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Consider the Possibilistic C-Means (PCM) objective function:

$$J = \sum_{i=1}^{N} \sum_{j=1}^{M} u_{ij}^{m} \cdot d(x_i, \theta_j) + \sum_{j=1}^{M} \eta_j \sum_{i=1}^{N} (1 - u_{ij})^m$$

Let's find the $u_{ij}$'s which optimize $J$ by differentiating $J$ w.r.t. $u_{ij}$ and setting the result equal to zero:

$$\frac{\partial J}{\partial u_{ij}} = 0 = m\, u_{ij}^{m-1}\, d(x_i, \theta_j) + \eta_j\, m (1 - u_{ij})^{m-1} (-1)$$

$$\Rightarrow \quad \eta_j\, m (1 - u_{ij})^{m-1} = m\, u_{ij}^{m-1}\, d(x_i, \theta_j)$$

Raise both sides to the power $\frac{1}{m-1}$:

$$\eta_j^{\frac{1}{m-1}} (1 - u_{ij}) = u_{ij}\, d^{\frac{1}{m-1}}$$

$$1 - u_{ij} = u_{ij} \left(\frac{d}{\eta_j}\right)^{\frac{1}{m-1}}$$

$$1 = u_{ij} \left[1 + \left(\frac{d}{\eta_j}\right)^{\frac{1}{m-1}}\right]$$

$$\Rightarrow \quad u_{ij} = \frac{1}{1 + \left(\frac{d(x_i, \theta_j)}{\eta_j}\right)^{\frac{1}{m-1}}}$$