# Cognitive Science

## Counting to Infinity: Does learning the syntax of the count list predict knowledge that numbers are infinite?

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| **Corresponding Author:** | Junyi Chu<br>Massachusetts Institute of Technology<br>Cambridge, MA UNITED STATES |
| **Corresponding Author Secondary Information:** | |
| **Corresponding Author's Institution:** | Massachusetts Institute of Technology |
| **Corresponding Author's Secondary Institution:** | |
| **First Author:** | Junyi Chu |
| **First Author Secondary Information:** | |
| **Order of Authors:** | Junyi Chu |
| | Pierina Cheung |
| | Rose M Schneider |
| | Jessica Sullivan |
| | David Barner |
| **Order of Authors Secondary Information:** | |
| **Abstract:** | By around the age of 5½, many children in the US judge that numbers never end, and that it is always possible to add +1 to a set. These same children also generally perform well when asked to label the quantity of a set after 1 object is added (e.g., judging that a set labeled “five” should now be “six”). These findings suggest that children have implicit knowledge of the “successor function”: every natural number, n, has a successor, n+1, such that numbers are infinite. Here, we explored how children discover this recursive function, and whether it might be related to discovering productive morphological rules that govern language-specific counting routines (e.g., the rules in English that represent base 10 structure). We tested 4- and 5-year-old children’s knowledge of counting with three tasks, which we then related to (1) children’s belief that 1 can always be added to any number (the successor function), and (2) their belief that numbers never end (infinity). Children who exhibited knowledge of a productive counting rule were significantly more likely to believe that numbers are infinite (i.e., there is no largest number), though such counting knowledge wasn’t linked to knowledge of the successor function, per se. Our findings suggest that children as young as four years of age are able to implement rules defined over their verbal count list to generate number words beyond their spontaneous counting range, an insight which may support reasoning over their acquired verbal count sequence to infer that numbers never end. |

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Dear Editors:

We are delighted to submit our manuscript titled "Counting to Infinity: Does learning the syntax of the count list predict knowledge that numbers are infinxite?" to *Cognitive Science* as a regular article. Despite experiencing only particular number words, children eventually believe that numbers go on forever. Here we report the first investigation into how children may discover the infinite nature of numbers by noticing productive morphological rules that generate successive number words.

We measured 4-5-year-old children's familiarity with the count list and separately, their ability to use the decade+unit rule to generate numbers outside their familiar count list. Critically, **children's knowledge that numbers never end depended not on the length of their familiar count list, but instead, was predicted by their knowledge of this productive decade rule.** These results suggest that children's conceptions of number are influenced by their linguistic knowledge of number words: Upon knowing that the numbers one through nine are repeated iteratively across decades, children not only learn to generate successive number words given some starting point, but also gain access to the insight that one can *always* continue counting, and thus, that numbers never end.

The interaction between numerical reasoning, word learning, and concept acquisition is a topic of great interest in fields ranging from experimental psychology and language development, to math education and computational models of learning. Our results, suggesting that syntactic knowledge may lead human learners to discover that the natural numbers form an infinite class, should be of interest to researchers in all these fields. The results also suggest the important role that language learning plays in the acquisition of abstract concepts such as the successor relation and infinity. We believe that the originality, broad impact, and accessibility of these findings make *Cognitive Science* an appropriate venue for this work; we hope you will agree.

Given their expertise in number learning, language acquisition, and conceptual development, we would like to suggest as reviewers Elizabeth Gunderson, Steven Piantadosi, Barbara Sarnecka, Hilary Barth, Darko Odic, and Daniel Ansari. Due to a potential conflict of interest, we ask that Charles Yang not serve as a reviewer here. All authors have approved the final version for submission and the manuscript has not been previously published nor submitted to another journal for publication. The experimental work was approved by our Institutional Review Board and conducted with informed consent of parents. If you have any questions about the manuscript, Junyi Chu will be serving as the corresponding author.

Please let us know if there is any additional information we can provide. Thank you for your time and attention to the manuscript.

Sincerely,
Junyi Chu, Pierina Cheung, Rose M. Schneider, Jessica Sullivan, and David Barner.

# Counting to Infinity: Does learning the syntax of the count list predict knowledge that numbers are infinite?

Junyi Chu[a,1,*], Pierina Cheung[b], Rose M. Schneider[a], Jessica Sullivan[c], & David Barner[a]

[a]University of California San Diego, La Jolla, CA 92093, USA

[b]National Institute of Education, Nanyang Technological University, Singapore

[c]Skidmore College, Saratoga Springs, NY 12866, USA

Word Count: 9743

* Corresponding author.

Email address: junyichu@mit.edu

[1] Present address: Department of Brain and Cognitive Sciences, Massachusetts Institute of Technology, Cambridge, MA 02138, USA

## Abstract

By around the age of 5½, many children in the US judge that numbers never end, and that it is always possible to add +1 to a set. These same children also generally perform well when asked to label the quantity of a set after 1 object is added (e.g., judging that a set labeled "five" should now be "six"). These findings suggest that children have implicit knowledge of the "successor function": every natural number, *n*, has a successor, *n*+1, such that numbers are infinite. Here, we explored how children discover this recursive function, and whether it might be related to discovering productive morphological rules that govern language-specific counting routines (e.g., the rules in English that represent base 10 structure). We tested 4- and 5-year-old children's knowledge of counting with three tasks, which we then related to (1) children's belief that 1 can always be added to any number (the successor function), and (2) their belief that numbers never end (infinity). Children who exhibited knowledge of a productive counting rule were significantly more likely to believe that numbers are infinite (i.e., there is no largest number), though such counting knowledge wasn't linked to knowledge of the successor function, *per se*. Our findings suggest that children as young as four years of age are able to implement rules defined over their verbal count list to generate number words beyond their spontaneous counting range, an insight which may support reasoning over their acquired verbal count sequence to infer that numbers never end.


*Keywords*: Count list, Infinity, Conceptual change, Successor function, Highest count, Decade rule

# 1. Introduction

Human learners draw on a finite set of experiences to acquire information about the world, but nevertheless acquire systems of rules that permit the generation of unbounded representational outputs. For example, natural language is often touted as an example of how humans make "infinite use of finite means": a finite lexicon and system of combinatorial rules allows children to generate an unbounded number of possible utterances (Chomsky, 1965; Humboldt, 1836/1999, p.91). Similarly, numerate humans learn a set of symbols and combinatorial rules that permit an unbounded set of mathematical expressions. For example, the English base-10 numeral system expresses 80 numbers (from twenty to ninety-nine) by composing just 17 unique decade and unit words (twenty through ninety, and one through nine). The expressive power of this system is exponential and in fact, unbounded: to express larger numbers, one simply needs to learn the appropriate words representing powers of ten (e.g. hundred, thousand, million, etc.), and recursively compose them following the syntactic rules of numerals (Cheung, Dale, & Le Corre, 2016; Hurford, 1975). As we describe below, although children initially believe that numbers are finite – and have limited knowledge of both the structure and meaning of number words – they ultimately come to believe that numbers never end. In the present study, we investigate how children learn that numbers are infinite, and whether the rule-governed structure of the verbal count list might play a role in children's inference that numbers form an infinite class.

When children first begin learning about numbers in early childhood, their knowledge is clearly item-based and finite. At around the age of two, English-speaking children in the US begin to recite a subset of the verbal count list (one, two, three, four, etc.), but often can't count beyond ten at this point (Fuson, 1988; Fuson & Hall, 1983; Gelman & Gallistel, 1978). Moreover, these number words appear to lack meanings at this early stage: When asked to give a number (e.g., to give one fish), children initially give a random amount (e.g. Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990). Some months later, children appear to acquire an exact meaning for the word one, and can give one object when asked, while failing to reliably give two when asked. These children are often called one-knowers. Another 6 to 9 months later the children become two-knowers (and can reliably give two objects), then three-knowers several months after that. One by one, children add meanings to their number words in a way that suggests the lack of a productive logic governing these meanings (Sarnecka & Lee, 2009).

While children initially lack a productive rule for understanding number words, a breakthrough appears to happen at around the age of 3 and a half or 4 (in US English-speaking groups), when children appear to realize that they can correctly give any requested number by counting and giving all objects that are implicated in their count. These children are often called "Cardinal Principle Knowers" or CP-knowers. While the time-course varies across language groups, the basic developmental sequence - i.e., of progressing through discrete number-knower stages and finally using the counting routine to productively generate any quantity within the child's count list - has been reported for children learning a variety of languages around the world (Almoammer et al., 2013; Barner, Libenson, Cheung, & Takasaki, 2009; Le Corre, Li, Huang, Jia, & Carey, 2016; Piantadosi, Jara-Ettinger, & Gibson, 2014; Sarnecka, Kamenskaya, Yamana, Ogura, & Yudovina, 2007). Also, in bilingual preschoolers, children progress through discrete number-knower stages independently in each language, but generally become CP-knowers at the same time in both languages (Wagner, Kimura, Cheung, & Barner, 2015). What remains unclear, however, is what, exactly, children learn when they become CP-knowers.

By some accounts, becoming a CP-knower not only involves mastery of an enumeration procedure for set sizes within one's familiar count list, but also an inductive leap in understanding the meaning and structure of natural numbers. Multiple researchers have argued that, to learn how counting works, children construct a type of analogical mapping between the verbal count list and the ordered set of cardinalities that the list represents, beginning with the numbers one, two, and three (Carey, 2004; Gentner, 2010; Schaeffer et al., 1974; Wynn, 1992; for review, see Marchand & Barner, 2018). For example, Wynn (1992) argues that "in order to learn the counting system, children must implicitly make the analogy between the magnitudinal relationships of their own representations of numerosities, and the positional relationships of the number words." (p. 250). Similarly, according to Carey (2004): "Children may here make a wild analogy– that between the order of a particular quantity within an ordered list, and that between this quantity's order in a series of sets related by additional individuals. These are two quite different bases of ordering – but if the child recognizes this analogy, she is in the position to make the crucial induction: For any word on the list whose quantificational meaning is known, the next word on the list refers to a set with another individual added." (p. 67). Following this logic, Sarnecka and Carey (2008) created a task that they called the Unit Task, in which children were told, e.g., "Ok. I'm putting FOUR frogs in the box", then saw 1 or 2 items added, and were then asked, "Now is it FIVE or SIX?" (trials included 4+1, 4+2, 5+1, 5+2). They found that children identified as CP-knowers by Wynn's Give-a-Number task succeeded on 67% of trials overall, whereas subset knowers performed at chance. Based on this, they concluded that becoming a CP-knower involves more than acquiring a procedural rule, and instead marks the moment at which children acquire the successor function.

On this account, the ability to accurately count objects therefore reflects the ability to link cardinal representations with implicit knowledge of a type of logical rule, called the "successor function", described by logicians and philosophers of mathematics including von Leibniz (1704/1996), Peirce (1881), Dedekind (1888/1963), and Peano (1889) in efforts to construct axioms to define the natural numbers. One such system, commonly referred to as the Peano-Dedekind axioms, include principles akin to those in (1), inter alia (though various notational variants exist):

(1) i. 1 is a natural number.

ii. All natural numbers exhibit logical equality (e.g., x=x; if x=y, then y=x, etc.).

iii. For every natural number n, S(n) (the successor of n) is a natural number.

iv. Every natural number has a successor.

Critically, because the Peano axioms state that every natural number has a successor, they generate an infinite number of numbers. Consequently, a child who has implicit knowledge of such rules would be expected to believe that it is always possible to add 1 to a number, and also that numbers never end. Thus, this account predicts that becoming a CP-knower represents a shift from representing numbers as a finite sequence of individual words to understanding them as products of a rule - the successor function - that generates an infinite set of positive integers (for discussion, see Sarnecka & Carey, 2008).

Does becoming a CP-knower involve learning to reason about cardinalities in terms of a recursive successor function? The results of Sarnecka and Carey (2008) leave open this question, since they don't test whether children who succeed on the Unit task generalize this knowledge to all numbers in their count list, let alone to all possible numbers. Also, while they showed that CP-knowers outperformed subset knowers on the Unit task, they didn't show that success on the

Unit task, and thus the ability to reason about cardinalities in terms of successor relations, was a prerequisite to becoming a CP-knower. Instead, they tested only the very smallest numbers - four and five - leaving open the possibility that children's knowledge was item-based, rather than governed by an abstract logical rule. Also, the study left open the precise timeline according to which successor function knowledge is acquired.

Several subsequent studies have suggested that, if children acquire a rule akin to the successor function, this likely occurs several years later than predicted by Sarnecka and Carey (2008). First, Davidson, Eng, and Barner (2012) tested a large group of CP-knowers with the Unit Task, but included a slightly wider range of numbers, extending from 4 to 25. As a proxy for experience with number words, Davidson et al. asked children to count as high as they could, and binned them into low, medium, and high counters, analyzing Unit Task performance only for numbers within each child's count list. They found that almost all low counters (who could count up to 19) performed at 50% on the Unit Task for numbers within their count range despite being CP-knowers, and that only the highest counters (who could count beyond 30) performed systematically well on small numbers. For larger numbers, all groups performed relatively poorly, even when those large numbers were within their count range. Similar results were found in a study of bilingual learners (Wagner et al., 2015), and in a training study which found that many CP-knowers lacked successor function knowledge, though they were more likely than subset knowers to show improvement over 2-3 weeks of training (Spaepen, Gunderson, Gibson, Goldin-Meadow, & Levine, 2018). Finally, in a study of slightly older children, Cheung, Rubenson, and Barner (2017) found that CP-knowers did not perform reliably above-chance on the Unit Task for all numbers in their count list until around the age of 5 and a half - almost two years after having become CP-knowers. Thus, children learn to reason productively about successor relations only after a protracted time period, at which point performance is correlated with, but lagging behind, the length of their count list. This raises the possibility that learning about successor relations depends on some gradual inductive inference or additional knowledge about count words that are not captured in these studies.

Critically, as noted above, the successor function doesn't merely state that for a particular number, *n*, there is a successor. Instead, it states that every number has a successor, such that numbers are infinite. Given this, Cheung et al. (2017) tested the relation between knowledge of how counting implements the successor function (i.e. as measured by the Unit task) and children's developing knowledge of numerical infinity. To do this, they asked children not only to reason about successor relations for familiar numbers, but also asked children their beliefs about numbers in general. Specifically, Cheung et al. (2017) tested children using the Unit Task and an infinity battery first reported by Gelman and colleagues (e.g., Evans, 1983; Hartnett & Gelman, 1998). In this battery, children were asked about the largest number they could name and whether it was the largest possible number, or whether it might be possible to repeatedly add 1 to it. This "successor question" tested their explicit understanding that the successor function applies to all numbers. Children also answered an "endless question" about whether counting would get them to the end of numbers, or if numbers went on forever. Like the earliest studies on this topic, Cheung et al. (2017) found that children initially believe that numbers are finite, and that it's not always possible to add 1, but that by around the age of 6, many undergo a transition and begin to claim that numbers never end. Further, they found that this knowledge emerged shortly after children were able to reason about successor relations and pass the Unit task for large numbers in their count list.

How might children learn, from knowledge of a finite count list, that numbers exhibit a successor function, and are infinite? Previous studies have found that children's ability to identify successors for known numbers, as measured by the Unit task, is related to how high they can count (Cheung et al., 2017; Davidson et al., 2012). For example, Cheung et al. (2017) found that children who could count up to at least 80 (many of whom could count even higher) were able to identify the cardinal value of successors for a wide range of known numbers within their count list, whereas children with lower highest counts could only do so reliably for the smallest numbers. This observation suggests at least two mutually compatible explanations for the relationship between counting experience and successor function knowledge. The first possibility is that there is no direct link between how high a child can count and successor knowledge, and that these two outcomes are correlated because they both result from general exposure to number. The second possibility is that there is a more direct causal link between the two: that children's understanding of how count words are syntactically structured might inform their intuitions regarding successor relations and infinity, and that this structure is only apparent after children have learned to count to relatively large numbers. Specifically, Cheung et al. (2017) noted that when children learn to count in English, they are required to learn a recursive base 10 structure wherein they first count from one to nine, then repeat this one through nine structure with varying degrees of regularity for higher decades, which themselves are generated by multiplying 1-9 by 10 (for related proposals, see Barner, 2017; Yang, 2016).

Consistent with this, previous studies have found that children who can count beyond 100 are better able to decompose numbers into decades and ones, whereas children who can't count as high appear to store the count list as a memorized string (Fuson, Richards, & Briars, 1982; Siegler & Robinson, 1982). Also relevant is that children who can't yet count all the way to 100 nevertheless make errors which suggest some knowledge of rules that structure counting. For example, when asked to count as high as they can, many children stop at decade transitions (Fuson et al., 1982; Siegler & Robinson, 1982; Wright, 1994), with the most frequent being 29 and 39 (Gould, 2017). If children were merely reciting a memorized and unstructured list as they do the alphabet, we might expect the distribution of their errors to be random rather than at decade transitions. These errors suggest that children have memorized an initial list, e.g., up to 20 or 30, and use some form of rule, discussed below, to generate numbers up to the next decade transition (which requires memorized knowledge, since in English decade labels are irregular, and can't be generated from a rule alone). Consistent with this, children exposed to languages with relatively transparent base-10 counting systems, like Mandarin or Cantonese Chinese, appear to count higher and make fewer errors than children learning less transparent counting systems, like English or Welsh (Dowker, Bala, & Lloyd, 2008; Miller, Smith, Zhu, & Zhang, 1995; Miller & Stigler, 1987). Such evidence suggests that children make use of the linguistic structure of their count list to learn rules governing counting. An open question - and the main focus of the present study - is whether learning that number words are compositionally structured might facilitate insights into the conceptual structure of numbers, such as learning the successor function and infinity.

Critically, knowing that numbers exhibit a successor relation might not automatically lead children to believe that numbers never end. As noted by Cheung et al. (2017), many children in their study believed that it's always possible to add 1 to a set but nevertheless believe that numbers must ultimately end (children they called Successor Only Knowers), whereas only a handful of children held the opposite pattern of beliefs - that you can't always add 1, but that numbers are nevertheless infinite (what they called Endless Only Knowers). Previous studies find the same pattern but report no Endless Only knowers at all (Evans, 1983; Hartnett & Gelman, 1998). Cheung et al. (2017) interpreted this pattern as evidence for a developmental sequence whereby children learn some kind of bounded (item-based) successor rule that applies to a finite list, and only later learn that numbers never end. For example, children may first learn that known numbers exhibit a successor relation by empirically noticing this relation between familiar numbers, but may make the induction that this function is recursive by learning that *number words* can be productively generated via the base-10 rules that govern counting. On this hypothesis, experience with counting may be related to both learning the successor function and that numbers are infinite, but for different reasons – e.g., in the first instance providing opportunities for item-based observation regarding relations between known numbers, and in the second instance allowing children to discover rules that generate an unbounded set of numerical symbols.

While previous studies have shown that how high a child can count is related to their ability to identify successor relations of specific numbers (Cheung et al., 2017; see also Davidson et al., 2012), they have not tested whether counting is related to the belief that numbers are endless, independently of the belief that every number has a successor. This is important, because learning rules that govern the structure of number words may allow children to make a broad inference about numbers – e.g., by learning that number words can be decomposed into decades and ones, children may realize that such rules can generate an infinite set of number words, which may in turn form the basis for the belief that numbers are infinite. Notably, although this relation between counting and infinity might hinge on first acquiring knowledge of successor relations, this needn't necessarily be the case. For example, children might learn that it's possible to generate indefinitely many numbers based purely on the syntax of the count list, and thereby infer that numbers must be infinite, even without yet understanding how this recursive rule relates to cardinality. Given this possibility, it's important to test how children's knowledge of the structure of counting relates both to learning that it's always possible to add 1 to a number, and separately to learning that numbers are infinite.

In the present study, we had two goals. First, we sought to measure children's acquisition of productive syntactic rules that govern counting – what we called "decade productivity", described below, to determine when such knowledge emerges, and how it is related to general counting experience (e.g., highest count). Second, we asked whether such knowledge of such productive rules was related to both acquisition of successor function knowledge and the belief that numbers are infinite. To do this, we presented 4- and 5-year-old children with three tasks. First, to assess children's decade productivity, we tested them with the "Highest Count" task. A critical difference between this study and previous work is how we evaluated children's highest count. Specifically, in previous studies of successor function knowledge, a child's highest count was defined as the highest number to which they could count before their first error. While this is likely a good first-pass measure of how much training children have received, it leaves open whether a particular child has acquired a productive rule or has simply memorized their count sequence. For this reason, we explored not only how high children could count without error, but also which number they stopped on, and what they did when provided with the next number. As already noted, previous studies find that children's highest counts are not randomly distributed, and that instead their most common first error occurs at decade transitions, compatible with having learned a decade rule (i.e., that 2-digit numbers are formed by composing a decade label, "N-ty", with a unit label between *one* and *nine* – e.g., *forty-nine*). To test whether children count up to decades (e.g., up to 29, 39, or 49) by exploiting such a rule, we provided children who stopped at a decade transition with the next decade term (e.g., 30, 40, or 50) and asked whether they could then continue counting. We reasoned that children who have acquired a productive decade rule should be able to count higher once decade terms are provided - i.e., they should know that for any decade label N-ty, the next number in the counting sequence should be N-ty-one, followed by N-ty-two, etc., providing evidence for a structure like the one in (2):

| | | | | | | |
|---|---|---|---|---|---|---|
| (2) | Verbal Numeral: | thir | | ty | | five |
| | Arabic Numeral: | 3 | | 10 | | 5 |
| | Compositional Rule: | unit | * | decade | + | unit |

As a second test of whether children have acquired a productive rule for generating numbers, we used the Next Number task, in which children were told a number (e.g., “fifty-seven”) and asked to generate the next number in the count sequence (i.e., “What comes next?”). We reasoned that children who understand the decade structure of counting should not represent the count list as a single memorized string, and therefore should be able to generate the next number for any decade - i.e., they should exhibit knowledge of how the verbal count list implements the successor function. Based on this, we reasoned that productive knowledge of counting as measured by the Highest Count task should be correlated with performance on the Next Number task. Finally, to explore the second goal of this study, using these two tests of counting productivity we asked whether any of them predicted children’s intuitions about infinity in a qualitative Infinity Interview. In particular, we tested how our counting measures were related to children’s belief that it’s always possible to add 1 (i.e., successor function knowledge), and that numbers never end.

# 2. Method

## 2.1 Participants

We tested 122 4- and 5-year-old children (M = 5;0, SD = 7 months, range = 4;0 to 5;11) recruited from preschools and museums. A stopping rule was defined such that 30 children were tested in each 6-month age bin within this range of ages. An additional 25 children participated but were excluded for the following reasons: their primary language was not English ($n$ = 4), they did not complete the tasks ($n$ = 10), parental interference ($n$ = 1), experimenter error ($n$ = 3), or failing the practice trials on the Next Number Task ($n$ = 9).

## 2.2 Stimuli and Procedure

**2.2.1 Highest Count Task.** To test children's knowledge of the counting sequence up to 99, we first asked each child to count as high as they could. If the child failed to respond, the experimenter said, "Let's count together! One..." with rising intonation to encourage the child to continue counting alone. We allowed children to count until they stopped naturally, and recorded errors made along the way. Each skipped number (e.g., "12, 13, 15"), skipped sequence of numbers (e.g., "18, 19, 30, 31"), or substitution error (e.g., "4, 9, 6") was counted as one error. Children were also allowed to self-correct or restart counting with no penalty (e.g., if a child counted "1, 2, 4, no, 1, 2, 3, 4, 5", 3 would not count as an error.). To avoid underestimating children's counting ability due to lapses in attention, we always reminded children of the last number they had said whenever they stopped counting. For instance, a child who stopped at 25 was prompted with, "So what's after 25?" Children were allowed to continue counting and receive as many reminders as necessary; these pauses and reminders were not considered errors. Using this method, we obtained children's Initial Highest Count (IHC), which was the highest number children counted to before making any errors, either on their own or with reminders. We also obtained children's Final Highest Count (FHC), which was the highest number children ever counted to that was part of a 3-number consecutive sequence and allowing for up to 10 previous errors[2]. For some children, their Final Highest Count also included experimenter-provided decade prompts, as described below.

In this study, we were especially interested in testing whether children have a productive decade rule for counting, as measured by their ability to either (a) count to 99 on their own with 3 or fewer errors, or (b) extend their count sequence when provided with decade labels beyond their initial highest count. To measure the latter ability, during testing we first identified instances where children's first or second error occurred at a decade transition (e.g., stopping at 29, substituting '30' with 'twenty-ten', or skipping 30 altogether). We called these Decade-Change Errors and provided children with a corrective decade prompt. For instance, a child who made an error after 29 was told, "After 29 is 30. Can you keep counting? 29, 30..." Children who counted-on successfully were provided with decade prompts for any subsequent Decade-Change errors they made, with decade prompts ranging from 20 to 90. Children who made no Decade-Change Errors did not receive any decade prompts and were simply allowed to continue counting until they stopped naturally.[3] The task ended whenever a child reached 99 or if they said they could not continue counting, whichever was earlier.

In addition to obtaining children's Initial and Final Highest Counts and counting errors, we also used the Highest Count task to classify participants as Productive or Non-Productive Counters. Children who never received decade prompts were classified as Productive Counters if

[2] Additionally, we allowed for at most 3 errors in each decade.
[3] Upon coding the data, we noticed that some counting errors were not detected during the experiment, such that some children ($n$ = 10) received a decade prompt despite making mid-decade errors before their first Decade-Change Error. We classified these children as Non-Productive Counters irrespective of their subsequent counting progress, since they should not have received a decade prompt. Any counting past the erroneously provided decade prompt was excluded from our analyses.

they counted to 99 with three or fewer errors, or as Non-Productive Counters otherwise (i.e. stopping earlier or reached 99 with four or more errors). Children who received decade prompts were classified children as Productive Counters if they could count at least two decades beyond their initial Decade-Change Error without making more than three errors in those two decades, including any decade-transition errors that elicited a decade prompt. For example, a child whose initial error was at 29 but continued counting with decade prompts to 49 or higher was classified as a Productive Counter, but if they continued to only 39 or made too many errors before getting to 49, they were classified as a Non-Productive Counter. Thus, our Productivity classification allowed for two decade prompts and only one additional error within the two decades past the decade-change error.

**2.2.2 Next Number Task.** In this task, children were provided with a number and asked, "What comes next?". Children received two practice items (*one*, *five*) to ensure they understood the task, and corrective feedback was provided on these practice items if needed. For example, when asked, "*Five*. What comes next?", children who answered *four* were invited to count and figure out the correct answer (e.g., "No, *four* comes before *five*. What comes after *five*? Can you count and find out?"). All children in the final dataset successfully answered these practice questions before continuing.

Since we were interested in individual differences between children, all children received the same test items in a fixed order, ranging from 20 to 90: 23, 40, 62, 70, 37, 29, 86, 59. These test items were designed to cover the range of counting abilities up to 99 and to utilize the decade rule for forming number words. No feedback was provided during the test trials.

**2.2.3 Infinity Interview.** Following the protocol of previous studies of infinity knowledge (Cheung et al., 2017; Evans, 1983; Evans & Gelman, 1982; Hartnett & Gelman, 1998), we also assessed children's understanding of infinity by probing two types of belief: (1) that there is no biggest number, and (2) that it is always possible to add 1 to any number. To probe this, we asked six questions as follows:

1. *"What is the biggest number you can think about?"* If the child did not answer, the experimenter probed them by asking how high they could count.
2. *"Is that the biggest number there could ever be?"*
    a. If yes, move on.
    b. If no, "*Can you think of a bigger number? Is that the biggest number there could ever be?"* The experimenter repeated this exchange up to 4 times or until the child affirmed that they had produced the biggest number.
3. "*If I keep counting, will I ever get to the end of numbers, or do numbers go on forever? Why?*"
4. "*If we thought of a really big number, could we always add to it and make it even bigger, or is there a number so big we couldn't add any more? Why / Why not?*"
5. "*You said the biggest number you know is X. Tell me, is it possible to add one to X, or is X the biggest number possible? Why?*" For this question, X was the largest number the child had stated in the entire testing session.
6. "*Could I keep adding one? Why / Why not?*"
    a. If yes, "*What would happen if I kept adding one?*"

Each child was assigned a binary classification for each of two aspects of infinity understanding. Classifications were assigned by the first author and a coder blind to the hypotheses. The coding scheme and example transcripts are consistent with previous studies (Cheung et al., 2017; Evans, 1983; Hartnett & Gelman, 1998) and are provided in the

Supplementary Materials (A). First, we coded whether children believed that there was a highest number, such that numbers must end, or whether they believed that numbers go on forever, which we labeled as "Endless" knowledge. This coding was based on responses to questions 1-3 and 5. Second, we coded whether the child believed that it was always possible to add 1 to any number according to their responses to questions 4-6. We call this the "Successor" knowledge. Initial agreement was 84.0% for Endless Knowledge coding (Cohen's Kappa = .63, $p < .001$), and 80.9% for Successor Knowledge coding (Cohen's Kappa = .62, $p < .001$). Disagreements were resolved through consulting a third coder. Finally, we identified children as "Full Infinity knowers" if they endorsed both aspects of infinity understanding.

## 3. Results

Compatible with the first goal of this study, our first set of analyses examined children's acquisition of the rule-governed morphological structure of counting – what we called "decade productivity" – using data from the Highest Count and Next Number tasks. Second, compatible with our second goal, we analyzed children's performance on the Infinity interview, and asked how their responses to the Successor Knowledge and Endless Knowledge items were related to counting experience and decade productivity.

All analyses were conducted in R (version 3.6.0, R Core Team, 2019). Regression models were constructed using either the R base stats package or, for models contained mixed effects, using lme4 (Bates, Mächler, Bolker, & Walker, 2014). For ease of interpretation, predictor variables were mean-centered for analyses.[4] To test for the significance of specific independent variables, we conduct Likelihood Ratio Tests comparing models with and without particular

[4] Continuous variables were mean-centered and scaled by 1 standard deviation. Categorical variables (all binary in this paper) were also mean-centered and weighted by their group counts (i.e. weighted effect coding, see Grotenhuis et al., 2017). This allows regression coefficients to be interpreted as standardized main effects and to be compared across models.

effects of interest. For analyses containing within-subject measures, we construct mixed-effects models contain random intercepts for all relevant grouping units (e.g., subject, item).

### 3.1 Characterizing Decade Productivity

**3.1.1 Highest Count Task.** Fig. 1 presents the distribution of children's Initial Highest Count by productivity classification. Thirty-two children reached 99 on their Initial Highest Count and were therefore classified as Productive Counters. Of the remaining ninety participants, 49 children (54%) made a Decade-Change error on either their first error ($n$ = 37) or second error ($n$ = 12) and thus received Decade Prompts. Of these, 32 (65%) continued counting two decades further with fewer than 2 additional errors, and were classified as Productive, while 17 did not, and were classified as Non-Productive. Another 9 children were classified as Productive for counting past their Initial Highest Count to reach 99 with three or fewer errors, without receiving any Decade Prompts. Overall, 73 children were classified as Productive Counters ($M_{age}$ = 5;3, *SD* = 6 months) and 49 were classified as Non-Productive Counters ($M_{age}$ = 4;7, *SD* = 5 months). Unsurprisingly, Productive Counters had, on average, a higher Initial Highest Count (*Mean* = 69.2, *SD* = 28.8, *Median* = 65) than Non-Productive Counters (*Mean* = 21.4, *SD* = 14.7, *Median* = 15). This difference remained if we considered only Productive Counters with Initial Highest Count below 99 (*Mean* = 46.0, *SD* = 15.3, *Median* = 49). However, many Productive Counters ($n$=9) had an Initial Highest Count of 29 or lower (see Fig. 1).

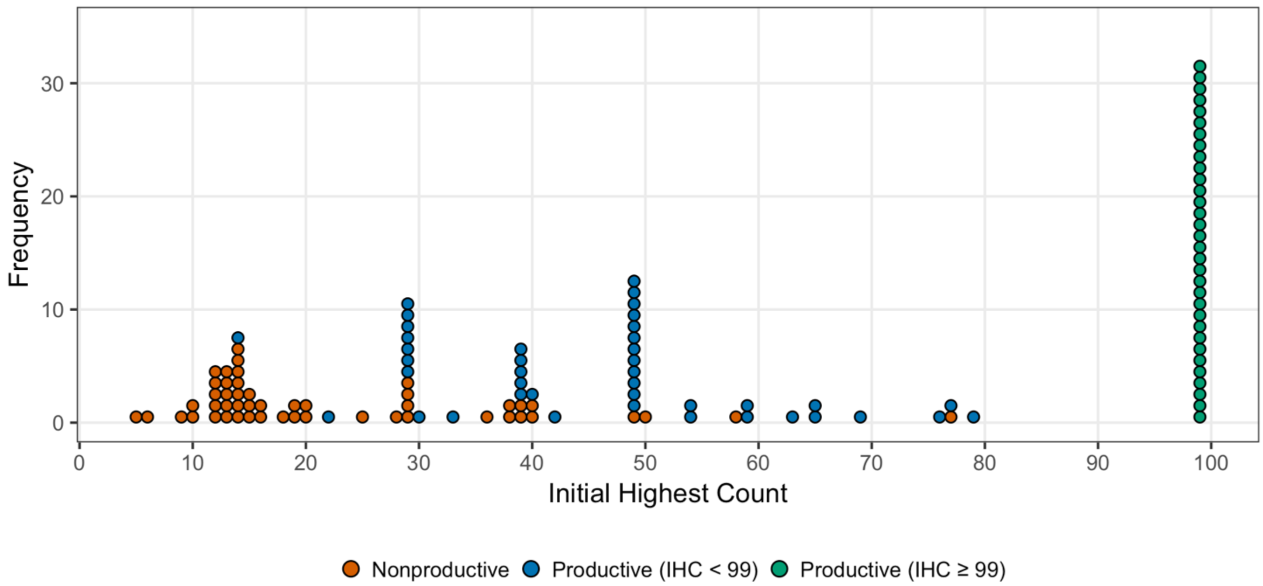


*Figure 1.* Distribution of children's Initial Highest Count by Decade Productivity.

Next, we analyzed precisely how far children could count past their initial error. Fig. 2a and 2b show participants' Initial Highest Count, Final Highest Count, any errors made, and provided decade prompts, if any. While our classification criteria for productive counting only required children to count past their initial errors by two decades, most of the Productive Counters were able to count several decades further, and many of them reached 99 (*Mean number of prompts* = 3.4, *SD* = 1.7, *Range* = 1 to 7). Almost half of the Productive Counters (*n*=32) received decade support on the Highest Count Task, and the remaining children (*n*=41) counted to at least 99 without assistance and without making more than three errors. Productive Counters had a median Final Highest Count of 99 (*M* = 96.4, *SD* = 9.5, *Range* = 49 to 99), which was 34 numbers higher than their median Initial Highest Count of 65. About a third of Non-Productive counters (*n*=15) received decade support (*Mean number of prompts* = 1.3, *SD* = 0.6, *Range* = 1 to 3), and the remaining children (*n*=34) initially made mid-decade errors and therefore did not receive decade prompts. Non-Productive Counters had a median Final Highest Count of 29 (*M* = 32.0, *SD* = 17.6, *Range* = 5 to 99), which was only 14 numbers past their median Initial Highest Count of 15. This difference is of course not surprising, since the ability to count at least 2 decades past the experimenter's first prompt was what defined the difference between Productive and Non-Productive children.

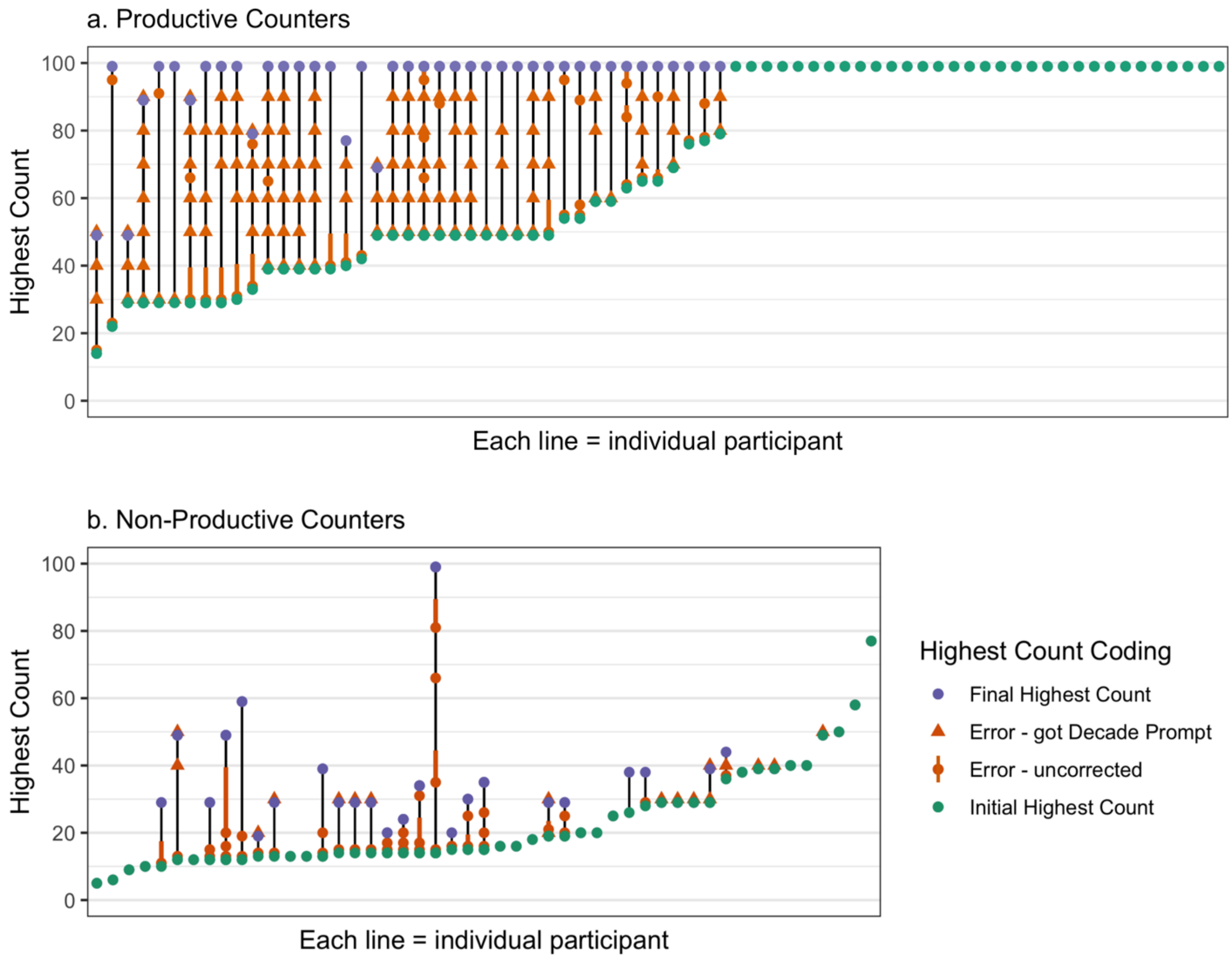


*Figure 2.* Highest count profiles for (a) Productive Counters and (b) Non-Productive Counters. Each line represents highest counts achieved by an individual child. *Initial Highest Count* (green dots) show highest number counted to before any error. *Final Highest Count* (purple dots) show highest number counted to as part of a consecutive 3-number sequence, allowing for up to a maximum of 10 errors throughout counting, and not more than three errors in any given decade. *Errors* include counting mistakes made on single numbers (orange dots and triangles) or over continuous intervals (orange lines). *Decade Prompts* (orange triangle) show decade terms provided by the experimenter after an error at a decade transition (e.g. stopping at 39). Black lines represent absolute gain from Initial to Final Highest Count.

In summary, we found that between the ages of 4-and-a-half and 5-and-a-half many children in our study exhibited evidence of having learned a productive decade rule. Overall, the average age of Productive Counters was 5 years, 3 months. Though some of these children (*n*=32) were classified as Productive on the basis of having counted up to 99 with minimal errors, many received this classification because, upon stalling on a decade transition, they were able to recover once provided with a decade label, compatible with the use of a rule. These data not only provide evidence that children who stop on decade transitions do likely count using productive rules, but also suggests that a child's initial highest count may not provide the best measure of their mastery of counting. Even many children who had quite low initial counts (e.g., below 30) were able to recover when provided a decade prompt, suggesting that some children may memorize only a small subset of the count list (e.g., less than 30) before extracting a rule.

**3.1.2 Next Number Task.** The Productivity classification, above, was one of two measures of decade rule knowledge that we explored in this study. We also tested this using the Next Number task. Here, we asked how our two candidate measures of decade rule knowledge were related to one another.

First, we found that Productive Counters (71% correct; $SD$ = 27%) significantly outperformed Non-Productive Counters (28%; $SD$ = 26%) on the Next Number task ($t(120) = -8.76$, $p < .001$). However, recall that Productive Counters included children who could count to 99 on their own (IHC $\geqslant$ 99) and those who could not (IHC < 99). As shown in Fig. 3, Productive Counters with IHC $\geqslant$ 99 performed close to ceiling on the Next Number task ($M$ = 91%, $SD$ = 15%), likely because all the items tested were within their familiar count sequence. To obtain a stronger, more conservative test of our hypothesis that Productive Counters could utilize a decade rule to succeed on this task, we excluded Productive Counters with IHC $\geqslant$ 99 from the following analyses. A mixed effects logistic regression predicting trial-level accuracy from Productivity, Initial Highest Count and age (with random intercepts for subject and item magnitude), found that although accuracy was twice as high among Productive Counters (IHC<99) (M=56%, SD=26%) compared to Non-Productive Counters (*M* = 28%, *SD* = 26%), this difference did not meet the threshold of statistical significance ($\beta$ = 0.76, 95% CI=[-0.09, 1.6], $\chi^2(1) = 3.06$, $p = 0.08$), though there was a significant effect of Initial Highest Count ($\beta$ = 0.85, 95% CI=[0.42, 1.28], $\chi^2(1) = 14.5$, $p < 0.001$). Age was also not a significant predictor ($\chi^2(1) = -0.009$, $p = 1$). [5]

This first analysis suggests the Next Number task and children's ability to count-up from counting errors (i.e., our Productivity classification), capture different aspects of counting knowledge. Whereas Initial Highest Count was a strong predictor of Next Number performance in this dataset, our Productivity classification found that children with a variety of initial highest counts were productive. Why might this be? One likely reason is that a child's score on the Next Number task reflected their performance for both small and large numbers, whereas to be classified as Productive, a child might be tested with only relatively small numbers. Also, the Next Number task may place greater working memory demands on children – since it requires counting up from arbitrary points in the count list – potentially making a more strongly routinized count list more valuable, requiring less working memory resources for the retrieval of numbers.

[5] We obtain similar results when all Productive counters are included: there was a significant effect of Initial Highest Count, but not Productivity or Age. Full analysis results are reported in Supplementary Materials (B).

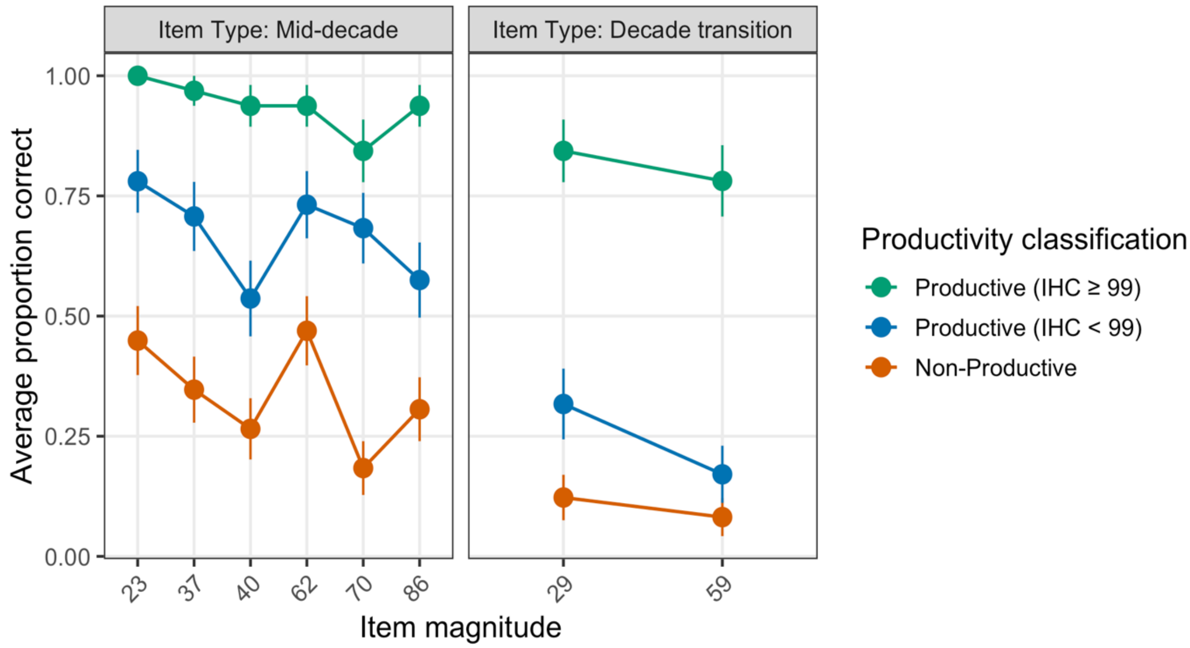


*Figure 3.* Average proportion of correct response on Next Number task by item and Decade Productivity status. Error bars indicate standard error about the mean.

We further explored how the two tasks are related by conducting a *post-hoc* analysis testing whether Productive Counters (IHC<99) show a selective advantage on the Next Number task for Mid-Decade items, where a productive decade rule ("*N*-ty-one, *N*-ty-two, …") would be most beneficial, compared to decade transitions, where this rule would be less beneficial. This analysis found that all participants performed better on Mid-Decade items (e.g., 23) than on Decade Transition items (e.g., 29, see Fig. 3). There was a significant main effect of Item Type ($\beta$ = 2.28, $\chi^2(1) = 12.57$, $p < .001$), with greater accuracy on Mid-Decade items ($M$ = 47%, $SD$ = 35%) than on Decade Transition items ($M$ = 16%, $SD$ = 30%). Adding a Productivity by Item type (Decade transition/Mid-decade) interaction did not significantly improve model fit ($\chi^2(1)$ = 0.67, $p$ = .41), indicating that both Productive Counters and Non-Productive Counters found mid-decade items easier than decade transition items.[6]

[6] Again, we obtain similar results when all participants are included in the analysis.

Next, we reasoned that, if a memorized count list is what allows children to generate successors on the Highest Count and Next Number tasks, then children should perform better on items within their Initial Highest Count than beyond it. If, instead, children have acquired a rule that generates the decade structure, then we might find an interaction between Productivity Group and Item Range, such that Productive Counters perform well both within and beyond their Initial Highest Count, while Non-Productive Counters are only able to generate successors within their Initial Highest Count. To test this prediction, we conducted a logistic mixed effects regression predicting trial-level accuracy from Initial Highest Count, Productivity, Item Range (Within/Beyond IHC), age, and the interaction of Productivity and Trial Type, with random intercepts for subject and item magnitude. For this analysis, we excluded children with Initial Highest Count $\geqslant 99$ because all numbers tested on this task would be within their counting range.[7] Model comparison by Likelihood Ratio Test found no significant main effect of either Productivity ($p$ = 0.08) or Item Range ($p$ = 0.68). Critically, there was a significant interaction effect of Productivity and Item Range ($\beta = -1.17$, 95% CI = [-2.20, -0.14], $\chi^2(1) = 5.00$, $p = .025$). Planned contrasts indicate that performance on numbers within children's Initial Highest Count was similar for Productive Counters (IHC<99) ($M$ = 53%, $SD$ = 31%) and Non-Productive Counters ($M$ = 55%, $SD$ = 32%; $p$ = .88 by $t$-test) whereas accuracy for numbers beyond their Initial Highest count was significantly greater among Productive Counters ($M$ = 58%, $SD$ = 31%) than among Non-Productive Counters ($M$ = 27%, $SD$ = 26%; $t(88) = -5.24$, $p < .001$; see Fig. 4).

[7] We do not perform this analysis including all participants, because Productive counters with Initial Highest Count $\geq 99$ did not receive any trials outside their Initial Highest Count

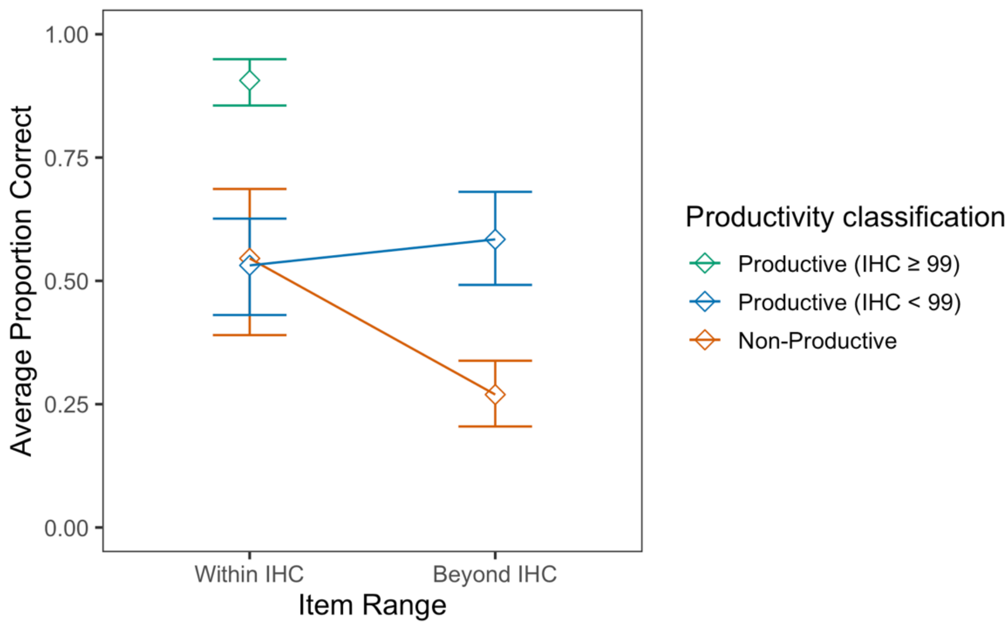


*Figure 4.* Average proportion correct for each participant on the Next Number task by Decade Productivity Status (between subjects) and Item Range (within subjects). *Within IHC*: correct answer is within a particular participant's Initial Highest Count; *Beyond IHC*: correct answer is greater than that participant's Initial Highest Count. Error bars show bootstrapped 95% confidence intervals about the mean. The largest number tested on this task was 86, thus, all trials were within IHC for productive counters with IHC ≥ 99 (in blue).

Collectively, analyses contrasting children's highest count behaviors and their performance on the Next Number task suggest that these measures capture slightly different phenomena. Both require children to draw on knowledge of the numeral sequence, but only the Next Number task requires children to generate number sequences from arbitrary points in the count list, without the benefit of the "momentum" afforded by the count routine for small and large numbers. Previous work finds that children's ability to count-up from an arbitrary position in the count list is significantly affected by an experimenter verbally rehearsing preceding numbers, and thereby providing momentum (e.g. Fuson, Richards, & Briars, 1982; Siegler & Robinson, 1982). In this sense, the Next Number task is a more difficult task, and perhaps a more stringent test of children's knowledge of the decade rule.

**3.2 Characterizing Infinity Knowledge**

In the next set of analyses, we report children's successor function knowledge and beliefs about infinity. In the following section, we then probe how these are related to children's performance on the Highest Count and Next Number tasks.

We coded whether children thought every number has a successor ("Successor" knowledge) and separately, whether they thought numbers never end ("Endless" knowledge). Following previous work (Cheung et al., 2017), we also identified children who endorsed both beliefs as having "Full Infinity" knowledge (though we should note that, naturally, there is much more to fully grasping the notion of infinity than these two pieces of knowledge).

About half of our sample (48%, $n$ = 59) demonstrated no knowledge of infinity, claiming that there was a biggest number and that it was not possible to keep adding one. Non-Productive counters were more likely to fall in this category than Productive counters (77% of Non-Productive vs. 33% of Productive counters, $\chi^2(1) = 10.58$, $p = .001$). A smaller fraction of our sample (19%, $n$ = 23) exhibited full knowledge of infinity, claiming that there was no biggest number and that we could always keep adding one. Productive counters were more likely to exhibit full infinity knowledge than Non-Productive counters (25% vs. 7%, $\chi^2(1) = 7.34$, $p = .006$). The remaining children (33%, $n$ = 40) had partial knowledge of infinity – claiming either that you could always add one ($n$ = 29) or that there was no biggest number ($n$ = 11), but not both. Table 1 reports the frequency of these outcome classifications.

Table 1
*Frequency of infinity knowledge in productive and non-productive counters*

| Classification | Non-Productive Counters (N = 49) | Productive Counters, IHC < 99 (N = 41) | Productive Counters, IHC ≥ 99 (N = 32) | Total (N = 122) |
|---|---|---|---|---|
| No Infinity knowledge | 33 (67%) | 18 (44%) | 8 (25%) | 59 (48%) |
| Only Successor Knowledge | 12 (24%) | 10 (24%) | 7 (22%) | 29 (24%) |
| Only Endless Knowledge | 1 (2%) | 4 (10%) | 6 (19%) | 11 (9%) |
| Full Infinity Knowledge | 3 (6%) | 9 (22%) | 11 (34%) | 23 (19%) |

### 3.3 Predictors of Infinity Knowledge

In this section we address how our measures of counting and productivity knowledge (Initial Highest Count, Productivity Group, Next Number task) were related to beliefs about infinity. Because the two components of infinity knowledge might develop independently, as indicated by the partial knowers who have one component knowledge but not the other, we conducted separate analyses to predict children's belief that every number has a successor (Successor Knowledge) and belief that numbers never end (Endless knowledge). This allowed us to identify possible similarities or differences in relevant factors for acquiring each component of Infinity knowledge. We also predicted children's status as Full Infinity Knowers, following Cheung et al. (2017). For our initial models, predictors included Initial Highest Count, Productivity Group (Productive vs. Non-Productive Counters), and Next Number accuracy (total score out of 8). Any predictor in these initial models that significantly predicted the outcome variable relative to a base model (with age as the only predictor) was added to a final model, to allow comparison among the predictors. Models were constructed hierarchically, with model comparisons performed at every step using a Likelihood Ratio Test, and with models selected on the basis of a significant chi-squared statistic and reduced AIC value. All models were constructed with the following formula: glm(infinity-successor / infinity-endless / infinity-full ~ (predictor) + age (centered), family = binomial).

Once again, our analyses excluded Productive Counters with IHC ≥ 99, so as to obtain a stronger, more conservative test of our hypothesis that Productive knowledge of counting might relate to beliefs about infinity.[8] Thus, we first compared Productive Counters with IHC < 99 against Non-Productive Counters (total *N* = 90). Interestingly, in predicting children's Successor Knowledge, none of the three predictors explained a significant proportion of additional variance compared to the base model. In contrast, for models predicting children's possession of Endless Knowledge, Productivity Group explained significant additional variance relative to the base model ($\chi^2(1) = 5.52$, $p = .019$; AIC = 84.92), though other measures of counting ability (Initial Highest Count, Next Number accuracy) did not explain additional variance when controlling for age (see Table 3 for details). This final model thus included only Productivity and Age as predictors, and estimated that Productive counters were more likely than Non-Productive Counters to have Endless Knowledge ($\beta = 1.63$, $p = .03$, OR = 5.08 [95% CI = 1.30, 23.50]), while Age was not a significant predictor ($p = 0.94$). Finally, we constructed models predicting children's status as Full Infinity Knowers. None of our counting measures (Initial Highest Count, Productivity group, or Next Number accuracy) improved model fit compared to the base model with only age as a predictor. For details about model fits and model comparison results, see Tables 2-4.

[8] Analyses incorporating the full sample are included in the Supplementary Materials (C). Those analyses qualitatively mirror the results presented here, though with some differences: None of the counting productivity measures were predictive of Successor knowledge. All three measures, however, were predictive of children's Endless knowledge, with Productivity group showing the largest effect. No single counting measure remained significant when entered into a full model predicting Endless knowledge, suggesting that the three measures explain overlapping variance. Next Number performance explained significant additional variance in predicting Full Infinity Knowledge, but the coefficient was not significantly different from 0 (p = .059), suggesting overlapping variance with age.

In summary, we found that children who were classified as Productive, based on the Highest Count task, were significantly more likely than Non-Productive children to believe that numbers never end. Other measures, such as Initial Highest Count and Next Number performance, were not as strongly related to Infinity knowledge. In addition, none of our measures of counting knowledge were related to children's Successor Knowledge as measured by the infinity interview. Thus, in this study we find that one measure of children's productive counting rules predicts a belief that numbers are endless, but none of these measures predict the belief that it's always possible to add +1 to a number. This suggests that children may learn successor relations between numbers independent of learning productive counting rules – e.g., that this +1 rule describes the finite set of that they know, but that a morphological decade rule may explain why children believe that numbers never end – i.e., because number *words* can be productively generated.

Table 2

*Regression models for predicting Successor knowledge on the Infinity Interview (Participants IHC <99, N=90)*

| Models | Coefficient Estimates (ß) | | | | Summary statistics | | |
|---|---|---|---|---|---|---|---|
| | Age | IHC | Next Number accuracy | Productivity Group | Loglikelihood [a] | AIC | $R^2_{Nagelkerke}$ |
| **Base Model** | | | | | | | |
| Age | 0.296 | | | | -58.76 | 121.51 | 0.027 |
| **Initial Models** | | | | | | | |
| Age + IHC | 0.598 * | -0.520 | | | -57.05 | 120.10 | 0.074 |
| Age + Next Number accuracy | 0.358 | | -0.155 | | -58.55 | 123.10 | 0.033 |
| Age + Productivity Group | 0.156 | | | 0.496 | -58.32 | 122.64 | 0.040 |

*Notes*. Coefficients were compared against 0 using *t*-tests. Model comparisons done using Likelihood Ratio Tests.

[a] Each initial model was compared against the base model.

* $p < 0.05$

Table 3

*Regression models for predicting Endless knowledge on the Infinity Interview (Participants IHC <99, N=90)*

| Models | Coefficient Estimates (ß) | | | | Summary statistics | | |
|---|---|---|---|---|---|---|---|
| | Age | IHC | Next Number accuracy | Productivity Group | Loglikelihood [a] | AIC | $R^2_{Nagelkerke}$ |
| **Base Model** | | | | | | | |
| Age | 0.451 | | | | -42.22 | 88.44 | 0.049 |
| **Initial Models** | | | | | | | |
| Age + IHC | 0.261 | 0.349 | | | -41.63 | 89.26 | 0.070 |
| Age + Next Number accuracy | 0.287 | | 0.466 | | -41.04 | 88.07 | 0.090 |
| Age + Productivity Group | 0.025 | | | 1.625** | -39.46* | 84.92 | 0.142 |

*Notes*. Coefficients were compared against 0 using *t*-tests. Model comparisons done using Likelihood Ratio Tests.

[a] Each initial model was compared against the base model.

* $p < 0.05$, ** $p < 0.01$

Table 4

*Regression models for predicting Full Infinity knowledge on the Infinity Interview (Participants IHC <99, N=90)*

| Models | Coefficient Estimates (ß) | | | | Summary statistics | | |
|---|---|---|---|---|---|---|---|
| | Age | IHC | Next Number accuracy | Productivity Group | Loglikelihood [a] | AIC | $R^2_{Nagelkerke}$ |
| **Base Model** | | | | | | | |
| Age | 0.527 | | | | -33.91 | 71.81 | 0.058 |
| **Initial Models** | | | | | | | |
| Age + IHC | 0.481 | 0.084 | | | -33.88 | 73.76 | 0.059 |
| Age + Next Number accuracy | 0.417 | | 0.307 | | -33.51 | 73.03 | 0.073 |
| Age + Productivity Group | 0.207 | | | 1.229 | -32.71 | 71.43 | 0.104 |

*Notes*. Coefficients were compared against 0 using *t*-tests. Model comparisons done using Likelihood Ratio Tests.

[a] Each initial model was compared against the base model.

## 4. Discussion

Given only finite experience with discrete quantities, number words, and counting, how do children learn that *every* natural number has a successor, and that numbers are endless? In this paper, we had two goals. First, we sought to characterize children's acquisition of productive morphological rules, and when this knowledge emerges in development. Second, we asked how such knowledge might be related to (a) their knowledge of the successor function (i.e., that it's possible to add +1 to any number), and (b) their beliefs regarding infinity (i.e., that numbers never end). Prior research suggests that how high children can count is related to their ability to identify successor relations for known numbers (e.g., Cheung et al., 2017), leading to the suggestion that counting experience causes children to notice the recursive base-10 structure of the count list, which in turn provides a basis for learning about successor relations and for generating unbounded number words (Barner, 2017; Cheung et al., 2017; Rule, Dechter, & Tenenbaum, 2015; Yang, 2016). Learning a rule that generates successive number words might lead children to the belief that all numbers have successors, and that numbers never end. Our study found multiple pieces of evidence that some 4- and 5-year-old children, but not others, use a productive rule when counting. Also, we found that Productive counters differed from Non-Productive counters with respect to their understanding of numerical infinity, though, interestingly, not their successor function knowledge, *per se*.

Several results provide evidence that some, but not all, 4- and 5-year-old children use a productive decade rule when counting (such that Productive Counters had a mean age of 5;3). First, when asked to count as high as they can, many children stopped at decade transitions, with about half making a decade-transition error as their first or second error. Whereas a memorized sequence predicts that errors should be randomly distributed over the count list, a decade rule predicts that they should occur disproportionately for irregular words that are not generated by a rule, such as decade labels (e.g., *twenty, thirty, fifty*). Furthermore, we found that when provided with decade prompts, many (65%) of the children who made decade transition errors could continue counting, often counting two or more decades further. Second, data from the Next Number task found that some children exhibited knowledge of a decade rule beyond their Initial Highest Count, whereas some children did not, and could only name next numbers for numbers within their initial count. This, too, suggests that while some children's count list was purely memorized, unabetted by a decade rule, other children benefitted from a rule that allowed them to identify next numbers on trials outside their familiar count routine. Third, we found some evidence that these two measures of productivity were related to one another, though imperfectly so. First, although Productive Counters had higher average accuracy on the Next Number task than Non-Productive Counters, the strongest predictor of Next Number performance was children's Initial Highest Count, not their Productivity classification. Second, whereas Productive and Non-Productive children performed similarly on the Next Number task for numbers within their Initial Highest Count, Productive children performed significantly better than Non-Productive children for numbers outside their Initial Highest Count, resulting in a significant interaction (and compatible with the use of a productive rule). The fact that Initial Highest Count was a stronger predictor of Next Number performance than Productivity suggests that, though in their own ways compelling measures of children's decade rule knowledge, these tasks draw on different constructs, perhaps because only the Next Number task requires children to count-up from arbitrary numbers without the benefit of momentum afforded by the count routine, and requires knowledge of both small and large numbers (Fuson et al., 1982; Siegler & Robinson, 1982).

In addition to characterizing several measures of counting productivity, our second goal was to explore how such measures might be related to children's intuitions about infinity. Overall, we found that children's belief that numbers never end was predicted by Productivity classification, but not other measures of counting proficiency, suggesting that children's ability to count-up from a decade label provided by the experimenter is the best predictor of whether they think numbers never end. Interestingly, we also found that no measures of counting proficiency were related to children's successor function knowledge, as measured by the infinity interview. Specifically, children's judgment that it's always possible to add 1 to a number didn't appear to be related to how high they could count, whether they could readily count-up from a decade label, or whether they could identify the next number in an arbitrary position in the count list. While previous studies have found that children with full infinity knowledge often have a highest count around 100 (e.g., Cheung et al., 2017; Hartnett & Gelman, 1998), the present study is the first to investigate how counting proficiency relates to successor and endless knowledge of infinity separately. This combination of results suggests that children's knowledge of successor relations may be acquired separately from their intuition that numbers never end, either because the successor function is not necessarily the basis by which children learn about infinity, or because successor function knowledge is initially defined over a finite set of numbers, and only later rendered fully recursive – e.g., by learning that recursive morphological rules of generate an infinite set of number words, and thus that the successor function can also be infinitely applied.

A starting point for this work was the observation in previous studies (Cheung et al., 2017) that children's acquisition of generalized successor function knowledge appears to be related to how high children can count. Critically, however, Cheung et al. only tested how counting abilities are related to children's reasoning about specific numbers, as measured by the

Unit task (Sarnecka & Carey, 2008), but not whether counting was related to the two beliefs about how numbers behave in general. Our work tested whether counting – specifically learning a productive decade rule – might explain more general intuitions regarding the successor function and infinity. We reasoned that counting might be related to such intuitions in two broad ways. First, it might be the case that, as children are increasingly exposed to numbers, they acquire more knowledge about how those numbers operate (including but not restricted to successor relations), which they may generalize to all numbers, without making a specific connection between learning morphological rules of counting and discovering that numbers are infinite. An alternative, however, is that counting abilities might relate to knowledge of infinity specifically because the morphological rules that govern counting provide rules for generating ever larger numbers. Such rules might provide the basis for the belief that numbers never end. That is, learning the morphological rules may allow children to reason that number words can be productively generated, and thus conclude that numbers are endless. On this view, counting may be separately related to the belief that every number has a successor and to the belief that numbers are endless.

Our data are compatible with this distinction between successor function and infinity knowledge. Children may learn item-based successor relations early on, and may even believe that all numbers have a successor despite believing that only a finite number of numbers exist – i.e., that the function is bounded to a finite list. Alternatively, children may have beliefs about numbers that support certain logical inferences, but not be aware of the entailments of these beliefs – e.g., that a fully recursive function necessitates that numbers never end. As evidence for this, Harnett and Gelman (1998, see also Hartnett, 1991) note that children's beliefs about whether or not numbers are infinite can change over the course of a single testing session, simply by asking them what the highest number is, and whether it's possible to add 1 to it repeatedly. For instance, some children in their study were able to recognize the inconsistency between their belief that there was a largest number and their belief that it is always possible to add one, and by the end of the testing session, had given up on either of these beliefs.

Our finding that children may acquire a productive counting rule by the time they have memorized a count list as short as 29 (see Figs. 1 and 2) presents a challenge to current computational models of number word learning. For example, Yang (2016) developed a "Tolerance Principle" which states that children invoke rules for explaining regularities in linguistic input when the number of exceptions or irregularities are below some threshold. According to his model, if there are $N$ different linguistic tokens in the input, then a regular rule will be preferred only if the number of exceptions is below $N/\ln(N)$. In English, the early number words from 1-20 are exceptions to the decade rule, so the Tolerance Principle predicts that children would have to acquire a count list of at least length 72 before inducing a regular rule. Similar estimates in the 60-70 range were obtained by Rule, Dechter and Tenenbaum (2015) using a Bayesian architecture for inferring word to quantity mappings. However, our data suggests that children can acquire a productive counting rule with much less data: the median Initial Highest Count of Productive Counters was only 49 (ignoring those who reached 99 on their own). Future work should reconcile these empirical findings within computational models of number word learning.

Future work might also explore how intuitions regarding infinity may come from sources beyond the count list. One possible source, for example, is experience with formal mathematical operations like addition, or the recursive addition of zeros in Arabic notation, which could help children appreciate ways of generating endlessly more quantities. For example, Singer and Voica

(2008) describe how a group of 5th-6th-graders explained that rational numbers are infinite, because infinitely many digits could be added after the decimal point. Another possible source of intuitions is by analogy to quantities other than number, such as space, time, or geometry. One early study by Evans (1983) found that knowledge about infinity in number, time, and space developed similarly between Kindergarten and 3rd Grade (e.g., from bounded to unbounded), but this research did not test if such beliefs were correlated within individual children (see Hartnett & Gelman, 1998 for similar findings). Current studies in our labs are exploring this possibility. More recently, Smith, Solomon and Carey (2005) found that 1st-2nd graders' intuitions matter being infinitely divisible preceded their intuitions about the infinite divisibility of rational numbers. Numbers can be infinite in many ways (Monaghan, 2001); correspondingly, there is much room for future research into children's intuitions about infinity in different conceptual domains.

In conclusion, we suggest that sometime around 5 years of age, children learn to generate number words beyond their spontaneous counting range by implementing a recursive base-10 rule defined over their verbal count list. This insight may support an inductive inference over their acquired verbal count sequence, which facilitates conceptual insights into the infinite nature of the natural numbers. Our results suggest that understanding the logic of natural numbers is closely related to understanding the syntactic logic of the verbal numerals, suggesting one route by which learning language can impact learning about number concepts.

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**A: Detail methods and example transcripts**

We describe a detailed coding scheme for classifying responses to the Infinity Interview. We first provide coding rules (A1), followed by an example transcript for each of the four possible Infinity knowledge classifications (A2-A5). The Infinity Interview was described in the main text (section 2.2.3) with results presented in section 3.2.

**A1: Coding scheme for Infinity Interview**

The Infinity Interview consisted of 6 questions presented in the same order:

1. *"What is the biggest number you can think about?"* If the child did not answer, the experimenter probed them by asking how high they could count.
2. *"Is that the biggest number there could ever be?"*
    a. If yes, move on.
    b. If no, "*Can you think of a bigger number? Is that the biggest number there could ever be?"* The experimenter repeated this exchange up to 4 times or until the child affirmed that they had produced the biggest number.
3. "*If I keep counting, will I ever get to the end of numbers, or do numbers go on forever? Why?*"
4. "*If we thought of a really big number, could we always add to it and make it even bigger, or is there a number so big we couldn't add any more? Why / Why not?*"
5. "*You said the biggest number you know is X. Tell me, is it possible to add one to X, or is X the biggest number possible? Why?*" For this question, X was the largest number the child had stated in the entire testing session.
6. "*Could I keep adding one? Why / Why not?*"
    a. If yes, "*What would happen if I kept adding one?*"

We coded successor and endless knowledge of infinity separately, as two binary variables.

To have endless knowledge of infinity, participants had to respond that there was no biggest number (Q1, Q2, Q5) or have provided four responses to the request to provide a bigger number (Q2). Participants who claimed to have provided the biggest number at some point in the interview (Q1, Q2, Q5) or said that numbers end (Q3) were classified as not exhibiting Endless knowledge of infinity.

To have successor knowledge of infinity, participants had to respond that it was possible to add 1 to some big number (Q4 and Q5) or claim that it was possible to keep adding one to that big number (Q6).

**A2: Infinity transcript for a child classified as having no knowledge of infinity**

E = Experimenter
C = Child (Age: 4 years 1 month)

E: What is the biggest number you can think about?
C: One hundred.
E: Is that the biggest number there could ever be?
C: Yes.
E: If I keep counting, will I ever get to the end of numbers, or do numbers go on forever?
C: End.
E: Why?
C: Long and get to one hundred.
E: If we thought of a really big number, could we always add to it and make it even bigger, or is there a number so big we couldn't add any more?
C: No.
E: Why?
C: Already so big.
E: So you said that the biggest number you know is one hundred. Is it possible to add 1 to one hundred, or is one hundred the biggest number possible?
C: Yes.
E: Why?
C: There's also a hundred seventy.
E: Could I keep adding 1?
C: No.
E: Why?
C: It already has one.

**A3: Infinity transcript for a child classified as having only Successor knowledge of infinity**

E = Experimenter
C = Child (Age: 5 years 6 months)

E: What is the biggest number you can think about?
C: Sixty one.
E: Is that the biggest number there could ever be?
C: No, one hundred fifty nine sixty.
E: Is that the biggest number there could ever be?
C: I don't know.
E: Well, can you think of a bigger number?
C: Eighteen ninety sixty one.
E: Is that the biggest number there could ever be?
C: No.
E: Can you think of a bigger number?
C: No.
E: If I keep counting, will I ever get to the end of numbers, or do numbers go on forever?
C: Forever.
E: Why?
C: End.
E: Why?
C: actually, you won't (get to the end of numbers) my friend can count to 100 and after and I learnt to count to 100 but not really.
E: Do you think she could count forever?
C: I don't know.
E: If we thought of a really big number, could we always add to it and make it even bigger, or is there a number so big we couldn't add any more?
C: Yes.
E: Why?
C: It makes it bigger.
E: So you said that the biggest number you know is eighteen ninety sixty one. Is it possible to add 1 to eighteen ninety sixty one, or is eighteen ninety sixty one the biggest number possible?
C: Yes.
E: Why?
C: Eighty nine.
E: Could I keep adding 1?
C: Yes
E: Why?
C: It makes it bigger and bigger and bigger.
E: What would happen if I kept adding 1?
C: It will get too big.
E: Do we have to stop or could we keep adding 1?
C: Stop.

**A4: Infinity transcript for a child classified as having only Endless knowledge of infinity**

E = Experimenter
C = Child (Age: 4 years 9 months)

E: What is the biggest number you can think about?
C: One hundred.
E: Is that the biggest number there could ever be?
C: No, three hundred.
E: Is that the biggest number there could ever be?
C: No, four hundred.
E: Is that the biggest number there could ever be?
C: Yes.
E: If I keep counting, will I ever get to the end of numbers, or do numbers go on forever?
C: Forever.
E: Why?
C: There's a lot of numbers and they never end.
E: If we thought of a really big number, could we always add to it and make it even bigger, or is there a number so big we couldn't add any more?
C: Couldn't add more.
E: Why?
C: Because we couldn't add nine hundred to it.
E: So you said that the biggest number you know is four hundred. Is it possible to add 1 to four hundred, or is four hundred the biggest number possible?
C: No.
E: Why?
C: Take a long time.
E: Could I keep adding 1?
C: I don't know.

**A5: Infinity transcript for a child classified as having Full knowledge of infinity**

E = Experimenter
C = Child (Age: 5 years 10 months)

E: What is the biggest number you can think about?
C: A Jillion.
E: Is that the biggest number there could ever be?
C: No, infinity.
E: If I keep counting, will I ever get to the end of numbers, or do numbers go on forever?
C: Forever.
E: Why?
C: There's lots of numbers in the world.
E: If we thought of a really big number, could we always add to it and make it even bigger, or is there a number so big we couldn't add any more?
C: Yes.
E: Why?
C: Every number has a partner. Like 5 goes with 10s, twenties goes thirties, forties goes fifties, and sixties goes seventies, eighty goes ninety, one hundred goes one hundred and one.
E: So you said that the biggest number you know is infinity. Is it possible to add 1 to infinity, or is infinity the biggest number possible?
C: Yes.
E: Why?
C: I can if I want to.
E: Could I keep adding 1?
C: Yes.
E: What would happen if I kept adding 1?
C: I'd get to infinity, it'll be a million and one.

**B: Predictors of accuracy on Next Number task with all participants**

This analysis is referenced in Section 3.1.2 of the main text (Footnote 5). In the main text, we reported analyses excluding participants with an Initial Highest Count of 99. Here, we compare Non-Productive Counters and Productive Counters in terms of their accuracy on the Next Number task when all participants are included.

A mixed effects logistic regression predicting item-level accuracy from Productivity, Initial Highest Count and age (with random intercepts for subject and item magnitude), found no significant effect of Productivity ($p = 0.12$) or Age ($p = 0.89$). However, there was a significant effect of Initial Highest Count ($\beta = 1.70$, 95% CI=[1.24, 2.16], $\chi^2(1) = 54.49$, $p < 0.001$).

Next, we conducted a post-hoc analysis testing whether Productive counters may show a selective advantage on the Next Number task for mid-decade items but not decade transition items. This analysis found a significant main effect of Item Type ($\beta = 2.11$, 95% CI=[1.24, 2.97], $\chi^2(1) = 11.41$, $p < .001$), with greater accuracy on Mid-Decade items ($M = 61\%$, $SD = 36\%$) than on Decade Transition items ($M = 34\%$, $SD = 42\%$). Adding a Productivity by Item type (Decade transition/Mid-decade) interaction did not significantly improve model fit ($\chi^2(1) = 0.05$, $p = .83$), indicating that both Productive Counters and Non-Productive Counters found mid-decade items easier than decade transition items.

**C: Predictors of Infinity knowledge with all participants**
These analyses (C1-C3) are referenced in the main text, Section 3.3 (Footnote 8).

In the main text, we exclude from analysis Productive Counters who had counted to 99 on their own without error. That allows for a more conservative test of the hypothesis that Productive knowledge of counting might relate to beliefs about the successor function and infinity. Here, we provide analysis results when all participants are included, and note any qualitative differences from the analysis in the main text.

We conduct separate analyses to predict children's belief that every number has a successor (C1: Successor Knowledge,) and belief that numbers never end (C2: Endless Knowledge). We also predict children's status as Full Infinity Knowers, following Cheung et al. (2017) (C3: Full Knowledge). For our initial models, predictors included Initial Highest Count, Productivity Group (Productive vs. Non-Productive Counters), and Next Number accuracy (total score out of 8). Any predictor in these initial models that significantly predicted the outcome variable relative to a base model (with age as the only predictor) was added to a final model, to allow comparison among the predictors. Models were constructed hierarchically, with model comparisons performed at every step using a Likelihood Ratio Test, and with models selected on the basis of a significant chi-squared statistic and reduced AIC value. All models were constructed with the following formula: glm(infinity-successor / infinity-endless / infinity-full ~ (predictor) + age (centered), family = binomial.

**C1: Regression analyses predicting Successor Knowledge of Infinity**

Similar to analyses with the partial sample, initial models predicting Successor Knowledge of Infinity found that none of the three predictors explained a significant proportion of additional variance compared to the base model.

Table C1

*Regression models for predicting Successor knowledge on the Infinity Interview (All participants, N=122)*

| Models | Coefficient Estimates (ß) | | | | Summary statistics | | |
|---|---|---|---|---|---|---|---|
| | Age | IHC | Next Number accuracy | Productivity Group | Loglikelihood [a] | AIC | $R^2_{Nagelkerke}$ |
| **Base Model** | | | | | | | |
| Age | 0.342 | | | | -81.54 | 167.08 | 0.037 |
| **Initial Models** | | | | | | | |
| Age + IHC | 0.304 | 0.074 | | | -81.48 | 168.96 | 0.038 |
| Age + Next Number accuracy | 0.286 | | 0.126 | | -81.36 | 168.72 | 0.041 |
| Age + Productivity Group | 0.163 | | | 0.661 | -80.52 | 167.03 | 0.059 |

*Notes.* Coefficients were compared against 0 using *t*-tests. Model comparisons done using Likelihood Ratio Tests.

[a] Each initial model was compared against the base model.

**C2: Regression analyses predicting Endless Knowledge of Infinity**

For models predicting children's possession of Endless knowledge, all three counting measures explained significant additional variance relative to the base model. Controlling for age, children were more likely to have Endless knowledge if they were Productive counters ($\beta$ = 1.70, $p$ = .004, OR = 5.46 [95% CI = 1.69, 21.58]), had greater accuracy on the Next Number task ($\beta$ = 0.69, $p$ = .006, OR = 2.00 [95% CI = 1.21, 3.46]), or had greater Initial Highest Counts ($\beta$ = 0.67, $p$ = .006, OR = 1.96 [95% CI = 1.22, 3.20]).

To evaluate the relative contribution of each predictor, we constructed a full model which included all three predictors while controlling for age. This model explained significant additional variance compared to the base model with only age ($\chi^2(3) = 11.31$, $p$ = 0.01), but did not improve model fit relative to any of the initial models. In addition, none of the predictor coefficients in this full model were significantly different from zero, suggesting that these measures of counting ability may explain overlapping variance in predicting Endless Knowledge among our sample.

Table C2

*Regression models for predicting Endless knowledge on the Infinity Interview (All participants, N=122)*

| Models | Coefficient Estimates (ß) | | | | Summary statistics | | |
|---|---|---|---|---|---|---|---|
| | Age | IHC | Next Number accuracy | Productivity Group | Loglikelihood [a] | AIC | $R^2_{Nagelkerke}$ |
| **Base Model** | | | | | | | |
| Age | 0.707** | | | | -66.66 | 137.31 | 0.125 |
| **Initial Models** | | | | | | | |
| Age + IHC | 0.403 | 0.675** | | | -62.75** | 131.50 | 0.207 |
| Age + Next Number accuracy | 0.448 | | 0.693** | | -62.89** | 131.79 | 0.204 |
| Age + Productivity Group | 0.329 | | | 1.698** | -62.46** | 130.93 | 0.212 |
| **Full Model** | | | | | | | |
| Age + IHC + Next Number accuracy + Productivity Group | 0.251 | 0.251 | 0.291 | 1.092 | -61.00* | 131.99 | 0.242 |

*Notes.* Coefficients were compared against 0 using *t*-tests. Model comparisons done using Likelihood Ratio Tests.
[a] Each Initial model and the Full model were compared against the base model.
* p < 0.05, ** p < 0.01

**C3: Regression analyses predicting Full Knowledge of Infinity**

For initial models predicting Full Infinity knowledge, accuracy on the Next Number task explained significant additional variance relative to the base model ($\chi^2(1) = 3.85$, $p = .0496$). However, the model estimated coefficient for Next Number accuracy did not meet the threshold for significance ($\beta = 0.57$, $p = 0.059$, OR = 1.77 [95% CI = 1.00, 3.34]).

Table C3

*Regression models for predicting Full Infinity knowledge on the Infinity Interview (All participants, N = 122)*

| Models | Coefficient Estimates (ß) | | | | Summary statistics | | |
|---|---|---|---|---|---|---|---|
| | Age | IHC | Next Number accuracy | Productivity Group | Loglikelihood [a] | AIC | $R^2_{Nagelkerke}$ |
| **Base Model** | | | | | | | |
| Age | 0.752** | | | | -54.36 | 112.71 | 0.120 |
| **Initial Models** | | | | | | | |
| Age + IHC | 0.560 | 0.400 | | | -53.31 | 112.62 | 0.145 |
| Age + Next Number accuracy | 0.532 | | 0.573 | | -52.43* | 110.86 | 0.166 |
| Age + Productivity Group | 0.480 | | | 1.219 | -52.78 | 111.57 | 0.158 |

*Notes.* Coefficients were compared against 0 using *t*-tests. Model comparisons done using Likelihood Ratio Tests.

[a] Each initial model was compared against the base model.