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ELSEVIER Applied Mathematics and Computation 147 (2004) 741–752

APPLIED MATHEMATICS AND COMPUTATION

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Four alternative patterns of the Hilbert curve

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Abstract

The second pattern of the Hilbert curve was shown by Moore in 1900. In this article, we present four more patterns. These patterns, together with Hilbert's original pattern and Moore's pattern, comprise a complete set of the Hilbert curve.

Keywords: Hilbert curve; Hilbert order; Space-filling curve; Spatial ordering

1. Introduction

A space-filling curve is a continuous mapping from the one-dimensional space onto the two-dimensional space. Hilbert discovered such a curve [5]. Usually, a parameter called *resolution* is used to describe the domain granularity of the concerned curve. The Hilbert's space-filling curve of resolution 1, 2, 3, and 4 are depicted in Figs. 1–4.

The elegance and complexity of the Hilbert curve have been appreciated by several researchers for many years ([1–8]). Although the literature has almost exclusively referred to the original pattern discovered by Hilbert, there exist other patterns. Moore reported an alternative pattern [9]. The graphical representation of Moore's pattern of resolution 1 is the same as shown in Fig. 1, and the representations of resolution 2, 3, and 4 are depicted in Figs. 5–7

In this paper, we present other four patterns of the Hilbert curve, named pattern $L1, L2, L3$, and $L4$, respectively.

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0096-3003/$ - see front matter
doi:10.1016/S0096-3003(02)00808-1

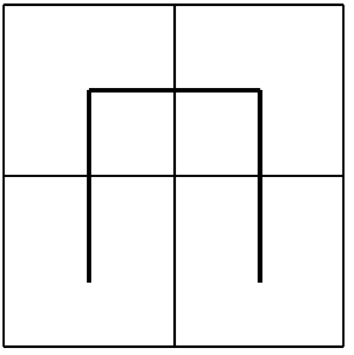

Fig. 1. Hilbert's pattern of resolution 1.

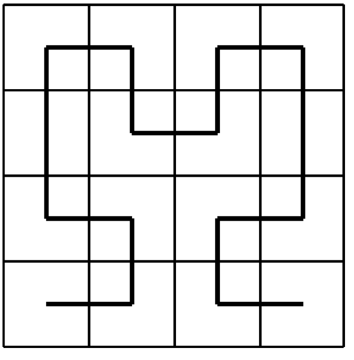

Fig. 2. Hilbert's pattern of resolution 2.

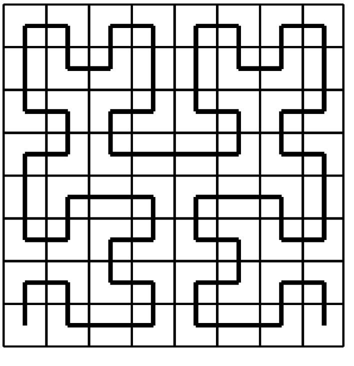

Fig. 3. Hilbert's pattern of resolution 3.

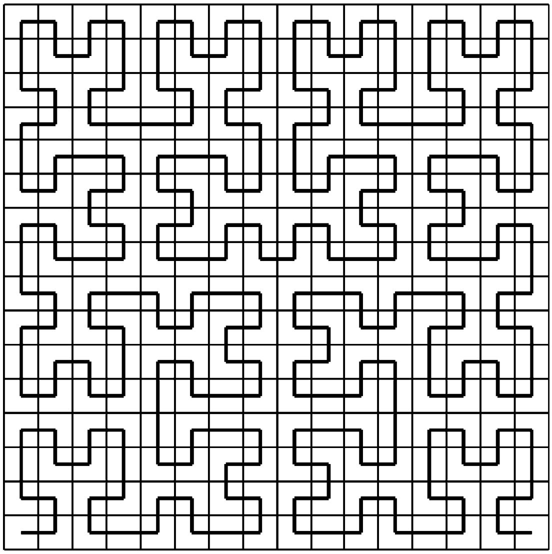

Fig. 4. Hilbert's pattern of resolution 4.

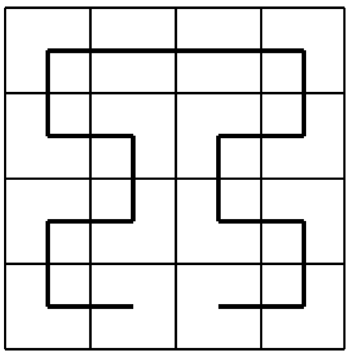

Fig. 5. Moore's pattern of resolution 2.

The configurations of these patterns can analytically be described in terms of complex algebra. Let $\mathbf{C}$ be the complex space. Consider the region $\mathbf{C}_+ = \{z|\Re(z) \geqslant 0, \Im(z) \geqslant 0, z \in \mathbf{C}\}$. We will construct the concerned patterns in $\mathbf{C}_+$. Given resolution r, consider a square D with its lower-left corner at the origin and with side 2^r, partitioned into four quadrants and indexed by 0, 1, 2, and 3, respectively. The sequence 0, 1, 2, 3 also specifies L patterns of resolution 1 (Fig. 8). The reference path for establishing the configuration of L patterns of resolution 1 in D is the same as shown in Fig. 1.

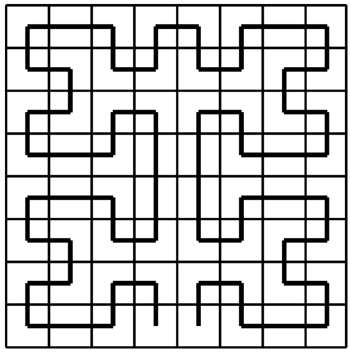

Fig. 6. Moore's pattern of resolution 3.

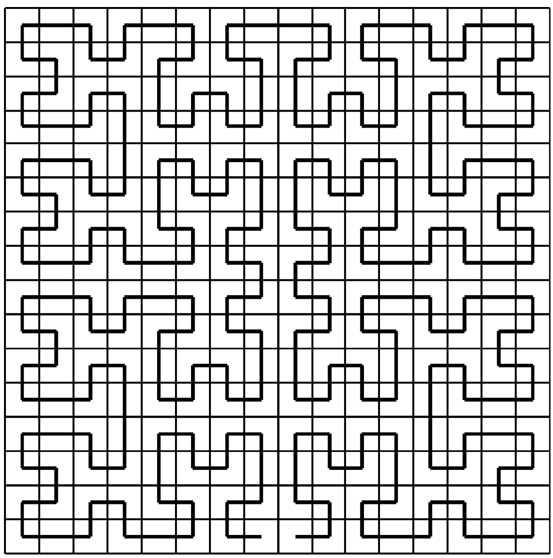

Fig. 7. Moore's pattern of resolution 4.

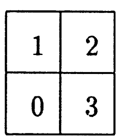

Fig. 8. Indexing L patterns of resolution 1.

2. Configuration of pattern $L1$

The generation of the configuration of pattern $L1$ from resolution 1 to 2 is determined by the following transformations:

$$T_0(z) = (1 + j - z)/2$$
$$T_1(z) = (j + z)/2$$
$$T_2(z) = (1 + j + z)/2$$
$$T_3(z) = 1 + (j - z)/2$$

where j is the imaginary number $\sqrt{-1}$ and T_k is the transformation that maps D onto its quadrant k $(k = 0, 1, 2, 3)$.

The generation of the configuration from resolution k to $k+1$ $(k \geqslant 2)$ is determined by the following transformations:

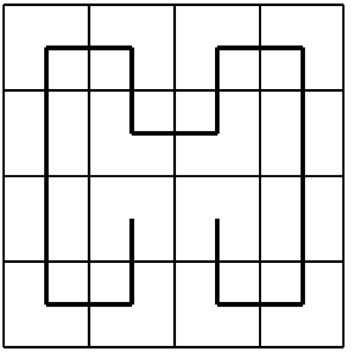

Fig. 9. Pattern $L1$ of resolution 2.

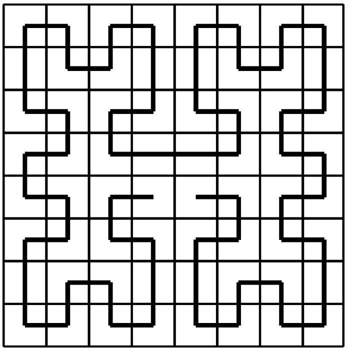

Fig. 10. Pattern $L1$ of resolution 3.

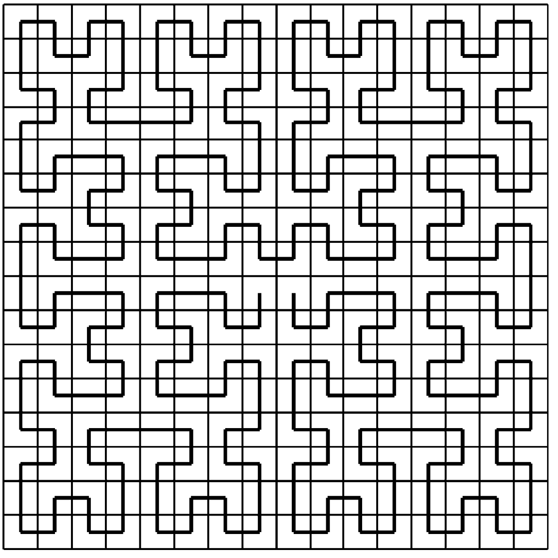

Fig. 11. Pattern $L1$ of resolution 4.

$$T_0(z) = j\bar{z}/2$$
$$T_1(z) = (j + z)/2$$
$$T_2(z) = (1 + j + z)/2$$
$$T_3(z) = 1 + j(1 - \bar{z})/2$$

The graphical representations of pattern $L1$ of resolution 2, 3, and 4 are depicted in Figs. 9–11.

3. Configuration of pattern $L2$

The generation of the configuration of pattern $L2$ from resolution 1 to 2 is determined by the following transformations:

$$T_0(z) = (j + \bar{z})/2$$
$$T_1(z) = [1 + j(1 + z)]/2$$
$$T_2(z) = [1 + j(2 - z)]/2$$
$$T_3(z) = (1 + j + \bar{z})/2$$

The transformations of the configuration from resolution k to $k + 1 (k \geqslant 2)$ are the same as for pattern $L1$.

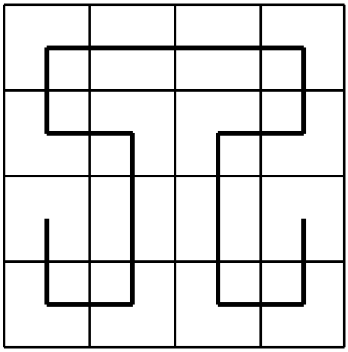

Fig. 12. Pattern $L2$ of resolution 2.

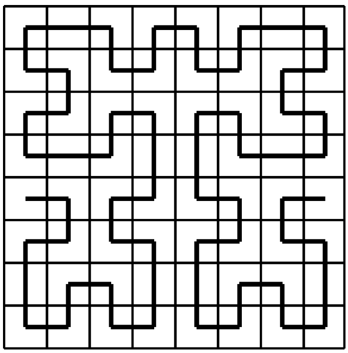

Fig. 13. Pattern $L2$ of resolution 3.

The graphical representations of pattern $L2$ for the resolutions 2, 3, and 4 are depicted in Figs. 12–14.

4. Configuration of pattern $L3$

The generation of the configuration of pattern $L3$ from resolution 1 to 2 is determined by the following transformations:

$$T_0(z) = j\bar{z}/2$$
$$T_1(z) = (j + z)/2$$
$$T_2(z) = (1 + j + z)/2$$
$$T_3(z) = 1 + (j - z)/2$$

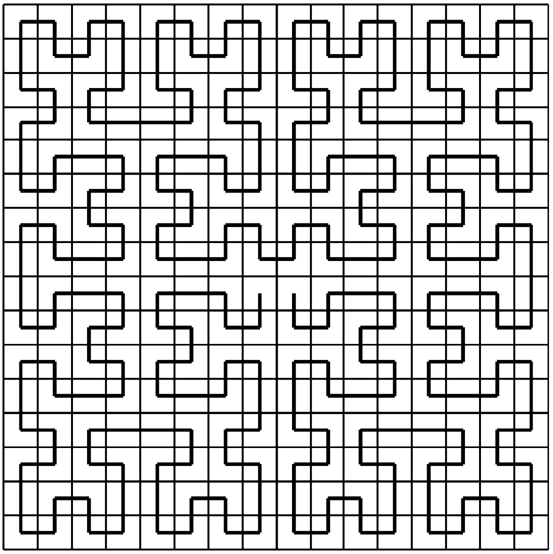

Fig. 14. Pattern $L2$ of resolution 4.

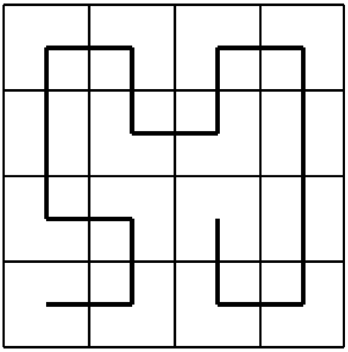

Fig. 15. Pattern $L3$ of resolution 2.

The transformations of the configuration from resolution k to $k+1 (k \geqslant 2)$ are the same as for pattern $L1$.

The graphical representations of pattern $L3$ for the resolutions 2, 3, and 4 are depicted in Figs. 15–17.

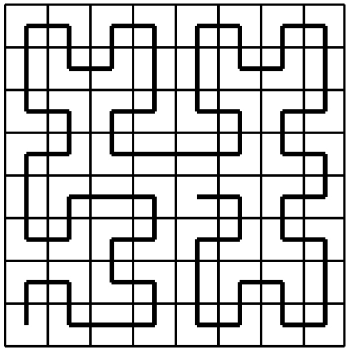

Fig. 16. Pattern $L3$ of resolution 3.

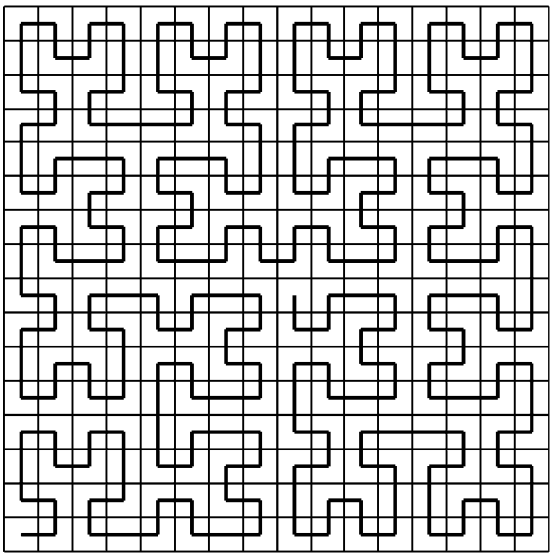

Fig. 17. Pattern $L3$ of resolution 4.

5. Configuration of pattern $L4$

The generation of the configuration of pattern $L4$ from resolution 1 to 2 is determined by the following transformations:

$$T_0(z) = (j + \bar{z})/2$$

$$T_1(z) = [1 + j(z + 1)]/2$$

$$T_2(z) = [1 + j(2 - z)]/2$$

$$T_3(z) = [1 + j(1 - z)]/2$$

The transformations of the configuration from resolution k to $k + 1 (k \geqslant 2)$ are the same as for pattern $L1$.

The graphical representations of pattern $L4$ for the resolutions 2, 3, and 4 are depicted in Figs. 18–20.

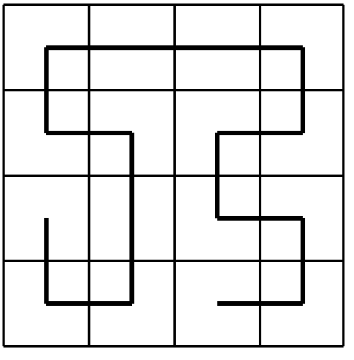

Fig. 18. Pattern $L4$ of resolution 2.

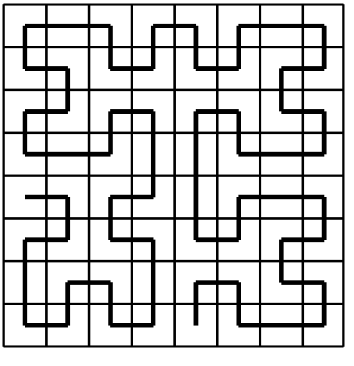

Fig. 19. Pattern $L4$ of resolution 3.

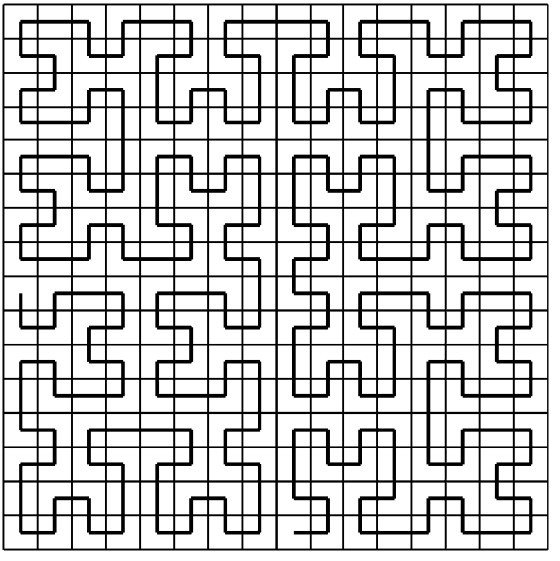

Fig. 20. Pattern $L4$ of resolution 4.

6. Conclusion

Interest in Hilbert's space-filling curve has been high. Besides the one discovered by Hilbert himself, there exist other patterns. Moore reported such a pattern in 1900. In this paper, we present four more patterns. In a two-dimensional lattice system of resolution 2, there are in total six possibilities of the quadrant-exhaustive and rook-connected patterns. In this sense, Hilbert's original pattern, Moore's pattern, and patterns $L1$ through $L4$ comprise a complete set of the Hilbert curve. The applicability of these alternative patterns may be paradigm-dependent. For example, the finite-element-method (FEM) in electromagnetic field analysis would benefit from a particular pattern, since there are various boundary conditions that may make sense to 'wrap around' the finite grid in defining a region. For the issue on computational complexity, it is anticipated that these alternative patterns behave similarly since the structure difference among them lies only in the iteration from resolution 1 to resolution 2.

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