Chapter 2
Hypergraphs: First Properties

In the first chapter we saw that hypergraphs generalize standard graphs by defining edges between multiple vertices instead of only two vertices. Hence some properties must be a generalization of graph properties In this chapter, we introduce some basic properties of hypergraphs which will be used throughout this book.

2.1 Graphs versus Hypergraphs

2.1.1 Graphs

A *multigraph*, $\Gamma = (V; E)$ is a hypergraph such that the rank of Γ is at most two. The hyperedges are called *edges*. If the hypergraph is simple, without loop, it is a *graph*. Consequently any definition for hypergraphs holds for graphs. Given a graph Γ, we denote by $\Gamma(x)$ the *neighborhood* of a vertex x, i.e. the set formed by all the vertices which form a edge with x:

$$\Gamma(x) = \{y \in V : \{x, y\} \in E\}$$

In the same way, we define the *neighborhood of* $A \subseteq V$ as

$$\Gamma(A) = \bigcup_{x \in A} \Gamma(x).$$

The *open neighborhood* of A is

$$\Gamma^o(A) = \Gamma(A) \setminus A.$$

An induced subgraph generated by $V^{'} \subseteq V$ is denoted by $\Gamma(V^{'})$.
A graph $\Gamma = (V; E)$ is *bipartite* if

A. Bretto, *Hypergraph Theory*, Mathematical Engineering,
DOI: 10.1007/978-3-319-00080-0_2,

$$V = V_1 \cup V_2 \text{ with } V_1 \cap V_2 = \emptyset$$

and every edge joins a vertex of V_1 to a vertex V_2.
It is well known that a graph $\Gamma = (V; E)$ is bipartite if and only if it does not contain any cycle with an odd length [Wes01, Vol02].
A graph is *complete* if any pair of vertices is an edge. A *clique* of a graph $\Gamma = (V; E)$ is a complete subgraph of Γ.
The maximal cardinality of a clique of a graph Γ is denoted by $\omega(\Gamma)$.
Remember that a graph is *chordal* if each of its cycles of four or more vertices has a *chord*, that is, an edge joining two non-consecutive vertices in the cycle.
For more informations about graphs see [Bol98, BLS99, BFH12, CL05, CZ04, GY06].

2.1.2 Graphs and Hypergraphs

Let $H = (V; E = (e_i)_{i \in I})$ be a hypergraph such that $E \neq \emptyset$. The *line-graph* (or *representative graph*, but also *intersection graph*) of H is the graph $L(H) = (V'; E')$ such that:

1. $V' := I$ or $V' := E$ when H is without repeated hyperedge;
2. $\{i, j\} \in E'$ $(i \neq j)$ if and only if $e_i \cap e_j \neq \emptyset$.

Figure 2.1 illustrates this definition.
Some properties of hypergraphs can be seen on the line-graph, for instance it is easy to show that:

Lemma 2.1 *The hypergraph H is connected if and only if $L(H)$ is.*

Proposition 2.1 *Any non trivial graph Γ is the line-graph of a linear hypergraph.*

Proof Let $\Gamma = (V; E)$ be a graph with $V = \{x_1, x_2, \dots x_n\}$. Without loosing generality, we suppose that Γ is connected (otherwise we treat the connected components one by one). We can construct a hypergraph $H = (W; X)$ in the following way:

- the set of vertices is the set of edges of Γ, i. e. $W := E$. It is possible since Γ is simple;
- the collection of hyperedges X is the family of X_i where X_i is the set of edges of Γ having x_i as incidence vertex.

So we can write:

$$H = (E; X = (X_1, X_2, \dots, X_n))$$

with:

$$X_i = \{e \in E : x_i \in e\} \text{ where } i \in \{1, 2, 3, \dots n\}$$

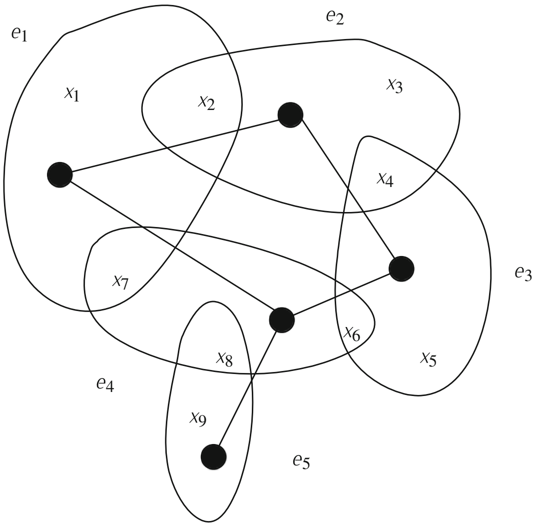

Fig. 2.1 Figure above shows a hypergraph $H = (V; E)$, where $V = \{x_1, x_2, x_3, \ldots, x_9, \}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, and its representative. The vertices of $L(H)$ are the *black dots* and its edges are the *curves* between these dots

Notice that if Γ has only one edge then

$$V = \{x_1,\ x_2\} \text{ and } X_1 = X_2.$$

It is the only case where H has a repeated hyperedge.

If $|E| > 1$, if $i \neq j$ and $X_i \cap X_j \neq \emptyset$; there is exactly one, (since Γ is a simple graph) $e \in E$ such that $e \in X_i \cap X_j$ with $e = \{x_i, x_j\}$. It is clear that Γ is the line-graph of H (Fig. 2.2). □

This proposition is illustrated in Fig. 6.3.
Let $H = (V; E)$ be a hypergraph, the *2-section* of H is the graph, denoted by $[H]_2$, which vertices are the vertices of H and where two distinct vertices form an edge if and only if they are in the same hyperedge of H. An example of 2-section is given in Fig. 2.3.
We can generalize the 2-section in the following way:
Given a hypergraph $H = (V; E)$ with $|V| = n$ and $|E| = m$, we build a *labeled-edge multigraph* denoted by $G[H]_2$ and called *generalized 2-section* as follows:

$$V(G[H]_2) = V$$

and the vertices x and y are connected by an edge, labeled with e, when $\{x, y\} \subseteq e$.

We frequently denote by (xy, e), the labelled-edges of $G[H]_2$, where xy is an edge and e is one hyperedge label of xy. Note that the total number of edges xy in $G[H]_2$ is

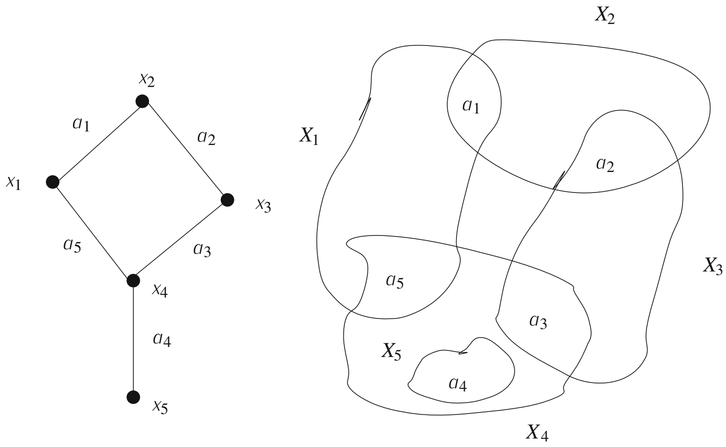

Fig. 2.2 Figure above illustrates Proposition 2.1

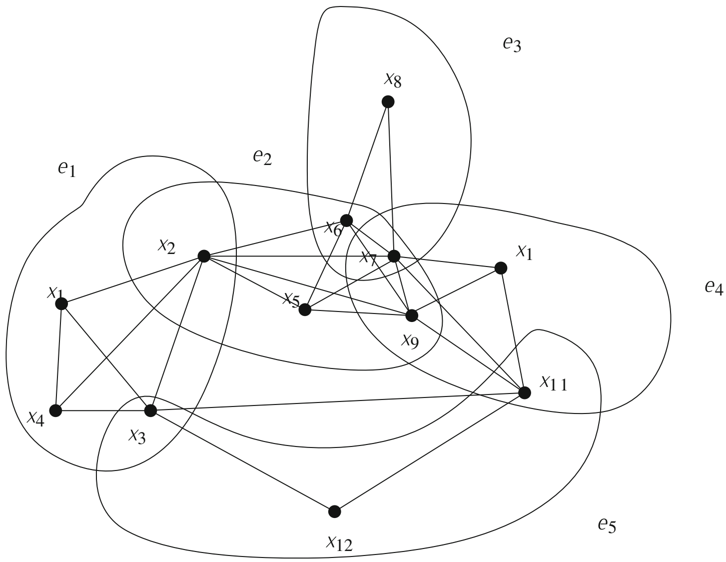

Fig. 2.3 Figure above shows the 2-section of a hypergraph

$$\sum_{i=1}^{m}(|e_i|(|e_i|-1)/2,$$

which is of order bounded by

$$O(mr(H)^2).$$

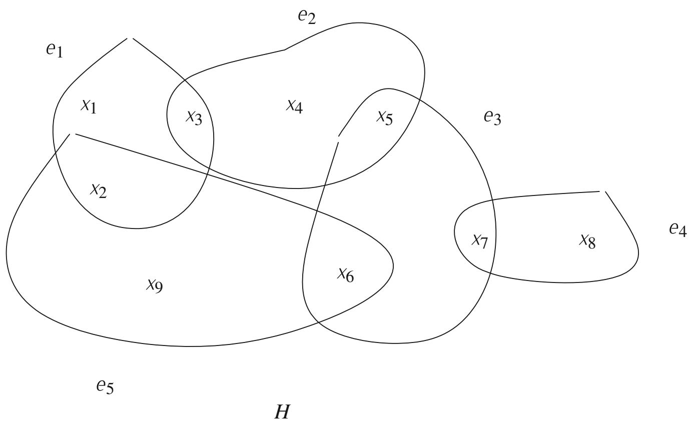

Fig. 2.4 Above a hypergraph which has nine vertices and five hyperedges

Furthermore, the maximal degree $\Delta(G[H]_2)$ of a vertex in $G[H]_2$ is clearly bounded by

$$r(H)\Delta(H).$$

The *incidence graph* of a hypergraph $H = (V; E)$ is a bipartite graph $IG(H)$ with a vertex set $S = V \sqcup E$, and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in e$.

Let $H = (V; E)$ be a hypergraph, the *degree of a hyperedge*, $e \in E$ is its cardinality, that is $d(e) = |e|$ (Fig. 2.4).

Proposition 2.2 *Let $H = (V; E)$ be a hypergraph, we have :*

$$\sum_{x \in V} d(x) = \sum_{e \in E} d(e).$$

Proof Let $IG(H)$ be the incidence graph of H. We sum the degrees in the part E and in the part V in $IG(H)$. Since the sum of the degrees in these two parts are equal we obtain the result (Fig. 2.5). □

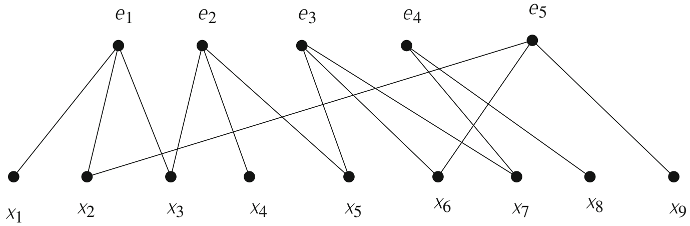

Fig. 2.5 The incidence graph associated with the hypergraph H

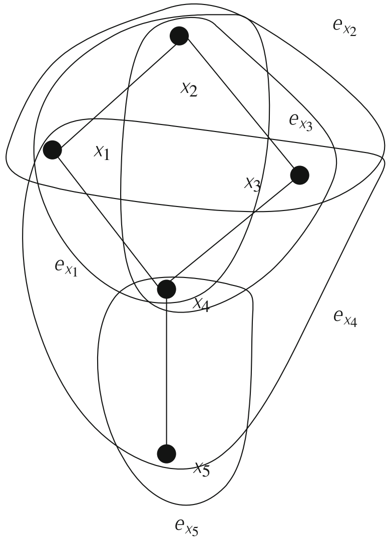

Fig. 2.6 Figure above shows a neighborhood hypergraph $H_\Gamma = (V, (e_x = \{x\} \cup \Gamma(x)))$ associated with a graph Γ

Proposition 2.3 *The dual H^* of a linear hypergraph without isolated vertex is also linear.*

Proof Let H be a linear hypergraph. Assume that H^* is not linear. There is two distinct hyperedges X_i and X_j of H^* which intersect with at least two vertices e_1 and e_2. The definition of duality implies that x_i and x_j belong to e_1 and e_2 (the hyperedges of H standing for the vertices e_1, e_2 of H^* respectively) so H is not linear. Contradiction (Fig. 2.6). □

We have seen several methods to associate a graph to a hypergraph, the converse can be done also. Indeed, let $\Gamma = (V; E)$ be a graph, we can associate a hypergraph called *neighborhood hypergraph* to this graph (Fig. 2.7):

$$H_\Gamma = (V, (e_x = \{x\} \cup \Gamma(x))_{x \in V}).$$

We can also associate a hypergraph without repeated hyperedge called *without repeated hyperedge neighborhood hypergraph*:

$$H_\Gamma = (V, \{e_x = \{x\} \cup \Gamma(x) : x \in V\}).$$

We will say that the hyperedge e_x is generated by x. This concept is illustrated Fig. 2.6.

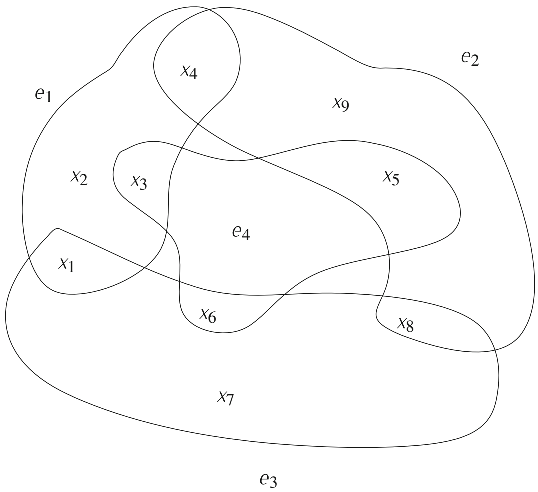

Fig. 2.7 Intersecting family

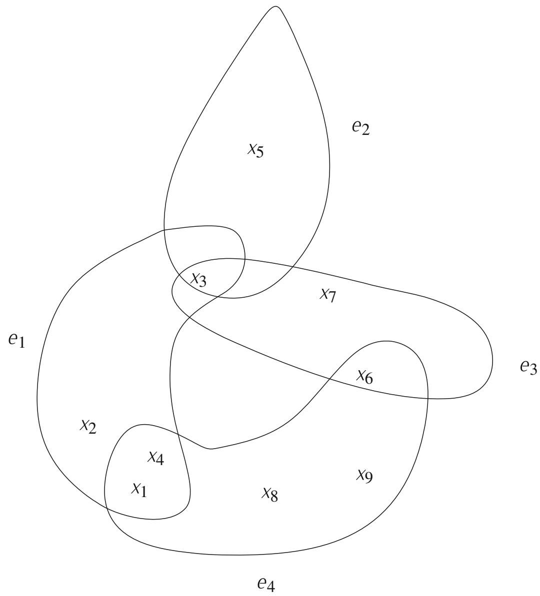

Fig. 2.8 The hypergraph above has not the Helly property since the intersecting family e_1, e_3, e_4 has an empty intersection, that is, $e_1 \cap e_3 \cap e_4 = \emptyset$

2.2 Intersecting Families, Helly Property

2.2.1 Intersecting Families

Let $H = (V; E = (e_i)_{i \in I})$ be a hypergraph. A subfamily of hyperedges $(e_j)_{j \in J}$, where $J \subseteq I$ is an *intersecting family* if every pair of hyperedges has a non empty intersection. The maximum cardinality of $|J|$ (of an intersecting family of H) is denoted by $\Delta_0(H)$.

Remember that a *star* $H(x)$ centered in x is the family of hyperedges $(e_j)_{j \in J}$ containing x. The maximum cardinality of $|J|$ is denoted by $\Delta(H)$. Since a star is an intersecting family, obviously we have $\Delta_0(H) \geq \Delta(H)$. An intersecting family with 3 hyperedges e_1, e_3, e_3 and $e_1 \cap e_3 \cap e_3 = \emptyset$ is called a *triangle*. **In the sequel sometimes we will write** $e_i \cap e_j$ **for** $V(e_i) \cap V(e_j)$.

2.2.2 Helly Property

The Helly property plays a very important role in the theory of hypergraphs as the most important hypergraphs have this property [BUZ02, Vol02, Vol09]. A hypergraph has the *Helly property* if each intersecting family has a non-empty intersection (belonging to a star). It is obvious that if a hypergraph contains a triangle it has not the Helly property. A hypergraph having the Helly property will be called *Helly hypergraph*.
A hypergraph has the *strong Helly property* if each partial induced subhypergraph has the Helly property. The hypergraph shown in Fig. 2.9 has the Helly property but it has not the strong Helly property.

In the sequel, we write e_{uv} to express that the hyperedge e_{uv} contains the vertices u, v.

We can characterize the strong Helly property by the following:

Theorem 2.1 *Let H be a hypergraph. Any partial induced subhypergraph of H has the Helly property if and only if for any three vertices x, y, z and any three hyperedges e_{xy}, e_{xz}, e_{yz} of H, where $x \in e_{xy} \cap e_{xz}$, $y \in e_{xy} \cap e_{yz}$, $z \in e_{xz} \cap e_{yz}$ there exists $v \in \{x, y, z\}$ such that $v \in e_{xy} \cap e_{xz} \cap e_{yz}$.*

Proof Assume that any partial induced subhypergraph of H has the Helly property. Then, for any three hyperedges e_{xy}, e_{xz}, e_{yz} of H, where

$$x \in e_{xy} \cap e_{xz}, \ \ y \in e_{xy} \cap e_{yz}, \ \ z \in e_{xz} \cap e_{yz},$$

just take the partial subhypergraph $H(Y)$ induced by the set $Y = \{x, y, z\}$ to see that there is a vertex $v \in \{x, y, z\}$ such that:

$$v \in e_{xy} \cap e_{xz} \cap e_{yz}.$$

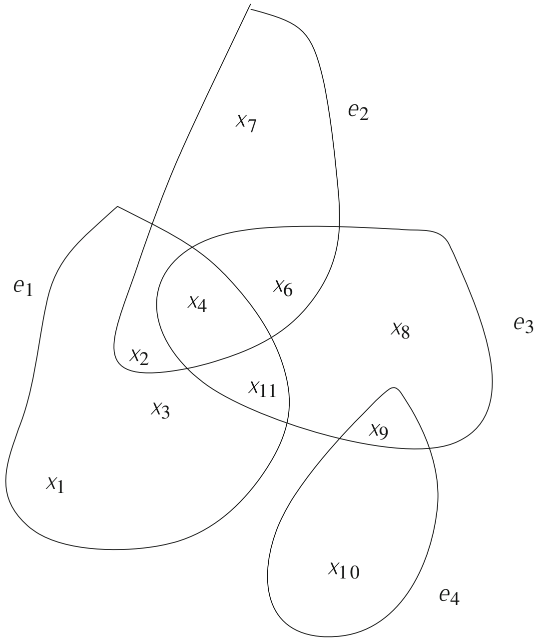

Fig. 2.9 The hypergraph above has the Helly property but not the strong Helly property because the induced subhypergraph on $Y = V \setminus \{x_4\}$ contains the triangle: $e'_1 = e_1 \cap Y$, $e'_2 = e_2 \cap Y$, $e'_3 = e_3 \cap Y$

We prove the reversed implication by induction on ℓ, the maximal size of an intersecting family of an induced subhypergraph of H. The assertion is clearly true for $\ell = 3$. Assume that for $i = 3, 4, \dots, \ell$ any partial induced subhypergraph of H with intersecting families of at most ℓ hyperedges has the Helly property. Let

$$e_1, e_2, \dots, e_{\ell+1}$$

be an arbitrary intersecting family of hyperedges of H. By induction,

$$\exists x \in \cap_{i \neq 1} e_i,\ \exists y \in \cap_{i \neq 2} e_i,\ \exists z \in \cap_{i \neq 3} e_i.$$

As $\{e_1, e_2, e_3\}$ is an intersecting family, there is a vertex

$$\xi \in \{x, y, z\}$$

which is in the intersection $e_1 \cap e_2 \cap e_3$. Hence, $\xi \in \cap_i e_i$ and the assertion holds for $(\ell + 1)$. $\square$

By using the same arguments than in the proof of Theorem 2.1, we can deduce the following GILMORE's characterization of the Helly property:

Corollary 2.1 *(GILMORE) A hypergraph H has the Helly property if and only if, for any three vertices x, y, z, the family of all hyperedges containing at least two of these vertices has a nonempty intersection.*

From this characterization we can deduce the following algorithms:

Algorithm 2: StrongHelly

Data: $H = (V; E)$ a hypergraph and $G[H]_2$ its generalized 2-section
Result: H has or has not the strong Helly property
begin
  foreach $(xy, e_1) \in E(G[H]_2)$ **do**
    foreach *pair of edges* $(xz, e_2), (yz, e_3) \in E(G[H]_2)$ **do**
      if $x \notin e_1 \cap e_2 \cap e_3$ *and* $y \notin e_1 \cap e_2 \cap e_3$ *and* $z \notin e_1 \cap e_2 \cap e_3$ **then**
        output(**the strong Helly property does not hold**.)
      end
    end
  end
end

For the Helly property we have the following algorithm:

Algorithm 3: Helly

Data: $H = (V; E)$ a hypergraph and $G[H]_2$ its generalized 2-section
Result: H has or has not the Helly property
begin
  foreach *pair of vertices* x, y *of* H **do**
    X_{xy} := all hyperedges containing both x and y;
    foreach *vertex* v *of* H **do**
      if x *and* y *are both neighbors of* v **then**
        X_{xv} := all hyperedges containing both x and v
        X_{yv} := all hyperedges containing both y and v
        $X := X_{xy} \cup X_{xv} \cup X_{yv}$;
        if *the intersection of all elements of* X *is empty* **then**
          output(**the Helly property**
          does not hold)
        end
      end
    end
  end
end

2.3 Subtree Hypergraphs

let $H = (V; E)$ be a hypergraph. This hypergraph is called a *subtree hypergraph* if

- there is a tree Γ with vertex set V such that each hyperedge $e \in E$ induces a subtree in Γ.

Notice that, for the same hypergraph we may have several generated trees using the above method. Moreover if $H = (V; E)$ is not a subtree hypergraph, for any tree on V, there is at least one hyperedge which induces a disconnected subgraph.

Conversely, let $\Gamma = (V; A)$ be a tree, i.e. a connected graph without cycle. We build a connected hypergraph H in the following way:

- the set of vertices of H is the set of vertices of Γ;
- the set of hyperedges is a family $E = (e_i)_{i \in \{1,2,\ldots,m\}}$ of subset V such that the induced subgraph $\Gamma(V(e_i))$ is a subtree of Γ, (subgraph which is a tree).

Notice that, for the same tree we may have several hypergraphs generated by the method above. An example of subtree hypergraph is given in Fig. 2.10.

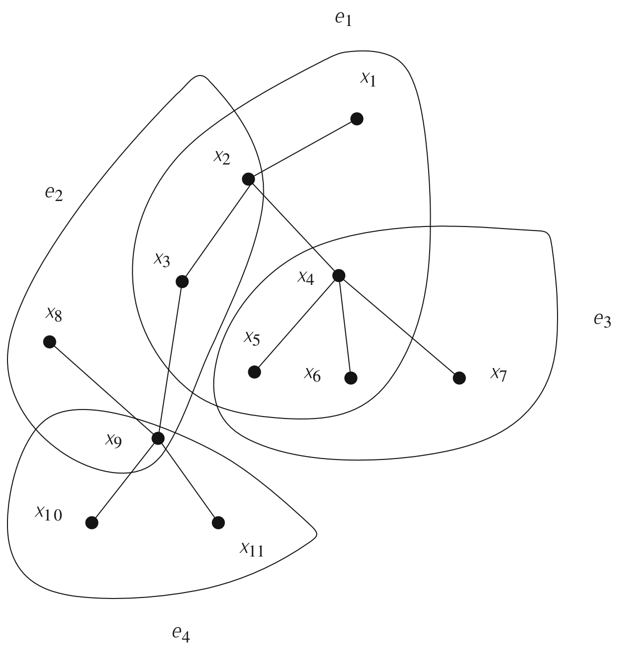

Fig. 2.10 A subtree hypergraph associated with a tree

Proposition 2.4 *Let $\Gamma = (V; A)$ be a tree and H be a subtree hypergraph associated with Γ, H has the Helly property.*

Proof We are going to use Corollary 2.1. In a tree Γ, there is exactly one path denoted by $Pa[x, y]$ between two vertices x, y, otherwise Γ would contain a cycle. Let u, v, w be three vertices of H. The paths

$$Pa[u, v], \ \ Pa[v, w] \text{ and } Pa[w, u]$$

have one common vertex, otherwise Γ would contain a cycle. Consequently, any family of hyperedges for which every hyperedge contains at least two of these vertices u, v, w has a nonempty intersection. □

Proposition 2.5 *Let $\Gamma = (V; A)$ be a tree and H be a subtree hypergraph, associated with Γ then $L(H)$ is chordal.*

Proof Let $\Gamma = (V, A)$ be a tree and $H = (V; E)$ be a subtree hypergraph associated with it.
If $|V| = 1$, H has just one vertex and one hyperedge. So, the linegraph of H has just one vertex and it is a clique, hence it is chordal.
Assume now that the assertion is true for any tree Γ with $n - 1$ vertices, $n > 1$.

Let Γ be a tree with n vertices. Let $x \in V$ be a leaf (a vertex with a unique neighbor y). Remember that in a tree with at least 2 vertices there are at least 2 leaves. Let

$$\Gamma^{'} = (V \setminus \{x\}; A')$$

where $\Gamma^{'}$ is the subgraph on $V \setminus \{x\}$; and

$$H'(V \setminus \{x\}) = (V \setminus \{x\}; E'), \ \ |V| > 1.$$

The graph

$$\Gamma^{'} = (V \setminus \{x\}; A')$$

is a tree and

$$H' = (V \setminus \{x\}; E')$$

is an induced subtree hypergraph associated with $\Gamma^{'}$.
By induction, $L(H')$ is chordal.
If $|E| = |E'|$ then

$$L(H) \simeq L(H')$$

($\{x\}$ is not a hyperedge of H and all hyperedges containing x contain the neighbor y of x in Γ) and $L(H)$ is chordal.
If $|E| \neq |E'|$ then

$$\{x\} \in E \text{ and } |E| > |E'|.$$

It is easy to show that the neighborhood of $\{x\}$ in $L(H)$ is a clique (this neighborhood stand for the hyperedges containing x (excepted $\{x\}$)). So any cycle passing through $\{x\}$ is chordal in $L(H)$ and so $L(H)$ is chordal. □

Using Propositions 2.4, 2.5, it can be shown ([Sla78]) that

Theorem 2.2 *The hypergraph H is a subtree hypergraph if and only if H has the Helly property and its line graph is chordal.*

The dual of a subtree hypergraph is a concept used in relational database theory [Fag83].
From Proposition 2.6 and Proposition 2.7 below we have:

Corollary 2.2 *The dual of a hypergraph H is conformal and its 2-section is chordal if and only if H is a subtree hypergraph.*

2.4 Conformal Hypergraphs

A hypergraph H is *conformal* if any maximal clique (for the inclusion) of the 2-section $[H]_2$ is a hyperedge of H.
Figure 2.11 shows the 2-section of a hypergraph H. It may be noticed that this hypergraph is not conformal.

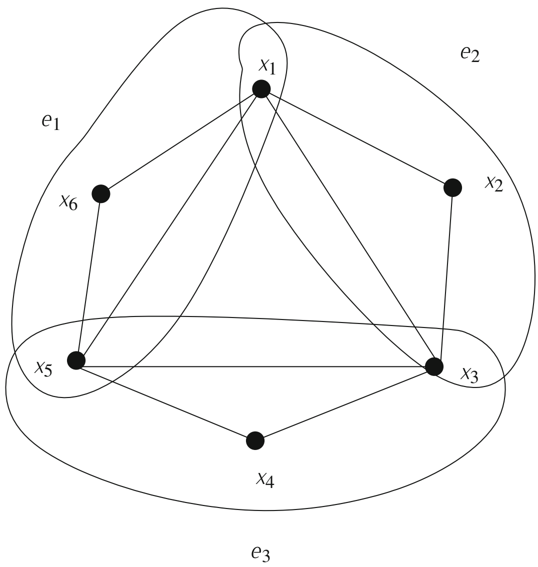

Fig. 2.11 The hypergraph above is not conformal since the maximal clique $\{x_1, x_3, x_5\}$ is not a hyperedge. It may be noticed that if we add this clique as a hyperedge, the hypergraph becomes conformal but does not have the Helly property

Proposition 2.6 *A hypergraph is conformal if and only if its dual has the Helly property.*

Proof Let $H = (V; E)$ be a hypergraph. Assume that H is conformal.
Let

$X = \{X_1^*, X_2^*, X_3^*, \ldots X_k^*\}$ be a maximal intersecting family of H^*.

$$\text{For all } i, j \in \{1, 2, \ldots, k\}, X_i^* \cap X_j^* \neq \emptyset,$$

which implies that there is a hyperedge $e_{i,j} \in E$ which contains x_i, x_j (the vertices of H standing for the hyperedges X_i^*, X_j^* respectively) for all $i, j \in \{1, 2, \ldots, k\}$. Hence the family X stands for a set of vertices of a maximal clique K_k of $[H]_2$. Since H is conformal, the clique K_k is contained in a hyperedge e which stands for a vertex of H^*, consequently

$$e \in \bigcap_{j \in \{1,2,\ldots k\}} X_j^*$$

and X is a star in H^*.

Conversely, assume that H^* has the Helly property. Let K_k be a maximal clique of $[H]_2$. By definition of the 2-section, for all $x_i, x_j \in K_k$ there is a hyperedge which contains these two vertices. So the set of vertices of K_k stands for an intersecting family X of H^* which is included into a star since H has the Helly property. Hence there is a vertex of H^* which is common to any element of X. But this vertex stands for a hyperedge of H which contains any vertex of K_k. So H is conformal. □

Proposition 2.7 *The line graph $L(H)$ of a hypergraph H is the 2-section of H^*, i.e.*

$$L(H) \simeq [H^*]_2.$$

Moreover the two following statements are equivalent, where Γ is a graph:

(i) *H verifies the Helly property and Γ is the line graph of H.*
(ii) *Maximal hyperedges (for inclusion) of H^* are maximal cliques of Γ.*

Proof The vertices of both $L(H)$ and H^* are the hyperedges of H. A pair of vertices e_i, e_j of $L(H)$ is an edge if and only if the corresponding hyperedges have a non-empty intersection. So these two vertices belong to the same hyperedge of H^*. Consequently $\{e_i, e_j\}$ is an edge of $[H^*]_2$. The converse inclusion is done in a similar way. Hence $L(H)$ is isomorphic to $[H^*]_2$ (modulo loops, since H^* may have some).

Assume that H has the Helly property. Hence H^* is conformal by Proposition 2.6. So (i) implies that $\Gamma = [H^*]_2$ has the maximal hyperedges of H^* as maximal cliques. In the same way we have (ii) implies (i). □

2.5 Stable (or Independent), Transversal and Matching

Let $H = (V; (e_i)_{i \in I})$ be a hypergraph without isolated vertex.
A set $A \subseteq V$ is a *stable or an independent* (resp. a *strong stable*) if no hyperedge is contained in A (resp. $|A \cap V(e_i)| \leq 1$, for every $i \in I$).
The *stability number* $\alpha(H)$ (resp. the *strong stability number* $\alpha'(H)$) is the maximum cardinality of a stable (resp. of a strong stable).

A set $B \subseteq V$ is a *transversal* if it meets every hyperedge i.e.

$$\text{for all } e \in E, \;\; B \cap V(e) \neq \emptyset.$$

The minimum cardinality of a transversal is the *transversal number*. It is denoted by $\tau(H)$.

A *matching* is a set of pairwise disjoint hyperedges of H.
The *matching number* $\nu(H)$ of H is the maximum cardinality of a matching.
A *hyperedge cover* is a subset of hyperedges:

$$(e_j)_{j \in J}, \;\; (J \subseteq I) \text{ such that: } \bigcup_{j \in J} e_j = V.$$

The *hyperedge covering number*, $\rho(H)$ is the minimum cardinality of a hyperedge cover.
Figure 2.12 illustrates these definitions and numbers.

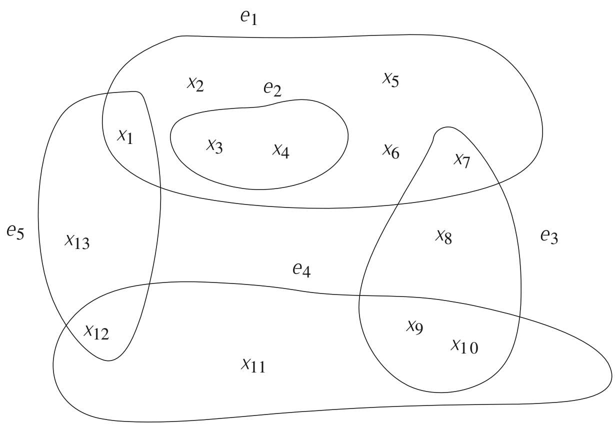

Fig. 2.12 The set $\{x_1; x_3; x_5; x_9; x_{11}; x_{13}\}$ is a stable of the hypergraph above but it is not a strong stable. The set $\{x_3; x_8; x_{11}; x_{13}\}$ is a transversal; $\tau(H) = 3$, $\rho(H) = 4$ and $\nu(H) = 3$. It is conformal and it has the Helly property

2.5.1 Examples:

(1) The problem of scheduling the presentations in a conference is an example of the maximum independent set problem. Let us suppose that people are going to present their works, where each work may have more than one author and each person may have more than one work.
The goal is to assign as many presentations as possible to the same time slot under the condition that each person can present at most one work in the same time slot.
We construct a hypergraph with a vertex for each work and a hyperedge for each person, it is the set of works that he (or she) presents. Then a maximum strong independent set represents the maximum number of presentations that can be given at the same time.
(2) The problem of hiring a set of engineers at a factory is an example of the minimum transversal set problem.
Let us suppose that engineers apply for positions with the lists of proficiency they may have, the factory management then tries to hire the least possible number of engineers so that each proficiency that the factory needs is covered by at least one engineer.
We construct a hypergraph with a vertex for each engineer and an hyperedge for each proficiency, then a minimum transversal set represents the minimum group of engineers that need to be hired to cover all proficiencies at this factory.

Lemma 2.2 *Let $H = (V; E)$ be a hypergraph without isolated vertex. We have the following properties.*

(i) $\nu(H) \leq \tau(H)$.
(ii) $\rho(H) = \tau(H^*)$.
(iii) $\alpha'(H) = \nu(H^*)$.
(iv) $\alpha'(H) \leq \rho(H)$.

Proof Notice that for T a transversal and C a matching, we have:

$$|T \cap V(e)| \leq 1 \text{ for each } e \in C,$$

consequently

$$|C| \leq |T|.$$

So

$$\nu(H) \leq \tau(H).$$

A hyperedge minimum covering of H becomes a transversal in H^* and conversely every minimum transversal of H^* becomes a minimum covering of H.

Indeed the elements of a hyperedge covering in H becomes a set of vertices which meets every hyperedge in H^*. So

$$\rho(H) = \tau(H^*).$$

In the similar way

$$\alpha'(H) = \nu(H^*)$$

and so (ii) is proved.
Hence

$$\alpha'(H) = \nu(H^*) \leq \tau(H^*) = \rho(H)$$

and (iii) is proved. □

2.6 König Property and Dual König Property

The hypergraph H has the *König property* if

$$\nu(H) = \tau(H)$$

and the *dual König property* if and only if

$$\alpha'(H) = \rho(H).$$

The hypergraph in Fig. 2.13 has the Konig property and it has also the dual Konig property since

$$\alpha^{'}(H) = \rho(H) = 2.$$

Proposition 2.8 *Let $\Gamma = (V; E)$ be a tree and let H be a subtree hypergraph associated with Γ. Then H has the König property, i.e.*

$$\nu(H) = \tau(H).$$

Proof Let $V^{'} \subseteq V$ such that the induced subgraph $\Gamma(V^{'})$ is a tree which contains a minimal transversal T of H in such a way that $|V^{'}|$ is minimum.
A leaf x_1 of $\Gamma(V^{'})$ belongs to T, otherwise $\Gamma(V^{'} \setminus \{x_1\})$ would be a tree which contains T contradicting the fact that $|V^{'}|$ is minimum.

The family

$$E(H_1) = (e \in E(H), V(e) \cap T = \{x_1\})$$

is non empty.
Indeed, T being a minimal transversal, there is $e \in E(H)$ such that

$$V(e) \cap T \ni x_1.$$

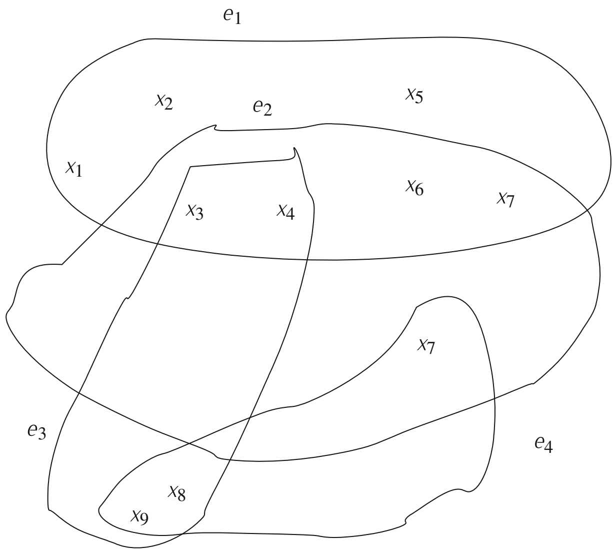

Fig. 2.13 In the hypergraph above we have: $\tau(H) = 2$, $\rho(H) = 2$, $\nu(H) = 2$ and $\alpha'(H) = 2$. So this hypergraph has the König property and the dual König property

If we assume now that, for all $e_i \in H(x_1)$, there is $x_i \in T$, $x_i \neq x_1$, such that $\{x_1, x_i\} \in e_i$ then $T \setminus \{x_1\}$ would be a transversal, contradicting the minimality of T. Now, since

$$T \setminus \{x_1\} \subseteq V' \setminus \{x_1\}$$

where $\Gamma(V' \setminus \{x_1\})$ is a tree, there is a connected component $\Gamma(V \setminus \{x_1\})$ of Γ which contains $T \setminus \{x_1\}$.

Let H' be the partial hypergraph obtained by deleting all hyperedges which contains x_1, that is, $E(H') = E(H) \setminus H(x_1)$. Clearly H' has a transversal:

$$T' \subseteq T \setminus \{x_1\} \subseteq V' \setminus \{x_1\}$$

such that

$$\tau(H') = |T| - 1.$$

Since T' is a transversal and because the hyperedges of H' are subtrees, we have

$$V(E(H')) \subseteq V \setminus \{x_1\}.$$

By induction hypothesis

$$\tau(H') = |T| - 1 = \nu(H').$$

There is

$$e_1 \in E(H_1), \text{ such that } V(e_1) \cap V' = \{x_1\}.$$

Indeed otherwise, for all $e \in E(H_1)$, we would have $|V(e_1) \cap V'| \geq 2$ and $V' \setminus \{x_1\}$ would contain a transversal. Hence it would contain a minimal transversal of H; consequently $|V^{'}|$ would not be minimum. So

$$V(e_1) \cap V' \setminus \{x_1\} = \emptyset.$$

Now let C' be a maximum matching of H', $C' \cup \{e_1\}$ is a matching of H with a cardinality $|T|$, consequently $\nu(H) \geq \tau(H)$. From Lemma 2.2 we get

$$\nu(H) = \tau(H).$$

□

2.7 linear Spaces

We remind the reader that a linear space is a hypergraph in which each pair of distinct vertices is contained in precisely one edge. A trivial linear space is a hypergraph with only one hyperedge which contains all vertices.

Theorem 2.3 *If a non-trivial, non-empty linear space has n vertices and m edges then $m \geq n$.*

Proof Assume that $H = (V; E)$ is a linear space, with $|V| = n$ and $|E| = m$. Suppose $1 < m \leq n$. Choose a vertex $v \in V$ and $e \in E$ such that $v \notin e$. Since H is a linear space we have: $d(v) \geq |e|$. So from this and $m \geq n$, it follows:

$$\frac{1}{n(m - d(v)} \geq \frac{1}{m(n - |e|)}.$$

Hence by Adding these inequalities for all pairs $v \notin e$ we have:

$$1 = \sum_{v \in V} \sum_{e \not\ni v} \frac{1}{n(m - d(v))} \geq \sum_{e \in E} \sum_{v \notin e} \frac{1}{m(n - |e|)} = 1.$$

Indeed the inner sums are never empty since $1 < m$. Moreover
For the first inner sum:

- fix a vertex v, there are exactly $m - d(v)$ hyperedges which do not contain v.

For the second inner sum:

- fix a hyperedge e, there are exaclty $n - |e|$ vertices which are not in e.

Therefore we have:

$$\sum_{v\in V}\sum_{e\not\ni v}\frac{1}{n(m-d(v))}=\sum_{e\in E}\sum_{v\notin e}\frac{1}{m(n-|e|)}.$$

Consequently:

$$\sum_{v\in V}\frac{1}{n}=\sum_{e\in E}\frac{1}{m};$$

hence

$$\frac{n-1}{n}-\frac{m-1}{m}=\frac{1}{n}-\frac{1}{m};$$

which implies

$$n-m=m-n;$$

so, $n=m$. □

References

[Wes01] D.B. West, *Introduction to Graph Theory*, 2nd edn. (Prentice-Hall, Upper Saddle River, 2001)

[Vol02] V.I. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms, and Applications*. American Mathematical Society. Fields Institute Monographs, 2002

[Bol98] B. Bollobas, *Modern Graph Theory* (Springer Verlag, New York, 1998)

[BLS99] A. Brandstädt, V. Bang Le, J.P. Spinrad, *Graph Classes: A Survey* (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1999)

[BFH12] A. Bretto, A. Faisant, F. Hennercart, *Elément de théorie des graphes*. IRIS (Springer, Paris, 2012)

[CL05] G. Chartrand, L. Lesniak, *Graphs and Digraphs*, 4th edn. (CRC Press, Boca Raton, 2005)

[CZ04] G. Chartrand, P. Zhang, *Introduction to Graphs Theory*. Walter Rudin Student Series in Advanced Mathematics (McGraw-Hill, New York, 2004)

[GY06] J.L. Gross, J. Yelleni, *Graph Theory and its Applications*, 2nd edn. Discrete Mathematics and its Applications (Chapman and Hall/CRC, New York, 2006)

[BUZ02] A. Bretto, S. Ubeda, Y. Zerovnik, A polynomial algorithm for the strong helly property. IPL. Inf. Process. Lett. **81**(1), 55–57 (2002)

[Vol09] V.I. Voloshin, *Introduction to Graph and Hypergraph Theory* (Nova Science Publishers, New York, 2009)

[Sla78] P.J. Slater, A characterization of soft hypergraphs. Can. Math. Bull. **21**, 335–337 (1978)

[Fag83] R. Fagin, Degrees of acyclicity for hypergraphs and relational database systems. J. Assoc. Comput. Mach. **30**, 514–550 (1983)

http://www.springer.com/978-3-319-00079-4

Hypergraph Theory
An Introduction
Bretto, A.
2013, XIII, 119 p., Hardcover
ISBN: 978-3-319-00079-4