# Heat Transfer - Midterm

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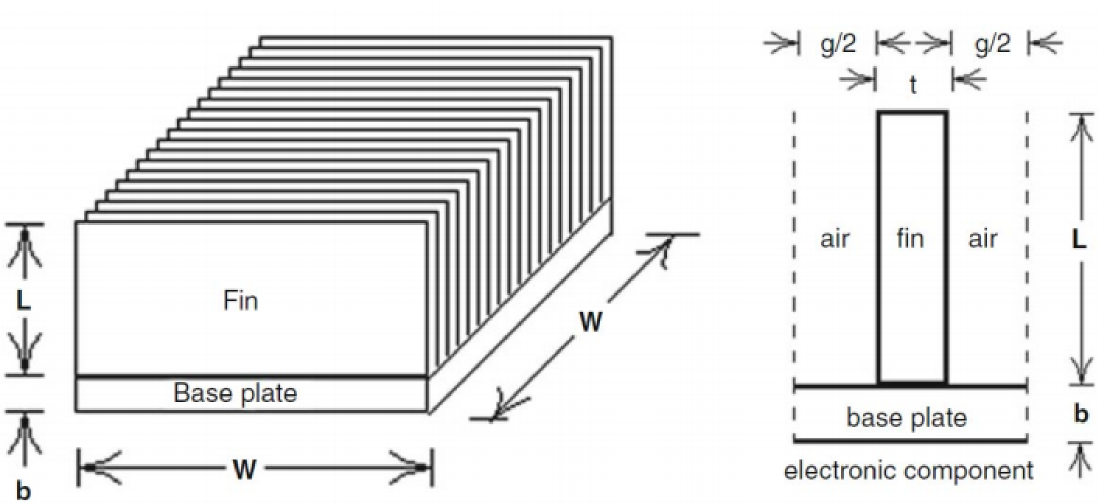
## 1 Problem Statement

A "heat sink" is mounted onto a 6 cm x 6 cm electronic component that generates 250 W of heat when operating. The heat sink has a 0.6 cm thick ($b$) based plate that fits exactly onto the component. Sixteen rectangular fins of width $w = 6$ cm, thickness $t = 0.10$ cm and length $L = 4$ cm are mounted onto the base plate with an air gap of $g = 0.275$ cm between each fin. The fins are cooled by a fan that provides sufficient ventilation to maintain the air temperature in the gaps at ambient temperature ($T_\infty = 25$°C). Compare the transient performance of heat sinks of cooper, aluminum, and steel, with the thermal properties below:

| Material | $k$ $(W/m/K)$ | $\rho$ $(kg/m^3)$ | $c$ $(J/kg/K)$ |
|---|---|---|---|
| Copper | 400 | 8900 | 390 |
| Aluminum | 200 | 2700 | 910 |
| Steel | 20 | 7900 | 490 |

Write code to simulate when the component is powered on, allowed to reach an effective steady-state, and then turned off and allowed to cool to ambient. Analyze a single fin by breaking it into 3 equal-sized nodes and include a $4^{th}$ node for the base. Assume the fin is exposed to ambient with a fixed heat transfer coefficient, $h = 50 \frac{\mathrm{W}}{\mathrm{m^2K}}$. Neglect radiation since the fins "see" each other.



## 2 Impedances

In order to simplify the problem, the symmetry of the fin will be utilized. The system will model half of one fin and the impedances will be calculated accordingly. Using the material properties defined, the dimensions of the component, and the operating conditions of the system, the thermal impedances of the component are found.

The conductive resistance through the fin is $R_k = \frac{\Delta x}{kA} = \frac{l}{kw(t/2)} = \frac{2l}{kwt}$. The convective resistance of the base plate to ambient is $R_{h,base} = \frac{1}{hA} = \frac{1}{hw(g/2)} = \frac{2}{hwg}$. The convective resistance of the fin to ambient is $R_h = \frac{1}{hA} = \frac{1}{hwl}$. The convective resistance of the end of the fin to ambient is $R_h = \frac{1}{hA} = \frac{1}{hw(t/2)} = \frac{2}{hwt}$.

The capacitance of the base is $C_{base} = \rho c(\frac{g}{2})(\frac{t}{2})b = \frac{\rho cgtb}{4}$ and the resulting impedance is $Z_{C,base} = \frac{1}{sC_{base}}$. The capacitance of the fin is $C = \frac{\rho c(lwt)}{2}$ and the resulting impedance is $Z_C = \frac{1}{sC}$.

As indicated in the diagrams above, $t$ is the thickness of the fin, $w$ is the width of the fin, $l$ is the length of the fin, $g$ is the air gap between each fin, $b$ is the thickness of the base plate, $h$ is the convective heat transfer coefficient, $k$ is the conductive coefficient of the material, $c$ is the thermal capacitance of the material, and $\rho$ is the density of the material.

Furthermore, the value of the current source used in the diagram is $q = 250/16$ W. This is the total heat generated by the electronic device divided by the number of fins.

## 3 Equivalent Circuits

The entire system can be modelled with the circuit below using the impedances defined in the previous section. When the electronic component is on, the switch is closed. When the electronic component is off, the switch is open (assume current goes to some other location).

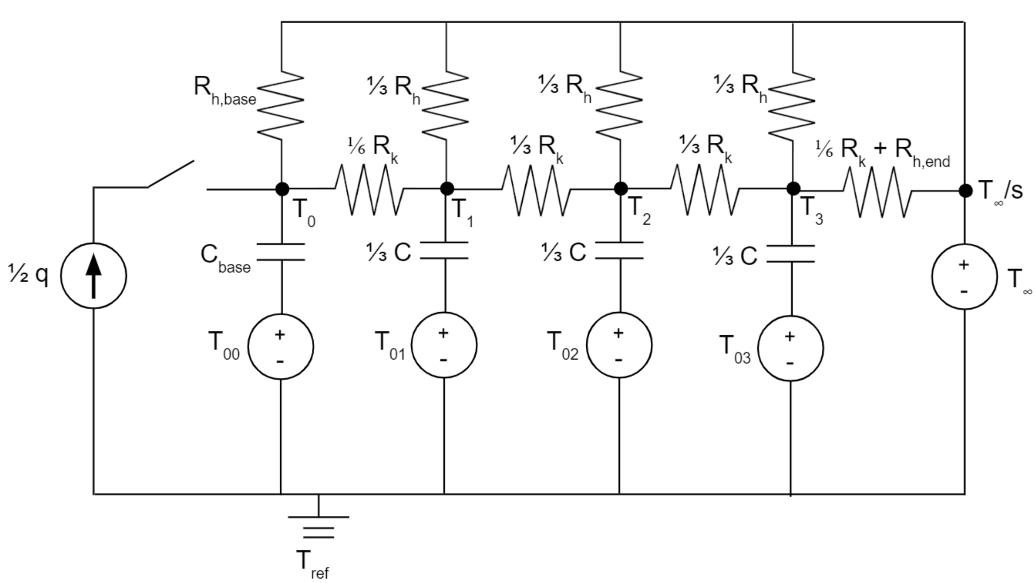



Figure 1: Equivalent Circuit

Before finding the response of the system when the component is turned off, the initial conditions need to be found. This can be done by modelling the electronic component as a current source and treating the elements with heat capacitance as an open circuit. The steady-state system is modelled with the circuit below:

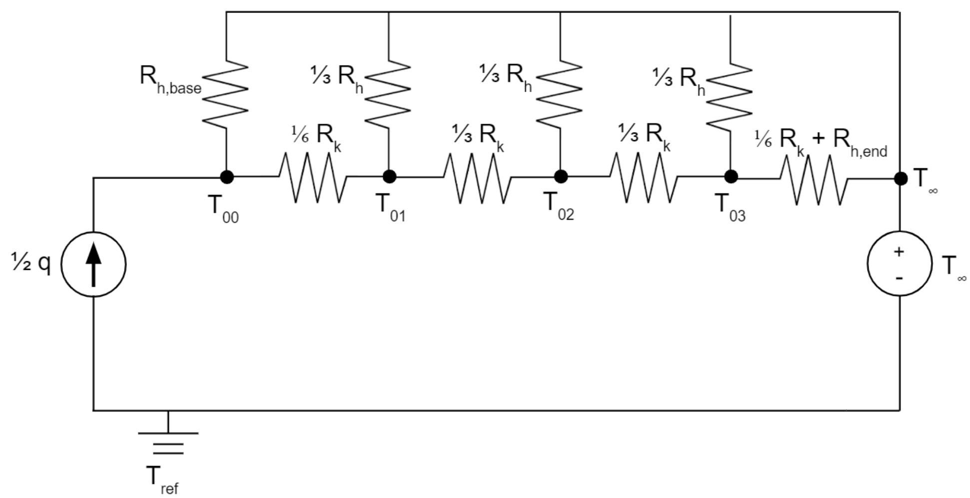



Figure 2: Electronic Component at Steady-State

Once the values of $T_{00}$, $T_{01}$, $T_{02}$, and $T_{03}$ are found, the transient response of the system once the electronic component is turned off can be found. The system is then modelled with the following circuit:

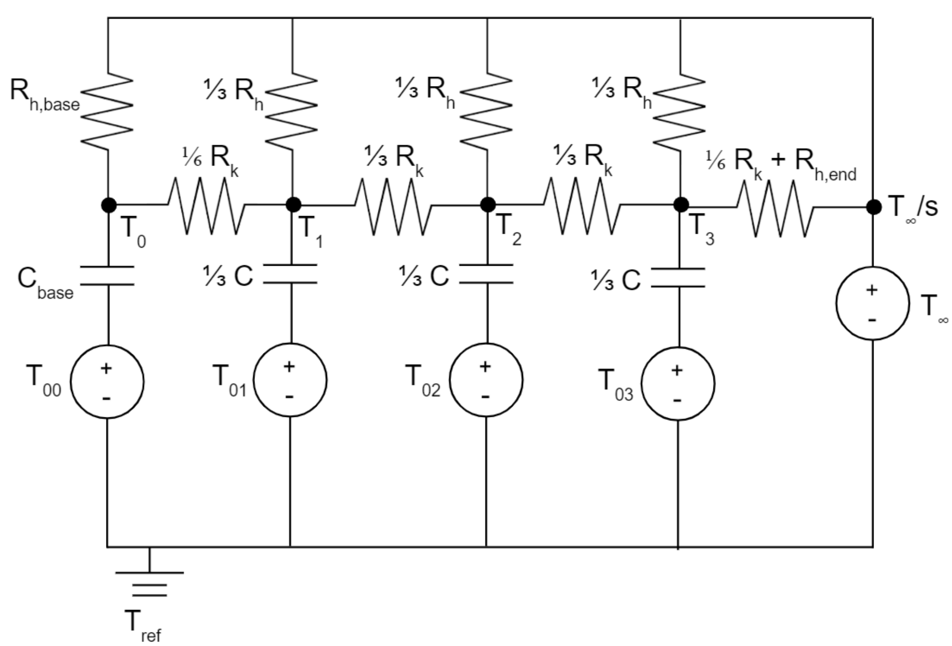



Figure 3: Electronic Component Turned Off After Reaching Steady-State

# 4 Nodal Equations

The transient initial conditions $T_{00}$, $T_{01}$, $T_{02}$, and $T_{03}$ can be found by solving for the temperatures indicated in Figure 2. The nodal equations used to find these temperatures are

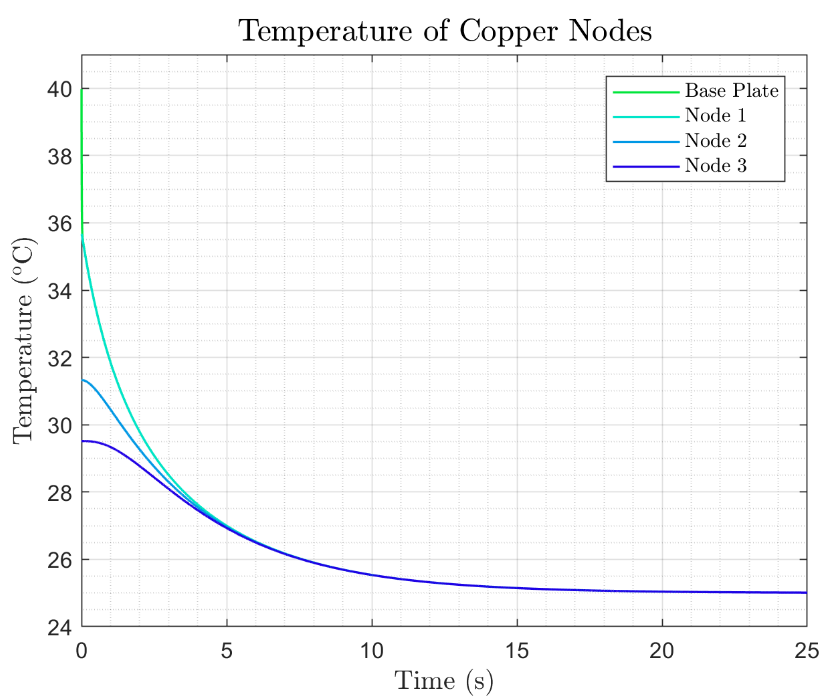
$$\begin{pmatrix} \frac{1}{R_{h,base}} + \frac{6}{R_k} & -\frac{6}{R_k} & 0 & 0 \\ -\frac{6}{R_k} & \frac{9}{R_k} + \frac{3}{R_h} & -\frac{3}{R_k} & 0 \\ 0 & -\frac{3}{R_k} & \frac{6}{R_k} + \frac{3}{R_h} & -\frac{3}{R_k} \\ 0 & 0 & -\frac{3}{R_k} & \frac{3}{R_k} + \frac{3}{R_h} + \frac{1}{\frac{R_k}{6} + R_{h,end}} \end{pmatrix} \begin{pmatrix} T_{00} \\ T_{01} \\ T_{02} \\ T_{03} \end{pmatrix} = \begin{pmatrix} \frac{q}{2} + \frac{T_\infty}{R_{h,base}} \\ \frac{3T_\infty}{R_h} \\ \frac{3T_\infty}{R_h} \\ \frac{3T_\infty}{R_h} + \frac{T_\infty}{\frac{R_k}{6} + R_{h,end}} \end{pmatrix}$$

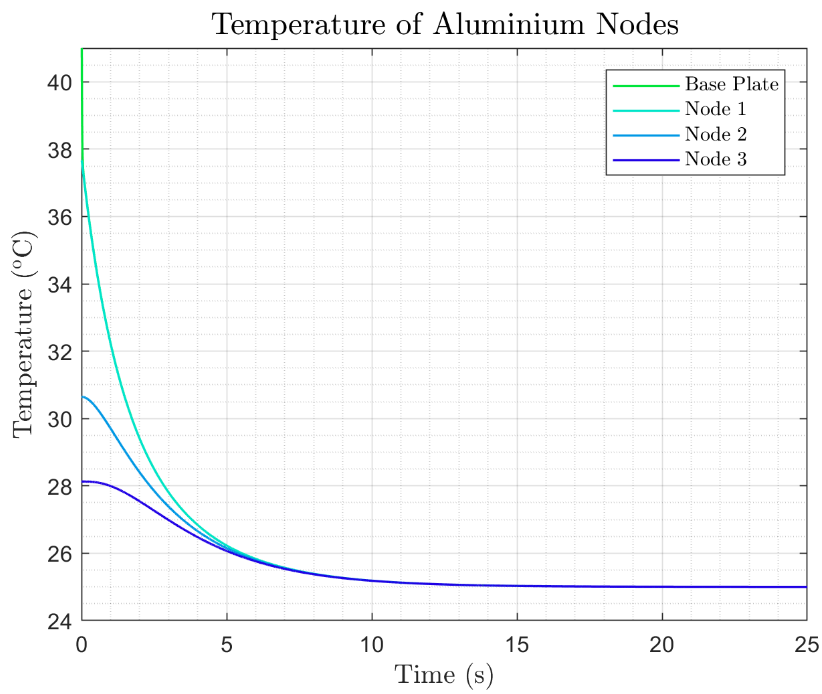
Once the initial conditions are found, the temperatures of the nodes indicated in Figure 3 can be solved using nodal analysis. The resulting equations in the Laplace domain are

$$\begin{pmatrix} \frac{1}{R_{h,base}} + \frac{6}{R_k} + \frac{1}{Z_{C,base}} & -\frac{6}{R_k} & 0 & 0 \\ -\frac{6}{R_k} & \frac{9}{R_k} + \frac{3}{R_h} + \frac{1}{3Z_C} & -\frac{3}{R_k} & 0 \\ 0 & -\frac{3}{R_k} & \frac{6}{R_k} + \frac{3}{R_h} + \frac{1}{3Z_C} & -\frac{3}{R_k} \\ 0 & 0 & -\frac{3}{R_k} & \frac{3}{R_k} + \frac{3}{R_h} + \frac{1}{3Z_C} + \frac{1}{\frac{R_k}{6} + R_{h,end}} \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix} = \frac{1}{s} \begin{pmatrix} \frac{T_{00}}{Z_{C,base}} + \frac{T_\infty}{R_{h,base}} \\ \frac{T_{01}}{3Z_C} + \frac{3T_\infty}{R_h} \\ \frac{T_{02}}{3Z_C} + \frac{3T_\infty}{R_h} \\ \frac{T_{03}}{3Z_C} + \frac{T_\infty}{R_{h,end} + \frac{R_k}{6}} \end{pmatrix}$$

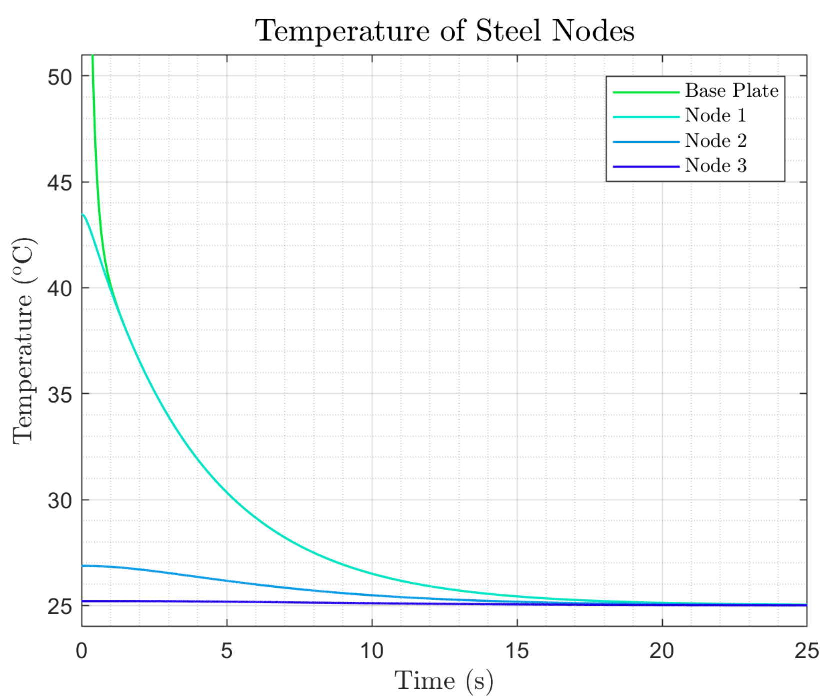
Once MATLAB is used to solve the systems, the temperatures at each of the nodes and the base plate can be found and plotted against time.

# 5 Plots





Temperature of Aluminium Nodes
Base Plate
Node 1
Node 2
Node 3
Temperature (°C)
Time (s)
40
38
36
34
32
30
28
26
24
0
5
10
15
20
25

Temperature of Steel Nodes
Base Plate
Node 1
Node 2
Node 3
Temperature (°C)
Time (s)
50
45
40
35
30
25
0
5
10
15
20
25

# 6 Final Results

- The steady-state temperature for all three materials is the ambient temperature, **25° C**.
- It takes approximately **15 seconds** for the copper system to reach its steady-state temperature.
- It takes approximately **12 seconds** for the aluminum system to reach its steady-state temperature.
- It takes approximately **20 seconds** for the steel system to reach its steady-state temperature.
- The temperature of the nodes are most similar for aluminium and least similar for steel.

It can be seen that the temperatures of the base plate and each fin equalize to one steady-state value for all three materials. However, the time it takes for the three systems to reach their steady-state values varies. For example, aluminum, which has the highest thermal capacitance value, lowest density, and a thermal conductivity higher than steel, reaches steady-state the fastest. This is because the density and thermal capacitance affect the system's ability store thermal energy. Because aluminum has the lowest $\rho c$ value, it holds heat the worst among the three materials. Copper has the second lowest $\rho c$ value, and steel has the highest $\rho c$ value. These align with the behavior of their respective heat sinks, which take the second-longest and longest to reach steady-state, respectively.

The difference in temperature between the nodes during the transient response is determined by the conductive coefficient of each material. Steel, which has the lowest conductive coefficient and therefore the highest conductive resistance, has a large difference in temperature between each node. Aluminum has the second smallest conductive coefficient and thus the temperature of the nodes are more similar than the steel nodes. Copper has the highest conductive coefficient and therefore the smallest change in temperature between nodes.

These results allow one to see the costs and benefits of using these three materials for a heat sink system. The initial temperature of the base plate indicates the fins ability to cool the electronic component at steady-state and the transient response time indicates the ability of the fins to cool down once the component is powered off. However, in real life, while certain materials may have better thermal properties, factors such as cost, weight, and availability of the material must also be considered. This problem highlights how different material properties affect the transient performance of a system. It also highlights the importance of taking a complex, three dimensional system, and simplifying it for analysis purposes. While certain assumptions, such as neglecting radiation, may cause slight differences between the results and the actual transient performance of the system, the calculated results give a close estimate of how the system behaves, thus allowing for further design changes and manipulation of the system to achieve desired behavior.

# 7 MATLAB Code

## Contents



## Midterm Resubmission - ME342 - March 28, 2020

```
clear all; close all; clc;
```

## Define Parameters

```
% Dimensions
t = 0.001; % m
w = 0.06; % m
l = 0.04; % m
g = 0.00275; % m
b = 0.006; % m
F = 16; % number of fins

% Thermal Properties
h = 50; % W/m^2K
% Material chosen by commenting out some variables
    % % Copper
    % k = 400; % W/mK
    % rho = 8900; % kg/m^3
    % c = 390; % J/kgK
    % % Aluminium
    % k = 200; % W/mK
    % rho = 2700; % kg/m^3
    % c = 910; % J/kgK
% Steel
k = 20; % W/mK
rho = 7900; % kg/m^3
c = 490; % J/kgK

% System Conditions
Q = 250/F; % W
T_inf = 25+273.15; % K
```

## Impedances

```
% Note: These are calculated for half of one fin.

% Resistances
R_k = (2*l)/(k*w*t); % K/W
R_h = 1/(h*l*w); % K/W
R_h_base = 2/(h*w*g); % K/W
R_h_end = 2/(h*w*t); % K/W

% Capacitances
s = tf('s');
C = (c*l*w*t*rho)/2; % J/K
C_base = (c*g*t*b*rho)/4; % J/K
Z_C = 1/(s*C);
Z_C_base = 1/(s*C_base);
```

## Find Initial Conditions

```
% Define terms of admittance matrix in steady-state
mho_0_11 = (1/R_h_base) + (6/R_k);
mho_0_22 = (9/R_k) + (3/R_h);
mho_0_33 = (6/R_k) + (3/R_h);
mho_0_44 = (3/R_k) + (3/R_h) + (1/((R_k/6) + R_h_end));
mho_0_12 = -6/R_k;
mho_0_23 = -3/R_k; mho_0_34 = -3/R_k;

% Define admittance matrix
mho_0 = [mho_0_11, mho_0_12, 0, 0;
         mho_0_12, mho_0_22, mho_0_23, 0;
         0, mho_0_23, mho_0_33, mho_0_34;
         0, 0, mho_0_34, mho_0_44];

% Define terms of forcing vector
q_0_1 = (Q/2) + (T_inf/R_h_base);
q_0_2 = 3*T_inf/R_h;
q_0_3 = 3*T_inf/R_h;
q_0_4 = (3*T_inf/R_h) + (T_inf/((R_k/6) + R_h_end));

% Define forcing vector
q_0 = [q_0_1; q_0_2; q_0_3; q_0_4];

% Find initial temperatures
T_0 = mho_0\q_0;
```

```
% Define variable names for intial temperatures
T_00 = T_0(1);
T_01 = T_0(2);
T_02 = T_0(3);
T_03 = T_0(4);
```

## Find Transient Response

```
% Define terms of admittance matrix
mho_11 = (6/R_k) + (1/R_h_base) + (1/Z_C_base);
mho_22 = (9/R_k) + (3/R_h) + (1/(3*Z_C));
mho_33 = (6/R_k) + (3/R_h) + (1/(3*Z_C));
mho_44 = (3/R_k) + (3/R_h) + (1/((R_k/6) + R_h_end)) + (1/(3*Z_C));
mho_12 = -6/R_k; mho_23 = -3/R_k; mho_34 = -3/R_k;

% Define admittance matrix
mho = [mho_11, mho_12, 0, 0;
       mho_12, mho_22, mho_23, 0;
       0, mho_23, mho_33, mho_34;
       0, 0, mho_34, mho_44];

% Define terms of forcing vector
q_1 = (T_00/Z_C_base) + (T_inf/R_h_base);
q_2 = (T_01/(3*Z_C)) + (3*T_inf/R_h);
q_3 = (T_02/(3*Z_C)) + (3*T_inf/R_h);
q_4 = (T_03/(3*Z_C)) + (3*T_inf/R_h) + T_inf/((R_k/6)+R_h_end);

% Define forcing vector
q = [q_1; q_2; q_3; q_4]/s;

% Find temperatures at nodes
T = mho\q;

% Convert to time domain and switch to Celcius
time = linspace(0,25,10000);
t_0 = impulse(T(1),time) - 273.15;
t_1 = impulse(T(2),time) - 273.15;
t_2 = impulse(T(3),time) - 273.15;
t_3 = impulse(T(4),time) - 273.15;
```

## Plot Temperature of Nodes

```
% Color
four_colors = [0, 229, 59; 0, 229, 201; 0, 152, 229; 31, 0, 229]/255;
set(groot,'defaultAxesColorOrder',four_colors)
```

```
figure(1)
plot(time,t_0,time,t_1,time,t_2,time,t_3,'LineWidth',1)
title('Temperature of Steel Nodes','interpreter','latex','FontSize',14)
xlabel('Time (s)','interpreter','latex','FontSize',12)
ylabel('Temperature ($^{\mathrm o}$C)','interpreter','latex','FontSize',12)
key = legend('Base Plate','Node 1','Node 2','Node 3');
set(key,'Interpreter','latex');
grid on
grid minor
xlim([0,time(end)]);
ylim([24,51]);
```