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ECE416 Adaptive Filters
Problem Set II: Optimum Filtering
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Note: Problems refer to Haykin 5th ed, or 4th ed. (problem numbers are the same). Also, here use MATLAB as a “calculator” and do not use built-in functions to compute FPEF coefficients, AR models and the like. For example, code up the Levinson-Durbin recursion on your own, don’t use a canned routine. Also, some of these problems require referencing the notes given out that give exact formulas for the correlation, power spectrum and innovations filters for an AR(2) process.

1. Problem 2.18. Some comments:

   (a) For the Wiener filter, assume the signal $u[n]$ has the form as given under hypothesis H_1, so $\vec{u} = \vec{s} + \vec{v}$ where $\vec{s}$ is deterministic. The desired signal is $s[k]$ for some fixed k. By finding "R" and "p" you can find the Wiener filter, and it will have the form as given in the text. You will note regardless of the choice of $s[k]$, all these vectors are the same up to a scaling factor. And that is the point of the problem: the solutions to (b) and (c) yield the same vector, up to a scaling factor.

   (b) Haykin gives you a hint. Use it to transform the SNR into a Rayleigh quotient of a matrix. Please don’t compute gradients- you should know how to maximize a Rayleigh quotient!

   (c) You have (in my notes or in the book) the formula for the complex Gaussian pdf. **Remark:** The maximum-likelihood (ML) decision rule for choosing the hypothesis would be to compare Λ to 1 (or $\log \Lambda$ to 0). If you assume this, you will get a specific value for the threshold η involving quantities that are presumed know (e.g., R_v). However, if you use say a Neyman-Pearson test, then the threshold η will be an adjustable parameter that determines the probability of false alarm (probability of choosing H_1 when H_0 is true) and probability of detection (probability of choosing H_1 when H_1 is true). Either way, the point is the form of the test will be to compare $w^H u$ to a threshold, where w again has the same form as the other parts of the problem.

2. Consider an AR(2) process where the poles are 0.8 and 0.6. The exact model is:

$$x[n] = v[n] - a_1 x[n-1] - a_2 x[n-2]$$

   where v is unit variance white noise.

   (a) Find a_1 and a_2, and compute the PSD $S(\omega)$ (you will graph it later, superimposed with certain estimates). Also compute $r[m]$ (exact values) up to order 10.

   (b) Generate 10^3 samples of x, estimate $\hat{r}[m]$ up to order 10.

   (c) Find the maximum absolute error between the exact and estimated correlation. Also compute the spectral norm of the difference between the actual correlation matrix and the estimated one (for order 10).

(d) Use the Levinson-Durbin algorithm to compute the reflection coefficients up to order 10 using both the exact and estimated values of r. Find the maximum absolute error between these values.

(e) Examine the second order FPEF. It can be interpreted as providing an estimate of a_1, a_2, and thus the pole values. Find the maximum absolute error for the AR coefficients, and for the poles.

(f) In theory, the reflection coefficients for order beyond 2 should be zero. Check that for both the exact and approximate cases.

(g) Plot the powers of the prediction errors P_m for $m = 0$ to 10.

(h) Compute $\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log S(\omega)\,d\omega\right)$ (numerically using the theoretical PSD) and check that it is what it should be.

(i) Compute $\frac{1}{M}\log(\det R_M)$ where R_M is the $M \times M$ correlation matrix, for $M = 1$ up to 11. Plot these values. Are they strictly decreasing? Do they converge?

3. Consider the following ARMA(1,1) process:

$$u[n] = -0.5u[n-1] + v[n] - 0.2v[n-1]$$

where $v[n]$ is unit variance white noise. Consider the problem of forming an AR(3) model for this process.

(a) Generate 1000 samples to estimate the correlation $r[m]$ for $|m| \leq 3$. Formulate the corresponding correlation matrix, compute its eigenvalues, and determine the bound on μ for a steepest descent approach.

(b) Use the Levinson-Durbin algorithm to find the 3rd order FPEF, exactly. Specify the coefficients of the filter, and also the reflection coefficients.

(c) Now instead apply the steepest descent method for the values $\mu = 0.1\mu_{\max}$, $0.5\mu_{\max}$ and $0.9\mu_{\max}$ for enough iterations until the tap weight vector $w[n]$ approximates the ideal $w_0[n]$ so that each coefficient is within 10^{-3} of the correct answer. Compare the number of iterations required for each case.

(d) This problem is a modification of 4.14. There, Haykin expects you to determine the exact AR(3) coefficients. A look at the solution key gives the trick:

$$1 + x + x^2 + \cdots + x^N \approx \frac{1}{1-x}$$

for small x. If the ideal innovation filter is:

$$H(z) = \frac{1 + bz^{-1}}{1 + az^{-1}}$$

then by using the approximation:

$$1 + bz^{-1} \approx \frac{1}{1 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots}$$

(you should determine the β_i's) and from that you can approximate:

$$H(z) \approx \frac{1}{A_M(z)}$$

where $A_M(z)$ is a polynomial of degree M. Do this to obtain an AR(3) model, and use the coefficients of the 3rd order FPEF, running Levinson-Durbin BACKWARDS, to find the correlation $r[m]$. Are you getting the same results? (Based on the discussion in the course, there is no reason to presume in advance that this method of approximation yields the AR(M) model, i.e., the M^{th} order FPEF).

4. Here we examine ideal beamformers for sensor arrays. Let $s(\Theta)$ denote a steering vector *normalized to unit length.* If w is the beamformer vector, then the *array pattern* is defined as:

$$A(\Theta) = \left|w^H s(\Theta)\right|^2$$

We would normally graph this in dB. In this problem we consider a uniform linear array (for reference, aligned along the z-axis). Take the locations of the array, for $1 \leq m \leq M$, as:

$$\vec{r}_m = (m-1)\, d\hat{a}_z$$

If $\vec{k}(\Theta)$ is the incident wavenumber vector, then:

$$\vec{k}(\Theta) \cdot \vec{r}_m = \frac{2\pi d}{\lambda} \cos\theta$$

We assume $M = 20$ and $d/\lambda = 0.5$ in this problem. The array pattern will only depend on a single angle parameter, θ (the angle between the incident AOA and the axis of the array), not the azimuthal angle ϕ. Thus, $A(\theta)$ is a function of a single variable, with $0 \leq \theta \leq 180°$. $\theta = 90°$ is called the broadside AOA, and all the sensors pick up the wave in phase. If $\theta = 0°$ or $180°$, this is called the endfire AOA and the phase difference between the sensors is maximal.

Consider three uncorrelated sources at angles $\theta = 10°,\ 30°,\ 50°$. If we take the $10°$ wave having unit power, then assume the $30°$ source is $5dB$ lower, the $50°$ source is $15dB$ lower, and the background noise is $25dB$ lower.

(a) Compute the (theoretical) R matrix for this environment.

(b) For each source, compute the array patterns for the MVDR and GSC beamformers (i.e., take a source as the "primary", forcing a distortionless response in that direction; for the case of the GSC, impose exact nulls in the directions of the other sources). For each source:

1. Plot the MVDR and GSC, either superimposed or as subplots in the same Figure, whichever looks better. Make sure to scale the vertical axis reasonably.
2. The GSC beamformer puts exact nulls ($-\infty dB$) at the interfering sources. The MVDR beamformer will try to suppress the signal in those directions, but won't put exact nulls. Compute the amount of attenuation the MVDR places in those directions.

3. Since the GSC places exact nulls, you would expect the peak gain in the sidelobes, or in the mainlobe (remember the peak gain may not be 1, although you are enforcing 1 in the primary source direction) to be larger than that of the MVDR. Compare these peak values (you can read it off your graphs).

(c) We now want to consider the "invisible region". Recall the electrical angle is defined as:

$$\hat{\omega} = \frac{2\pi d}{\lambda} \cos\theta$$

As such, the array pattern can be interpreted as $A(\hat{\omega}) = |W(\hat{\omega})|^2$ where $W(\hat{\omega})$ is the frequency response of an FIR filter with coefficients $\{w_m^*\}$. Since $\cos\theta$ ranges from -1 to 1, if $d/\lambda > 0.5$, we get *spatial aliasing:* different AOAs θ may give rise to the same $\hat{\omega}$, and hence the beamformer can't distinguish among them. If $d/\lambda < 0.5$, then only a limited band of $\hat{\omega}$ corresponds to a physical AOA θ; the range $\{-2\pi d/\lambda < \hat{\omega} < 2\pi d/\lambda\}$ is called the *visible region;* the portion of the $[-\pi, \pi]$ interval outside this range is called the INVISIBLE region. Now, compute the MVDR and GSC beamformers for each of the 3 sources assuming $d/\lambda = 0.4$. Graph the array patterns as a function of $\hat{\omega}$ (not θ), over the full range $\{-\pi < \hat{\omega} < \pi\}$, and on your graph clearly label the boundary of the visible region (e.g., draw vertical lines on your graphs). For each case (you basically have 6 beamformers all together), compute the peak value of the array pattern in the invisible region.