PSET #2:

let $u(n) \in \mathbb{C}^M$ be data samples, $v(n) \in \mathbb{C}^M$ be zero-mean Gaussian noise, $s(n)$, $1 \le n \le M$ be samples of the signal.

hypothesis 1 (H_1): $u(n) = s(n) + v(n)$

hypothesis 0 (H_0): $u(n) = v(n)$

(a) Prove $w_o = \frac{s_k}{1 + s^H R_v^{-1} s} R_v^{-1} s$

Lemma: $A = B^{-1} + C D^{-1} C^H$, $A^{-1} = B - BC(D + C^H B C)^{-1} C^H B$

$$R_u = E[uu^H] = E[(s+v)(s^H + v^H)] = E[ss^H + sv^H + vs^H + vv^H]$$

$$= \underbrace{E[ss^H]}_{\text{deterministic}} + \underbrace{E[sv^H] + E[vs^H]}_{\text{uncorrelated}} + E[vv^H]$$

$$= ss^H + E[vv^H] = ss^H + R_v$$

$$w_o = R^{-1} p = (ss^H + R_v)^{-1}(s s_k)$$ (lemma)

$$w_o = (ss^H + R_v)^{-1} p = \left[R_v^{-1} - R_v^{-1} s (I + s^H R_v^{-1} s)^{-1} s^H R_v^{-1}\right] s s_k$$

$$= \left[R_v^{-1} - \frac{R_v^{-1} s s^H R_v^{-1}}{1 + s^H R_v^{-1} s}\right] s s_k$$

$$= \left[\frac{1 + s^H R_v^{-1} s}{1 + s^H R_v^{-1} s} - \frac{s^H R_v s}{1 + s^H R_v^{-1} s}\right] R_v^{-1} s s_k$$

$$= \left[\frac{s_k}{1 + s^H R_v^{-1} s}\right] R_v^{-1} s$$

(b) $\rho = \frac{E[(w^H s)^2]}{E[(w^H v)^2]}$, show that $w_{SN} = R_v^{-1} s$

$$\rho = \frac{E[w^H s s^H w]}{E[w^H v v^H w]} = \frac{w^H s s^H w}{w^H R_v w}$$

let $x = R^{1/2} w \rightarrow x^H = w^H R^{1/2} \rightarrow w = R^{-1/2} x$

$$\rho = \frac{w^H s s^H w}{x^H x} = \frac{x R^{-1/2} s s^H R^{-1/2} x}{x^H x}$$

The eigenvector cooresponding to the largest eigenvalue is $x = R^{-1/2} s \rightarrow R^{-1/2} s = R^{1/2} w \rightarrow w_{SN} = R^{-1} s$

(c) $f_v(v) = \frac{1}{(2\pi)^{M/2} \det(R_v)^{1/2}} \exp\left[\frac{-1}{2} v^T R_v^{-1} v\right]$

$$f_u(u|H_0) = \frac{1}{(2\pi)^{M/2} \det(R_u)^{1/2}} \exp\left[\frac{-1}{2} u^T R_u^{-1} u\right]$$

$$f_u(u|H_1) = \frac{1}{(2\pi)^{M/2} \det(R_u)^{1/2}} \exp\left[\frac{-1}{2} (u-s)^T R_u^{-1} (u-s)\right]$$

$$\Lambda = \frac{f_u(u|H_1)}{f_u(u|H_0)} = \exp\left[-\frac{1}{2}(u-s)^T R_u^{-1}(u-s) + \frac{1}{2} u^T R_u^{-1} u\right]$$

$$\Lambda = \exp\left[\frac{1}{2}\left(\cancel{u^T R_u^{-1} u} - \cancel{u R_u^{-1} u} - s^T R_u^{-1} u - u^T R_u^{-1} s + s^T R_u^{-1} s\right)\right]$$

$$\Lambda = \exp\left[\frac{1}{2}\left(\cancel{u^T R_u^{-1} u} - \cancel{u R_u^{-1} u} - s^T R_u^{-1} u - u^T R_u^{-1} s + s^T R_u^{-1} s\right)\right]$$

$$\ln(\Lambda) = \frac{1}{2}\left[-s^T R_u^{-1} u - u^T R_u^{-1} s + s^T R_u^{-1} s\right]$$

$$= \frac{1}{2}\left[-w_{ml}^H u - u^H w_{ml} + s^T R_u^{-1} s\right]$$ $R_u^{-1} s = w_{ml}$

$$= -w_{ml}^H u + \frac{1}{2} s^T R_u^{-1} s$$ ← let $\eta = \frac{1}{2} s^T R_u^{-1} s$

when $\ln(\Lambda) = 0$ $\Lambda = 1$, equal likelihood

$$\underbrace{w_{ml}^H u < \eta}_{H_1} \qquad \underbrace{w_{ml}^H u > \eta}_{H_0}$$