BIOS 26210: Lab Exercise 4 (Chapter 3 Exercise)

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Problem 1

1a.

For the **ODE** $\dot{N} = f(N) = rN(N/U - 1)(1 - N/K)$, at the equilibrium we have $\dot{N} = 0$.
Then, $N^* = 0, N^* = U$ or $N^* = K$

For the derivative of $f(N)$, $f'(N) = -\frac{3rN^2}{UK} + \frac{2rN}{U} + \frac{2rN}{K} - r$:

when $N^* = 0, f'(N) = -r$ the equilibrium is stable.

when $N^* = U, f'(N) = -\frac{3rU}{K} + 2r + \frac{2rU}{K} - r$. If $f'(U) > 0$, $1 - U/K > 0$
Hence, if $U < K$ the equilibrium is not stable; if $U > K$ the equilibrium is stable;
if $U = K$ we need more information to determine whether the equilibrium is stable.

when $N^* = K, f'(N) = -\frac{3rK}{U} + \frac{2rK}{U} + 2r - r$ If $f'(N) > 0$, $1 - \frac{K}{U} > 0$
Hence, if $U > K$ the equilibrium is not stable; if $U < K$ the equilibrium is stable;
if $U = K$ we need more information to determine whether the equilibrium is stable.

For the **ODE** $\dot{N} = g(N) = rN^2(1 - N/K)$, at the equilibrium we have $\dot{N} = 0$
Then $N^* = 0$ or $N^* = K$.

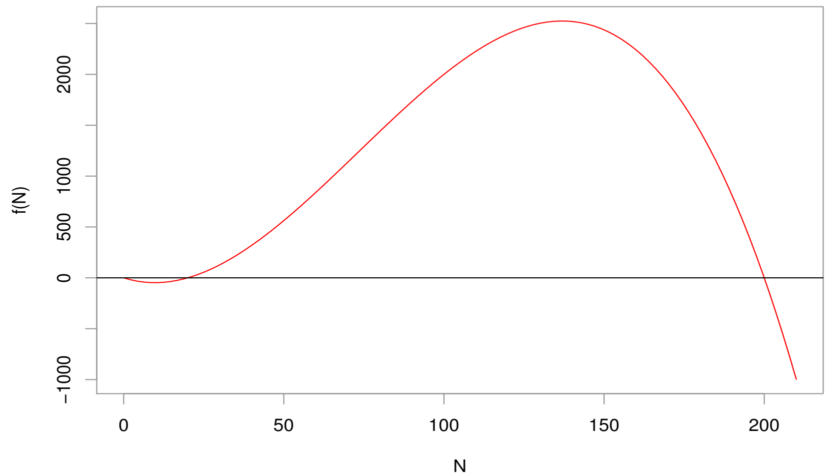
For the derivative of $g(N)$, $g(N)' = 2rN - \frac{3rN^2}{K}$

when $N^* = 0, f'(N) = 0$ the equilibrium can be either stable or unstable.

when $N^* = K, f'(N) = 2rK - 3rK = -rK < 0$. The equilibrium is stable.

1b.

Code: **HW4-1b_SiTang.R**



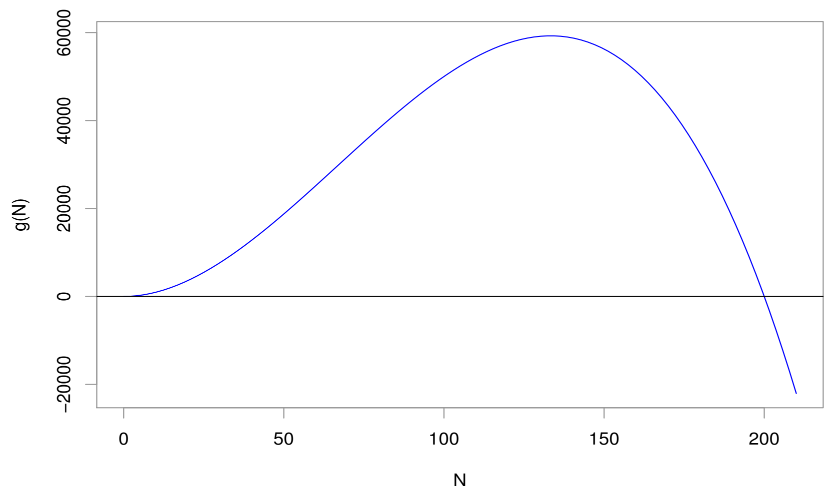


Fig. 4-1 Plots of $f(N)$ and $g(N)$

For model $f(N)$, the population grows when $20 \leq N \leq 200$ and decays elsewhere, while for model $g(N)$, the population grows when $0 \leq N \leq 200$ and decays elsewhere.

1c.

The main difference between these two models is that there is an interval $[0, 20]$ in model $f(N)$ where the population decays, while in model $g(N)$ the population keeps growing until it reaches a population of 200. The first model $f(N)$ take the effect when there only exists a small number of population into account: the growth rat is negative at this population size. Since the individuals are so rare, the population might go extinction due to the accumulation of deleterious genes in the population, which results from the mating between close relatives.

Problem 2

2a.

The differential equation model for this population is:

$$\dot{N} = f(N, t) = rN - \sin(\frac{4t}{2\pi})$$

2b.

The fixed point is obtained by solving the equation $f(N, t) = rN - \sin(\frac{4t}{2\pi}) = 0$, $N^* = \frac{1}{r}\sin(\frac{4t}{2\pi})$. The derivative of $f(N)$ is $f'(N) = r > 0$, therefore the 'fixed' point is not stable.

2c.

Code: **HW4-2c_SiTang.R**
(See next page Fig. 4-2 for the direction field of the model for the bacteria culture).

The fixed points appears periodically over time and are not stable. It is consistent with the result derived above, where $N^* = \frac{1}{r}\sin(\frac{4t}{2\pi})$, which is a periodic function, and N^* is not stable.

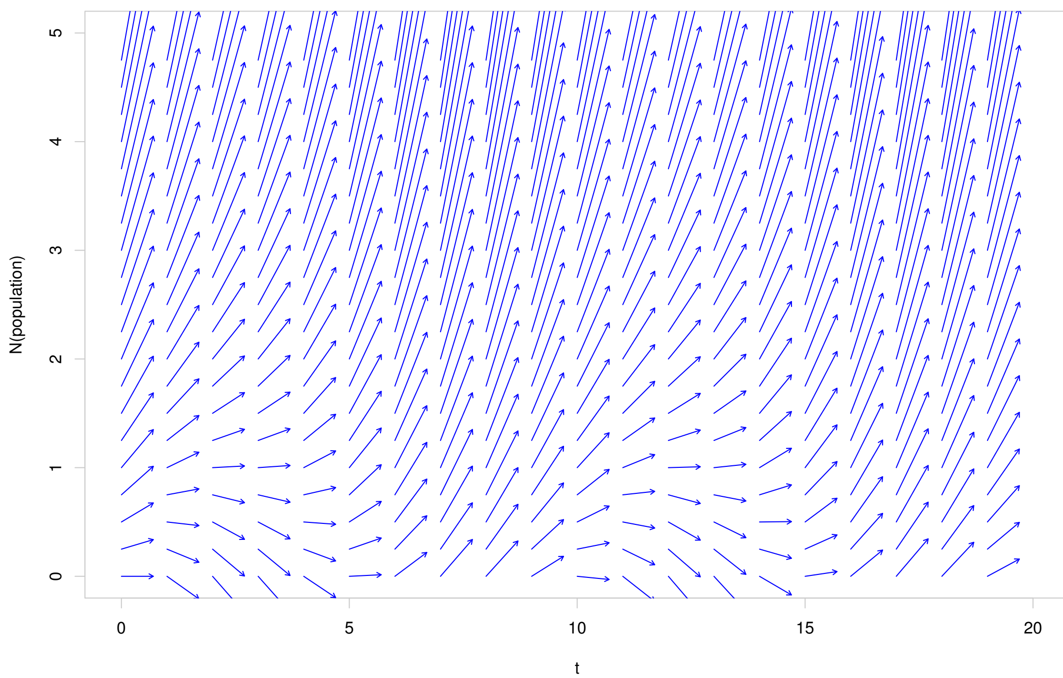


Fig4-2 The direction filed of the model for the bacteria culture ($r = 1$)

2d.

The differential equation $\dot{N} = \frac{dN}{dt} = f(N,t) = rN - \sin(\frac{4t}{2\pi})$ is solved as follows:

$$\begin{aligned} \frac{dN}{dt} - rN &= -\sin(\frac{4t}{2\pi}) \\ \frac{dN}{dt} \cdot e^{-rt} - rNe^{-rt} &= -e^{-rt} \cdot \sin(\frac{4t}{2\pi}) \\ \frac{d}{dt}(Ne^{-rt}) &= -e^{-rt} \cdot \sin(\frac{4t}{2\pi}) \end{aligned}$$

The integral of the left side is: $\int \frac{d}{dt}(Ne^{-rt})dt = Ne^{-rt}$

For the right side of the equation, let $A(t) = sin(\frac{2t}{\pi})e^{-rt}$, $B(t) = cos(\frac{2t}{\pi})e^{-rt}$ and $C(t) = B(t) + iA(t) = e^{-rt}e^{i2t/\pi} = e^{(\frac{2i}{\pi} - r)t}$, then

$$
\begin{aligned}
\int C(t)dt &= \frac{e^{(\frac{2i}{\pi}-r)t}}{\frac{2i}{\pi}-r} + C_1 \\
&= \frac{(\frac{2i}{\pi}+r)e^{\frac{2i\cdot t}{\pi}} \cdot e^{-rt}}{-\frac{4}{\pi^2}-r^2} + C_1 \\
&= \left[\text{Real Part of } \int C(t)dt\right] + \frac{1}{-\frac{4}{\pi^2}-r^2} e^{-rt} \left\{(\frac{2}{\pi} cos\frac{2t}{\pi} + rsin\frac{2t}{\pi})\right\} i + C_2 \\
\text{Then } \int A(t)dt &= [\text{Imaginary Part}] \int C(t)dt = \frac{1}{-\frac{4}{\pi^2}-r^2} e^{-rt} (\frac{2}{\pi} cos\frac{2t}{\pi} + rsin\frac{2t}{\pi}) + C_3
\end{aligned}
$$

So :

$$
\begin{aligned}
Ne^{-rt} &= \frac{-1}{-\frac{4}{\pi^2}-r^2} e^{-rt} (\frac{2}{\pi} cos\frac{2t}{\pi} + rsin\frac{2t}{\pi}) + C_4 \\
&= \frac{1}{\frac{4}{\pi^2}+r^2} e^{-rt} (\frac{2}{\pi} cos\frac{2t}{\pi} + rsin\frac{2t}{\pi}) + C_4 \qquad \text{(since } r = 1) \\
N &= \frac{1}{\frac{4}{\pi^2}+1} (\frac{2}{\pi} cos\frac{2t}{\pi} + sin\frac{2t}{\pi}) + C_4
\end{aligned}
$$

When $t = 0$, $N = N_0$, then $C_4 = N_0 - \frac{2\pi}{4+\pi^2}$. So,

$$
N = \frac{1}{\frac{4}{\pi^2}+1} (\frac{2}{\pi} cos\frac{2t}{\pi} + sin\frac{2t}{\pi}) + (N_0 - \frac{2\pi}{4+\pi^2})
$$

Plot the solution with three different initial conditions as in Fig4-3. (See next page).

Code: **HW4-2d_SiTang.R**

From Fig4-3 the population of the bacterial changes periodically, so does the fixed points. It is consistent with the analysis of the direction field in Problem 2(c).

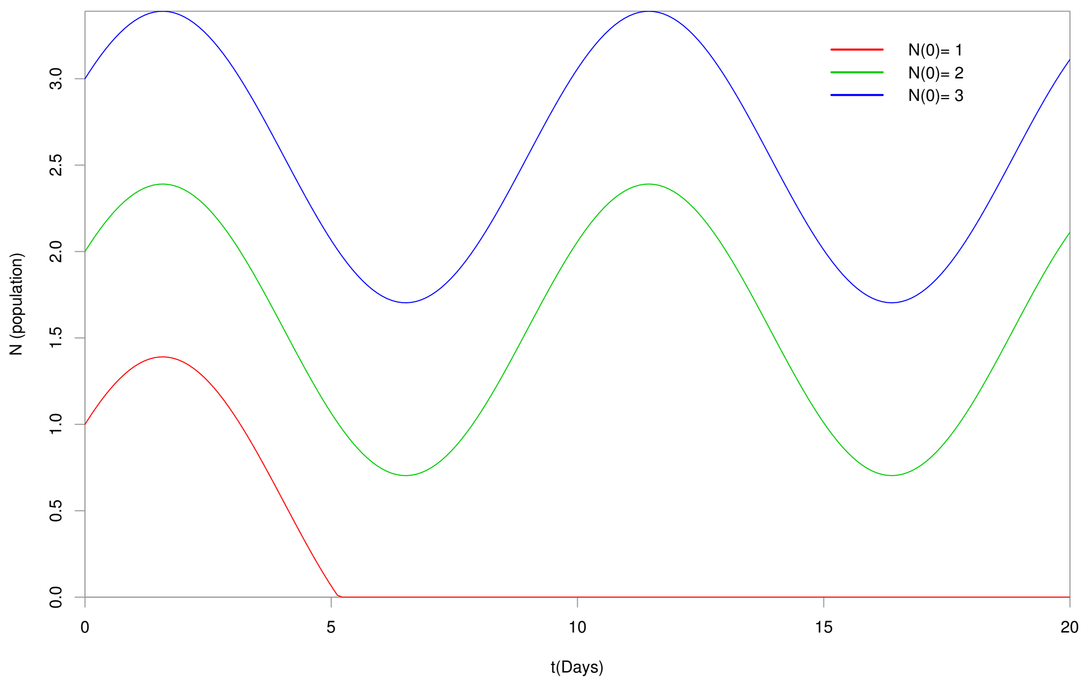


Fig4-3 The plot of model for the bacteria culture, with three different initial conditions

Problem 3

3a.

Since at the fixed point of $\dot{V} = f(V) = \frac{1}{C}[-g_{Na}(V - V_{Na}) - g_K(V - V_K)]$, we have

$$-g_{Na}(V - V_{Na}) - g_K(V - V_K) = 0$$

Then $V^* = \frac{g_{Na}V_{Na}+g_K V_K}{g_{Na}+g_K} = -68.3mV$. The derivative of $f(V)$ is:
$f'(V) = -\frac{1}{C}(g_{Na} + g_K) = -673.33\ \text{mmho} \cdot \text{cm}^2/\mu\text{F} < 0$. So the fixed point is stable.

3b.

Code: **HW4-3b_SiTang.R**

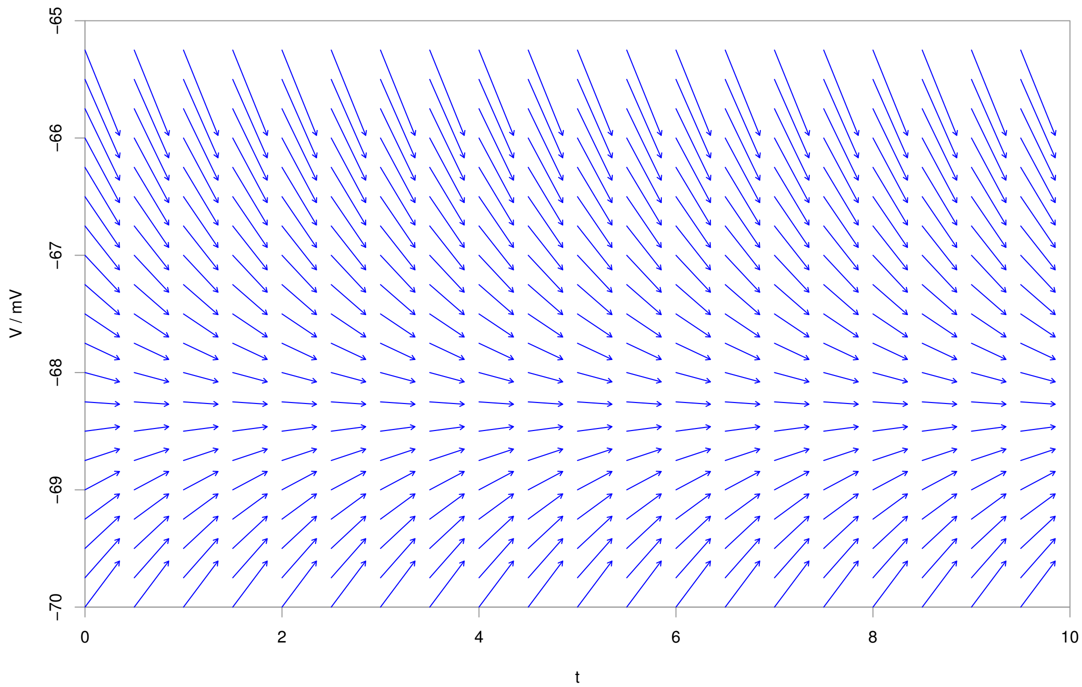


Fig4-4 The direction field of the model for two voltage-dependent ion channels.

From the Fig4-4, we could see that the fixed point of this model is around V=-68.3 mV, and the direction of flow is decreasing above $V = -68.3mV$ and increasing below $V = -68.3$, suggesting that the fixed point is stable.

3c.

Code: **HW4-3c_SiTang.R**

$$
\begin{aligned}
\dot{V} &= \frac{1}{C}(-g_{Na}(V - V_{Na}) - g_K(V - V_K)) \\
&= \frac{1}{C}(g_{Na}V_{Na} + g_K V_K) - \frac{1}{C}(g_K + g_{Na})V \\
&= C_0 - C_1 V \\
\text{And} \qquad C_0 &= \frac{1}{C}(g_{Na}V_{Na} + g_K V_K) = -46012.66\ m\text{mho} \cdot m\text{V} \cdot \text{cm}^2/\mu\text{F} \\
C_1 &= \frac{1}{C}(g_K + g_{Na}) = 673.33\ m\text{mho} \cdot \text{cm}^2/\mu\text{F}
\end{aligned}
$$

To solve this **ODE**, multiply both sides by $e^{C_1 t}$, we have,

$$\begin{aligned}
\frac{e^{C_1 t} dV}{dt} + C_1 V e^{C_1 t} &= C_0 e^{C_1 t} \\
\frac{d(e^{C_1 t V})}{dt} &= C_0 e^{C_1 t} \\
V &= \frac{C_0}{C_1} + C_2 e^{-C_1 t} \\
&= (-68.336 + C_2 e^{-673.33t})\ m\text{V}
\end{aligned}$$

Let $t = 0$, then $V_0 = -68.336 + C_2$ and $C_2 = (V_0 + 68.336)m\text{V}$. Therefore:

$$V = [-68.336 + (V_0 + 68.336)e^{-673.33t}]\ m\text{V}$$

Consider three different initial conditions of (1).$\{V_0 = -100mV\}$, (2).$\{V_0 = -68.33mV\}$ and (3). $\{V_0 = 0mV\}$, plot three solution curves respectively as follows.

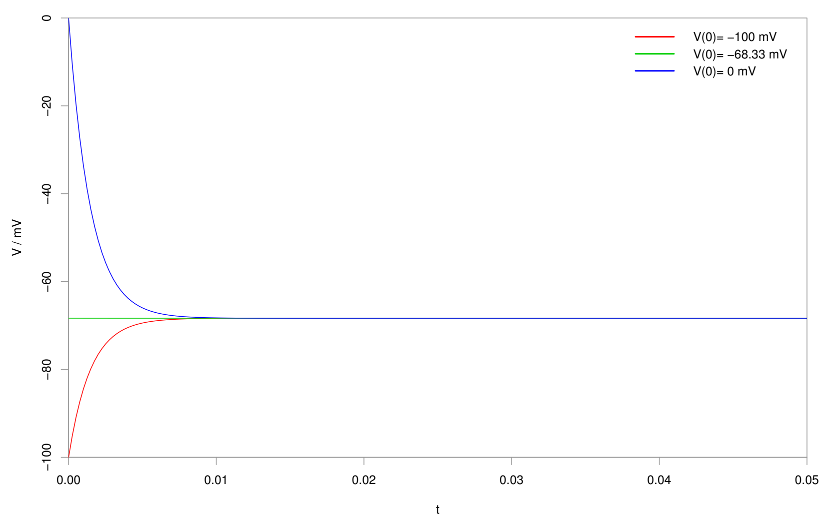


Fig4-5. The plot of the general solution of Voltage, with three different initial conditions

From the plot of the solution, we know that the voltage of the ion channel will reach a fixed point $V^* = -68.33\ mV$ and stay at this point in the long term. Actually, we do not need to have the analytical solution to come to this conclusion. From the analysis of the fixed points in this model above, we already know that there is a stable fixed point at $V^* = -68.33\ mV$.

Problem 4

4a. Pseudocode:

```
define function F(x: REAL)
var root : REAL ← Solve(F(x)) // Solve the roots of F(x) ==0

for i : INT in 1 to (length(root)-1)
begin
    median[i] : REAL ← (root[i] + root[i+1] )/2
     // Calculate F(x) value between two adjacent roots and draw the arrow
    if( F(median) > 0 ) DrawArrow[ (root[i], 0 ), ( root[i+1], 0 ) ]
    if( F(median) < 0 ) DrawArrow[ (root[i+1], 0 ), ( root[i], 0 ) ]
end

low : REAL ← root[1] - 5
high : REAL ← root[i+1] +5

//Calculate the F(x) value beyond the leftmost/minimum root and rightmost/maximum root
// and draw the arrow
if( F(low) > 0 ) DrawArrow[ (root[1]-1, 0), (root[1], 0) ]
if( F(low) < 0 ) DrawArrow[ (root[1], 0), (root[1]-1, 0) ]
if( F(high) > 0 ) DrawArrow[ (root[i+1], 0), (root[i+1] + 1, 0) ]
if( F(high) < 0 ) DrawArrow[ (root[i+1]+1, 0), (root[i+1], 0) ]
```

4b.

Code: **HW4-4b_SiTang.R**

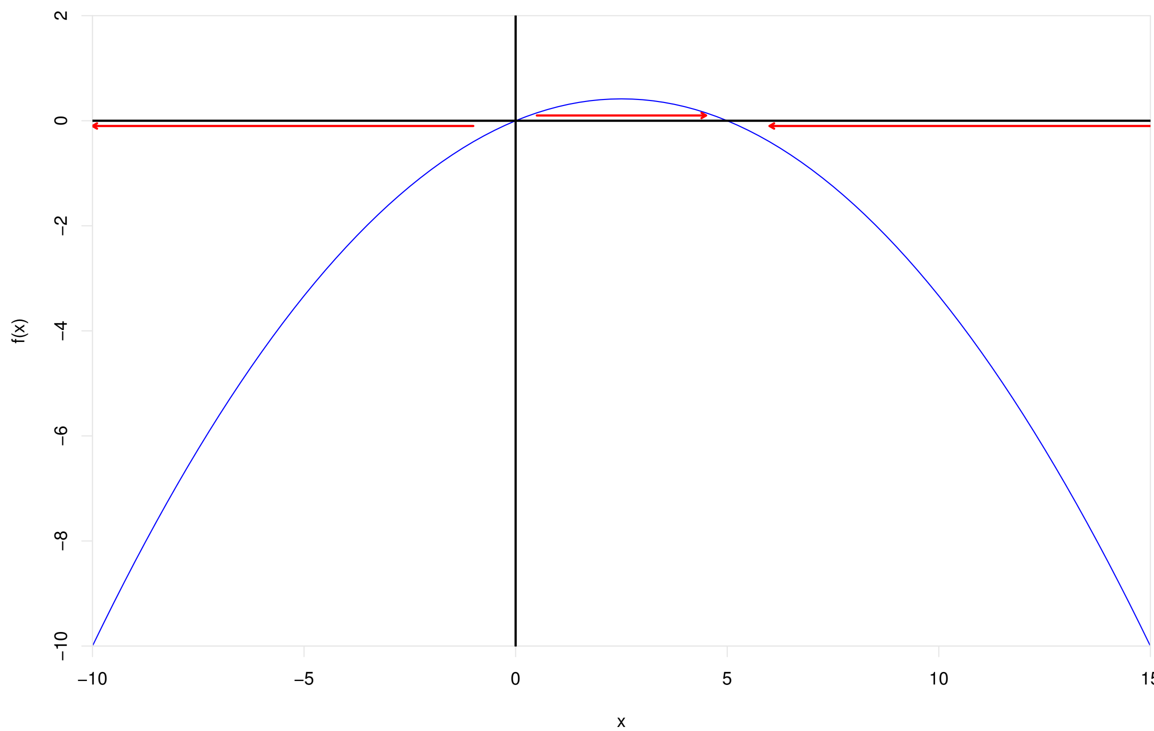


Fig4-6. The plot of flow of the logistic model $f(x) = \frac{x}{3}(1 - \frac{x}{5})$