Problem Set #1:

1. Let A be square invertible matrix, α scalar, u vector. Simplify $(A+\alpha uu^H)^{-1}$.

Lemma: $(B^{-1}+CD^{-1}C^H)^{-1} = B - BC(D+C^HBC)^{-1}C^HB$

Let $B^{-1}=A$ or $B=A^{-1}$, $D^{-1}=\alpha$ or $D=\frac{1}{\alpha}$, $C=u, C^H=u^H$

$(B^{-1}+CD^{-1}C^H)^{-1} = (A+\alpha uu^H)^{-1} \rightarrow$ apply lemma

$(A+\alpha uu^H)^{-1} = A^{-1} - A^{-1}u\left(\frac{1}{\alpha}+u^HA^{-1}u\right)^{-1}u^HA^{-1}$

$$= \boxed{A^{-1} - \left(\frac{1}{\alpha}+u^HA^{-1}u\right)^{-1}A^{-1}uu^HA^{-1}}$$

Take special case $A=\delta I$. Simply the expression and determine the condition α for this to be invertible.

$$(\delta I+\alpha uu^H)^{-1} = \frac{1}{\delta}I - \left(\frac{1}{\alpha}+\frac{1}{\delta}u^Hu\right)^{-1}\left(\frac{1}{\delta}\right)Iuu^H\left(\frac{1}{\delta}\right)I$$

$$= \left(\frac{1}{\delta}\right)I - \frac{1}{\delta^2}\left(\frac{\delta+\alpha u^Hu}{\alpha\delta}\right)^{-1}uu^H$$

$$= \boxed{\left(\frac{1}{\delta}\right)I - \frac{\alpha}{\delta(\delta+\alpha u^Hu)}uu^H}$$

$\delta+\alpha u^Hu \neq 0 \Leftrightarrow \alpha u^Hu \neq -\delta \Leftrightarrow \boxed{\alpha \neq -\delta(u^Hu)^{-1}}$

2. Let $\{u_i\}_{i=1}^r$ = lin. indep. $M\times 1$ vects, $\{v_i\}_{i=1}^r$ lin. indep $N\times 1$

Show that $M\times N$ matrix $A=\sum_{i=1}^r u_iv_i^H$ has rank r exactly.

$\left[\sum_{i=1}^r(u_iv_i^H)\right]x=0 \Leftrightarrow v_i^Hx=0\ \forall i \Leftrightarrow x\in\left(\operatorname{span}\{v_i\}_{i=1}^r\right)^{\perp}$

So $N_A=\left(\operatorname{span}\{v_i\}_{i=1}^r\right)^{\perp}$, $N_A^{\perp}=\operatorname{span}\{v_i\}_{i=1}^r$, r-dimensional

$\therefore N_A^{\perp}, R_A$ are r-dim, A is rank r

3. Let x,u be $M\times 1$ col. vects. Find optimal α s.t. $x\approx\alpha u$.

(a) Find $u^{\#}$, show that $\alpha=\frac{\langle x,u\rangle}{|u|^2}$

$\underset{(M\times1)}{x} = \underset{(M\times1)(1\times1)}{u\alpha} \rightarrow u^Hx=u^Hu\alpha \rightarrow \alpha=\frac{u^Hx}{u^Hu}=\frac{\langle x,u\rangle}{|u|^2}$

(b) Let $D=\operatorname{diag}\{d_i\}$

$[DA]_{ij}=\sum_{k=1}^N d_{ik}a_{kj} = d_{ii}a_{ij} \leftarrow$ scales rows

$[AD]_{ij}=\sum_{k=1}^N a_{ik}d_{kj} = a_{ij}d_{jj} \leftarrow$ scales cols

DA scales the rows, AD scales the columns

switched U to A b/c notation was easier

(c) Let $A\in\mathbb{C}^{M\times N}$ with orthogonal columns $\{a_i\}$. Find the SVD of A, use that to find $A^{\#}$, use this to find $\{\alpha_i\}$ s.t. $x\approx\sum\alpha_i a_i$

$A^HA = V\Sigma U^HU\Sigma V^H = V\Sigma^2V^H$

$$\begin{bmatrix} a_1^H \\ \vdots \\ a_N^H\end{bmatrix}\underbrace{\begin{bmatrix} a_1 & \cdots & a_N\end{bmatrix}}_{M\times1\text{ col. vect}} = \begin{bmatrix}|a_1|^2 & & & \\ & |a_2|^2 & & \\ & & \ddots & \\ & & & |a_N|^2\end{bmatrix}$$

$$V=I_{(N\times N)},\quad \Sigma=\begin{bmatrix}|a_1| & & & \\ & |a_2| & & \\ & & \ddots & \\ & & & |a_N|\end{bmatrix} \swarrow \text{by inspection}$$

$$A=U\Sigma V^H \rightarrow AV=U\Sigma \rightarrow Av_i=\sigma_iu_i \rightarrow u_i=\frac{Av_i}{\sigma_i}$$

$$u_i=\frac{Ae_i}{|a_i|}=\frac{a_i}{|a_i|} \longrightarrow U=\underbrace{\begin{bmatrix}\frac{a_1}{|a_1|} & \cdots & \frac{a_N}{|a_N|}\end{bmatrix}}_{M\times1}$$

$$A^{\#}=(A^HA)^{-1}A^H=\begin{bmatrix}\frac{1}{|a_1|^2} & & \\ & \ddots & \\ & & \frac{1}{|a_N|^2}\end{bmatrix}\begin{bmatrix}a_1^H \\ \vdots \\ a_N^H\end{bmatrix}$$

$$=\begin{bmatrix}a_1^H/|a_1|^2 \\ \vdots \\ a_N^H/|a_N|^2\end{bmatrix}$$

$$x\approx\sum\alpha_ia_i = A\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{bmatrix} \rightarrow \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{bmatrix}\approx A^{\#}x$$

$$\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{bmatrix}=\begin{bmatrix}a_1^H/|a_1|^2 \\ \vdots \\ a_N^H/|a_N|^2\end{bmatrix}x=\begin{bmatrix}a_1^Hx/|a_1|^2 \\ \vdots \\ a_N^Hx/|a_N|^2\end{bmatrix}$$

* P is an orthogonal projection?

$\alpha_i=\frac{a_i^Hx}{|a_i|^2}=\frac{\langle x,a_i\rangle}{|a_i|^2}$ (consistant with part b)

4. Let U be a linear subspace of $\mathbb{C}^M$, P be the $M\times M$ projection onto U. $P=P^H$ and $P=P^2$.

(a) Prove the converse: If P is $M\times M$ Hermetian with $P=P^2$, then it is the projection matrix onto a subspace.

- $\forall x,y$, $\langle Px,(I-P)y\rangle = y^H(I-P^H)Px = (y^H-y^HP)(Px)$
  $= y^HPx - y^HP^2x = 0 \rightarrow R_P$ and R_{I-P} are orthogonal
- $\forall v\in\mathbb{C}^M$, $v=v_1+v_2$, $v_1\in R_P$ $v_2\in R_{I-P}$ or $v_1=Px_1$
  $v_2=(I-P)x_2$ for some x_1,x_2
- $Pv=P(v_1+v_2)=Pv_1+Pv_2=P(Px_1)+P(I-P)x_2$
  $=P^2x_1+Px_2-P^2x_2=Px_1+Px_2-Px_2=Px_1=v_1$

  so P is an orthogonal projection onto U

(b) Let P = orth. projection onto U. $H=I-2P$. Consider $x=x_1+x_2$, $x_1\in U$, $x_2\in U^{\perp}$. Express Hx in terms of x_1 and x_2.

$Hx=(I-2P)x=x-2Px=(x_1+x_2)-2x_1=\boxed{x_2-x_1}$