligned}\Big|\partial^n_{k_i}\frac{1}{-\vartheta_\varepsilon(k)+\lambda}\Big| &\lesssim \sum_{\{m\}}\Big|\frac{1}{-\vartheta_\varepsilon(k)+\lambda}\Big|^{1+m_1+\dots+m_n}\prod_{j=1}^n\Big(2^{2j}\Big)^{m_j}\\ &\lesssim \sum_{\{m\}}\Big|\frac{1}{\frac{3}{2}(2\pi)^2\nu_0 2^{2j}+\lambda}\Big|^{1+m_1+\dots+m_n}(2^{2j})^{m_1+\dots+m_n}\\ &\lesssim \frac{1}{\frac{1}{2}(2\pi)^2\nu_0 2^{2j}+\lambda} \lesssim \frac{1}{2^{2j}+\lambda},\end{aligned}$$

as requested for (24). The estimate (22) follows similarly.

Step 3. We pass to the small-scale estimates, namely for j such that $2^j\varepsilon > \kappa_0$. Here we will need tighter control on the decay of $\widehat{\chi}(k)$: since χ is not smooth, the decay at infinity is not faster than any polynomial and is quantified in Lemma 5.1. We now estimate (25), for $s \in (d, d+1)$, by:

$$\begin{aligned}&\Big(\int_{\mathbb{R}^d}\frac{1}{(1+|x|)^s}\,\mathrm{d}x\Big)\sup_{x\in\mathbb{R}^d}\Big[(1+|x|^s+2^{-2j}|x|^{s+(d+1)})\Big|\mathcal{F}^{-1}_{\mathbb{R}^d}\Big[e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\Big]\Big|(x)\Big]\\ &\quad\lesssim \|e^{\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^\infty} + \|(1-\Delta)^{\frac{s}{2}}e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^q(\mathbb{R}^d)} + \sum_{i=1}^d 2^{-j(d+1)}\|\partial^{2(d+1)}_{k_i}e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^\infty},\end{aligned}$$

for any $q \in (1,\infty)$. As for the first term, since $|\hat{\chi}(k)| < 1$ for $k \neq 0$ and it decays to zero at infinity, up to reducing the value of $c > 0$ we can assume that:

$$\vartheta_\varepsilon(2^jk) \le -c\varepsilon^{-2}.$$

This is sufficient to show:

$$\|e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^\infty} \lesssim e^{-ct\varepsilon^{-2}},$$

which is a bound of the required order. As for the second term, denote with H^s_p and $\Lambda^s_{p,q}$ the Bessel potential spaces and Sobolev spaces respectively, following the notation of [51, Section 2.3.5], for which the embedding $\Lambda^{s'}_{p,q} \subseteq H^s_p$ holds, it $s' > s$. Then since $d+1 > s$ one has:

$$\begin{aligned}\|(1-\Delta)^{\frac{s}{2}}e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^q} &=: \|e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{H^s_q(\mathbb{R}^d)}\\ &\lesssim \|e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{\Lambda^{d+1}_{\infty,\infty}} := \sum_{|\alpha|\le d+1}\|D^\alpha e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^\infty},\end{aligned}$$

where we write $|\alpha| = \alpha_1+\cdots+\alpha_d$ for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $D^\alpha = \frac{\mathrm{d}^{|\alpha|}}{\mathrm{d}^{\alpha_1}x_1 \ldots \mathrm{d}^{\alpha_n}x_n}$. Now bounding these derivatives is similar to bounding the last term:

$$\sum_{i=1}^{d} 2^{-j(d+1)} \|\partial_{k_i}^{2(d+1)} e^{t\vartheta_\varepsilon(2^j \cdot)} \psi(\cdot)\|_{L^\infty},$$

so we concentrate on the latter, which has the added difficulty of containing derivatives of higher order, counterbalanced by the factor $2^{j(d+1)}$. Here observe that for $1 \le n \le 2(d+1)$:

$$\partial_{k_i}^n e^{t\vartheta_\varepsilon(2^j k)} = \partial_{k_i}^{n-1} \big[e^{t\vartheta_\varepsilon(2^j k)} 2\hat{\chi}(2^j \varepsilon k)[\partial_{k_i}\hat{\chi}](2^j \varepsilon k) \big] \cdot (2^j \varepsilon) \cdot (t\varepsilon^{-2}).$$

Iterating the above procedure, we apply Faá Di Bruno's formula again to obtain

$$\big|2^{-j(d+1)} \partial_{k_i}^n e^{t\vartheta_\varepsilon(2^j k)}\big| \lesssim 2^{-j(d+1)} e^{t\vartheta_\varepsilon(2^j k)} \sum_{\{m\}} \prod_{\ell=1}^{n} \big(\partial_{k_i}^{\ell-1} \big[2\hat{\chi}(\cdot)[\partial_{k_i}\hat{\chi}(\cdot)] \big] \big|_{2^j \varepsilon k} \cdot (2^j \varepsilon)^\ell \big)^{m_\ell} \cdot (t\varepsilon^{-2})^{m_\ell}.$$

In view of Lemma 5.1 crucially:

$$\sup_{k \in \mathrm{supp}(\psi)} |\partial_{k_i}^{\ell-1} \big[2\hat{\chi}(\cdot)[\partial_{k_i}\hat{\chi}(\cdot)] \big] \big|_{2^j \varepsilon k}| \lesssim \frac{1}{1+|2^j\varepsilon|^{d+1}}.$$

Hence, as before up to further reducing the value of $c > 0$:

$$\begin{aligned} \|\partial_{k_i}^n e^{t\vartheta_\varepsilon(2^j \cdot)}\|_{L^\infty} &\lesssim e^{-ct\varepsilon^{-2}} 2^{-j(d+1)} (2^j\varepsilon)^n \sum_{\{m\}} \prod_{\ell=1}^{n} \langle 2^j \varepsilon \rangle^{-m_\ell(d+1)} \\ &\lesssim e^{-ct\varepsilon^{-2}} 2^{-j(d+1)} (2^j\varepsilon)^{n-(d+1)} \lesssim e^{-ct\varepsilon^{-2}}, \end{aligned}$$

since at least one of the elements of the sequence m_ℓ is strictly positive and since $n \le 2(d+1)$. This concludes the proof of (23). Regarding the resolvent, one can follow mutatis mutandis the previous discussion until one has, as before, to bound:

$$\sum_{i=1}^{d} 2^{-j(d+1)} \Big\| \partial_{k_i}^{2(d+1)} \frac{\psi(\cdot)}{-\vartheta_\varepsilon(2^j \cdot) + \lambda} \Big\|_\infty \lesssim \sum_{i=1}^{d} \sum_{n=0}^{2(d+1)} 2^{-j(d+1)} \Big\| \partial_{k_i}^n \frac{1}{-\vartheta_\varepsilon(2^j \cdot) - \lambda} \Big\|_{L^\infty}.$$

Then again, with Faá di Bruno's formula:

$$\begin{aligned} \Big| \partial_{k_i}^n \frac{1}{-\vartheta_\varepsilon(2^j k) + \lambda} \Big| &\lesssim \sum_{\{m\}} \Big| \frac{1}{-\vartheta_\varepsilon(2^j k) + \lambda} \Big|^{1+m_1+\cdots+m_n} \prod_{\ell=1}^{n} \big| \partial_{k_i}^{\ell-1} (\hat{\chi}(\cdot)\partial_{k_i}\hat{\chi}(\cdot))|_{2^j\varepsilon k} \big|^{m_\ell} \cdot (2^j\varepsilon)^{\ell m_\ell} \\ &\lesssim \frac{1}{\varepsilon^{-2}+\lambda} \sum_{\{m\}} \Big| \frac{1}{\varepsilon^{-2}+\lambda} \Big|^{m_1+\cdots+m_n} \prod_{\ell=1}^{n} \Big(\frac{1}{1+|2^j\varepsilon|} \Big)^{m_\ell(d+1)} (2^j\varepsilon)^{\ell m_\ell} \\ &\lesssim \frac{1}{\varepsilon^{-2}+\lambda} 2^{j(d+1)}. \end{aligned}$$

Plugging this into the previous formula provides us the correct bound. Similarly one can also treat the small-scale estimate for (22). □

The previous proposition motivates the introduction of cut-off operators as follows.

Definition 5.3. *Let* $\daleth \colon \mathbb{R}^d \to \mathbb{R}$ *be a smooth radial function with compact support which is constant outside of an annulus* $A_r^R = \{x \in \mathbb{R}^d \colon r \le |x| \le R\}$ *for some* $0 < r < R$ *and such that:*

$$\daleth(x) = 1, \ \forall x \in A_0^r, \qquad \daleth(x) = 0, \ \forall x \in A_R^\infty.$$

Define

$$\mathcal{P}_\varepsilon = \daleth(\varepsilon D), \qquad \mathcal{Q}_\varepsilon = (1-\daleth)(\varepsilon D).$$

We say that $\mathcal{P}_\varepsilon$ is a projection on ***large scales****, since those Fourier modes describe a function macroscopically, whereas $\mathcal{Q}_\varepsilon$ is a projection on* ***small scales****. We furthermore will use for $j \geq -1, j \in \mathbb{Z}$ the notation:*

$$j \gtrsim \varepsilon^{-1}, \qquad j \lesssim \varepsilon^{-1},$$

if there exists a constant $C > 0$ indepdendent of j, ε such that $2^j \varepsilon \geq C$ (or, repsectively, $2^j \varepsilon \leq C$).

The next lemma states that the cut-off operators are bounded.

Lemma 5.4. *Consider $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$. For $\daleth$ as in Definition 5.3 one can bound uniformly over $\varepsilon \in (0,1)$:*

$$\|\mathcal{P}_\varepsilon \varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}, \qquad \|\mathcal{Q}_\varepsilon \varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

Proof. Let $\widehat{\daleth}(x) = \mathcal{F}^{-1}_{\mathbb{R}^d} \daleth(x)$. By an application of the Poisson summation formula (Lemma B.1) and a scaling argument:

$$\begin{aligned}\|\daleth(\varepsilon D)\varphi\|_{\mathcal{C}^\alpha_p} &= \sup_{j \geq -1} 2^{j\alpha} \|(\mathcal{F}^{-1}_{\mathbb{T}^d}[\daleth(\varepsilon \cdot)]) * \Delta_j \varphi\|_{L^p} \lesssim \|\mathcal{F}^{-1}_{\mathbb{T}^d}[\daleth(\varepsilon \cdot)]\|_{L^1(\mathbb{T}^d)} \|\varphi\|_{\mathcal{C}^\alpha_p} \\ &\lesssim \|\varepsilon^{-d} \widehat{\daleth}(\varepsilon^{-1} \cdot)\|_{L^1(\mathbb{R}^d)} \|\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.\end{aligned}$$

The same argument shows that $(1 - \daleth(a \cdot))$ is bounded. □

5.1. **Elliptic regularity.** In this subsection we prove Theorem 3. The theorem is a direct consequence of the lemma and the proposition that follow.

Lemma 5.5. *Fix any $\alpha \in \mathbb{R}, \zeta > 0, p \in [1, \infty]$. Uniformly over $\varphi \in \mathcal{C}^\alpha_p$ and $\varepsilon \in (0, 1/2)$:*

$$\|\mathcal{A}_\varepsilon \mathcal{P}_\varepsilon \varphi\|_{\mathcal{C}^{\alpha-2}_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

Moreover, as $\varepsilon \to 0$

$$\mathcal{A}_\varepsilon \varphi \to \nu_0 \Delta \varphi \quad \text{in} \quad \mathcal{C}^{\alpha-2-\zeta}_p,$$

where

$$\nu_0 = \frac{1}{12} \quad \text{for} \quad d = 1, \qquad \nu_0 = \frac{1}{4\pi} \quad \text{for} \quad d = 2.$$

Proof. On large scales, Proposition 5.2 and Lemma 5.4 implie that

$$\|\mathcal{A}_\varepsilon \mathcal{P}_\varepsilon \varphi\|_{\mathcal{C}^{\alpha-2}_p} \lesssim \|\mathcal{P}_\varepsilon \varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

Moreover on small scales the same results guarantee that for any $\zeta \geq 0$:

$$\|\mathcal{Q}_\varepsilon \mathcal{A}_\varepsilon \varphi\|_{\mathcal{C}^{\alpha-2-\zeta}_p} \lesssim \varepsilon^{-2} \sup_{j \gtrsim \varepsilon^{-1}} 2^{j(\alpha-2-\zeta)} \|\Delta_j \mathcal{Q}_\varepsilon \varphi\|_{L^p} \lesssim \varepsilon^\zeta \|\varphi\|_{\mathcal{C}^\alpha_p},$$

which tends to 0 as ε tends to 0 if $\zeta > 0$. Combining those two observations provides the first bound and guarantees compactness in $\mathcal{C}^{\alpha-2-\zeta}_p$. Convergence follows since, by Lemma 5.1,

$$\mathcal{F}_{\mathbb{T}^d}[\mathcal{A}_\varepsilon \mathcal{P}_\varepsilon \varphi](k) = \daleth(\varepsilon k) \frac{\hat{\chi}^2(\varepsilon k) - 1}{\varepsilon^2} \hat{\varphi}(k) \to -(2\pi)^2 \nu_0 |k|^2 \hat{\varphi}(k) = \mathcal{F}_{\mathbb{T}^d}[\nu_0 \Delta \varphi](k).$$

□

The regularity gain provided by the operator $\mathcal{A}_\varepsilon$ can be described as follows (for the proof of Theorem 3 we require the result only for $\delta = 0$).

Proposition 5.6. *Fix any $\alpha \in \mathbb{R}$, $\delta \in [0, 1]$ and $p \in [1, \infty]$. Uniformly over $\lambda > 1, \varepsilon \in (0, 1/2)$ and $\varphi \in \mathcal{C}^\alpha_p$ the following estimates hold:*

$$\lambda^{-\delta} \|\mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon + \lambda)^{-1} \varphi\|_{\mathcal{C}^{\alpha+2(1-\delta)}_p} + \lambda^{-\delta} \varepsilon^{-2(1-\delta)} \|\mathcal{Q}_\varepsilon(-\mathcal{A}_\varepsilon + \lambda)^{-1} \varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

Moreover, as ε tends to 0,

$$\mathcal{P}_\varepsilon \big(- \mathcal{A}_\varepsilon - \lambda \big)^{-1} \varphi \to (\nu_0 \Delta - \lambda)^{-1} \varphi$$

where the convergence is in $\mathcal{C}^{\alpha+2-\zeta}_p$ for any $\zeta > 0$ and ν_0 is as in Lemma 5.5.

Proof. Consider the large-scale estimate. Proposition 5.2 and Lemma 5.4 guarantee that for $j \lesssim \varepsilon^{-1}$:

$$\|\Delta_j \mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon+\lambda)^{-1}\varphi\|_{L^p} \lesssim \frac{1}{2^{2j}+\lambda} 2^{-\alpha j}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} \lesssim 2^{-2j(1-\delta)-\alpha j}\lambda^{-\delta}\|\varphi\|_{\mathcal{C}^\alpha_p},$$

which is a bound of the correct order. All other bounds follow similarly, and the proof of the convergence is analogous to the one in Lemma 5.5. □

5.2. **Parabolic regularity.** In this subsection we study the regularization effect of the semigroup $e^{t\mathcal{A}_\varepsilon}$. This discussion requires certain spaces of time-dependent functions, which we introduce in the following. Let us fix $T>0$ an arbitrary time horizon. All function spaces will implicitly depend on T. For time dependent functions taking values in a Banach space $\mathcal{X}$ the α-Hölder norm (with $\alpha\in(0,1)$) is defined as

$$\|f\|_{C^\alpha\mathcal{X}} = \sup_{t\in[0,T]}\|f(t)\|_{\mathcal{X}} + \sup_{t,s\in[0,T]}\frac{\|f(t)-f(s)\|_{\mathcal{X}}}{|t-s|^\alpha}.$$

It is convenient to incorporate a blow-up at time $t=0$. This reflects the fact that the regularization of the semigroup occurs only at strictly positive times.

$$\mathcal{E}^\gamma\mathcal{C}^\alpha_p = \{f\colon (0,T]\to\mathcal{C}^\alpha_p \mid \|f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p} = \sup_{t\in[0,T]} t^\gamma\|f(t)\|_{\mathcal{C}^\alpha_p}<\infty\},$$

and one can combine the previous spaces in the following way:

$$\mathscr{L}^{\gamma,\alpha}_p = \{f\in\mathcal{E}^\gamma\mathcal{C}^\alpha_p \mid \|f\|_{\mathscr{L}^{\gamma,\alpha}_p} = \|f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p} + \|t\mapsto t^\gamma f(t)\|_{C^{\alpha/2}L^p}<\infty\}.$$

Now we state the main result of this section, the parabolic Schauder estimates.

Proposition 5.7. *Fix $p\in[1,\infty], T>0, \gamma\in[0,1)$ and $\alpha\in(-2,0), \beta\in[\alpha,\alpha+2)\cap(0,2)$. Uniformly over $\varphi\in\mathcal{C}^\alpha_p$ and $f\in\mathcal{E}^\gamma_T\mathcal{C}^\alpha_p$ and locally uniformly over $T>0$:*

$$\|t\mapsto\mathcal{P}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{\mathscr{L}^{(\beta-\alpha)/2,\beta}_p} \lesssim \|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}, \tag{26}$$

$$\left\|t\mapsto\int_0^t \mathcal{P}_\varepsilon e^{(t-s)\mathcal{A}_\varepsilon}f(s)\,\mathrm{d}s\right\|_{\mathscr{L}^{\gamma,\alpha+2}_p} \lesssim \|\mathcal{P}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}. \tag{27}$$

In addition, let $\zeta_1,\zeta_2\in[0,1)$ such that $\zeta_1+\zeta_2<1$ and $\delta_1,\delta_2,\delta_3\in[0,1]$ such that $\delta_1+\delta_2+\delta_3=1$. Then:

$$\|t\mapsto t^{\zeta_1+\zeta_2}\mathcal{Q}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{C^{\zeta_1}\mathcal{C}^\alpha_p} \lesssim \varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}, \tag{28}$$

$$\left\|t\mapsto t^\gamma\int_0^t e^{(t-s)\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon f(s)\,\mathrm{d}s\right\|_{C^{\delta_1}\mathcal{C}^\alpha_p} \lesssim \varepsilon^{2\delta_2}T^{\delta_3}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}. \tag{29}$$

with constants independent of f,φ,T.

In many steps the proof mimics proofs in [26] and [28], to which we refer the reader for simple proofs of classical Schauder estimates in the setting of stochastic PDEs.

Proof. Step 1. We begin with large scales, namely (26). By Proposition 5.2:

$$\begin{aligned}\sup_{j\geq-1}2^{\beta j}\|\Delta_j\mathcal{P}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{T}^d)} &\lesssim \sup_{j\geq-1} e^{-ct2^{2j}}2^{(\beta-\alpha)j}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}\\ &= t^{-\frac{\beta-\alpha}{2}}\sup_{j\geq-1}e^{-ct2^{2j}}(t2^{2j})^{\frac{\beta-\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} \lesssim t^{-\frac{\beta-\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}.\end{aligned}$$

Therefore

$$\|t\mapsto\mathcal{P}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{\mathcal{E}^{(\beta-\alpha)/2}\mathcal{C}^\beta_p} \lesssim \|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}.$$

Similarly, for (27)

$$\sup_{j\geq -1} 2^{j(\alpha+2)}\Big\| \int_0^t \Delta_j e^{(t-s)\mathcal{A}_\varepsilon}\mathcal{P}_\varepsilon f(s)\,\mathrm{d}s\Big\|_{L^p(\mathbb{T}^d)} \lesssim \|\mathcal{P}_\varepsilon f\|_{\mathcal{E}^\gamma_T\mathcal{C}^\alpha_p} \sup_{j\geq -1} 2^{j2}\int_0^t e^{-cs2^{2j}}(t-s)^{-\gamma}\,\mathrm{d}s.$$

which can be bounded by $\|\mathcal{P}_\varepsilon f\|_{\mathcal{E}^\gamma_T\mathcal{C}^\alpha_p}$ by the same arguments as in the proof of [26, Lemma A.9]. We still need to address the temporal regularity for both terms. Again, Proposition 5.2 leads to:

$$\begin{aligned}\|(e^{t\mathcal{A}_\varepsilon}-\mathrm{Id})\mathcal{P}_\varepsilon\varphi\|_{L^p(\mathbb{T}^d)} &= \Big\|\int_0^t e^{s\mathcal{A}_\varepsilon}\mathcal{A}_\varepsilon\mathcal{P}_\varepsilon\varphi\,\mathrm{d}s\Big\|_{L^p(\mathbb{T}^d)}\\ &\lesssim \int_0^t s^{-1+\frac{\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}\,\mathrm{d}s \simeq t^{\frac{\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}.\end{aligned} \tag{30}$$

To conclude the proof of both (26) and (27) it is now sufficient to follow the same steps as in [28, Lemma 6.6].

Step 2. We turn our attention to the small scale bounds (28) and (29). Fix $\zeta_1=\delta_1=0$ first. With calculations in the same spirit as in the Step 1, we arrive at:

$$\|\mathcal{Q}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{\mathcal{C}^\alpha_p} = \sup_{j\geq -1} 2^{\alpha j}\|\Delta_j\mathcal{Q}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d)} \lesssim e^{-ct\varepsilon^{-2}}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} \lesssim (t\varepsilon^{-2})^{-\delta}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}.$$

For the inequality (29), if $\delta_3>0$ the spatial bound follows from the previous result. If $\delta_3=0$, we observe that

$$\Big\|\int_0^t \mathcal{Q}_\varepsilon e^{(t-s)\mathcal{A}_\varepsilon}f(s)\,\mathrm{d}s\Big\|_{\mathcal{C}^\alpha_p} \lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}\int_0^t e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim \varepsilon^2 t^{-\gamma}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}.$$

The last bound in the above inequality is obtained in the same spirit as [26, Lemma A.9]. Namely, choose $\lambda\in(0,t/2)$ and split the integral at time λ. We note that

$$\int_0^\lambda e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s \leq \int_0^\lambda (t-s)^{-\gamma}\,\mathrm{d}s = t^{-\gamma+1}\int_0^{\lambda/t}(1-s)^{-\gamma}\,\mathrm{d}s \lesssim t^{-\gamma}\lambda,$$

since, as $\lambda/t\leq 1/2$, $1-(1-\lambda/t)^{(1-\gamma)}\lesssim \lambda/t$. We then observe that for any $\varrho\in(0,1)$,

$$\begin{aligned}\int_\lambda^t e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s &\lesssim \int_\lambda^t (s\varepsilon^{-2})^{-(1+\varrho)}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim t^{-\gamma-\varrho}\varepsilon^{2(1+\varrho)}\int_{\lambda/t}^1 s^{-(1+\varrho)}(1-s)^{-\gamma}\,\mathrm{d}s\\ &\lesssim t^{-\gamma}\varepsilon^{2(1+\varrho)}\lambda^{-\varrho}.\end{aligned}$$

If $\varepsilon^2\leq t/2$, choosing $\lambda=\varepsilon^2$ provides the result. Otherwise, one simply notes that

$$\int_0^t e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim t^{1-\gamma} \lesssim t^{-\gamma}\varepsilon^2.$$

Step 3. We now investigate the full temporal regularity for (28) and (29), that is, we allow for $\zeta_1,\delta_1>0$. We first observe that for $\delta\in[0,1)$

$$\|(e^{t\mathcal{A}_\varepsilon}-\mathrm{Id})\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} = \Big\|\int_0^t e^{s\mathcal{A}_\varepsilon}\mathcal{A}_\varepsilon\mathcal{Q}_\varepsilon\varphi\,\mathrm{d}s\Big\|_{\mathcal{C}^\alpha_p} \lesssim \|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}\int_0^t (s\varepsilon^{-2})^{-\delta}\varepsilon^{-2}\,\mathrm{d}s = \|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}\varepsilon^{2(\delta-1)}t^{1-\delta}. \tag{31}$$

For $\delta\in[0,1)$ and $\zeta=\zeta_1+\zeta_2\in[0,1)$, the temporal regularity of the first terms can be established via

$$\begin{aligned}\|t^\zeta e^{t\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi - s^\zeta e^{s\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} &\lesssim (t^\zeta-s^\zeta)t^{-\zeta_2}\varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} + s^\zeta\|(e^{(t-s)\mathcal{A}_\varepsilon}-\mathrm{Id})e^{s\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}\\ &\lesssim (t^\zeta-s^\zeta)t^{-\zeta_2}\varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} + s^\zeta(t-s)^{1-\delta}\varepsilon^{2(\delta-1)}\|e^{s\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p}\\ &\lesssim [(t^\zeta-s^\zeta)t^{-\zeta_2}\varepsilon^{2\zeta_2} + (t-s)^{1-\delta}\varepsilon^{2(\delta-1)}\varepsilon^{2\zeta}]\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} \lesssim (t-s)^{\zeta_1}\varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p},\end{aligned}$$

where in the last step we set $\delta=1-\zeta_1$ and notice that $(t^\zeta-s^\zeta)t^{-\zeta_2}\lesssim(t-s)^{\zeta_1}$.

The bound for (29) follows similar pattern. For simplicity write $V(t) = \int_0^t e^{(t-s)\mathcal{A}_\varepsilon} \mathcal{Q}_\varepsilon f(s)\,\mathrm{d}s$. Then

$$\|t^\gamma V(t) - s^\gamma V(s)\|_{\mathcal{C}^\alpha_p} \le (t^\gamma - s^\gamma)\|V(t)\|_{\mathcal{C}^\alpha_p} + s^\gamma \Big\| \int_s^t e^{(t-r)\mathcal{A}_\varepsilon} \mathcal{Q}_\varepsilon f(r)\mathrm{d}r \Big\|_{\mathcal{C}^\alpha_p} + s^\gamma \|(e^{(t-s)\mathcal{A}_\varepsilon} - \mathrm{Id})V(s)\|_{\mathcal{C}^\alpha_p}.$$

The only term for which the estimation does not follow the already established pattern is the one in the middle, for which we observe that

$$\begin{aligned} s^\gamma \Big\| \int_s^t e^{(t-r)\mathcal{A}_\varepsilon} \mathcal{Q}_\varepsilon f(r)\,\mathrm{d}r \Big\|_{\mathcal{C}^\alpha_p} &\lesssim s^\gamma \int_s^t ((t-r)\varepsilon^{-2})^{-\delta_2} r^{-\gamma}\,\mathrm{d}r \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} \\ &\lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} \varepsilon^{2\delta_2} s^\gamma t^{-\delta_2 - \gamma + 1} \int_{s/t}^1 (1-r)^{-\delta_2} r^{-\gamma}\,\mathrm{d}r \lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} \varepsilon^{2\delta_2} t^{1-\delta_2} \int_{s/t}^1 (1-r)^{-\delta_2}\,\mathrm{d}r \\ &\lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} \varepsilon^{2\delta_2} t^{1-\delta_2} (1-s/t)^{1-\delta_2} \le \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} \varepsilon^{2\delta_2} (t-s)^{1-\delta_2} \le \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} \varepsilon^{2\delta_2} T^{\delta_3} (t-s)^{\delta_1}, \end{aligned}$$

which completes the proof of the proposition. □

The following result is essentially a by-product of the previous proof, but deserves a separate statement, for later use.

Lemma 5.8. *Consider $\alpha, \beta \in \mathbb{R}$ and $p \in [1, \infty]$ with $\gamma := \alpha - \beta \in [0, 2]$. Then uniformly over φ:*

$$\|(e^{t\mathcal{A}_\varepsilon} - \mathrm{Id})\varphi\|_{\mathcal{C}^\beta_p} \lesssim t^{\frac{\gamma}{2}} \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

Proof. The proof follows from Proposition 5.7. Indeed, Equation (30) implies that for $j \lesssim \varepsilon^{-1}$ one has:

$$2^{j\beta} \|(e^{t\mathcal{A}_\varepsilon} - \mathrm{Id})\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} 2^{i\beta} \|\Delta_j \varphi\|_{\mathcal{C}^\gamma_p} \lesssim t^{\frac{\gamma}{2}} \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

While a slight modification (to L^p spaces) of (31) guarantees that for $j \gtrsim \varepsilon^{-1}$:

$$2^{j\beta} \|(e^{t\mathcal{A}_\varepsilon} - \mathrm{Id})\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} 2^{j\beta} \varepsilon^{-\gamma} \|\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} 2^{j\alpha} \|\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

This concludes the proof. □

6. Semidiscrete Parabolic Anderson Model

This section is devoted to the proof of Theorem 4. This theorem is an approximation result for the continuous Anderson Hamiltonian in dimension $d = 1$ and $d = 2$. Before we proceed, let us collect some basic ideas of the proof that will follow. First, recall that (see Appendix B) given two distributions $\varphi \in \mathcal{C}^\alpha, \psi \in \mathcal{C}^\beta$, their product can be decomposed as $\varphi\psi = \varphi \varolessthan \psi + \varphi \odot \psi + \psi \varolessthan \varphi$, where the central term $\varphi \odot \psi$, called resonant product, is well-defined if $\alpha + \beta > 0$ (while both other terms are always defined).

The proof of the theorem concentrates on the two-dimensional case, since here the resolvent equation $(-\nu_0 \Delta + \lambda)u = (u - \infty 1_{\{d=2\}})\xi$ is a *singular* stochastic PDE, in the sense that the expected regularity of u, namely $\mathcal{C}^{1-\kappa}$, for any $\kappa > 0$, is not sufficient to define the product with the distribution ξ, of regularity $\mathcal{C}^{-1-\kappa}$ (in $d = 1$ the product is still well-defined and the discussion that follows is not required). In the construction of the Hamiltonian in $d = 2$ we follow the results in [2] that rely on paracontrolled calculus. In a nutshell (we refer the reader to [26] and [28] for a more in-depth discussion), this approach follows the Ansatz that the solution u to the previous equation is of the form $u = u' \varolessthan X_\lambda + u^\sharp$, the previous being a paraproduct (see Appendix B.2) with X_λ solving $(-\nu_0 \Delta + \lambda)X_\lambda = \xi$, and $u^\sharp \in \mathcal{C}^{1+\kappa}$ (we will call a u of this form *paracontrolled*). This should be interpreted as a "Taylor expansion" in terms of functionals of the noise, and the reason why the rest term is expected to be of better regularity is encoded in the concept of subcriticality, introduced in [29]. Now, for paracontrolled u the previously ill-defined product can be rewritten as $u\xi = (u' \varolessthan X_\lambda)\xi + u^\sharp \xi$. While the last term is now well-defined, a commutator estimate (see Lemma B.10) guarantees that the resonant product can be approximated as $(u' \varolessthan X_\lambda) \odot \xi \simeq u'(X_\lambda \odot \xi)$. The latter resonant product $X_\lambda \odot \xi$ remains still ill-defined in terms of regularity, but one can make sense of it through some Gaussian computations (since X_λ is also Gaussian), up to renormalization. By this we mean that the product lives in two levels of the Wiener chaos. While the second chaos part turns out to be well-defined, the zeroth chaos is actually diverging. Eventually, one can rigorously define a distribution $X_\lambda \diamond \xi$ that formally can be written as $X_\lambda \odot \xi - \infty = X_\lambda \odot \xi - \mathbb{E}\big[X_\lambda \odot \xi\big]$, which lives in the second Wiener chaos and explains the ∞ appearing in the equation.

In the cartoon we have just sketched, we hope to explain that the theories for singular stochastic PDEs have two critical ingredients. First, some stochastic computations guarantee the existence of certain products of random distributions. Second, given a realization of these distributions, an purely analytic argument, based on regularity estimates and a Taylor-like expansion guarantee the existence of a solution to the PDE.

In the present setting we concentrate on semidiscrete approximations of the Anderson Hamiltonian, that is we will prove that u as a above is the limit $u = \lim_{\varepsilon\to 0} u_\varepsilon$, with $(-\mathcal{A}_\varepsilon + \lambda)u_\varepsilon = (\xi_\varepsilon - c_\varepsilon 1_{\{d=2\}})u_\varepsilon$. Following the previous explanation we will first prove some stochastic estimates and then pass to the main analytic result. The next definition introduces the space in which we will control the stochastic terms.

Definition 6.1. *Let $d = 2$ and fix any $\kappa > 0$. For any $\varepsilon \in (0, 1/2)$ we will call an enhanced noise a vector of distributions*

$$\boldsymbol{\xi}_\varepsilon = (\xi_\varepsilon, X_{\varepsilon,\lambda}, Y_{\varepsilon,\lambda}),$$

for which the following norm is finite:

$$\begin{aligned}|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa} := \sup_{\zeta\in[0,1]} \varepsilon^{\zeta}\Big\{ \|\xi_\varepsilon\|_{\mathcal{C}^{-(1-\zeta)-\frac{\kappa}{2}}} + \|\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-(1-\zeta)+2-\frac{\kappa}{2}}} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-(1-\zeta)-\frac{\kappa}{2}}} \Big\} \\ + \varepsilon\|\xi_\varepsilon\|_{L^\infty} + \varepsilon^{-1}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{L^\infty} + \|Y_{\varepsilon,\lambda}\|_{\mathcal{C}^{-\frac{\kappa}{2}}}.\end{aligned}$$

Proposition 6.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a sequence of random functions $\xi_\varepsilon \colon \mathbb{T}^d \to \mathbb{R}$ as in Assumption 1.5. Furthermore, in dimension $d = 2$, for $\lambda > 1$, define*

$$X_{\varepsilon,\lambda} = (-\mathcal{A}_\varepsilon + \lambda)^{-1}\xi_\varepsilon, \qquad \xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda} = \xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda} - c_\varepsilon,$$

where

$$c_\varepsilon = \sum_{k\in\mathbb{Z}^2} \frac{\widehat{\chi}^2(\varepsilon k)\widehat{\chi}_Q(\varepsilon k)}{-\vartheta_\varepsilon(k) + \lambda}, \quad \textit{with} \quad c_\varepsilon \simeq \log\frac{1}{\varepsilon}.$$

For any $\kappa > 0$ one can bound in dimension $d = 1$:

$$\sup_{\varepsilon\in(0,1/2)} \mathbb{E}\Big[\sup_{\zeta\in[0,1]} \varepsilon^{\frac{\zeta}{2}}\|\xi_\varepsilon\|_{\mathcal{C}^{-\frac{1}{2}(1-\zeta)-\frac{\kappa}{2}}} + \varepsilon\|\xi_\varepsilon\|_{L^\infty}\Big] < \infty.$$

And in dimension $d = 2$, again for any $\kappa > 0$, with $\boldsymbol{\xi}_\varepsilon = (\xi_\varepsilon, X_{\varepsilon,\lambda}, \xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda})$:

$$\sup_{\varepsilon\in(0,1/2)} \mathbb{E}\big[|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa}\big] < \infty.$$

Moreover, there exists a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, supporting space white noise ξ on $\mathbb{T}^d$, and a sequence of random functions $\overline{\xi}_\varepsilon \colon \mathbb{T}^d \to \mathbb{R}$ such that $\xi_\varepsilon = \overline{\xi}_\varepsilon$ in distribution and such that for almost all $\omega \in \overline{\Omega}$:

$$\overline{\xi}_\varepsilon(\omega) \to \xi(\omega) \quad \textit{in} \quad \mathcal{C}^{-\frac{d}{2}-\kappa}.$$

In addition, in dimension $d = 2$, there exists also a random distribution $\overline{\xi} \diamond X_\lambda$ such that:

$$\mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon + \lambda)^{-1}\overline{\xi}_\varepsilon(\omega) \to (-\Delta + \lambda)^{-1}\xi(\omega) \quad \textit{in} \quad \mathcal{C}^{-\frac{d}{2}+2-\kappa}, \quad \xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda} \to \overline{\xi} \diamond X_\lambda(\omega) \quad \textit{in} \quad \mathcal{C}^{-\kappa}.$$

The proof of this result is rather technical, and for the sake of readability deferred to the end of this section. Having fixed the correct probability space, we are now in position to prove Proposition 1.19. We will work under the following convention.

Assumption 6.3. *Up to changing the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we assume that for all $\kappa > 0$ and almost all $\omega \in \Omega$ the convergences in Proposition 6.2 hold true. If $d = 2$ we additionally assume that:*

$$\sup_{\varepsilon \in (0,1/2)} |||\boldsymbol{\xi}_\varepsilon(\omega)|||_{\varepsilon,\kappa} < \infty.$$

Proof of Proposition 1.19. The Hamiltonian $\mathcal{H}(\omega)$ has been constructed in dimension $d = 1$ in [25] (albeit with Dirichlet boundary conditions, but the construction for periodic boundary conditions is identical) and in dimension $d = 2$ in [2], for almost all $\omega \in \Omega$ (with the latter space satisfying the convention above). In both cases $\mathcal{H}(\omega)$ is an unbounded, selfadjoint operator on L^2, that is:

$$\mathcal{H}(\omega) \colon \mathcal{D}(\mathcal{H}(\omega)) \subset L^2 \to L^2.$$

In particular, in $d = 2$ [2, Proposition 4.13] implies that the operator $\mathcal{H}(\omega)$ admits compact resolvents (cf. [25, Section 2] for the analogous discussion in $d = 1$). Hence the spectrum of $\mathcal{H}(\omega)$ is discrete and the eigenvalues converge to $-\infty$. By a classical result, see [44, Theorem 3.3], the semigroup generated by $\mathcal{H}(\omega)$, denoted by $e^{t\mathcal{H}(\omega)}$, is compact. Moreover, as a consequence of strong maximum principle (in $d = 2$ such result for singular stochastic PDEs is proven in [9, Theorem 5.1 and Remark 5.2], the semigroup $e^{t\mathcal{H}(\omega)}$ is strictly positive: that is, for any non-zero function f such that $\forall x \in \mathbb{T}^d e^{t\mathcal{H}(\omega)} f(x) > 0$. Therefore since $e^{t\mathcal{H}(\omega)}$ is a compact, strictly positive operator, the Krein-Rutman Theorem implies that the largest eigenvalue of $\mathcal{H}(\omega)$ has multiplicity one and the associated eigenfunction is strictly positive.

To conclude the analysis of spectral properties of $\mathcal{H}(\omega)$ it remains to show the regularity of the eigenfunctions, and their density. For the sake of clarity, this part of the proof can be found in Lemma 6.4 below. □

Lemma 6.4. *Under Assumption 6.3, fix $\omega \in \Omega$. Consider the Anderson Hamiltonian $\mathcal{H}(\omega)$ as in the previous Proposition. Define the domain:*

$$\mathcal{D}_\omega = \{\textit{Finite linear combinations of } \{e_k(\omega)\}_{k \in \mathbb{N}}\}.$$

Such domain is dense in $C(\mathbb{T}^d)$ and $\mathcal{D}_\omega \subseteq \mathcal{C}^{2-\frac{d}{2}-\kappa}$ for any $\kappa > 0$. Moreover, for $\varphi \in C^\infty$ and $\zeta < 1$ there exists a sequence $\varphi^k \in \mathcal{D}_\omega$ with $\lim_{k\to\infty} \varphi^k = \varphi$ in $\mathcal{C}^\zeta$.

Proof. Since $\omega \in \Omega$ is fixed, we avoid writing it to lighten the notation. As the statement regarding the approximation of φ in $\mathcal{C}^\zeta$ implies density in $C(\mathbb{T}^d)$ we also restrict to just proving the latter. First, we require some better understanding of the parabolic Anderson semigroup. Here we make use of some known regularization results.

Step 1. Consider the operator $\mathcal{H}$ as in Proposition 1.19: Taking an exponential one can construct the semigroup:

$$e^{t\mathcal{H}} \colon L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d).$$

This semigroup inherits some of the regularizing properties of the heat semigroup, namely, for $T > 0$ and $p \in [1, \infty]$ it can be extended so that:

$$\sup_{0<t\le T} t^\gamma \|e^{t\mathcal{H}}\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\beta_p}, \tag{32}$$

for α and β satisfying:

$$\gamma > \frac{\alpha - \beta}{2}, \qquad \beta + 2 > \frac{d}{2}, \qquad \alpha < 2 - \frac{d}{2}, \qquad \alpha > \beta.$$

The first constraint is essentially identical to the one appearing in Schauder estimate (cf. Proposition 5.7), the second one guarantees that the product $e^{t\Delta}\varphi \cdot \xi$ is a well-defined product of distributions, while the third constraint is due to the fact that $\int_0^t e^{(t-s)\Delta}\xi \, \mathrm{d}s$ has always worse regularity than $2 - \frac{d}{2}$. We will not prove these results: instead we refer to [45, Proposition 3.1] and the reference therein (the cited proposition is set on the entire space, with the added complication of weights at infinity. Such case contains the current setting by extending the noise periodically). The same results guarantee that in the case $\beta > 2 - \frac{d}{2}$ and for $\zeta < 2 - \frac{d}{2}$ one has:

$$\sup_{0 \le t \le T} \| e^{t\mathcal{H}} \varphi \|_{\mathcal{C}^{\zeta}_p} \lesssim \| \varphi \|_{\mathcal{C}^{\beta}_p}. \tag{33}$$

Step 2. Applying iterativrly Equation (32) and Besov embeddings implies that $e_k \in \mathcal{C}^{2-\frac{d}{2}-\kappa}$ for any $\kappa > 0$. Hence the embedding $\mathcal{D}_\omega \subseteq \mathcal{C}^{2-\frac{d}{2}-\kappa}$ is established. Now we prove the statement regarding the approximability of φ. For any $\varphi \in C^\infty$ and $\zeta = 1 - \kappa < 1$ (for some $\kappa > 0$) one has:

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t e^{s\mathcal{H}} \varphi \, \mathrm{d}s = \varphi \quad \text{in} \quad \mathcal{C}^\zeta.$$

This can be seen as follows: Equation (33) implies that

$$\sup_{0 \le t \le T} \left\| \frac{1}{t} \int_0^t e^{s\mathcal{H}} \varphi \, \mathrm{d}s \right\|_{\mathcal{C}^{\zeta'}} < \infty,$$

for $\zeta < \zeta' < 2 - \frac{d}{2}$. This guarantees compactness in $\mathcal{C}^\zeta$. Projecting on the eigenfunctions e_k one sees that any limit point is necessarily φ. Hence fix any $\varepsilon > 0$ and choose $t(\varepsilon)$ such that

$$\left\| \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} e^{s\mathcal{H}} \varphi \, \mathrm{d}s - \varphi \right\|_{\mathcal{C}^\zeta} < \frac{\varepsilon}{2}.$$

Define $\Pi_{\le N} \varphi = \sum_{k=0}^N \langle \varphi, e_k \rangle e_k$. Since the projection commutes with the operator, the proof is complete if we can show that there exists an $N(\varepsilon)$ such that:

$$\left\| \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} e^{s\mathcal{H}} (\Pi_{\le N(\varepsilon)} \varphi - \varphi) \, \mathrm{d}s \right\|_{\mathcal{C}^\zeta} \le \frac{\varepsilon}{2}.$$

Here we use (32) to bound for general $\psi \in L^2$:

$$\begin{aligned} \left\| \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} e^{s\mathcal{H}} \psi \, \mathrm{d}s \right\|_{\mathcal{C}^\zeta} &\lesssim \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} \left(\frac{s}{2} \right)^{-\left(\frac{1}{2} - \frac{\kappa}{4}\right)} \| e^{\frac{s}{2}\mathcal{H}} \psi \|_{\mathcal{C}^{-\frac{\kappa}{2}}} \, \mathrm{d}s \\ &\lesssim \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} \left(\frac{s}{2} \right)^{-\left(\frac{1}{2} - \frac{\kappa}{4}\right)} \| e^{\frac{s}{2}\mathcal{H}} \psi \|_{\mathcal{C}_2^{\frac{d}{2} - \frac{\kappa}{2}}} \, \mathrm{d}s \\ &\lesssim \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} s^{-1 + \frac{\kappa}{4} + \frac{\kappa}{8}} \, \mathrm{d}s \| \psi \|_{L^2} \lesssim t(\varepsilon)^{-1 + \frac{3\kappa}{8}} \| \psi \|_{L^2}, \end{aligned}$$

where we additionally applied Besov embeddings. Choosing $N(\varepsilon)$ such that $\| \Pi_{\le N} \varphi - \varphi \|_{L^2} \lesssim t(\varepsilon)^{1 - \frac{3\kappa}{8}} \frac{\varepsilon}{2}$, the proof is complete. □

Now we pass to the main result of this section.

Proof of Theorem 4. As in the previous proof, we fix $\omega \in \Omega$, the latter satisfying Assumption 6.3, but to lighten the notation we avoid writing explicitly the dependence on ω in what follows. We restrict our attention to dimension $d = 2$. In dimension $d = 1$ the proof is similar, but simpler. For $\lambda \in \mathbb{R}$ define

$$\mathcal{H}_{\varepsilon,\lambda} \colon L^2 \to L^2, \qquad \mathcal{H}_{\varepsilon,\lambda} \psi = (\mathcal{A}_\varepsilon + (\xi_\varepsilon - c_\varepsilon) \Pi_\varepsilon^2 - \lambda) \psi.$$

Let us assume that there exists a $\bar{\lambda} > 0$ such that for all $\lambda \ge \bar{\lambda}$ and $\varepsilon \in (0, 1/2)$ the operator $\mathcal{H}_{\varepsilon,\lambda}$ is invertible and

$$\lim_{\varepsilon \to 0} \| \mathcal{H}_{\varepsilon,\lambda}^{-1} - (\mathcal{H} - \lambda)^{-1} \|_{B(L^2, L^2)} = 0 \tag{34}$$

where $B(X, Y)$ is the space of bounded linear operators between two Banach spaces X, Y with the standard operator norm. By the continuity of the spectrum, see [33, Chapter 4, Theorem

3.16], and (34), it follows that for any $k \in \mathbb{N}$ there exists a $\varepsilon_0(k)$ and eigenvalues and associated an associated eigenfunction $(\lambda_k^\varepsilon, e_k^\varepsilon) \in \mathbb{R} \times L^2$ such that

$$\lim_{\varepsilon \to 0} \lambda_k^\varepsilon = \lambda_k, \qquad \lim_{\varepsilon \to 0} e_k^\varepsilon = e_k \quad \text{in} \quad L^2.$$

Hence, to conclude the proof we have to prove (34) as well as the convergence of the eigenfunctions in $\mathcal{C}^{1-\kappa}$ (the previous argument guarantees convergence only in L^2). The strategy of the proof is a perturbation of the proof in [2] and is based on a fixed point argument. In Step 1 we solve the resolvent equation through such fixed point argument, uniformly over ε and λ large enough. The precise estimates for this fixed point are discussed in Step 2 and the convergence to the continuous Anderson Hamiltonian is established in Step 3. In the fourth step we show that $\Pi_\varepsilon e_k^\varepsilon \to e_k \in \mathcal{C}^{1-\kappa}$. Throughout the proof the parameter $\kappa > 0$ will be chosen small enough, so that all computations hold.

Step 1. Fix $p \in [1, \infty]$ as well as $\varphi \in \mathcal{C}_p^{-1+2\kappa}$. In dimension $d = 1$, solving the resolvent equation $\mathcal{H}_{\varepsilon,\lambda}\psi = \varphi$ is equivalent to solving the fixed point problem

$$\psi = M_{\varphi,\lambda}(\psi) := (-\mathcal{A}_\varepsilon + \lambda)^{-1}[(\xi_\varepsilon - c_\varepsilon)\Pi_\varepsilon^2 \psi - \varphi]. \tag{35}$$

In dimension $d = 2$ we will not prove directly that $M_{\varphi,\lambda}$ is a contraction (while in $d = 1$ this is the case: the arguments that follow are not required and Proposition 5.6 allows to find a fixed point $\psi \in \mathcal{C}_p^{\frac{3}{2}-\kappa}$). To find the fixed point we look for a *paracontrolled* solution. Consider a space $\mathcal{D}_\varepsilon \subseteq L^p(\mathbb{T}^d) \times L^p(\mathbb{T}^d)$ which, for a pair $(\psi', \psi^\sharp)$ is characterized by the norm

$$\|(\psi', \psi^\sharp)\|_{\mathcal{D}_\varepsilon} := \|\psi'\|_{\mathcal{C}_p^{1-\kappa}} + \|\mathcal{P}_\varepsilon \psi^\sharp\|_{\mathcal{C}_p^{1+\kappa}} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon \psi^\sharp\|_{\mathcal{C}_p^{-1+2\kappa}}.$$

A function ψ is associated to a pair $(\psi', \psi^\sharp)$ by

$$\psi = \psi' \varolessthan [(-\mathcal{A}_\varepsilon + \lambda)^{-1}\xi_\varepsilon] + \psi^\sharp.$$

With an abuse of notation, we identify the pair $(\psi', \psi^\sharp)$ with the function ψ and write $\|\psi\|_{\mathcal{D}_\varepsilon} = \|(\psi', \psi^\sharp)\|_{\mathcal{D}_\varepsilon}$. Define a map (note the presence of ψ') $\overline{M}_{\varphi,\lambda} \colon \mathcal{D}_\varepsilon \to L^p$ as

$$\overline{M}_{\varphi,\lambda}(\psi) := (-\mathcal{A}_\varepsilon + \lambda)^{-1}[\xi_\varepsilon \Pi_\varepsilon^2 \psi - c_\varepsilon \psi' - \varphi].$$

The map $\overline{M}_{\varphi,\lambda}$ can be extended to a map from $\mathcal{D}_\varepsilon$ into itself by

$$\mathcal{M}_{\varphi,\lambda}(\psi) = (M'_{\varphi,\lambda}(\psi),\ M^\sharp_{\varphi,\lambda}(\psi)) := (\Pi_\varepsilon^2\psi,\ \overline{M}_{\varphi,\lambda}(\psi) - (\Pi_\varepsilon^2\psi) \varolessthan [(-\mathcal{A}_\varepsilon + \lambda)^{-1}\xi_\varepsilon]) \in \mathcal{D}_\varepsilon,$$

The fixed point of $\mathcal{M}_\varphi$ solves (35) as well, since the fixed point satisfies

$$\psi' = \Pi_\varepsilon^2 \psi.$$

Step 2. In the course of the proof we repeatedly make use of the elliptic Schauder estimates of Proposition 5.6 and the paraproduct estimates of Lemma B.8, without stating them explicitly every time. The aim is to control

$$\|\mathcal{M}_{\varphi,\lambda}(\psi)\|_{\mathcal{D}_\varepsilon} = \|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}} + \|\mathcal{P}_\varepsilon\ M^\sharp_{\varphi,\lambda}(\psi)\|_{\mathcal{C}_p^{1+\kappa}} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon\ M^\sharp_{\varphi,\lambda}(\psi)\|_{\mathcal{C}_p^{-1+2\kappa}}.$$

Via Lemma B.6

$$\begin{aligned} \|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}} &\lesssim \|\Pi_\varepsilon^2\big[\psi' \varolessthan X_{\varepsilon,\lambda}\big]\|_{\mathcal{C}_p^{1-\kappa}} + \|\mathcal{P}_\varepsilon\psi^\sharp\|_{\mathcal{C}_p^{1+\kappa}} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon\psi^\sharp\|_{\mathcal{C}_p^{-1+2\kappa}} \\ &\lesssim \lambda^{-\frac{\kappa}{4}}\|\psi'\|_{\mathcal{C}_p^{1-\kappa}}(\|\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{1-\frac{\kappa}{2}}} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}) \\ &\quad + \|\mathcal{P}_\varepsilon\psi^\sharp\|_{\mathcal{C}_p^{1+\kappa}} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon\psi^\sharp\|_{\mathcal{C}_p^{-1+2\kappa}}. \end{aligned} \tag{36}$$

To tackle the norms $M^\sharp$, first rewrite as

$$\begin{aligned} M^\sharp_\varphi(\psi) = (-\mathcal{A}_\varepsilon + \lambda)^{-1}\Big\{ -\varphi + [\xi_\varepsilon \odot \Pi_\varepsilon^2\psi^\sharp] + \big\{\xi_\varepsilon \odot [\Pi_\varepsilon^2(\psi' \varolessthan X_{\varepsilon,\lambda})] - c_\varepsilon\psi'\big\} \\ + \xi_\varepsilon \varogreaterthan \Pi_\varepsilon^2\psi + C_{\varepsilon,\lambda}(\Pi_\varepsilon^2\psi, \xi_\varepsilon)\Big\}, \end{aligned}$$

where $C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon)$ is the commutator

$$C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon)=(-\mathcal{A}_\varepsilon+\lambda)^{-1}[(\Pi^2_\varepsilon\psi)\varolessthan\xi_\varepsilon]-[(\Pi^2_\varepsilon\psi)\varolessthan(-\mathcal{A}_\varepsilon+\lambda)^{-1}(\xi_\varepsilon)].$$

Combining the Schauder estimates with the smoothing properties of Π_ε and the paraproduct estimates one finds that

$$\begin{aligned}\lambda^{\frac{\kappa}{2}}\big(\|\mathcal{P}_\varepsilon M^\sharp_\varphi(\psi)\|_{\mathcal{C}^{1+\kappa}_p}+\varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon M^\sharp_\varphi(\psi)\|_{\mathcal{C}^{-1+2\kappa}_p}\big)&\lesssim\|\varphi\|_{\mathcal{C}^{-1+2\kappa}_p}+\|\Pi^2_\varepsilon\psi^\sharp\|_{\mathcal{C}^{1+\kappa}_p}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}\\&\quad+\|\xi_\varepsilon\odot[\Pi^2_\varepsilon(\psi'\varolessthan X_{\varepsilon,\lambda})]-c_\varepsilon\psi'\|_{\mathcal{C}^{-1+2\kappa}_p}+\|C_{\varepsilon,\lambda}(\psi,\xi_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}.\end{aligned}$$

To treat $\|\xi_\varepsilon\odot[\Pi^2_\varepsilon(\psi'\varolessthan X_{\varepsilon,\lambda})-c_\varepsilon\psi']\|_{\mathcal{C}^{-1+2\kappa}_p}$, we introduce (cf. B.9) the commutators

$$C^\Pi_\varepsilon(f,g)=\Pi^2_\varepsilon(f\varolessthan g)-f\varolessthan\Pi^2_\varepsilon g,\qquad C^\odot(f,g,h)=f\odot(g\varolessthan h)-g(f\odot h).$$

By Lemma B.11

$$\begin{aligned}\|\xi_\varepsilon\odot C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{-1+2\kappa}_p}&\le\|\xi_\varepsilon\odot C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^\kappa_p}\\&\lesssim\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}\|\mathcal{P}_\varepsilon C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{1+\kappa}_p}+\|\xi_\varepsilon\|_{\mathcal{C}^{-1+\kappa}}\|\mathcal{Q}_\varepsilon C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{1-\frac{\kappa}{2}}_p}\\&\lesssim\|\psi'\|_{\mathcal{C}^{1-\kappa}_p}|||\boldsymbol{\xi}_\varepsilon|||^2_{\varepsilon,\kappa}.\end{aligned}$$

We deduce that

$$\begin{aligned}\|\xi_\varepsilon\odot[\psi'\varolessthan(\Pi^2_\varepsilon X_{\varepsilon,\lambda})]-c_\varepsilon\psi'\|_{\mathcal{C}^{-1+2\kappa}_p}\lesssim&\|C^\odot(\xi_\varepsilon,\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{-1+2\kappa}_p}\\&+\|\psi'(\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}-c_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}.\end{aligned}$$

By Lemma B.10

$$\begin{aligned}\|C^\odot(\xi_\varepsilon,\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{-1+2\kappa}_p}&\le\|C^\odot(\xi_\varepsilon,\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{2\kappa}_p}\\&\lesssim\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}\|\psi'\|_{\mathcal{C}^{1-\kappa}_p}\|\Pi^2_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{1-\frac{\kappa}{2}}}\lesssim\|\psi'\|_{C^{1-\kappa}_p}|||\boldsymbol{\xi}_\varepsilon|||^2_{\varepsilon,\kappa}.\end{aligned}$$

Similarly

$$\|\psi'(\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}-c_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}\lesssim\|\psi'\|_{\mathcal{C}^{1-\kappa}_p}\|\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}-c_\varepsilon\|_{\mathcal{C}^{-1+2\kappa}}\le\|\psi'\|_{\mathcal{C}^{1-\kappa}_p}|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa}.$$

The estimate for $C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon)$ follows from Lemma B.12, by noticing that

$$\begin{aligned}\|C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}&\le\|\mathcal{P}_\varepsilon C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}+\|\mathcal{Q}_\varepsilon C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}\\&\lesssim\|\Pi^2_\varepsilon\psi\|_{\mathcal{C}^{1-\kappa}_p}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}+\varepsilon^2\|\Pi^2_\varepsilon\psi\|_{\mathcal{C}^{1-\kappa}_p}\|\xi_\varepsilon\|_{\mathcal{C}^{-1+2\kappa}}\lesssim\|\psi\|_{\mathcal{D}_\varepsilon}|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa}.\end{aligned}$$

Step 3. Estimates from Step 2 combined with linearity guarantee that that for $n\in\mathbb{N},n\ge2$ there exists a $C>0$ such that

$$\begin{aligned}\|\mathcal{M}_\varphi(\psi)\|_{\mathcal{D}_\varepsilon}&\le C\Big[\|\varphi\|_{B^{-1+2\kappa}_{p,q}}+\|\psi\|_{\mathcal{D}_\varepsilon}(1+|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa})^2\Big]\\\|\big[\mathcal{M}_\varphi(\psi)-\mathcal{M}_\varphi(\tilde\psi)\big]^n\|_{\mathcal{D}_\varepsilon}&\le C\Big[\lambda^{-\frac{\kappa}{4}}\|\psi-\tilde\psi\|_{\mathcal{D}_\varepsilon}(1+|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa})^2\Big].\end{aligned}$$

Note that we require $n\ge2$, since in (36) we do not have a small factor in front of the rest term with $\psi^\sharp$. In particular, there exists a $\bar\lambda(\sup_\varepsilon|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa})$ such that for $\lambda>\bar\lambda$ the map $\mathcal{M}_\varphi$ admits a unique fixed point, which we denote by $\mathcal{H}^{-1}_{\varepsilon,\lambda}\varphi$. Moreover, by the Banach fixed point theorem

$$\|\mathcal{H}^{-1}_{\varepsilon,\lambda}\varphi\|_{\mathcal{D}_\varepsilon}\lesssim\|\mathcal{M}^2_\varphi(0)\|_{\mathcal{D}_\varepsilon}\lesssim\|\varphi\|_{\mathcal{C}^{-1+2\kappa}_p}(1+|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa})^2,$$

implying that $\mathcal{H}^{-1}_{\varepsilon,\lambda}\in B(\mathcal{C}^{-1+2\kappa}_p,\mathcal{D}_\varepsilon)$, with the norm bounded uniformly in ε. Similar, but less involved calculations lead to a construction of the resolvent $\mathcal{H}^{-1}_\lambda=(\mathcal{H}-\lambda)^{-1}$ in the continuum for $\lambda\ge\bar\lambda$. This is a bounded operator $\mathcal{H}^{-1}_\lambda\in B(\mathcal{C}^{-1+2\kappa}_p,\mathcal{D}_0)$, where the latter is the Banach space defined by the norm (for $\psi=\psi'\varolessthan(-\Delta+\lambda)^{-1}\xi+\psi^\sharp$):

$$\|\psi\|_{\mathcal{D}_0}=\|\psi'\|_{\mathcal{C}^{1-\kappa}_p}+\|\psi^\sharp\|_{\mathcal{C}^{1+\kappa}_p}.$$

By linearity and computations on the line of to those in Step 2:

$$\lim_{\varepsilon\to 0}\sup_{\|\varphi\|_{\mathcal{C}_p^{-1+2\kappa}}\leq 1}\left\|(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)'-(\mathcal{H}_{\lambda}^{-1}\varphi)'\right\|_{\mathcal{C}_p^{1-\kappa}}+\left\|\mathcal{P}_{\varepsilon}(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^{\sharp}-(\mathcal{H}_{\lambda}^{-1}\varphi)^{\sharp}\right\|_{\mathcal{C}_p^{1+\kappa}}=0. \tag{37}$$

To show (34) it would be sufficient to show (in the particular case $p=2$) that $\mathcal{D}_\varepsilon\hookrightarrow L^p$, in the sense that $\|\psi\|_{L^p}\lesssim\|\psi\|_{\mathcal{D}_\varepsilon}$. Unfortunately, this is not the case, because $\mathcal{Q}_\varepsilon\psi^\sharp\in\mathcal{C}_p^{-1+2\kappa}$. So we need a better control on the regularity of $\psi^\sharp$, which we will obtain by using that $\varphi\in L^p$, namely:

$$\lambda^{\frac{\kappa}{2}}\varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon M_\varphi^\sharp(\psi)\|_{L^p}\lesssim\varepsilon^{3\kappa}\Big\{\|\xi_\varepsilon\Pi_\varepsilon^2\psi-c_\varepsilon\psi'-\varphi\|_{L^p}+\|(\Pi_\varepsilon^2\psi)\otimes X_{\varepsilon,\lambda}\|_{L^p}\Big\}$$

and since $c_\varepsilon\lesssim\log\frac{1}{\varepsilon}$ (see Proposition 6.2)

$$\|\xi_\varepsilon\Pi_\varepsilon^2\psi-c_\varepsilon\psi'-\varphi\|_{L^p}\lesssim\|\xi_\varepsilon\|_{\mathcal{C}^{-1+2\kappa}}\|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}}+\varepsilon^{-\kappa}\|\psi'\|_{L^p}+\|\varphi\|_{L^p}.$$

Similarly, since

$$\|f\otimes g\|_{L^p}\leq\|fg\|_{L^p}+\|g\otimes f\|_{L^p}+\|f\odot g\|_{L^p}\lesssim\|f\|_{\mathcal{C}_p^\kappa}\|g\|_{L^\infty}.$$

one has

$$\|(\Pi_\varepsilon^2\psi)\otimes X_{\varepsilon,\lambda}\|_{L^p}\lesssim\|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}}\|X_{\varepsilon,\lambda}\|_{L^\infty}.$$

Therefore the L^∞ bound on $\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}$ leads to

$$\lambda^{\frac{\kappa}{2}}\varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon M_\varphi^\sharp(\psi)\|_{L^p}\lesssim\|\varphi\|_{L^p}+\|\psi\|_{\mathcal{D}_\varepsilon}(1+|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa}). \tag{38}$$

In particular, the regularity of the resolvent map $\mathcal{H}_{\varepsilon,\lambda}^{-1}$ is enhanced by

$$\varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^\sharp\|_{L^p}\lesssim\|\varphi\|_{L^p}.$$

This leads to the embedding $\mathcal{D}_\varepsilon\hookrightarrow L^p$ which justifies

$$\begin{aligned}\|\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi-\mathcal{H}_{\lambda}^{-1}\varphi\|_{L^p}\leq{}&\|(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)'\otimes X_{\varepsilon,\lambda}-(\mathcal{H}_{\lambda}^{-1}\varphi)'\otimes(-\Delta+\lambda)^{-1}\xi\|_{L^p}\\&+\|\mathcal{P}_\varepsilon(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^\sharp-(\mathcal{H}_{\lambda}^{-1}\varphi)^\sharp\|_{L^p}+\|\mathcal{Q}_\varepsilon(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^\sharp\|_{L^p}.\end{aligned}$$

Letting $p=2$ and sending ε to 0 together with (37) and (38) proves (34).

Step 5. It remains to show that

$$\Pi_\varepsilon e_k^\varepsilon\to e_k\quad\text{in}\quad\mathcal{C}^{1-\kappa}.$$

Since the embedding $\mathcal{C}_p^\alpha\subseteq\mathcal{C}_p^{\alpha'}$ is compact for $\alpha>\alpha'$, and since κ can be chosen in arbitrary way, and we already established the convergence of e_k^ε in L^2 and hence also in the sense of distributions, it is sufficient to show that

$$\sup_\varepsilon\|\Pi_\varepsilon e_k^\varepsilon\|_{\mathcal{C}^{1-\kappa}}<\infty.$$

Due to the normalization we also already have a uniform bound in L^2:

$$\sup_\varepsilon\|\Pi_\varepsilon e_k^\varepsilon\|_{L^2}\leq 1.$$

Now, choose $\lambda>\bar{\lambda}$ (and hence $\lambda>\sup_\varepsilon\lambda_k^\varepsilon$), then one can rewrite:

$$e_k^\varepsilon=(\lambda_k^\varepsilon-\lambda)\mathcal{H}_{\varepsilon,\lambda}^{-1}e_k^\varepsilon.$$

So that

$$e_k^\varepsilon=(e_k^\varepsilon)'\otimes X_{\varepsilon,\lambda}+(e_k^\varepsilon)^\sharp,$$

and by all the bounds in the proof of the previous step:

$$\sup_\varepsilon\Big\{\|e_k^\varepsilon\|_{\mathcal{D}_\varepsilon}+\varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon(e_k^\varepsilon)^\sharp\|_{L^2}\Big\}<\infty,$$

where we use the space $\mathcal{D}_\varepsilon$ in the case $p=2$. Hence, applying Lemma B.6, one obtains:

$$\begin{aligned}\|\Pi_\varepsilon e_k^\varepsilon\|_{\mathcal{C}_2^{1-\kappa}} &\lesssim \|\Pi_\varepsilon\big[(e_k^\varepsilon)' \circledcirc X_{\varepsilon,\lambda}\big]\|_{\mathcal{C}_2^{1-\kappa}} + \|\mathcal{P}_\varepsilon(e_k^\varepsilon)^\sharp\|_{\mathcal{C}_2^{1-\kappa}} + \varepsilon^{-1}\|\mathcal{Q}_\varepsilon(e_k^\varepsilon)^\sharp\|_{L^2} \\ &\lesssim \varepsilon^{-1}\|(e_k^\varepsilon)'\|_{\mathcal{C}_2^{1-\kappa}}\|X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-\frac{\kappa}{2}}} + \|e_k^\varepsilon\|_{\mathcal{D}_\varepsilon} + \varepsilon^{-1}\|\mathcal{Q}_\varepsilon(e_k^\varepsilon)^\sharp\|_{L^2} \lesssim \|e_k^\varepsilon\|_{\mathcal{D}_\varepsilon}(1+|||\boldsymbol{\xi}_\varepsilon|||_{\varepsilon,\kappa}).\end{aligned}$$

Using the bounds on the noise terms, as well as the uniform bound we already established one thus has, by Besov embedding

$$\sup_\varepsilon \|\Pi_\varepsilon e_k^\varepsilon\|_{\mathcal{C}^{\frac{1}{2}-\kappa}} \lesssim \sup_\varepsilon \|\Pi_\varepsilon e_k^\varepsilon\|_{\mathcal{C}_2^{1-\kappa}} < \infty.$$

Iterating the entire procedure again in L^∞ instead of L^2, one obtains the required uniform bound in $\mathcal{C}^{1-\kappa}$. □

Before concluding, we provide the proof of the stochastic bounds we stated at the beginning of the section.

Proof of Proposition 6.2. We will prove in order the bounds for $\xi_\varepsilon, X_{\varepsilon,\lambda}$ and $\xi_\varepsilon \diamond \Pi_\varepsilon^2 X_{\varepsilon,\lambda}$. Eventually we address the convergence. Although only in the first case the dimension is allowed to be both $d=1$ and $d=2$, we will keep d as a parameter throughout the proof, for the sake of clarity.

Step 1. First, observe that by Assumption 2.1:

$$|\xi_\varepsilon(x)| \leq 2\varepsilon^{-\frac{d}{2}}.$$

This explains both the L^∞ bounds on ξ_ε and the bound in $\mathcal{C}^{-\frac{\kappa}{2}}$ (i.e. for $\zeta=1$). If we show that;

$$\sup_{\varepsilon\in(0,1/2)} \mathbb{E}\big[\|\xi_\varepsilon\|_{\mathcal{C}^{-\frac{d}{2}-\frac{\kappa}{2}}}\big] < \infty,$$

the bound for arbitrary ζ follows by interpolation, since by interpolation, from the definition of Besov spaces, for any $\zeta\in[0,1]$ and $\alpha,\beta\in\mathbb{R}$:

$$\|\varphi\|_{\mathcal{C}^{\zeta\alpha+(1-\zeta)\beta}} \leq \|\varphi\|_{\mathcal{C}^\alpha}^{\zeta}\|\varphi\|_{\mathcal{C}^\beta}^{1-\zeta}.$$

Hence let us consider the case $\zeta=0$. By Besov embedding, the required inequality follows if one can show that for any $p\in[2,\infty)$:

$$\sup_{\varepsilon\in(0,1/2)} \mathbb{E}\|\xi_\varepsilon\|^p_{B_{p,p}^{-\frac{d}{2}-\frac{\kappa}{4}}} < \infty.$$

Here in view of Assumption 1.5, and by the discrete Burkholder-Davis-Gundy inequality as well as Jensen's inequality one finds that:

$$\begin{aligned}\int_{\mathbb{T}^d}\mathbb{E}[|\Delta_j\varepsilon^{-\frac{d}{2}}s_\varepsilon|^p(x)]\,\mathrm{d}x &\lesssim \int_{\mathbb{T}^d}\bigg(\sum_{z\in\mathbb{Z}^d_\varepsilon}\varepsilon^d|\Delta_j\chi_{Q_\varepsilon}|^2(z+x)\bigg)^{p/2}\mathrm{d}x \\ &\leq \int_{\mathbb{T}^d}\bigg(\int_{\mathbb{T}^d}\mathrm{d}z\;|K_j(x+z)|^2\bigg)^{p/2}\mathrm{d}x \lesssim \|K_j\|^p_{L^2} \lesssim 2^{j\frac{dp}{2}},\end{aligned}$$

which is a bound of the required order.

As for $X_{\varepsilon,\lambda}$, by the elliptic Schauder estimates of Proposition 5.6:

$$\varepsilon^\zeta\|\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-(1-\zeta)+2-\frac{\kappa}{2}}} + \varepsilon^{\zeta-2}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-(1-\zeta)-\frac{\kappa}{2}}} \lesssim \varepsilon^\zeta\|\xi_\varepsilon\|_{\mathcal{C}^{-(1-\zeta)-\frac{\kappa}{2}}},$$

so that the required bound follows from the previous calculation. In addition, we need to bound $\varepsilon^{-1}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{L^\infty}$. Here:

$$\begin{aligned}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{L^\infty(\mathbb{T}^d)} &= \|\mathcal{F}^{-1}_{\mathbb{T}^d}[(1-\daleth)(\varepsilon\cdot)(-\vartheta_\varepsilon+\lambda)^{-1}(\cdot)\widehat{\xi}_\varepsilon(\cdot)]\|_{L^\infty(\mathbb{T}^d)} \\ &\leq \|\mathcal{F}^{-1}_{\mathbb{T}^d}[(1-\daleth)(\varepsilon\cdot)(-\vartheta_\varepsilon+\lambda)^{-1}(\cdot)]\|_{L^1(\mathbb{T}^d)}\|\xi_\varepsilon\|_{L^\infty(\mathbb{T}^d)} \\ &\lesssim \varepsilon^2\|\mathcal{F}^{-1}_{\mathbb{R}^d}[(1-\daleth)(\varepsilon\cdot)(-\widehat{\chi}^2+1+\varepsilon^2\lambda)^{-1}(\varepsilon\cdot)]\|_{L^1(\mathbb{R}^d)}\|\xi_\varepsilon\|_{L^\infty(\mathbb{T}^d)}\end{aligned} \tag{39}$$

where we applied the Poisson summation formula of Lemma B.1. Notice that

$$
\begin{aligned}
&\|\mathcal{F}^{-1}_{\mathbb{R}^d}[(1-\urcorner)(\varepsilon\cdot)(-\widehat{\chi}^2+1+\varepsilon^2\lambda)^{-1}(\varepsilon\cdot)]\|_{L^1(\mathbb{R}^d)}\\
&\quad\le \left\|\mathcal{F}^{-1}_{\mathbb{R}^d}\left[\frac{1-\urcorner(\varepsilon\cdot)}{1+\varepsilon^2\lambda}+(1-\urcorner)(\varepsilon\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\Big](\varepsilon\cdot)\right]\right\|_{L^1(\mathbb{R}^d)}\\
&\quad\le \left\|\mathcal{F}^{-1}_{\mathbb{R}^d}\left[\frac{1-\urcorner(\varepsilon\cdot)}{1+\varepsilon^2\lambda}\right]\right\|_{L^1(\mathbb{R}^d)}+\left\|\mathcal{F}^{-1}_{\mathbb{R}^d}\left[(1-\urcorner)(\varepsilon\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\Big](\varepsilon\cdot)\right]\right\|_{L^1(\mathbb{R}^d)}
\end{aligned}
$$

The first summand is bounded in $L^1(\mathbb{R}^d)$ uniformly over ε (with some abuse of notation for the Dirac δ function). As for the second observe that, for some $c>0$:

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}_{\mathbb{R}^d}\left[(1-\urcorner)(\varepsilon\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\Big](\varepsilon\cdot)\right]\right\|_{L^1(\mathbb{R}^d)}\\
&\qquad\le \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{\frac{d+1}{2}}\left|\int_{\mathbb{R}^d}e^{2\pi\iota\langle x,k\rangle}(1-\urcorner(k))\left[\frac{1}{-\widehat{\chi}^2(k)+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\right]\mathrm{d}k\right|\\
&\qquad\lesssim \sum_{0\le|\alpha|\le 2d}\int_{\mathbb{R}^d}\left|\left[D^\alpha\Big(\frac{1}{-\widehat{\chi}^2(k)+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\Big)\right]1_{\{|k|\ge c\}}\right|\mathrm{d}k,
\end{aligned}
$$

where with the sum we indicate all partial derivatives up to order $2d$. Now this term can be bounded by Lemma 5.1. Let us show this for $\alpha=0$ (the other cases are similar), where by a Taylor expansion:

$$
\begin{aligned}
\int_{\mathbb{R}^d}\left|\frac{1}{-\widehat{\chi}^2(k)+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\right|1_{\{|k|\ge c\}}\,\mathrm{d}k &\lesssim_c \left(\frac{1}{1+\varepsilon^2\lambda}\right)^2\int_{\mathbb{R}^d}\hat{\chi}^2(k)1_{\{|k|\ge c\}}\,\mathrm{d}k\\
&\lesssim \int_{\mathbb{R}^d}\frac{1}{1+|k|^{d+1}}\,\mathrm{d}k<\infty.
\end{aligned}
$$

Combining the last two observations with (39) leads to

$$\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{L^\infty(\mathbb{T}^d)}\lesssim\varepsilon^2\|\xi_\varepsilon\|_{L^\infty(\mathbb{T}^d)}\lesssim\varepsilon^{2-\frac{d}{2}},$$

which is of the required order.

Step 2. We now consider the bound on $\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}$. In this computation it is important to note that $d=2$. Define $\psi_0(k_1,k_2)$ and $\widehat{\xi}_\varepsilon(k)$ as

$$\psi_0(k_1,k_2):=\sum_{|i-j|\le 1}\varrho_i(k_1)\varrho_j(k_2),\qquad \widehat{\xi}_\varepsilon(k):=\mathcal{F}_{\mathbb{T}^d}\xi_\varepsilon(k).$$

Then

$$
\begin{aligned}
\mathbb{E}\big[\widehat{\xi}_\varepsilon(k_1)\widehat{\xi}_\varepsilon(k_2)\big]&=\int_{(\mathbb{T}^2)^2}e^{-2\pi\iota(k_1\cdot x_1+k_2\cdot x_2)}\chi_{Q_\varepsilon(x_1)}(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&=\int_{\mathbb{T}^2}e^{-2\pi\iota(k_1+k_2)\cdot x_1}\widehat{\chi}_Q(\varepsilon k_2)\,\mathrm{d}x_1=\widehat{\chi}_Q(\varepsilon k_1)1_{\{k_1+k_2=0\}}.
\end{aligned}
$$

Hence to compute the renormalization constant observe that

$$
\begin{aligned}
c_{\varepsilon,\lambda}=\mathbb{E}\big[\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}(x)\big]&=\int_{(\mathbb{Z}^2)^2}e^{2\pi\iota(k_1+k_2)\cdot x}\psi_0(k_1,k_2)\frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda}\mathbb{E}\big[\widehat{\xi}_\varepsilon(k_1)\widehat{\xi}_\varepsilon(k_2)\big]\;\mathrm{d}k_1\,\mathrm{d}k_2\\
&=\int_{\mathbb{Z}^2}\frac{\widehat{\chi}^2(\varepsilon k)\widehat{\chi}_Q(\varepsilon k)}{-\vartheta_\varepsilon(k)+\lambda}\;\mathrm{d}k.
\end{aligned}
$$

A similar calculation shows that actually $c_\varepsilon=\mathbb{E}\big[\xi_\varepsilon X_{\varepsilon,\lambda}\big]$ and the asymptotic $c_\varepsilon\simeq\log(1/\varepsilon)$ follows from a manipulation of the sum. We turn our attention to a bound for $\|\xi_\varepsilon\diamond\Pi^2_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-\frac{\kappa}{2}}}$.

As before, for $p \geq 2$, consider

$$
\begin{aligned}
\mathbb{E}\|\xi_\varepsilon \odot X_{\varepsilon,\lambda} - c_\varepsilon\|^p_{B^\alpha_{p,p}} &= \sum_{j\geq -1} 2^{\alpha j p}\mathbb{E}\|\Delta_j(\xi_\varepsilon \odot X_{\varepsilon,\lambda} - c_\varepsilon 1_{j=-1})\|^p_{L^p(\mathbb{T}^d)} \\
&= \sum_{j\geq -1} 2^{\alpha j p}\int_{\mathbb{T}^d} \mathbb{E}|\Delta_j(\xi_\varepsilon \odot X_{\varepsilon,\lambda} - c_\varepsilon 1_{j=-1})|^p(x)\,\mathrm{d}x.
\end{aligned} \tag{40}
$$

It is now convenient to introduce the notation:

$$
\mathcal{K}^\varepsilon_m(x) = \mathcal{F}^{-1}_{\mathbb{T}^2}\left(\varrho_m(\cdot)\frac{\widehat{\chi}^2(\varepsilon\cdot)}{-\vartheta_\varepsilon(\cdot)+\lambda}\right)(x).
$$

Then the integrand in (40) can be written as

$$
\begin{aligned}
&\mathbb{E}\big[|\Delta_j(\xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda})(x) - c_\varepsilon 1_{\{j=-1\}}|^p\big] \\
&\quad= \mathbb{E}\big[|\Delta_j(\xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda})(x) - \mathbb{E}\Delta_j(\xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda})(x)|^p\big] \\
&\quad= \mathbb{E}\left|\int_{\mathbb{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}\left(\int_{(\mathbb{T}^2)^2} K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\xi_\varepsilon(z_1)\diamond\xi_\varepsilon(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\right)\mathrm{d}y\right|^p,
\end{aligned} \tag{41}
$$

where, conveniently:

$$
\xi_\varepsilon(z_1)\diamond\xi_\varepsilon(z_2) = \xi_\varepsilon(z_1)\xi_\varepsilon(z_2) - \mathbb{E}\big[\xi_\varepsilon(z_1)\xi_\varepsilon(z_2)\big].
$$

Now we can write (41) as a discrete stochastic integral and apply Lemma C.1 to obtain

$$
\begin{aligned}
&\mathbb{E}\left|\int_{\mathbb{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}\sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2}\left(\int_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)} K_l(y-z_1)\mathcal{K}_m(y-z_2)\mathrm{d}z_1\,\mathrm{d}z_2\right)\xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2)\,\mathrm{d}y\right|^p \\
&\lesssim \left[\sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2}\varepsilon^{2d}\left|\int_{\mathbb{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}\left(\fint_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)} K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\mathrm{d}z_1\mathrm{d}z_2\right)\mathrm{d}y\right|^2\right]^{p/2} \\
&= \left[\sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2}\varepsilon^{2d}\left|\fint_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)}\int_{\mathbb{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}K_l(y-z_1)\mathcal{K}_m(y-z_2)\,\mathrm{d}y\,\mathrm{d}z_1\,\mathrm{d}z_2\right|^2\right]^{p/2} \\
&\leq \left[\int_{(\mathbb{T}^2)^2}\left|\int_{\mathbb{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\,\mathrm{d}y\right|^2\mathrm{d}z_1\,\mathrm{d}z_2\right]^{p/2},
\end{aligned}
$$

where the last step is an application of Jensen's inequality. Now, via Parseval's Theorem, the latter is bounded by

$$
\begin{aligned}
&\left[\int_{(\mathbb{Z}^2)^2}\left|\int_{\mathbb{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}e^{2\pi\iota k_1\cdot y}\varrho_l(k_1)e^{2\pi\iota k_2\cdot y}\varrho_m(k_2)\frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda}\,\mathrm{d}y\right|^2\mathrm{d}k_1\,\mathrm{d}k_2\right]^{p/2} \\
&\qquad\qquad= \left[\int_{(\mathbb{Z}^2)^2}\left|e^{2\pi\iota(k_1+k_2)\cdot x}\varrho_j(k_1+k_2)\psi_0(k_1,k_2)\frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda}\right|^2\mathrm{d}k_1\,\mathrm{d}k_2\right]^{p/2}.
\end{aligned}
$$

Now by Lemma 5.1:

$$
\frac{\widehat{\chi}^2(\varepsilon k)}{-\vartheta_\varepsilon(k)+\lambda} \lesssim \frac{\widehat{\chi}^2(\varepsilon k)}{|k|^2+\lambda}1_{\{|k\lesssim\varepsilon^{-1}|\}} + \frac{|k|^{-3}}{\lambda}1_{\{|k|\gtrsim\varepsilon^{-1}\}} \lesssim \frac{1}{\lambda+|k|^2}.
$$

Finally, taking into account the supports of the functions,

$$
\left[\int_{(\mathbb{Z}^2)^2}\left|\varrho_j(k_1+k_2)\psi_0(k_1,k_2)\frac{1}{1+|k_2|^2}\right|^2\mathrm{d}k_1\,\mathrm{d}k_2\right]^{p/2} \lesssim \left[2^{j2d}2^{-4j}\right]^{p/2} \leq 1,
$$

which provides a bound of the required order. With this we have concluded the proof of the regularity bound. We are left with a discussion of the convergence.

Step 3. What we established so far implies tightness of the following sequences of random variables in their respective spaces:

$$\xi_\varepsilon \in \mathcal{C}^{-\frac{d}{2}-\kappa}, \qquad \mathcal{P}_\varepsilon X_{\varepsilon,\lambda} \in \mathcal{C}^{1-\kappa}, \qquad \xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda} \in \mathcal{C}^{-\kappa}.$$

The next step is to show that the limiting points of ξ_ε and $\xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda}$ are unique in distribution. In particular, in view of Proposition 5.6, this would imply weak convergence also of $\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}$. Once we have proven weak convergence the required result concerning the almost sure convergence follows by Skorohod representation.

Convergence of ξ_ε to space time white noise ξ is an instance of central limit theorem (notice the normalization of variance in Assumption 1.5). We therefore focus our attention on the more involved Wick product $\xi_\varepsilon \diamond X_{\varepsilon,\lambda}$. For fixed $\varphi \in \mathcal{S}(\mathbb{T}^2)$

$$\begin{aligned}
&\langle \varphi, \xi_\varepsilon \diamond X_{\varepsilon,\lambda}\rangle \\
&= \int_{\mathbb{T}^2} \varphi(y) \sum_{|l-m|\le 1} \sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2} \Bigg(\int_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)} K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \Bigg) \xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2)\,\mathrm{d}y \\
&= \sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon} \Big\langle \varphi(\cdot), \sum_{|l-m|\le 1} \Pi_\varepsilon K_l(\cdot - x_1)\Pi_\varepsilon \mathcal{K}^\varepsilon_m(\cdot - x_2)\Big\rangle \xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2).
\end{aligned}$$

Consider a map $L_\varepsilon : \left(\mathbb{Z}^2_\varepsilon\right)^2 \to \mathbb{R}$ defined by

$$L_\varepsilon(x_1,x_2) := \langle \varphi(\cdot), \sum_{|l-m|\le 1} \Pi^Q_\varepsilon K_l(\cdot - x_1)\Pi^Q_\varepsilon \mathcal{K}_m(\cdot - x_2)\rangle 1_{\{(x_1,x_2)\in\mathbb{T}^2\times\mathbb{T}^2\}}.$$

This definition naturally extends to $\varepsilon = 0$, where L maps $(\mathbb{R}^2)^2$ to $\mathbb{R}$. Our goal is to show that

$$\sum_{(x_1,x_2)\in(\mathbb{Z}^2_\varepsilon)^2} L_\varepsilon(x_1,x_2)\xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2) \to \int_{(\mathbb{R}^2)^2} L(x_1,x_2)\xi(\,\mathrm{d}x_1)\diamond\xi(\,\mathrm{d}x_2), \tag{42}$$

where convergence holds in distribution and the limit is interpreted as an iterated stochastic integral in the second Wiener-Itô chaos. It is sufficient to verify the assumptions of Lemma C.2. That is, we have to show that there exists a $g \in L^2((\mathbb{R}^2)^2)$ such that:

$$\sup_{\varepsilon\in(0,1/2)} |1_{(\varepsilon^{-1}\mathbb{T}^2)^2}\mathcal{F}_{(\mathbb{Z}^2_\varepsilon)^2}L_\varepsilon| \le g, \qquad \lim_{\varepsilon\to 0} \|1_{(\varepsilon^{-1}\mathbb{T}^2)^2}\mathcal{F}_{(\varepsilon\mathbb{Z}^2)^2}L_\varepsilon - \mathcal{F}_{(\mathbb{R}^2)^2}L\|_{L^2((\mathbb{R}^2)^2)} = 0$$

For this purpose we calculate

$$\begin{aligned}
&1_{(\varepsilon^{-1}\mathbb{T}^2)^2}\mathcal{F}_{(\varepsilon\mathbb{Z}^2)^2}L_\varepsilon(k_1,k_2) \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2) \int_{(\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2)^2} e^{2\pi\iota(k_1\cdot x_1 + k_2\cdot x_2)} \langle \varphi(\cdot), \sum_{|l-m|\le 1} \Pi^Q_\varepsilon K_l(\cdot - x_1)\Pi^Q_\varepsilon \mathcal{K}_m(\cdot - x_2)\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2) \int_{(\mathbb{T}^2)^2} e^{2\pi\iota(k_1\cdot x_1 + k_2\cdot x_2)} \langle \varphi(\cdot), \sum_{|l-m|\le 1} K_l(\cdot - x_1)\mathcal{K}_m(\cdot - x_2)\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2) \int_{\mathbb{T}^2} \varphi(y) e^{2\pi\iota(k_1+k_2)\cdot y} \sum_{|l-m|\le 1} \varrho_l(-k_1)\varrho_m(-k_2) \frac{\widehat{\chi}^2(-\varepsilon k_2)}{-\vartheta_\varepsilon(-k_2)+\lambda}\,\mathrm{d}y \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2)(\mathcal{F}_{\mathbb{T}^2}\varphi)(k_1+k_2) \sum_{|l-m|\le 1} \varrho_l(k_1)\varrho_m(k_2) \frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda},
\end{aligned}$$

so that the required assumptions are naturally satisfied. Since φ is smooth, the latter term is bounded in L^2, uniformly over ε. In particular (42) follows. Hence the distribution of any limit point of $\langle \varphi, \xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda}\rangle$ is uniquely characterized and since φ is arbitrary this implies convergence in distribution of $\xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda}$. □

Appendix A. The SLFV in a random environment

In this section we provide a rigorous construction of the spatial Λ-Fleming-Viot process (SLFV) in a random environment. We work under the following assumptions.

Assumption A.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Fix $\varepsilon \in (0, 1/2)$ and $\mathfrak{u} \in (0,1), d = 1,2$ and $w^0 \colon \mathbb{T}^d \to [0,1]$ a measurable map and let $s_\varepsilon \colon \Omega \times \mathbb{T}^d \to (-1,1)$ be a measurable function.*

The natural state space of the spatial SLFV process is:

$$M = \{w \colon \mathbb{T}^d \to [0,1], \quad w \text{ measurable}\},$$

which is a metric space when endowed with the distance $d_M(u,w) = \sup_{x \in \mathbb{T}^d} |u(x) - w(x)|$. Then under the assumption above, for $x \in \mathbb{T}^d, \mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}$ and any function $w : \mathbb{T}^d \to [0,1]$ define the operator $\Theta^{\mathfrak{p}}_x \colon M \to M$ by

$$\begin{aligned} \Theta^{\mathfrak{p}}_x w(y) &= w(y) 1_{\{B^c_\varepsilon(x)\}}(y) + (\mathfrak{u} 1_{\{\mathfrak{p} = \mathfrak{a}\}} + (1-\mathfrak{u}) w(y)) 1_{\{B_\varepsilon(x)\}}(y) \\ &= w(y) + \mathfrak{u}(1_{\{\mathfrak{p}=\mathfrak{a}\}} - w(y)) 1_{\{B_\varepsilon(x)\}}(y). \end{aligned}$$

In the discussion below, let $\mathcal{B}(E)$ be the Borel sigma-algebra associated to some metric space E. We say that a probability measure $\mathbb{P}^\omega$ on $(E, \mathcal{B}(E))$ indexed by $\omega \in \Omega$ is a Markov kernel, if for any $A \in \mathcal{B}(E)$ the map $\omega \mapsto \mathbb{P}^\omega(A)$ is measurable. Then one can build the semidirect product measure $\mathbb{P} \ltimes \mathbb{P}^\omega$ on $\Omega \times E$ (with the product sigma-algebra), characterized, for $A \in \mathcal{F}, B \in \mathcal{B}(E)$, by:

$$\mathbb{P} \ltimes \mathbb{P}^\omega (A \times B) = \int_A \mathbb{P}^\omega(B) \mathbb{P}(\,\mathrm{d}\omega).$$

In the definition below we write:

$$s_+(x) = \max\{s(x), 0\}, \qquad s_-(x) = \max\{-s(x), 0\}.$$

Lemma A.2. *Under Assumption A.1, fix $\omega \in \Omega$. There exists a unique Markov jump process $t \mapsto w(t)$ in $\mathbb{D}([0,\infty); M)$ started in $w(0) = w^0$, associated to the generator*

$$\mathcal{L}(\varepsilon, s_\varepsilon(\omega), \mathfrak{u}) \colon C_b(M; \mathbb{R}) \to C_b(M; \mathbb{R}),$$

defined by

$$\mathcal{L}(f)(w) = \int_M (f(w') - f(w)) \mu(w, \,\mathrm{d}w'), \qquad f \in C_b(M; \mathbb{R}),$$

where the transition function $\mu \colon M \times \mathcal{B}(M) \to \mathbb{R}$ (depending on $s_\varepsilon(\omega), \mathfrak{u}, \varepsilon$) is defined by:

$$\mu(w, \,\mathrm{d}w') = 0 \quad \textit{unless there exist} \quad x \in \mathbb{T}^d, \mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\} \quad \textit{such that} \quad w' = \Theta^{\mathfrak{p}}_x w.$$

And if $w' = \Theta^{\mathfrak{p}}_x w$ for some $x \in \mathbb{T}^d, \mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}$:

$$\begin{aligned} \mu(w, \,\mathrm{d}w') = \bigg\{ & (1 - |s_\varepsilon(\omega, x)|) \Big[\Pi_\varepsilon w 1_{\{\mathfrak{p}=\mathfrak{a}\}} + (1 - \Pi_\varepsilon w) 1_{\{\mathfrak{p} = \mathfrak{A}\}} \Big](x) \\ & + (s_\varepsilon)_-(\omega, x) \Big[\big(\Pi_\varepsilon w\big)^2 1_{\{\mathfrak{p}=\mathfrak{a}\}} + \big(1 - \big(\Pi_\varepsilon w\big)^2\big) 1_{\{\mathfrak{p}=\mathfrak{A}\}} \Big](x) \\ & + (s_\varepsilon)_+(\omega, x) \Big[\Pi_\varepsilon w (2 - \Pi_\varepsilon w) 1_{\{\mathfrak{p}=\mathfrak{a}\}} + (1 - \Pi_\varepsilon w)^2 1_{\{\mathfrak{p}=\mathfrak{A}\}} \Big](x) \bigg\} \,\mathrm{d}x. \end{aligned}$$

The law $\mathbb{P}^\omega$ of w in $\mathbb{D}([0,\infty); M)$ is a Markov kernel and induces the semidirect product measure $\mathbb{P} \ltimes \mathbb{P}^\omega$ on $\Omega \times \mathbb{D}([0,\infty); M)$.

Proof. Note that μ defined as above is a Markov kernel on $M \times \mathcal{B}(M)$ (to be precise, here we have to observe that for fixed w the set $\{\Theta^{\mathfrak{p}}_x w, \quad x \in \mathbb{T}^d, \mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}\}$ is closed and hence measurable in M). Hence, the Markov process is constructed following [23, Section 4.2]. In addition, for $f \in C_b(M; \mathbb{R})$ measurable and bounded the map $\omega \mapsto \int_M f(w') \mu_\omega(w, \,\mathrm{d}w')$ is measurable (we made explicit the dependence of μ on ω). This implies, e.g. by [23, Equation 4.2.8], that the map $\omega \mapsto \mathbb{P}^\omega(A)$ is measurable, for $A \in \mathcal{B}(\mathbb{D}([0,\infty); M))$. So the proof is complete. $\square$

Lemma A.3. *Under Assumption A.1 fix $\omega \in \Omega$ and let w be the Markov process as in the previous result. For any $\varphi \in L^1(\mathbb{T}^d)$ the process $t \mapsto \langle w(t), \varphi \rangle$ satisfies the martingale problem of Lemma 1.3.*

Proof. In the discussion below we omit the dependence of $s_\varepsilon(\omega)$ on ε and ω, since such dependence is not relevant here. We will apply the generator to functions of the form $F_\varphi(w) = F(\langle w, \varphi \rangle)$, with $F \in C(\mathbb{R};\mathbb{R}), \varphi \in L^1(\mathbb{T}^d)$. For simplicity we divide the operator $\mathcal{L} = \mathcal{L}(\varepsilon, s, \mathfrak{u})$ in three parts:

$$\begin{aligned}\mathcal{L}(F_\varphi)(w) &:= \mathcal{L}^{\mathrm{neu}}(F_\varphi)(w) + \mathcal{L}^{\mathrm{sel}}(F_\varphi)(w) \\ &:= \mathcal{L}^{\mathrm{neu}}(F_\varphi)(w) + \mathcal{L}^{\mathrm{sel}}_{<}(F_\varphi)(w) + \mathcal{L}^{\mathrm{sel}}_{>}(F_\varphi)(w)\end{aligned}$$

(the first is the neutral part, the second two are the selective parts of the operator), where

$$\mathcal{L}^{\mathrm{neu}}(F_\varphi)(w) = \int\limits_{\mathbb{T}^d} (1-|s(x)|)\Big[\Pi_\varepsilon w[F_\varphi(\Theta^{\mathfrak{a}}_x w)-F_\varphi(w)]+(1-\Pi_\varepsilon w)[F_\varphi(\Theta^{\mathfrak{A}}_x w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

$$\mathcal{L}^{\mathrm{sel}}_{>}(F_\varphi)(w) = \int\limits_{\mathbb{T}^d} s_-(x)\Big[\big(\Pi_\varepsilon w\big)^2[F_\varphi(\Theta^{\mathfrak{a}}_x w)-F_\varphi(w)]+\big(1-\big(\Pi_\varepsilon w\big)^2\big)[F_\varphi(\Theta^{\mathfrak{A}}_x w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

$$\mathcal{L}^{\mathrm{sel}}_{<}(F_\varphi)(w) = \int\limits_{\mathbb{T}^d} s_+(x)\Big[\Pi_\varepsilon w(2-\Pi_\varepsilon w)[F_\varphi(\Theta^{\mathfrak{a}}_x w)-F_\varphi(w)]+(1-\Pi_\varepsilon w)^2[F_\varphi(\Theta^{\mathfrak{A}}_x w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

Now, in the special case of $F = \mathrm{Id}$, the neutral part of the generator takes form

$$\mathcal{L}^{\mathrm{neu}}(\mathrm{Id}_\varphi)(w) = \mathfrak{u}\varepsilon^d \int_{\mathbb{T}^d} (1-|s(x)|)[(\Pi_\varepsilon w)(\Pi_\varepsilon \varphi) - \Pi_\varepsilon(w\varphi)](x)\,\mathrm{d}x,$$

Analogously, the selective part can be written as

$$\mathcal{L}^{\mathrm{sel}}(\mathrm{Id}_\varphi)(w) = \mathfrak{u}\varepsilon^d \int_{\mathbb{T}^d} s(x)[\Pi_\varepsilon(w\varphi)-\big(\Pi_\varepsilon w\big)^2\Pi_\varepsilon\varphi](x) + 2s_+(x)[\Pi_\varepsilon w \Pi_\varepsilon \varphi - \Pi_\varepsilon(w\varphi)](x)\,\mathrm{d}x.$$

Adding those two we conclude that

$$\mathcal{L}(\mathrm{Id}_\varphi)(w) = \mathfrak{u}\varepsilon^d \int_{\mathbb{T}^d} [(\Pi_\varepsilon w)(\Pi_\varepsilon\varphi) - \Pi_\varepsilon(w\varphi)](x) + s(x)[(\Pi_\varepsilon w)(\Pi_\varepsilon\varphi) - (\Pi_\varepsilon w)^2\Pi_\varepsilon\varphi](x)\,\mathrm{d}x.$$

This justifies the drift in the required decomposition. To obtain the predictable quadratic variation of the martingale make use of Dynkin's formula, that is

$$\langle M^\varepsilon(\varphi)\rangle_t = \int_0^t \mathcal{L}(\mathrm{Id}^2_\varphi) - 2\big(\mathrm{Id}_\varphi \mathcal{L}(\mathrm{Id}_\varphi)\big)(X^\varepsilon_r)\,\mathrm{d}r.$$

Once again, it is natural to treat the terms involving $\mathcal{L}^{\mathrm{neu}}$ and $\mathcal{L}^{\mathrm{sel}}$ separately. For the neutral term:

$$\begin{aligned}&\big(\mathcal{L}^{\mathrm{neu}}(\mathrm{Id}^2_\varphi) - 2F_\varphi \mathcal{L}^{\mathrm{neu}}(\mathrm{Id}_\varphi)\big)(w) \\ &\qquad = \mathfrak{u}^2\varepsilon^{2d} \int_{\mathbb{T}^d} (1-|s(x)|)\Big[\Pi_\varepsilon w\big(\Pi_\varepsilon\varphi - \Pi_\varepsilon(w\varphi)\big)^2 + \big(1-\Pi_\varepsilon w\big)\big(\Pi_\varepsilon(w\varphi)\big)^2\Big](x)\,\mathrm{d}x,\end{aligned}$$

which can be written as

$$\mathfrak{u}^2\varepsilon^{2d} \int_{\mathbb{T}^d} (1-|s(x)|)\Big[\Pi_\varepsilon w\big[\big(\Pi_\varepsilon\varphi\big)^2 - 2\Pi_\varepsilon\varphi(x)\Pi_\varepsilon(w\varphi)\big] + \big[\Pi_\varepsilon(w\varphi)\big]^2\Big](x)\,\mathrm{d}x.$$

Analogous calculations for $\mathcal{L}^{\mathrm{sel}}_{<}$ lead to

$$\begin{aligned}&\big(\mathcal{L}^{\mathrm{sel}}_{<}(\mathrm{Id}^2_\varphi) - 2\mathrm{Id}_\varphi \mathcal{L}^{\mathrm{sel}}_{<}\mathrm{Id}_\varphi\big)(w) = \\ &\qquad = \mathfrak{u}^2\varepsilon^{2d} \int_{\mathbb{T}^d} s_-(x)\Big[(\Pi_\varepsilon w)^2\big[(\Pi_\varepsilon\varphi)^2 - 2\Pi_\varepsilon\varphi\Pi_\varepsilon(w\varphi)\big] + \big[\Pi_\varepsilon(w\varphi)\big]^2\Big](x)\,\mathrm{d}x.\end{aligned}$$

Whereas for $\mathcal{L}^{\mathrm{sel}}_{>}$ they lead to

$$\begin{aligned}&\big(\mathcal{L}^{\mathrm{sel}}_{>}(\mathrm{Id}^2_\varphi) - 2\mathrm{Id}_\varphi\mathcal{L}^{\mathrm{sel}}_{>}\mathrm{Id}_\varphi\big)(w)\\
&\qquad= \mathfrak{u}^2\varepsilon^{2d}\int_{\mathbb{T}^d} s_+(x)\Big[(\Pi_\varepsilon w)(2-\Pi_\varepsilon w)\big(\Pi_\varepsilon\varphi-\big(\Pi_\varepsilon(w\varphi)\big)^2 + \big(1-\Pi_\varepsilon w\big)^2\big(\Pi_\varepsilon w\big)^2\Big](x)\,\mathrm{d}x\\
&\qquad= \mathfrak{u}^2\varepsilon^{2d}\int_{\mathbb{T}^d} s_+(x)\Big[(\Pi_\varepsilon w)(2-\Pi_\varepsilon w)\big[\big(\Pi_\varepsilon\varphi\big)^2-2\Pi_\varepsilon\varphi\Pi_\varepsilon(w\varphi)\big]+\big[\Pi_\varepsilon(w\varphi)\big]^2(x)\Big]\,\mathrm{d}x.\end{aligned}$$

Summing neutral and selective terms one obtains

$$\begin{aligned}&\mathfrak{u}^2\varepsilon^{2d}\langle\Pi_\varepsilon w,(1-|s|)\Big[\big(\Pi_\varepsilon\varphi\big)^2-2\Pi_\varepsilon\varphi\Pi_\varepsilon(w\varphi)\Big]\rangle+\langle\big(\Pi_\varepsilon(w\varphi)\big)^2,(1-|s|)\rangle\\
&\quad+\mathfrak{u}\varepsilon^{2d}\langle(\Pi_\varepsilon w)^2, s_-\Big[(\Pi_\varepsilon\varphi)^2 - 2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(w\varphi)\big)\Big] + \langle\big(\Pi_\varepsilon(w\varphi)\big)^2, s_-\rangle\\
&\quad+\mathfrak{u}^2\varepsilon^{2d}\langle\Pi_\varepsilon w, s_+\Big[(2-\Pi_\varepsilon w)\Big((\Pi_\varepsilon\varphi)^2-2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(w\varphi)\big)\Big)\Big]\rangle+\langle\big(\Pi_\varepsilon(w\varphi)\big)^2, s_+\rangle,\end{aligned}$$

which can be written in the form from the statement of the Lemma.

□

Appendix B. Some analytic results

In this appendix we recall some of the analytic theory we require. First we concentrate on special properties of Besov spaces and the regularity of characteristic functions. Later we will address some relevant points in paracontrolled calculus.

B.1. **Besov spaces & characteristic functions.**

Let us begin by stating the Poisson summation formula (a proof is left to the reader, or can be found in many textbooks and web pages).

Lemma B.1. *For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it holds that:*

$$\mathcal{F}^{-1}_{\mathbb{T}^d}\varphi(x) = \sum_{z\in\mathbb{Z}^d}\mathcal{F}^{-1}_{\mathbb{R}^d}\varphi(x+z).$$

In particular, this implies for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the bound:

$$\|\mathcal{F}^{-1}_{\mathbb{T}^d}\varphi\|_{L^1(\mathbb{T}^d)} \le \|\mathcal{F}^{-1}_{\mathbb{R}^d}\varphi\|_{L^1(\mathbb{R}^d)}.$$

Recall that the Besov spaces $B^\alpha_{p,q}(\mathbb{T}^d)$ are defined via a dyadic partition of the unity $\{\varrho_j\}_{j\ge -1}$ such that for $j \ge 0$, $\varrho_j = \varrho(2^j\cdot)$ for a smooth function ϱ with compact support in an annulus.

Proposition B.2 (Besov embeddings). *For any $1 \le p_1 \le p_2 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$ the space $B^\alpha_{p_1,q_1}$ is continuously embedded in $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}$. In other words, there exists a constant $C > 0$ such that:*

$$\|\varphi\|_{B^{\alpha-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,q_2}} \le C\|\varphi\|_{B^\alpha_{p_1,q_1}}.$$

In addition, for $\alpha' < \alpha$ the embedding $B^\alpha_{p_2,q_2} \subseteq B^{\alpha'}_{p_1,q_1}$ is compact.

In certain cases, it will be convenient to use the following alternative characterization of certain Besov spaces.

Proposition B.3 (Sobolev-Slobodeckij norm). *For every $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and for every $p \in [1,\infty)$ define the the Sobolev-Slobodeckij norm for $\varphi \in \mathcal{S}'(\mathbb{T}^d)$ as:*

$$\|\varphi\|_{W^\alpha_p} := \|\varphi\|_{L^p} + \sum_{|m|=\lfloor\alpha\rfloor}\left(\int_{\mathbb{T}^d\times\mathbb{T}^d}\frac{|D^m\varphi(x)-D^m\varphi(y)|^p}{|x-y|^{d+(\alpha-\lfloor\alpha\rfloor)p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \in [0,\infty].$$

There exist constants a pair of constants $c(p), C(p) > 0$ such that for $\varphi \in \mathcal{S}'(\mathbb{T}^d)$

$$c\|\varphi\|_{B^\alpha_{p,p}} \le \|\varphi\|_{W^\alpha_p} \le C\|\varphi\|_{B^\alpha_{p,p}}.$$

For a proof consult e.g. [51] Theorem 2.5.7 and the discussion in Section 2.2.2. The next result states the regularizing properties of convolutions.

Lemma B.4. *For $p, q, r \in [1, \infty]$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and for any $\varphi, \psi \in \mathcal{S}'(\mathbb{T}^d)$:.*

$$\|\varphi * \psi\|_{C_r^{\alpha+\beta}} \lesssim \|f\|_{C_p^\alpha} \|g\|_{\mathcal{C}_q^\beta}.$$

Proof. By Young convolution inequality

$$\|\Delta_i(f * g)\|_{L^r} = \|\Delta_i f * \overline{\Delta}_i g\|_{L^r} \lesssim \|\Delta_i f\|_{L^p} \|\overline{\Delta}_i g\|_{L^q}, \tag{43}$$

where $\overline{\Delta}_i$ is associated with a dyadic partition of the unity different from the one we use for most of the proofs. Namely we require that it satisfies $\{\overline{\varrho_j}\}_{j \geq -1}$ such that $\varrho_j \overline{\varrho_j} = \varrho_j$. Then the bound follows immediately, since the Besov norms associated to different dyadic partitions are equivalent (cf. [3, Remark 2.17]). □

The following lemma is a special case of results obtained by [47]. The proof is included for completeness.

Lemma B.5. *Fix $\zeta \in [0, \frac{1}{p})$. Then, for $p \in [1, \infty)$:*

$$\sup_{\varepsilon \in (0,1/2)} \varepsilon^{\zeta + d - \frac{d}{p}} \|\chi_\varepsilon\|_{W_p^\zeta} < \infty.$$

Proof. We shall make use of the characterization of fractional Sobolev space in terms of Sobolev-Slobodeckij norm. A direct computation shows that

$$\begin{aligned} \|\chi_\varepsilon\|_{W_p^\zeta} &= \|\chi_\varepsilon\|_{L^p} + \left(\int_{\mathbb{T}^d \times \mathbb{T}^d} \varepsilon^{-dp} \frac{|1_{B_\varepsilon}(x) - 1_{B_\varepsilon}(y)|^p}{|x-y|^{d+\zeta p}} \mathrm{d}x \mathrm{d}y \right)^{1/p} \\ &\leq 1 + \left(2 \int_{B_\varepsilon} \int_{\mathbb{T}^d \setminus B_\varepsilon} \varepsilon^{-dp} \frac{|1_{B_\varepsilon}(x) - 1_{B_\varepsilon}(y)|^p}{|x-y|^{d+\zeta p}} \mathrm{d}x \mathrm{d}y \right)^{1/p}. \end{aligned}$$

Now let $d_\varepsilon(x)$ be the Euclidean distance of x from the boundary ∂B_ε and let $\overline{B}_{d_\varepsilon(x)}(y)$ be the ball of radius $d_\varepsilon(x)$ about y. Then the previous integral can be estimated by:

$$\begin{aligned} \left(\int_{B_\varepsilon} \int_{\mathbb{T}^d \setminus B_\varepsilon} \varepsilon^{-dp} \frac{|1_{B_\varepsilon}(x) - 1_{B_\varepsilon}(y)|^p}{|x-y|^{d+\zeta p}} \mathrm{d}x \mathrm{d}y \right)^{1/p} &\leq \left(\int_{B_\varepsilon} \int_{\mathbb{T}^d \setminus \overline{B}_{d_\varepsilon(y)}(y)} \varepsilon^{-dp} \frac{1}{|x-y|^{d+\zeta p}} \mathrm{d}x \mathrm{d}y \right)^{1/p} \\ = \left(\int_{B_\varepsilon} \int_{\mathbb{T}^d \setminus \overline{B}_{d_\varepsilon(y)}(0)} \varepsilon^{-dp} \frac{1}{|x|^{d+\zeta p}} \mathrm{d}x \mathrm{d}y \right)^{1/p} &\lesssim \left(\int_{B_\varepsilon} \varepsilon^{-dp} d_\varepsilon(y)^{-\zeta p} \mathrm{d}y \right)^{1/p} \\ \lesssim \left(\int_0^{c\varepsilon} \varepsilon^{-dp} (c\varepsilon - r)^{-\zeta p} r^{d-1} \mathrm{d}r \right)^{\frac{1}{p}} &\lesssim \varepsilon^{-d} \left(\varepsilon^{-\zeta p + d} \right)^{1/p} \leq \varepsilon^{-d-\zeta+d/p} \end{aligned}$$

□

Corollary B.6. *For $\zeta \in [0, 1), p \in [1, \infty]$ and $\alpha \in \mathbb{R}$*

$$\sup_{\varepsilon \in (0,1/2)} \varepsilon^\zeta \|\chi_\varepsilon * \varphi\|_{\mathcal{C}_p^{\alpha+\zeta}} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha}.$$

Proof. This is now a direct consequence of Lemmata B.4 and B.5 (the latter with $p = 1$). □

The rest of this subsection is devoted to the proof of Lemma 5.1.

Proof of Lemma 5.1. Let us start with the term involving the gradient. We have that for $i = 1, \ldots, d$:

$$(D\widehat{\chi})_i(0) = -2\pi\iota \fint_{B_1(0)} x_i e^{-2\pi\iota\langle k, x\rangle} \, \mathrm{d}x \Big|_{k=0} = 0.$$

For the term involving the Hessian, we observe that an analogous computation for $i \neq j$ shows that $(D^2\widehat{\chi})_{i,j}(0) = 0$. If $i = j$ we find that

$$(D^2\widehat{\chi})_{i,i}(0) = -(2\pi)^2 \fint_{B_1(0)} \mathrm{d}x \; x_i^2 e^{-2\pi\iota\langle k,x\rangle}\Big|_{k=0} =: -(2\pi)^2\nu_0,$$

with the value of ν_0 as in the statement. The two-sided inequality follows by a Taylor approximation. We are left with a bound on the decay of $\hat{\chi}$:

$$\Big|\frac{\mathrm{d}^n}{\mathrm{d}x_{i_1}\dots \mathrm{d}x_{i_n}}\hat{\chi}_B(k)\Big| \lesssim_\alpha (1+|k|)^{-\frac{d+1}{2}}.$$

For this purpose let $J_\nu(\cdot)$ be the Bessel function of the first kind with parameter ν, that is

$$J_\nu(k) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{k}{2}\right)^{2m+\nu}.$$

The Fourier transform of χ_Q is given, for some $c, C > 0$, by

$$\hat{\chi}_B(k) = c(d)\int_0^\pi \mathrm{d}t \;\sin^d(t) e^{-2\pi\iota|k|\cos(t)/4} = C|k|^{-d/2}J_{d/2}(\pi|k|/2) \tag{44}$$

Since $J_{\frac{1}{2}}(k) = \sqrt{\frac{2}{\pi k}}\sin k$, the bound for $d = 1$ is immediate. For $d = 2$, we make use of an asymptotic bound for Bessel functions:

$$\sup_{\varrho\geq 1} \varrho^{-1/2}|J_\nu(\varrho)| < +\infty.$$

We provide a proof of this bound in the next Lemma. The bound for the derivatives then follows from (44), the asymptotic result for Bessel functions, and the following pair of identities

$$\begin{aligned} \partial_x J_n(x) &= \frac{1}{2}(J_{n-1}(x)+J_{n+1}(x)), \qquad \forall n\in\mathbb{Z},\\ J_{-n}(\cdot) &= (-1)^n J_n(\cdot) \qquad \forall n\in\mathbb{N}_0. \end{aligned}$$

□

The following result is well-known (see e.g. [53], where many deeper results are presented). For completeness we provide a proof that satisfies all our purposes.

Lemma B.7. *Fix $\nu \in \mathbb{R}$. Then*

$$\sup_{\varrho\geq 1} \varrho^{-1/2}|J_\nu(\varrho)| < +\infty,$$

Proof. Through (44) and by changing variables $x = cos(t)$ we rewrite the Bessel function as

$$\int_{-1}^1 \mathrm{d}x \;(1-x^2)^{\frac{d-1}{2}}e^{\iota\varrho x} = 2\mathrm{Re}\Big(\int_0^1 \mathrm{d}x\;(1-x^2)^{\frac{d-1}{2}}e^{\iota\varrho x}\Big).$$

A change variables $x = 1-u^2$. yields

$$e^{i\varrho}\int_0^1 \mathrm{d}u\;\big(u^2(2-u^2)\big)^{\frac{d-1}{2}}e^{-\iota\varrho u^2}u = \frac{e^{i\varrho}}{\varrho^{\frac{d+1}{2}}}\int_0^{\sqrt{\varrho}}\mathrm{d}w\;\big(w^2(2-\frac{w^2}{\varrho})\big)^{\frac{d-1}{2}}e^{-\iota w^2}w$$

Observe that in order to obtain the desired bound it is now sufficient to show that the integral terms is bounded uniformly in ρ. After another change of variable $w = e^{-\iota\frac{\pi}{4}}z$ we obtain

$$\begin{aligned}&\int_0^{e^{\frac{\iota\pi}{4}}\sqrt{\varrho}} \mathrm{d}z\;\big(-\iota z^2(2+\iota z^2/\varrho)\big)^{\frac{d-1}{2}}e^{-z^2}z\\ &= \int_0^{\sqrt{\varrho}}\mathrm{d}z\;\big(-\iota z^2(2+\iota z^2/\varrho)\big)^{\frac{d-1}{2}}e^{-z^2}z + \int_0^{\pi/4}\mathrm{d}\varphi\;\big(-\iota\varrho e^{2\iota\varphi}(2+\iota e^{2\iota\varphi})\big)^{\frac{d-1}{2}}e^{-\varrho e^{2\iota\varphi}}\varrho e^{2\iota\varphi}\end{aligned}$$

The first integral can be trivially bounded uniformly over ϱ while the second one is tends to 0 as ρ tends to infinity since the exponential term dominates all the others. □

B.2. **Paraproducts & commutator estimates.** This section is devoted to products of distributions and commutator estimates, starting with the decomposition in paraproducts (through the symbol $\varolessthan$) and resonant products ($\odot$). For $\varphi, \psi \in \mathcal{S}'(\mathbb{T}^d)$ set

$$S_i\varphi := \sum_{j=-1}^{i-1} \Delta_j \varphi, \qquad \varphi \varolessthan \psi := \sum_{i \geq -1} S_{i-1}\varphi \Delta_i \psi, \qquad \varphi \odot \psi := \sum_{|i-j|\leq 1} \Delta_j \varphi \Delta_i \psi,$$

where the latter sum might not be well defined. Then, an *a priori* ill-posed product of φ and ψ can be written as

$$\varphi \cdot \psi = \varphi \varolessthan \psi + \varphi \odot \psi + \varphi \varogreaterthan \psi.$$

The following estimates are classical, see e.g. [3, Lemmata 2.82 and 2.85] and guarantee that the product is actually well-defined if the regularities α and β of φ and ψ satisfy $\alpha + \beta > 0$.

Lemma B.8. *Let $\alpha, \beta \in \mathbb{R}$ and fix $p, q, r \in [1, \infty]$ such that $1/r = 1/p + 1/q$. For $\varphi, \psi \in \mathcal{S}'(\mathbb{T}^d)$*

$$\begin{aligned} \|\varphi \varolessthan \psi\|_{\mathcal{C}^\alpha_r} &\lesssim \|\varphi\|_{L^p} \|\psi\|_{\mathcal{C}^\alpha_q}, \\ \|\varphi \varolessthan \psi\|_{\mathcal{C}^{\alpha+\beta}_r} &\lesssim \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha_q}, \quad \text{if} \quad \beta < 0, \\ \|\varphi \odot \psi\|_{\mathcal{C}^{\alpha+\beta}_r} &\lesssim \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha_q} \quad \text{if} \quad \alpha+\beta > 0. \end{aligned}$$

The rest of this subsection is deals with the following commutators.

Definition B.9. *For distributions $\varphi, \psi, \sigma \in \mathcal{S}'(\mathbb{T}^d)$ we define the (a-priori ill-posed) commutators*

$$\begin{aligned} C^{\odot}(\varphi, \psi, \sigma) &:= \varphi \odot (\psi \varolessthan \sigma) - \psi(\varphi \odot \sigma), \\ C^{\Pi}_{\varepsilon}(\varphi, \psi) &:= \Pi^2_\varepsilon(\varphi \varolessthan \psi) - \varphi \varolessthan \Pi^2_\varepsilon \psi, \\ C_{\varepsilon,\lambda}(\varphi, \psi) &:= (-\mathcal{A}_\varepsilon + \lambda)^{-1}(\varphi \varolessthan \psi) - \varphi \varolessthan (-\mathcal{A}_\varepsilon + \lambda)^{-1}\psi. \end{aligned}$$

The first commutator estimate is crucial, but by now well-known.

Lemma B.10 ([27], Lemma 14)**.** *For $\varphi, \psi, \sigma \in \mathcal{S}'(\mathbb{T}^d)$, $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma > 0$ and $p \in [1, \infty]$:*

$$\|C^{\odot}(\varphi, \psi, \sigma)\|_{\mathcal{C}^{\alpha+\gamma}_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha} \|\psi\|_{\mathcal{C}^\beta_p} \|\sigma\|_{\mathcal{C}^\gamma}.$$

We pass to the second estimate. Recall the operators $\mathcal{P}_\varepsilon, \mathcal{Q}_\varepsilon$ as in Definition 5.3.

Lemma B.11. *For $\varphi, \psi \in \mathcal{S}'(\mathbb{T}^d)$ and $\alpha \in \mathbb{R}, \beta > 0, p \in [1, \infty]$ it holds for every $\delta \in [0, \beta \wedge 1)$:*

$$\|\mathcal{P}_\varepsilon C^{\Pi}_\varepsilon(\varphi, \psi)\|_{\mathcal{C}^{\alpha+\delta}_p} \lesssim \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha}, \qquad \|\mathcal{Q}_\varepsilon C^{\Pi}_\varepsilon(\varphi, \psi)\|_{\mathcal{C}^\alpha_p} \lesssim \varepsilon^\delta \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha}.$$

Proof. Note that for any $i \geq 0$ there exists an annulus $\mathcal{A}$ (that is a set of the form $\{k \in \mathbb{R}^d \mid r \leq |k| \leq R\}$ for some $0 < r < R$) such that the Fourier transform of

$$\Pi^2_\varepsilon[S_{i-1}\varphi\Delta_i\psi] - S_{i-1}\varphi\Pi^2_\varepsilon\Delta_i\varphi$$

is contained in $2^i\mathcal{A}$. It is therefore sufficient to show that

$$\left\|\Pi^2_\varepsilon[S_{i-1}\varphi\Delta_i\psi] - S_{i-1}\varphi\Pi^2_\varepsilon\Delta_i\varphi\right\|_{L^p} \lesssim \varepsilon^\delta \|\varphi\|_{\mathcal{C}^\beta_p} \|\Delta_i\psi\|_{L^\infty}, \tag{45}$$

since this implies the required bound by estimating $\varepsilon^\delta \lesssim 2^{-\delta i}$ for i such that $\mathcal{P}_\varepsilon\Delta_i \neq 0$. To obtain (45), recall the Sobolev-Slobodeckij characterization of fractional spaces of Proposition B.3, so that for $\delta \in [0,1)$

$$\begin{aligned}\|\Pi_\varepsilon^2[S_{i-1}\varphi\Delta_i\psi] - S_{i-1}\varphi\Pi_\varepsilon^2\Delta_i\varphi\|_\infty &\leq \left(\int_{\mathbb{T}^d}\Big|\fint_{B_\varepsilon(x)}[S_{i-1}\varphi(y)-S_{i-1}\varphi(x)]\Delta_i\psi(y)\,\mathrm{d}y\Big|^p\,\mathrm{d}x\right)^{1/p}\\ &\lesssim \varepsilon^\delta\left(\int_{\mathbb{T}^d}\Big|\fint_{B_\varepsilon(x)}\frac{[S_{i-1}\varphi(y)-S_{i-1}\varphi(x)]}{|y-x|^\delta}\Delta_i\psi(y)\,\mathrm{d}y\Big|^p\,\mathrm{d}x\right)^{1/p}\\ &\lesssim \varepsilon^\delta\left(\int_{\mathbb{T}^d}\int_{B_\varepsilon(x)}\frac{|S_{i-1}\varphi(y)-S_{i-1}\varphi(x)|^p}{|y-x|^{d+\delta p}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p}\|\Delta_i\psi\|_\infty \lesssim \varepsilon^\delta\|S_{i-1}\varphi\|_{\mathcal{C}_p^\beta}2^{-\alpha i}\|\psi\|_{\mathcal{C}^\alpha},\end{aligned}$$

where the first inequality follows by Jensen's inequality and and we have used the embedding $B_{p,\infty}^\beta \subset B_{p,p}^\delta$. Now the result follows since:

$$\|S_{i-1}\varphi\|_{\mathcal{C}_p^\beta} \lesssim \|\varphi\|_{\mathcal{C}_p^\beta}.$$

This concludes the proof. □

Lemma B.12. *For $\varphi,\psi \in \mathcal{S}'(\mathbb{T}^d)$ and $\alpha \in (0,2)$, $\beta \in \mathbb{R}$ and $p \in [1,\infty]$*

$$\|\mathcal{P}_\varepsilon C_{\varepsilon,\lambda}(\varphi,\psi)\|_{\mathcal{C}_p^{\beta+2}} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon C_{\varepsilon,\lambda}(\varphi,\psi)\|_{\mathcal{C}_p^\beta} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha}\|\psi\|_{\mathcal{C}^\beta}$$

Proof. By the elliptic Schauder estimates in Proposition 5.6, it is sufficient to prove that

$$\|(-\mathcal{A}_\varepsilon+\lambda)C_{\varepsilon,\lambda}(\varphi,\psi)\|_{\mathcal{C}_p^\beta} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha}\|\psi\|_{\mathcal{C}^\beta}.$$

In turn to obtain this bound, since the quantities below are supported in an annulus $2^i\mathcal{A}$, it suffices to estimate for $i \geq 0$

$$\|S_{i-1}\varphi\Delta_i\psi - (-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi(-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi]\|_{L^p} \lesssim 2^{-i\beta}\|\phi\|_{\mathcal{C}_p^\alpha}\|\psi\|_{\mathcal{C}^\beta}. \tag{46}$$

Let $B_\varepsilon(\varphi,\psi)$ be defined as

$$B_\varepsilon(\varphi,\psi)(x) = \varepsilon^{-2}\fint_{B_\varepsilon(x)}\mathrm{d}y\,\fint_{B_\varepsilon(y)}\mathrm{d}z\,\,(\varphi(z)-\varphi(x))(\psi(z)-\psi(x)).$$

Then $\mathcal{A}_\varepsilon$ can be decomposed as

$$\mathcal{A}_\varepsilon(\varphi\cdot\psi) = \mathcal{A}_\varepsilon(\varphi)\cdot\psi + \varphi\cdot\mathcal{A}_\varepsilon(\psi) + B_\varepsilon(\varphi,\psi),$$

Hence proving Equation (46) reduces to finding a bound for

$$\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi](-\mathcal{A}_\varepsilon+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} + \|B_\varepsilon(S_{i-1}\varphi,(-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi)\|_{L^p}.$$

Starting with the first term, one has:

$$\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi](-\mathcal{A}_\varepsilon+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} \lesssim \|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi]\|_{L^p}\|(-\mathcal{A}_\varepsilon+\lambda)^{-1}[\Delta_i\psi]\|_{L^\infty}.$$

If $2^{-i} \geq \varepsilon$, since $\alpha < 2$, one can estimate via Proposition 5.2:

$$\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi]\|_{L^p} \leq \sum_{j=-1}^{i-1}\|(-\mathcal{A}_\varepsilon+\lambda)[\Delta_j\varphi]\|_{L^p} \lesssim \sum_{j=-1}^{i-1}2^{j(2-\alpha)}\|\varphi\|_{\mathcal{C}_p^\alpha} \lesssim 2^{i(2-\alpha)}\|\varphi\|_{\mathcal{C}_p^\alpha}.$$

If $2^{-i} \leq \varepsilon$ choose $i(\varepsilon)$ such that $2^{-i(\varepsilon)} \simeq \varepsilon$ (uniformly over ε). Then following the previous calculations and using that $\alpha > 0$:

$$\begin{aligned}\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi]\|_{L^p} &\leq \sum_{j=-1}^{i(\varepsilon)-1}\|(-\mathcal{A}_\varepsilon+\lambda)[\Delta_j\varphi]\|_{L^p} + \sum_{j=i(\varepsilon)}^{i-1}\|(-\mathcal{A}_\varepsilon+\lambda)[\Delta_j\varphi]\|_{L^p}\\ &\lesssim \varepsilon^{-(2-\alpha)}\|\varphi\|_{\mathcal{C}_p^\alpha}.\end{aligned}$$

By Proposition 5.7 moreover

$$\|(-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi\| \lesssim \left(2^{-2i}1_{\{2^{-i}\geq\varepsilon\}} + \varepsilon^2 1_{\{2^{-i}\leq\varepsilon\}}\right)2^{-\beta i}\|\psi\|_{\mathcal{C}^\beta},$$

which provides a bound of the required order for (46). Finally, we have to bound the term containing B_ε. If $2^{-i} \geq \varepsilon$

$$\|\nabla S_{i-1}\varphi\|_{L^p} \|\nabla(-\mathcal{A}_\varepsilon + \lambda)^{-1}\Delta_i \psi\|_{L^\infty} \lesssim 2^i \|S_{i-1}\varphi\|_{L^p} 2^{-(1+\beta)i} \|\psi\|_{\mathcal{C}^\beta} \lesssim 2^{-\beta i} \|\varphi\|_{\mathcal{C}^\alpha_p} \|\psi\|_{\mathcal{C}^\beta},$$

whereas if $2^{-i} \leq \varepsilon$

$$\begin{aligned} \|B_\varepsilon(S_{i-1}\varphi, (-\mathcal{A}_\varepsilon + \lambda)^{-1}\Delta_i\psi)\|_{L^p} &\lesssim \varepsilon^{-2} \|S_{i-1}\varphi\|_{L^p} \|(-\mathcal{A}_\varepsilon + \lambda)^{-1}\Delta_i \psi\|_{L^\infty} \\ &\lesssim 2^{-\beta i} \|\varphi\|_{L^p} \|\psi\|_{\mathcal{C}^\beta}. \end{aligned}$$

This bound is again of the correct order for (46) and hence the proof is concluded. □

Appendix C. Multiple discrete stochastic integrals

This appendix is devoted to results on discrete multiple stochastic integrals. The discussion is based on approach of [39, Section 5], which in turn based on [10]. The following lemma (see also [10, Theorem 2.3]) provides an estimate for the discrete multiple stochastic integrals. The definition of such integrals, in particular the definition of $\xi(z_1) \diamond \cdots \diamond (z_n)$, can be found at the beginning of Section 5 in [39].

Lemma C.1. *[[39], Lemma 5.1] Let ξ_ε satisfy Assumption 1.5. Fix $n \geq 1$. For $f \in L^2\big((\mathbb{Z}^d_\varepsilon)^n\big)$ define the discrete stochastic integral by*

$$\mathcal{J}_n f := \sum_{z_1,\ldots,z_n \in \mathbb{Z}^d_\varepsilon} \varepsilon^{-dn} f(z_1, \ldots, z_n) \xi(z_1) \diamond \cdots \diamond \xi(z_n).$$

Then for $p \geq 2$

$$\Big[\mathbb{E}\big|\mathcal{J}_n f\big|^p\Big]^{\frac{1}{p}} \lesssim \|f\|_{L^2((\mathbb{Z}^d_\varepsilon)^n)}.$$

The next lemma provides a convergence criterion for discrete multiple stochastic integrals to continuous multiple stochastic integrals. In the following $\mathcal{F}_{\mathbb{Z}^d_\varepsilon}$ indicates the natural discrete Fourier transform on $\mathbb{Z}^d_\varepsilon$. The definition can be found in the glossary of [39].

Lemma C.2. *[[39], Lemma 5.4] Let ξ_ε satisfy Assumption 1.5. Fix $n \geq 1$. Fix a sequence $f_\varepsilon \in L^2((\mathbb{Z}^d_\varepsilon)^n)$. Assume that there exists a function $g \in L^2\big((\mathbb{R}^d)^n\big)$ such that*

$$\sup_{\varepsilon \in (0,1/2)} |1_{(\varepsilon^{-1}\mathbb{T}^d)^n} \mathcal{F}_{(\mathbb{Z}^d_\varepsilon)^n} f_\varepsilon| \leq g,$$

and there exists an $f \in L^2\big((\mathbb{R}^d)^n\big)$ such that

$$\lim_{\varepsilon \to 0} \|1_{(\varepsilon^{-1}\mathbb{T}^d)^n} \mathcal{F}_{(\mathbb{Z}^d_\varepsilon)^n} f_\varepsilon - \mathcal{F}_{(\mathbb{R}^d)^n} f\|_{L^2((\mathbb{R}^2)^n)} = 0.$$

Then, if $\xi(\mathrm{d}z_1) \diamond \ldots \xi(\mathrm{d}z_k)$ denotes the Wiener-Itô integral against the Gaussian stochastic measure induced by a white noise ξ on $\mathbb{R}^d$, the following convergence holds in distribution

$$\lim_{\varepsilon \to 0} \mathcal{J}_n f_\varepsilon = \int_{(\mathbb{R}^d)^n} f(z_1, \ldots, z_k) \xi(\mathrm{d}z_1) \diamond \cdots \diamond \xi(\mathrm{d}z_k).$$

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