OBVP 公式推导

Cost Function：

$$J=\int_0^T g(x,u)=\int_0^T (1+a_x^2+a_y^2+a_z^2)dt \tag{1}$$

Robot State:

$$x=\begin{pmatrix} p_x & p_y & p_z & v_x & v_y & v_z \end{pmatrix}^T \tag{2}$$

Control Input:

$$u=\begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \tag{3}$$

System equation:

$$\dot{x}=f(x,u)=\begin{pmatrix} v_x & v_y & v_z & a_x & a_y & a_z \end{pmatrix} \tag{4}$$

OBVP Solving

1) Costate are defined according to the number of system variables

$$x=\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{pmatrix}^T \tag{5}$$

2) Define the Hamiltonian function

$$H(x,u,\lambda)=g(x,u)+\lambda^T f(x,u)$$

$$=(1+a_x^2+a_y^2+a_z^2)+\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \\ a_x \\ a_y \\ a_z \end{pmatrix} \tag{6}$$

$$=(1+a_x^2+a_y^2+a_z^2)+\lambda_1 v_x+\lambda_2 v_y+\lambda_3 v_z+\lambda_4 a_x+\lambda_5 a_y+\lambda_6 a_z$$

3) 根据边界条件：

$$\dot{\lambda}=-\nabla H(x^*,u^*,\lambda)=\begin{pmatrix} 0 & 0 & 0 & -\lambda_1 & -\lambda_2 & -\lambda_3 \end{pmatrix}^T \tag{7}$$

4) 根据（7）式得到下列一组含系数解，其中未知数α和β的下表数字 1,2,3 分别代表 x，y，z 轴

$$\lambda=\begin{bmatrix} 2\alpha_1 \\ 2\alpha_2 \\ 2\alpha_3 \\ -2\alpha_1 t-2\beta_1 \\ -2\alpha_2 t-2\beta_2 \\ -2\alpha_3 t-2\beta_3 \end{bmatrix} \tag{8}$$

5)　最优输入方程如下所示：

$$
\begin{aligned}
u^* &= \arg\min_{a(t)} H(x^*, u, \lambda) \\
&= \lambda_1 v_x^* + \lambda_2 v_y^* + \lambda_3 v_z^* + \left(\lambda_4 a_x + \lambda_5 a_y + \lambda_6 a_z\right) + (1 + a_x^2 + a_y^2 + a_z^2)
\end{aligned} \tag{9}
$$

为了使上式取到最小值，只需要分别让 a_x、a_y、a_z 相关项分别取最小即可，如下所示：

$$
u^* = \begin{bmatrix} a_x^* \\ a_z^* \\ a_z^* \end{bmatrix} = \begin{bmatrix} \arg\min\left(\lambda_4 a_x + a_x^2\right) \\ \arg\min\left(\lambda_5 a_y + a_y^2\right) \\ \arg\min\left(\lambda_6 a_z + a_z^2\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\lambda_4 \\ -\frac{1}{2}\lambda_5 \\ -\frac{1}{2}\lambda_6 \end{bmatrix} = \begin{bmatrix} \alpha_1 t + \beta_1 \\ \alpha_2 t + \beta_2 \\ \alpha_3 t + \beta_3 \end{bmatrix} \tag{10}
$$

6)　根据 u^*，我们可以通过积分得到 x^*

$$
x^* = \begin{bmatrix} p_x^* \\ p_y^* \\ p_z^* \\ v_x^* \\ v_y^* \\ v_z^* \end{bmatrix} = \begin{bmatrix} \int\int a_x^* dt \\ \int\int a_y^* dt \\ \int\int a_z^* dt \\ \int a_x^* dt \\ \int a_y^* dt \\ \int a_z^* dt \end{bmatrix} = \begin{bmatrix} \frac{1}{6}\alpha_1 t^3 + \frac{1}{2}\beta_1 t^2 + v_{x0} t + p_{x0} \\ \frac{1}{6}\alpha_2 t^3 + \frac{1}{2}\beta_2 t^2 + v_{y0} t + p_{y0} \\ \frac{1}{6}\alpha_3 t^3 + \frac{1}{2}\beta_3 t^2 + v_{z0} t + p_{z0} \\ \frac{1}{2}\alpha_1 t^2 + \beta_1 t + v_{x0} \\ \frac{1}{2}\alpha_2 t^2 + \beta_2 t + v_{y0} \\ \frac{1}{2}\alpha_3 t^2 + \beta_3 t + v_{z0} \end{bmatrix} \tag{11}
$$

其中，初始状态 $x(0) = \begin{pmatrix} p_{x0} & p_{y0} & p_{z0} & v_{x0} & v_{y0} & v_{z0} \end{pmatrix}^T$

7)　Cost function 可以写成如下形式：

$$
\begin{aligned}
J = T &+ (\frac{1}{3}\alpha_1^2 T^3 + \alpha_1 \beta_1 T^2 + \beta_1^2 T) + (\frac{1}{3}\alpha_2^2 T^3 + \alpha_2 \beta_2 T^2 + \beta_2^2 T) \\
&+ (\frac{1}{3}\alpha_3^2 T^3 + \alpha_3 \beta_3 T^2 + \beta_3^2 T)
\end{aligned} \tag{12}
$$

8)　通过待定系数法，可以求得 α 和 β

$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\beta_1\\\beta_2\\\beta_3\end{bmatrix}=\begin{bmatrix}\frac{6\left(v_{xf}-v_{x0}\right)}{T^2}+\frac{12v_{x0}}{T^2}+\frac{12\left(p_{xf}-p_{x0}\right)}{T^3}\\\frac{6\left(v_{yf}-v_{y0}\right)}{T^2}+\frac{12v_{y0}}{T^2}+\frac{12\left(p_{yf}-p_{y0}\right)}{T^3}\\\frac{6\left(v_{zf}-v_{z0}\right)}{T^2}+\frac{12v_{z0}}{T^2}+\frac{12\left(p_{zf}-p_{z0}\right)}{T^3}\\-\frac{2\left(v_{xf}-v_{x0}\right)}{T^2}-\frac{6v_{x0}}{T}+\frac{6\left(p_{xf}-p_{x0}\right)}{T}\\-\frac{2\left(v_{yf}-v_{x0}\right)}{T^2}-\frac{6v_{y0}}{T}+\frac{6\left(p_{yf}-p_{y0}\right)}{T}\\-\frac{2\left(v_{zf}-v_{z0}\right)}{T^2}-\frac{6v_{z0}}{T}+\frac{6\left(p_{zf}-p_{z0}\right)}{T}\end{bmatrix}\tag{13}$$

其中终止状态$x(T)=\begin{pmatrix}p_{xf} & p_{yf} & p_{zf} & v_{xf} & v_{yf} & v_{zf}\end{pmatrix}^T$

分析发现α和β在每个轴上具有对称性，故我们只分析一个轴，并假设末端速度为 0，即$v_{xf}=v_{yf}=v_{zf}=0$。我们可以得到如下式子：

$$\begin{bmatrix}\alpha_1\\\beta_1\\\alpha_1\beta_1\\\alpha_1^2\\\beta_1^2\end{bmatrix}=\begin{bmatrix}\frac{6}{T^3}(-2\Delta P_x+v_{x0}T)\\\frac{2}{T^2}(3\Delta P_x-2v_{x0}T)\\\frac{12}{T^5}(-6\Delta P_x^2+7\Delta P_x v_{x0}T-2v_{x0}^2T^2)\\\frac{36}{T^6}(4\Delta P_x^2-4\Delta P_x v_{x0}T+v_{x0}^2T^2)\\\frac{4}{T^4}(9\Delta P_x^2-12\Delta P_x v_{x0}T+4v_{x0}^2T^2)\end{bmatrix}\tag{14}$$

上式中$\Delta P_x=(p_{xf}-p_{x0})$。

9) 因为α和β只是 T 的函数，因此代价函数公式（12）也只是 T 的函数，我们对公式 12 求导可得：

$$\dot{J}=1-\frac{36}{T^4}\left(\Delta P_x^2+\Delta P_y^2+\Delta P_z^2\right)+\frac{24}{T^3}\left(\Delta P_x v_{x0}+\Delta P_y v_{y0}+\Delta P_z v_{z0}\right)+\frac{4}{T}\left(v_{x0}^2+v_{y0}^2+v_{z0}^2\right)$$

10)　让上式为 0，我们可以得到最优代价函数 J 是 T 的取值。

$$T^4-4\left(v_{x0}^2+v_{y0}^2+v_{z0}^2\right)+24\left(\Delta P_x v_{x0}+\Delta P_y v_{y0}+\Delta P_z v_{z0}\right)-36\left(\Delta P_x^2+\Delta P_y^2+\Delta P_z^2\right)=0$$

采用伴随矩阵求特征值的方法求解，把得到的正根进行比较，得出最优 T

参考 https://blog.csdn.net/fb_941219/article/details/102984587