Problem Set #4:

Kalman Filter:

$$\left.\begin{aligned} x(n+1) &= Ax(n) + v(n) \\ y(n) &= Cx(n) + w(n) \end{aligned}\right\} \begin{array}{l} v, w \ \emptyset\text{-mean uncorrelated} \\ \text{with covariance matrices } Q_v, Q_w \end{array}$$

$\mathcal{Y}_n = \text{span}\{y(k),\ 1 \le k \le n\}$, $\hat{u}(m|n) =$ projection of $u(m)$ on $\mathcal{Y}_n$

$\varepsilon(n,n-1) = x(n) - \hat{x}(n|n-1)$

Claim 1: $\varepsilon(n,n-1) \perp v(n)$

Lemma 1: $E[x(n)v^H(n)] = 0$

Base Case: $E[x(0)v^H(n)] = 0$ (given)

Assume: $E[x(k)v^H(n)] = 0$

Iterate: $E[x(k+1)v^H(n)] = E[(Ax(k) + v(k))v^H(n)]$

$$= \underbrace{AE[\cancelto{\emptyset}{x(k)v^H(n)}]}_{\text{assumed}} + \underbrace{E[\cancelto{\emptyset}{v(k)v^H(n)}]}_{\text{given}}$$

Lemma 2: $E[\hat{x}(n|n-1)v^H(n)] = 0$

- $\hat{x}(n|n-1)$ is the projection of $x(n)$ onto $\mathcal{Y}_{n-1}$
- $v(n)$ does not impact the state vector until time $n+1$, so $v(n) \perp y(k)$ for $k \le n \Rightarrow$ the projection of $v(n)$ onto $\mathcal{Y}_{n-1}$, $v(n|n-1)$, is zero
- Therefore, $\hat{x}(n|n-1) \perp v(n) \Rightarrow E[\hat{x}(n|n-1)v^H(n)] = 0$

$=$

$$E[\varepsilon(n,n-1)v^H(n)] = E[(x(n) - \hat{x}(n|n-1))v^H(n)]$$

$$= \underbrace{E[\cancelto{\emptyset}{x(n)v^H(n)}]}_{\text{lemma 1}} - \underbrace{E[\cancelto{\emptyset}{\hat{x}(n|n-1)v^H(n)}]}_{\text{lemma 2}} = 0$$

$\Rightarrow \varepsilon(n,n-1) \perp v(n)$

Claim 2: $\varepsilon(n,n-1) \perp w(n)$

Lemma 1: $E[x(n)w^H(n)] = 0$

Base Case: $E[x(0)w^H(n)] = 0$ (given)

Assume: $E[x(k)w^H(n)] = 0$

Iterate: $E[x(k+1)w^H(n)] = E[(Ax(k) + v(k))w^H(n)]$

$$= \underbrace{AE[\cancelto{\emptyset}{x(k)w^H(n)}]}_{\text{assumed}} + \underbrace{E[\cancelto{\emptyset}{v(k)w^H(n)}]}_{\text{given}} = 0$$

Lemma 2: $E[\hat{x}(n|n-1)w^H(n)] = 0$

- $\hat{x}(n|n-1)$ is the projection of $x(n)$ onto $\mathcal{Y}_{n-1}$
- $w(n)$ does not impact the output vector until time n, so $w(n) \perp y(k)$ for $k < n \Rightarrow$ the projection of $w(n)$ onto $\mathcal{Y}_{n-1}$, $w(n|n-1)$, is zero
- Therefore, $\hat{x}(n|n-1) \perp w(n) \Rightarrow E[\hat{x}(n|n-1)w^H(n)]$

$=$

$$E[\varepsilon(n,n-1)w^H(n)] = E[(x(n) - \hat{x}(n|n-1))w^H(n)]$$

$$= \underbrace{E[\cancelto{\emptyset}{x(n)w^H(n)}]}_{\text{lemma 1}} - \underbrace{E[\cancelto{\emptyset}{\hat{x}(n|n-1)w^H(n)}]}_{\text{lemma 2}} = 0$$

$\Rightarrow \varepsilon(n,n-1) \perp w(n)$