Manin's Conjecture for del Pezzo surfaces

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1 Introduction

As we have seen in the previous talks, a contemporary version of Manins's Conjecture (or Manin–Batyrev–Peyre Conjecture) can be formulated as follows.

Conjecture 1.1. *Let X be a Fano variety over a number field k, with $X(k) \neq \emptyset$, and let $\mathcal{L}$ be an adelically metrized ample line bundle on X. Then there exists a thin subset $Z \subset X(k)$ such that*

$$N(X(k) \setminus Z, \mathcal{L}, B) = \#\{P \in X(k) \setminus Z \mid H_{\mathcal{L}}(P) \leq B\} \sim C_{k,Z,\mathcal{L}} B^{a(X,\mathcal{L})} \log(B)^{b(X,\mathcal{L})-1}$$

as $B \to \infty$. Here $C_{k,Z,\mathcal{L}}$ is Peyre's constant as we saw last week, and a and b are the geometric invariants that we saw in week 1.

The goal for today is to see some explicit examples of Fano varieties that satisfy this conjecture, while giving an overview of known results in low dimensions. The simplest case is dimension 1, and we start by showing that Conjecture 1.1 holds in this case. As we will see, already in dimension 2, Conjecture 1.1 is still far from being proven.

1.1 Dimension 1 - the conjecture for projective spaces

In dimension 1, Fano varieties with a rational point are isomorphic to the projective line. Schanuel gave an asymptotic formula for the number of rational points of bounded height on projective spaces over number fields. Over $\mathbb{Q}$ his result states [Sch79, Corollary to Theorem 3]:

$$N(\mathbb{P}^n(\mathbb{Q}), \mathcal{O}(1), B) = \frac{2^n}{\zeta(n+1)} B^{n+1} + O_n(B^n \log(B)^{d_n}), \tag{1}$$

where $b_n = 1$ if $n = 1$, and $b_n = 0$ otherwise.

Remark 1.2.

- The height used in (1) induced by $\mathcal{O}(1)$ is the standard height in projective space: for a point $P = (x_0 : \cdots : x_n) \in \mathbb{P}^n$ with $x_i \in \mathbb{Z}$ mutually coprime, this gives $H_{\mathcal{O}^1}(P) = \max_i |x_i|$.
- We saw in the first talk that we have $\omega_{\mathbb{P}^n} = \mathcal{O}(-(n+1))$, and therefore we found $a(\mathbb{P}^n, \mathcal{O}(1)) = n+1$ and $b(\mathbb{P}^n, \mathcal{O}(1)) = 1$. So the order of B and $\log(B)$ in (1) agree with Conjecture 1.1.
- A clear short proof of (1) using Möbius inversion can be found in [Bro09, Theorem 1.2]. The proof in Schanuel is a lot more involved since he proves it for general number fields.

If we instead use the anticanonical height induced by $\omega_{\mathbb{P}^n}^{-1} = \mathcal{O}(n+1)$, then, since we have $H_{\mathcal{O}(n+1)}(P) = H_{\mathcal{O}(1)}(P)^{n+1}$, we have $N(\mathbb{P}^{n-1}(\mathbb{Q}), \omega_{\mathbb{P}^n}^{-1}, B) = N(\mathbb{P}^{n-1}(\mathbb{Q}), \mathcal{O}(1), B^{1/n+1})$, and we find

$$N(\mathbb{P}^n(\mathbb{Q}), \omega_{\mathbb{P}^n}^{-1}, B) \sim \frac{2^n}{\zeta(n+1)} B \tag{2}$$

as $B \to \infty$.

Remark 1.3. We also saw in the first talk that for the anticanonical sheaf we have $a(\mathbb{P}^{n-1}, \omega^1) = b(\mathbb{P}^{n-1}, \omega^1) = 1$, so the power of B and $\log B$ in (2) agree with Conjecture 1.1 as well.

Remark 1.4. The line bundle $\mathcal{O}(1)$ in projective space is very ample and induces the trivial embedding of the space in itself. The anticanonical embedding is the $n+1$-uple embedding of projective space. In the case of $\mathbb{P}^1$, this gives the two types of rational curves in the plane: they are either of degree 1 or degree 2.

Peyre has shown [Pey95, Proposition 6.1.1] that his constant agrees with the constant in (2), proving that Conjecture 1.1 holds for projective spaces, and in particular for all Fano varieties of dimension 1 (in fact, he shows this over all number fields). Here is an outline of the proof:

- By definition we have $C_{\text{Peyre}}(\mathbb{P}^n_{\mathbb{Q}}) = \alpha(\mathbb{P}^n_{\mathbb{Q}})\tau(\mathbb{P}^n_{\mathbb{Q}})$, where

  $$\alpha(\mathbb{P}^n_{\mathbb{Q}}) = \theta_{\omega_{\mathbb{P}^n}^{-1}}(\check{C}_{\text{eff}}(\mathbb{P}^n_{\mathbb{Q}}) \cap \mathcal{H}_{\omega_{\mathbb{P}^n}^{-1}}(1)) = \frac{1}{n+1}$$

  since $\omega_{\mathbb{P}^n}^{-1} = \mathcal{O}(n+1)$ [Pey95, (2.2.3)], and

  $$\tau(\mathbb{P}^n_{\mathbb{Q}}) = \boldsymbol{\omega_h}(\overline{\mathbb{P}^n_{\mathbb{Q}}}) = \lim_{s\to 1}(s-1)L_S(s, \text{Pic } \mathbb{P}^n_{\overline{\mathbb{Q}}})\boldsymbol{\omega_{h,\infty}}(\mathbb{P}^n_{\mathbb{R}}(\mathbb{R})) \prod_{p \text{ prime}} \frac{\boldsymbol{\omega_{h,p}}(\mathbb{P}^n_{\mathbb{Q}_p}(\mathbb{Q}_p))}{L_p(1, \text{Pic } \mathbb{P}^n_{\overline{\mathbb{Q}}})},$$

  where $\boldsymbol{h}$ is the height induced by $\omega_{\mathbb{P}^n}^{-1}$, and using:

  - the rank of Pic $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ is 1 (which is the power of $(s-1)$ in the expression above);
  - S is a finite set of places of $\mathbb{Q}$ such that there exists a model of $\mathbb{P}^n_{\mathbb{Q}}$ over $\mathbb{Z}_{\mathbb{S}}$ satisfying certain conditions [Pey95, Lemme 2.1.1]. We can take $S = \{\infty\}$ here.

- We have

$$\begin{aligned}
\boldsymbol{\omega_{h,\infty}}(\mathbb{P}^n_{\mathbb{R}}(\mathbb{R})) &= \int_{\mathbb{R}^n} \frac{1}{\max\{1, |x_1|^{n+1}, \dots, |x_n|^{n+1}\}}\, dx_1 \dots dx_n \\
&= 2^n \left(\int_{[0,1]^n} 1 \;\, dx_1 \dots dx_n + \int_{\mathbb{R}^n_{\geq 1}} \frac{1}{\max\{x_1^{n+1}, \dots, x_n^{n+1}\}}\, dx_1 \dots dx_n \right) \\
&= 2^n \left(1 + n \int_{\mathbb{R}^n_{\geq 1},\, x_n > \max\{x_1, \dots, x_{n-1}\}} \frac{1}{x_n^{n+1}}\, dx_1 \dots dx_n \right) \\
&= 2^n \left(1 + n \int_1^\infty \frac{1}{x_n^{n+1}} \left(\int_{\mathbb{R}^{n-1}_{\geq 1},\, \max\{x_1, \dots, x_{n-1}\} < x_n} 1\; dx_1 \dots dx_{n-1} \right) dx_n \right) \\
&= 2^n \left(1 + n \int_1^\infty \frac{1}{x_n^{n+1}} x_n^{n-1} dx_n \right) \\
&= 2^n(1+n)
\end{aligned}$$

- From [Pey95, Lemme 2.2.1] we have, for all primes p,

$$\omega_{h,p}(\mathbb{P}^n_{\mathbb{Q}_p}(\mathbb{Q}_p)) = \frac{\#\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p)}{p^n} = \frac{p^n + p^{n-1} + \cdots + p + 1}{p^n} = \sum_{k=0}^{n} \frac{1}{p^k}.$$

- We have $L_p(s, \mathrm{Pic}\ \mathbb{P}^n_{\overline{\mathbb{Q}}}) = (1 - \frac{1}{p^s})^{-1}$ since $\mathrm{Pic}\ \mathbb{P}^n_{\overline{\mathbb{Q}}} \cong \mathbb{Z}$ is the trivial representation [Jah14, p. 60], from which it also follows that

$$L_S(s, \mathrm{Pic}\ \mathbb{P}^n_{\overline{\mathbb{Q}}}) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p}} = \zeta(s),$$

and hence $\lim_{s\to 1}(s-1)L_S(s, \mathrm{Pic}\ \mathbb{P}^n_{\overline{\mathbb{Q}}}) = 1$.

- So we obtain

$$C_{\mathrm{Peyre}} = \frac{1}{n+1} 2^n (n+1) \prod \sum_{k=0}^{n} \frac{1}{p^k} \left(1 - \frac{1}{p}\right) = 2^n \prod \left(1 - \frac{1}{p^{n+1}}\right) = \frac{2^n}{\zeta(n+1)}.$$

2 Del Pezzo surfaces

Definition 2.1. A **del Pezzo surface** is a Fano variety of dimension two, that is, it is a *nice* (smooth, projective, geometrically integral) surface X with ample anticanonical divisor $-K_X$.
The **degree** of a del Pezzo surface X is the self-intersection number K_X^2.

As we will see, the degree of a del Pezzo surface is an integer between 1 and 9. The geometry of del Pezzo surfaces over algebraically closed fields is very well understood.

Definition 2.2. Let $r \leq 8$, and let $P_1, \ldots, P_r$ be points in $\mathbb{P}^2$. Then we say that $P_1, \ldots, P_r$ are **in general position** if no three of them lie on a line, no six of them lie on a conic, and no eight of them lie on a singular cubic with one of these eight points at the singularity.

Theorem 2.3. *Let k be an algebraically closed field, and let X be a del Pezzo surface over k. Then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, in which case X is of degree 8, or to $\mathbb{P}^2$ blown up at $r \leq 8$ points in general position, in which case the degree of X is $9 - r$.*

Proof. See [Man74], Theorem 24.4, Theorem 26.2, and Remark 26.3. □

For del Pezzo surfaces of degree at least 3, the anticanonical divisor is very ample.

Theorem 2.4. *For $r \leq 8$, let $P_1, \ldots, P_r$ be points in general position in $\mathbb{P}^2$. Let X be the blow-up of $\mathbb{P}^2$ in these points. Then $-K_X$ is ample, and very ample if $r \leq 6$.*

Proof. See [Man74], Theorem 24.5. □

Remark 2.5.

- If X is a del Pezzo surface of degree $d \geq 3$, then the linear system $|-K_X|$ determines an embedding in $\mathbb{P}^n$, with $n = \dim |-K_X| = d$ [Kol96, II.3.2.5.2], and the image of X under this embedding has degree $(-K_X)^2 = d$. So for $d \geq 3$, a del Pezzo surface of degree d is isomorphic to a surface of degree d in $\mathbb{P}^d$.

- For a del Pezzo surface X of degree 2, $-2K_X$ is very ample and defines an embedding as a degree 8 surface in $\mathbb{P}^6$. For degree 1, $-3K_X$ is very ample and defines an embedding as a degree 9 surface in $\mathbb{P}^6$.

Example 2.6.

- Del Pezzo surfaces of degree 3 are smooth cubic surfaces in $\mathbb{P}^3$.
- Del Pezzo surfaces of degree 4 are complete intersections of two quadrics in $\mathbb{P}^4$.

If X is a del Pezzo surface of degree d that is isomorphic to $\mathbb{P}^2$ blown up in $9-d$ points $P_1, \ldots, P_r$ in general position, then we have

$$\text{Pic } X \cong \mathbb{Z}^{10-d}.$$

More specifically, if E_i is the class of the exceptional curve corresponding to P_i, and L the class of the pullback of a line l in $\mathbb{P}^2$ not passing through any of the P_i, then $\{L, E_1, \ldots, E_r\}$ forms a basis for Pic X. We have

$$-K_X = 3L - \sum_{i=1}^{3} E_i.$$

Such a surface contains a finite number of **exceptional curves**, which are curves $C \subset X$ with $C^2 = C \cdot K_X = -1$. When $d \geq 3$, hence $-K_X$ is very ample, these curves correspond to lines on the image of the anticanonical embedding of X. Therefore we usually call exceptional curves **lines**. They can be described as follows.

Theorem 2.7. *Let $f : X \longrightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in the points $P_1, \ldots, P_r$. Then the image $f(C)$ of an exceptional curve $C \subset X$ is one of the following types.*

- *One of the points P_i;*
- *a line passing through two of the points P_i;*
- *a conic passing through five of the points P_i;*
- *a cubic passing through seven of the points P_i such that one of them is a double point;*
- *a quartic passing through eight of the points P_i such that three of them are double points;*
- *a quintic passing through eight of the points P_i such that six of them are double points;*
- *a sextic passing trough eight of the points P_i such that seven of them are double points and one is a triple point.*

Proof. [Man74, Theorem 26.2]. □

The number of exceptional curves on X thus depends on the degree d, and we obtain the following table.

d	1	2	3	4	5	6	7	8
exceptional curves on X	240	56	27	16	10	6	3	1

For example, we see here the famous result that a smooth cubic surface in $\mathbb{P}^3$ over an algebraically closed field contains 27 lines.

3 Manin's Conjecture for del Pezzo surfaces

Even though we understand the geometry of del Pezzo surfaces very well, it turns out that we are far from proving Manin's Conjecture for all such surfaces. We give an overview of what is known, combining what is stated in [Bro07, Bro09, Lou11, Jah14].

Remark 3.1. As we saw, del Pezzo surfaces contain a finite number of rational lines. Such lines should always be included in the exceptional set for Manin's Conjecture, since for a del Pezzo surface X containing a line L we have $\omega_X^{-1}|_L = \mathcal{O}_L(1)$, and as we saw earlier, for a rational line we have

$$N(L, \mathcal{O}(1), B) \sim CB^2.$$

In fact, in all results mentioned below, the exceptional set is taken to be the union of all rational lines on the surface.

Example 3.2. Let's start with a surface we saw last week: the blow-up X of the projective plane over $\mathbb{Q}$ in a rational point. We now know that this is a del Pezzo surface of degree 8 with one rational exceptional divisor E. Last week, we saw that we have

$$N(X \setminus E, \omega_X^{-1}, B) \sim \frac{8}{3\zeta(2)^2} B \log B.$$

We have $a(X, \omega_X^{-1}) = 1$ as usual, and Pic $X \cong \mathbb{Z}^2$, so $b(X, \omega_X^{-1}) = 2$. The powers of B and $\log B$ thus agree with Conjecture 1.1. To show that the constant agrees with Peyre's constant, we show how we can compute the latter for del Pezzo surfaces.

3.1 Peyre's Constant for split del Pezzo surfaces

Let X be a del Pezzo surface of degree at least 3. We show known techniques to compute

$$C_{\text{Peyre}}(X) = \alpha(X)\tau(X).$$

The constant $\alpha(X)$ is not difficult to compute explicitly when the pseudo-effective cone has a small number of generators. In [Jah14, Section A.II.5], this is done for several del Pezzo surfaces of degree 3, as well as for the surface in Example 3.2 and for $\mathbb{P}^1 \times \mathbb{P}^1$. As is shown there, $\alpha(X)$ depends on the Galois action on the lines on the surface.
The factor α has been computed using mathematical software for almost all del Pezzo surfaces [Der07, DJT08, DEJ14].

The tamagawa number $\tau(X)$ is discussed in [Lou11] for so-called **split** del Pezzo surfaces of degree at least 3.

Definition 3.3. [Lou11, Definition 2.5.1] A smooth projective geometrically rational surface X over a field k is **split** if every divisor over $\overline{k}$ is linearly equivalent to one defined over the k. For a del Pezzo surface X this is equivalent to X being either isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, or to the blow-up of $\mathbb{P}^2$ at a collection of rational points.

Lemma 3.4. *[Lou11, Lemma 5.3.3] Let X be a split del Pezzo surface with Picard rank ρ. Then we have*

$$\tau(X) = \boldsymbol{\omega}_\infty(X(\mathbb{R})) \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^\rho \left(1 + \frac{\rho}{p} + \frac{1}{p^2}\right),$$

where $\rho =$ rank Pic(X).

We sketch the proof:

- We can take the set S to be $S = \{\infty\}$, since X is the blow-up of $\mathbb{P}^2$ in $9-(\rho-1)$ points [Lou11, Lemma 5.3.2].

- Since X is split we have $\mathrm{Pic}\ X_{\overline{\mathbb{Q}}} \cong \mathbb{Z}^\rho$. This implies, as we saw with projective space, that we have $L_p(s, \mathrm{Pic}\ X_{\overline{\mathbb{F}_p}}) = \left(1-\frac{1}{p^s}\right)^{-\rho}$, and therefore it follows that we have $\lim_{s\to 1}(s-1)^\rho L_S(s, \mathrm{Pic}\ X_{\overline{\mathbb{Q}}}) = \lim_{s\to 1}(s-1)^\rho \zeta(s)^\rho = 1$.

- From [Pey95, Lemme 2.2.1] we have $\boldsymbol{\omega_p}(X(\mathbb{Q}_p)) = \frac{\# X_{\mathbb{F}_p}(\mathbb{F}_p)}{p^2}$. As is shown in [Lou11, Lemma 5.3.2], in the chosen model of X, the fiber $X_{\mathbb{F}_p}$ is isomorphic to $\mathbb{P}^2_{\mathbb{F}_p}$ blown up in $9-d$ smooth rational points. Since $\#\mathbb{P}^2_{\mathbb{F}_p}(\mathbb{F}_p) = p^2+p+1$, and blowing up a smooth point replaces the point by one copy of $\mathbb{P}^1_{\mathbb{F}_p}$, which has $p+1$ rational points, we obtain
$$\# X_{\mathbb{F}_p}(\mathbb{F}_p) = p^2+p+1+(\rho-1)p = p^2+\rho p+1.$$

Example 3.5. Let's come back to Example 3.2. For this surface X we have $\alpha(X) = \frac{1}{6}$ [Pey95, Proof Lemme 9.4.2],[Jah14, Example 5.6] and $\boldsymbol{\omega}_\infty(X(\mathbb{R})) = 16$ [Pey95, Théorème 9.6.1].
Since $\rho = 2$, using Lemma 3.4 we find

$$\begin{aligned} C_{\mathrm{Peyre}}(X) &= \frac{1}{6}\cdot 16 \prod_{p\ \mathrm{prime}} \left(1-\frac{1}{p}\right)^2 \left(1+\frac{2}{p}+\frac{1}{p^2}\right) \\ &= \frac{8}{3}\prod_{p\ \mathrm{prime}} \left(1-\frac{2}{p^2}+\frac{1}{p^4}\right) \\ &= \frac{8}{3}\prod_{p\ \mathrm{prime}} \left(1-\frac{1}{p^2}\right)^2 \\ &= \frac{8}{3\zeta(2)^2}. \end{aligned}$$

We conclude that Conjecture 1.1 holds for this del Pezzo surface of degree 8.

3.2 Known results for smooth surfaces

- Del Pezzo surfaces of degree at least 6 are toric varieties, for which Manin's Conjecture is known [BT98].

- For degree 5, Conjecture 1.1 is proven for a couple of surfaces [dlB02, dlBF04, Bro03].

- For degree 4, for a long time Conjecture 1.1 was not known even in a single case. In 2011 de la Bretèche and Browning provided the first example [dlBB11], which is the only known to date. This surface has two conic fibrations, and the result is proven by reducing the problem to counting rational points on the conic fibers.

- For degrees ≤ 3, Conjecture 1.1 is not proven for a single smooth del Pezzo surface.

  For cubic surfaces, the best upper bound is by Heath-Brown, who showed [HB97] that for a smooth cubic surface X containing 3 coplanar lines over $\mathbb{Q}$, when taking Z to be the set of lines in S, we have
$$N(X\setminus Z, \omega_S^{-1}, B) = O_{\varepsilon,S}(B^{4/3}+\varepsilon).$$

Example 3.6. The Fermat cubic surface is defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \subset \mathbb{P}^3.$$

It is obtained by blowing up the points

$$\begin{aligned} P_1 &= [1 : -\zeta_3^2 : -\zeta_3], & P_3 &= [0 : 1 : -\zeta_3], & P_5 &= [-\zeta_3 : 1 : 1], \\ P_2 &= [1 : -\zeta_3 : -\zeta_3^2], & P_4 &= [0 : 1 : -\zeta_3^2], & P_6 &= [-\zeta_3^2 : 1 : 1], \end{aligned}$$

where ζ_3 is a primitive cubic root of unity, and it contains the three lines

$$x_1 + x_2 = x_3 + x_4 = 0, \;\; x_1 + x_3 = x_2 + x_4 = 0, \;\; x_1 + x_4 = x_2 + x_3 = 0.$$

Under the conjecture that elliptic curves over $\mathbb{Q}$ with conductor C have rank of order $o(\log C)$ (it can be easily shown that the rank is of order $O(\log C)$, and partial results on this conjecture are known [Bro09, Page 27]), Heath-Brown proved his asymptotic for all smooth cubic surfaces [HB98]. His strategy is to cover the rational points of height at most B with plane sections of the surface, which reduces the problem to counting points of bounded height on elliptic curves.

The best unconditional result for smooth cubic surfaces is by Salberger, who announced at the conference *Géométrie arithmétique et variétés rationnelles* in Luminy in 2007 (see [Bro09, Page 27]; I could not find the paper itself) that for any smooth cubic surface X with Z the set of lines on X we have $N(X \setminus Z, \omega_S^{-1}, B) = O_\varepsilon(B^{\frac{7}{12}+\varepsilon})$ for any $\varepsilon > 0$. Moreover, Slater and Swinnerton-Dyer have shown [SSD98] a lower bound of $B(\log B)^{\rho-1}$ for surfaces that contain a pair of skew lines over $\mathbb{Q}$, where ρ is the Picard rank of the surface.

3.3 Singular surfaces

It turns out that Conjecture 1.1 is in general easier to prove for **singular del Pezzo surfaces**.

Definition 3.7. A **singular del Pezzo surface** X is a singular normal projective surface with only du Val singularities (rational double points), whose anticanonical divisor $-K_X$ is ample. It's degree is defined as K_X^2.

Theorem 3.8. *[Lou11, Theorem 2.2.4] Let X be a singular del Pezzo surface, and let $\tilde{X} \to X$ be a minimal desingularisation. Then $\tilde{X}$ is a smooth projective surface with nef and big anticanonical divisor. We call $\tilde{X}$ a* ***generalized del Pezzo surface****.*

Remark 3.9. When studying Conjecture 1.1 for a singular del Pezzo surface X, we take the a- and b-invariant for the anticanonical sheaf on the minimal desingularisation $\tilde{X}$ of X (that is, $a = 1$ and b is the rank of the Picard group of $\tilde{X}$), and we set $C_{\mathrm{Peyre}}(X) = C_{\mathrm{Peyre}}(\tilde{X})$ (Lemma 3.4 is actually more generally stated for generalized del Pezzo surfaces). The latter reflects that, outside exceptional curves, the counting problems for X and $\tilde{X}$ should be the same.

A singular del Pezzo surface X is often labeled by (d, D), where d is its degree and D is the Dynkin diagram related to its singularities, called the **singularity type** of X. This diagram corresponds to the -2-curves on the desingularisation $\tilde{X}$.

The singularity types over $\overline{\mathbb{Q}}$ of singular del Pezzo surfaces are known, and there is great interest in proving Conjecture 1.1 for each type. For degree ≥ 6, all singular del Pezzo surfaces are either toric or equivariant compactifications of $\mathbb{G}_a^2$, in which case Conjecture 1.1 is known to hold [BT98, CLT02]. For degrees 4 and 5, almost all singularity types have been proven, and several have been proven for degree 3. There is even one example of a singular del Pezzo surface of degree 2 that is shown to satisfy Conjecture 1.1 [BB13].
A great overview of all del Pezzo surfaces, possibly singular with singularity type over $\overline{\mathbb{Q}}$ and whether or not Conjecture 1.1 is proven, is given in [Lou11, Table 5.2].

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