Introduction to computational programming
Chapter 2 Lab Exercise
Cobweb plots and stability

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Overview

In this exercise we will build more complicated (cobweb) plots and continue our analysis of stability and fixed points.

Part I
Tutorial

1 Example: Cobweb plots

Lets start by examining the provided pseudocode for cobweb plot generation.

Pseudocode for producing a cobweb plot for an iterated map:

1. Define the iterated map function `F(x)`
2. Choose an initial condition x_0 and store it as the first element of the array x (we will start at $x[0]$, but in some languages the first index has to be 1)
3. Store in `y[0]` the value 0 (start the graph on the x-axis)

4. For i starting at 1, repeat until i exceeds the specified number of iterations
   (a) store in `y[2i-1]` the value of `F(x[2i-2])` (draw a vertical line to the graph of $y = f(x)$)
   (b) store in `y[2i]` the value of `y[2i-1]` (draw a horizontal line to the graph of $y = x$)
   (c) store in `x[2i-1]` the value `x[2i-2]` (draw a vertical line to the graph of $y = f(x)$)
   (d) store in `x[2i]` the value of `y[2i]` (draw a horizontal line to the graph of $y = x$)
   (e) increment i by 1
5. plot the graph of $F(x)$
6. plot the graph of $y = x$
7. plot the array y versus the array x

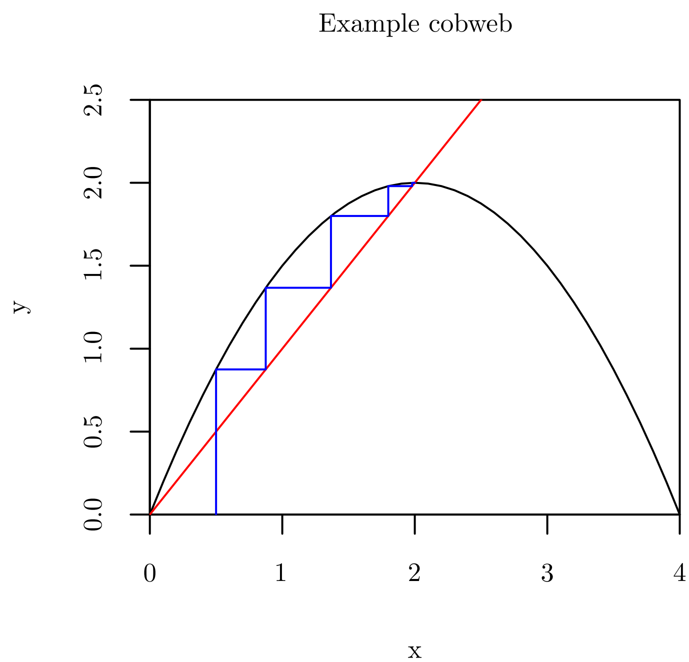
In the more compact form, this becomes

```
function F(x: real)    // iterated map function
var x0 : real          // initial condition
var N : integer        // maximum number of iterations
var X,Y : real array [1..N]
x[0] ← x0
Y[0] ← 0
for i ← 1 to N do
begin
   Y[2i-1] ← F(X[2i-2])
   Y[2i] ← F(Y[2i-1])
   X[2i-1] ← F(X[2i-2])
   X[2i] ← F(Y[2i])
end
plot F(x)
plot y = x
plot X,Y
```

Pseudocode listing 1 – Cobweb plot algorithm

The following example generates a “cobweb” plot for the logistic model given by $f(x) = x\left(2 - \frac{x}{2}\right)$

```
> cobImapFun <- function(x) {
+   return(2 * x - 0.5 * x ** 2);
+ }
> max_iter <- 50;
> y_vector <- x_vector <- numeric(
+   max_iter * 2);
> y_vector[1] <- 0;
> x_vector[1] <- 0.5;
> for(ii in 1:50) {
+   y_vector[2 * ii] <- cobImapFun(
+     x_vector[2 * ii - 1]);
+   y_vector[2 * ii + 1] <-
+     y_vector[2 * ii];
+   x_vector[2 * ii] <-
+     x_vector[2 * ii - 1];
+   x_vector[2 * ii + 1] <-
+     y_vector[2 * ii + 1];
+ }
> plot( -10:50 / 10,
+       cobImapFun(-10:50 / 10),
+       xlab='x',
+       ylab='y',
+       main='Example cobweb',
+       type='l',
+       xlim=c(0, 4),
+       ylim=c(0, 2.5),
+       xaxs='i',
+       yaxs='i');
> lines(-2:6, -2:6, col='red');
> lines(x_vector, y_vector,
+       col='blue');
```



Part II

Exercise

1. Find the equilibrium values of populations governed by the following equations, and determine their stability analytically. Is there is a stable nonzero equilibrium (carrying capacity) that the population may approach in the long run?

   (a) $N_{t+1} = 41N_t - 10N_t^2$

   (b) $N_{t+1} = 41N_t + 2N_t^2$

2. Generate cobweb plots for the following logistic models. Graphically identify all fixed points, and state whether they are "stable."

   (a) $f(x) = 3x - \frac{3x^2}{4} + 1$

   (b) $f(x) = 100x - 2x^2$

   (c) $f(x) = -100x + \frac{x^2}{2}$

3. Write a *functor* which takes as an argument a function describing a logistic model and generates a cobweb plot.

4. In physiology, maintaining a steady level of glucose in the bloodstream is necessary for the proper functioning of all organs. To study this process, define $G(t)$ to be the amount of glucose in the bloodstream of a person at time t. Assume that glucose is absorbed from the bloodstream at a rate proportional to the concentration $G(t)$, with rate parameter k.

   (a) Write down a differential equation to describe this situation. What kind of ODE is it?

   (b) Find the analytical solution for this equation, in terms of an initial value G_0 and the rate parameter k.

   (c) Let the initial glucose concentration be $G_0 = 100mg/dl$, and the glucose removal rate be $k = 0.01/min$. Use R to plot the solution as a graph over a reasonable time interval, with properly labeled axes.

   (d) What is the equilibrium concentration of blood sugar in this model? Is the equilibrium stable or unstable?

5. Now let us assume that glucose is added to to bloodstream at a constant rate a, independent of glucose concentration.

   (a) Write down a differential equation to describe this situation. What kind of ODE is it?

   (b) Find the analytical solution for this equation, in terms of an initial value G_0 and the parameters k and a.

   (c) Let the initial glucose concentration be $G_0 = 100mg/dl$, the glucose removal rate be $k = 0.01/min$, and $a = 4mg/dl/min$. Use R to plot the solution as a graph over a reasonable time interval, with properly labeled axes.

   (d) What is the equilibrium concentration of blood sugar in the model with glucose infusion? Is the equilibrium stable or unstable?

6. *Not required. A good exercise which relates to numerical approximation.*

   (a) *Binary Search.* Write a function which, given a sorted vector X of n unique integers, and an integer k, returns the index of k in vector X if $k \in X$ and -1 otherwise. Your solution should perform approximately $\log n$ comparisons in the "worst case" scenario. *Hint:* one example of a "worst case" is when the $X_0 = k$.

   (b) Modify the previous solution so that the requirement of *unique* integers may be relaxed. In the case that multiple solutions exist return a vector of all such solutions.