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EDGE-CONNECTIVITY AUGMENTATION OF SIMPLE GRAPHS*

KASPER SKOV JOHANSEN†, EVA ROTENBERG† , AND CARSTEN THOMASSEN†

Abstract. We consider the following variant of the edge-augmentation problem: Given a k-edge-connected graph with no loops or multiple edges, find a smallest edge set in the complement whose addition to G results in a $(k+1)$-edge-connected graph. We establish the following dichotomy for this problem: If the complement of G contains a matching covering all vertices of G-degree k (and possibly some more), then the complement also contains a matching whose addition to G results in a $(k+1)$-edge-connected graph. A smallest matching which augments the minimum degree can be found, in polynomial time, by Edmonds' matching algorithm, but it need not augment the edge-connectivity. Indeed, it is NP-hard to find a smallest edge-connectivity augmenting edge set, by a result of Tibor Jordan. On the other hand, if the complement of G contains no matching covering all vertices of G-degree k, then the complement has an minimum degree augmenting path system consisting of paths of length 1 or 2. Again we can find such a path system with as few edges as possible by Edmonds' matching algorithm. We can, in polynomial time, modify it to an edge-connectivity augmenting path system of paths of length 1 or 2 with the same number of edges, and this time it yields a smallest edge-connectivity augmenting set of edges. Combining these results, we conclude that a smallest edge-connectivity augmenting edge set in the complement of a k-regular, k-edge-connected simple graph has size $n - m(\overline{G})$, where n is the number of vertices of G and $m(\overline{G})$ is the size of a maximum matching in the complement of G. Another corollary is that the complement of every simple non-complete graph G with n vertices has a set of at most $2n/3$ edges whose addition to G results in a graph of larger edge-connectivity with equality holding if and only the complement of G is a disjoint union of 3-cycles.

Key words. Edge connectivity, graph augmentation, matchings.

MSC codes. 05C38, 05C40, 05C70.

1. Introduction.

Frank [3] gave a polynomial time algorithm for finding the smallest number of edges needed to add to a multigraph in order to increase the edge-connectivity. Both the edges of the graph and the added edges may be part of multiple edges. Jordan (see [1]) proved that the analogous problem for simple graphs (where neither the edges of the graph nor the added edges are allowed to be part of multiple edges) is NP-hard. Bang-Jensen and Jordan [1] gave a polynomially bounded algorithm for the special case where the edge-connectivity is fixed.

We shall here solve the problem in another special case, namely when the complement $\overline{G}$ has no matching covering the vertices of smallest degree. First we prove that, if the complement $\overline{G}$ of a simple k-edge-connected graph G contains a matching covering all vertices of G-degree k (and possibly some more), then $\overline{G}$ also contains a matching whose addition to G results in a $(k+1)$-edge-connected graph. By Edmonds' matching algorithm, the former matching can be found in polynomial time. In particular, if G is k-regular and k-edge-connected, and $\overline{G}$ has a perfect matching, then the minimum number of edges that must be added to increase the edge-connectivity is $n/2$. On the other hand, if $\overline{G}$ does not contains a matching covering all vertices of G-degree k, then the smallest number of edges in $\overline{G}$ whose addition to G results in a $(k+1)$-edge-connected graph equals $|V(G_k)| - m(\overline{G_k})$, where G_k is the subgraph of G induced by the vertices of degree k, and $m(\overline{G_k})$ is the size of a maximum matching in the complement of G_k. We point out that such a set of edges can be found in poly-

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nomial time. In particular, if G is k-regular and k-edge-connected, then the smallest number of edges in $\overline{G}$ whose addition to G results in a $(k+1)$-edge-connected graph equals $|V(G)| - m(\overline{G})$ because this expression coincides with the $n/2$ bound mentioned above when $\overline{G}$ has a perfect matching. Also, the complement of every simple non-complete graph with n vertices has a set of at most $2n/3$ edges whose addition to G results in a graph of larger edge-connectivity. Equality holds if and only if G is the complement of a collection of pairwise disjoint triangles.

2. Notation and terminology. A graph G may have multiple edges but no loops. If it has no multiple edges it is **simple**. The sets of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. The G**-degree** of a vertex $v \in V(G)$ is the number of edges incident with v. If all vertices have degree k, the graph is k**-regular**.

If G is simple, we denote by $\overline{G}$ the **complement** of G, that is, the graph consisting of the edges that are not in G. We denote by $G[X]$ the **induced subgraph** of $X \subseteq V(G)$ in G. A **matching** is a set M of edges, no two of which have an endvertex in common. We also consider the empty set to be a matching. An endvertex of an edge in M is **covered** by M. We denote by $m(G)$ the cardinality of a maximum matching in G. If A, B is a partition of $V(G)$, then the set of edges between A and B is a **cut**. If its number of edges is k it is also called a k**-cut**. We call A (respectively B) the **left side** (respectively **right side**) of the cut. A graph is k**-edge-connected** if $k > 0$ and it has no cut with fewer than k edges. Two cuts are **crossing** if all four intersections of their sides are non-empty.

Finally, a k**-path** (or k**-cycle**) is a path (or cycle) of length k. The d**-cube** is the graph whose vertex set consists of all points in $\{0,1\}^d$. Two vertices are joined by an edge if and only if their Hamming distance is 1, where their **Hamming distance** is the number of coordinates in which they differ. Note that the 2-cube is the 4-cycle.

3. Crossing cuts. The structure of k-cuts in a k-edge-connected graph is well understood, [2]. For the sake of completeness we include the proofs of the two lemmas below which are about crossing minimum cuts.

LEMMA 3.1. *Let S_1, S_2 be two crossing k-cuts in a k-edge-connected graph G, that is, $V(G)$ can be partitioned into four non-empty sets $V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1}$ such that S_1 consists of the edges leaving $V_{0,0} \cup V_{0,1}$ and S_2 consists of the edges leaving $V_{0,0} \cup V_{1,0}$. Then k is even, there are no edges between $V_{i,j}$ and $V_{p,q}$ when the Hamming distance of their indices is 2, and there are precisely $k/2$ edges between $V_{i,j}$ and $V_{p,q}$ when the Hamming distance is 1.*

Proof. As S_1, S_2 are k-cuts, it follows that $S_1 \cup S_2$ has at most $2k$ edges. As G is k-edge-connected, there are at least k edges leaving each $V_{i,j}$. Adding these numbers for all four $V_{i,j}$ results in at least $4k$, that is, at least $2k$ edges in $S_1 \cup S_2$. Combining these two counts, we conclude that $S_1 \cap S_2$ is empty, and it is now easy to complete the proof. □

If we contract each $V_{i,j}$ to a single vertex, we transform G into a 4-cycle (the 2-cube) where every edge is replaced by $k/2$ edges. In this case we say that G is a **blow up** of the 4-cycle. If S_1, S_2, S_3 are three pairwise crossing cuts in a graph G, then the eight intersections of their sides can be denoted $V_{i,j,p}$ (where each i, j, p is 0 or 1). That is, the indices may be thought of as vertices of the 3-cube, and the cuts S_1, S_2, S_3 are the three cuts separating one 4-cycle in the cube from another 4-cycle. With this notation we have:

LEMMA 3.2. *Let S_1, S_2, S_3 be three pairwise crossing k-cuts in a k-edge-connected* *graph G. Then G is a blow-up of a 6-cycle where each edge in the 6-cycle corresponds to $k/2$ edges in G, and each of S_1, S_2, S_3 has a 2-path of the 6-cycle on each side.*

Proof. Let the eight intersections of the sides of S_1, S_2, S_3 be denoted $V_{i,j,p}$ (where each i, j, p is 0 or 1). We think of the indices as the vertices of the 3-cube. Possibly some $V_{i,j,p}$ are empty. Lemma 3.1 implies that, if the Hamming distance between (i, j, p) and (i', j', p') is > 1, there is no edge between $V_{i,j,p}$ and $V_{i',j',p'}$. If the Hamming distance between (i, j, p) and (i', j', p') is 1, then at least one of $V_{i,j,p}$, $V_{i',j',p'}$ is non-empty (because S_1, S_2, S_3 are pairwise crossing). On the other hand, at least two of the eight sets $V_{i,j,p}$ must be empty since otherwise $S_1 \cup S_2 \cup S_3$ has at least $7k/2$ edges by the same count as in Lemma 3.1. If three of them are empty, then they correspond to vertices in the 3-cube with pairwise Hamming distance > 1. The only way for this to happen is if all three vertices have pairwise Hamming distance 2, and thus have some neighbour in common. Such a common neighbour must correspond to a non-empty $V_{i,j,p}$, but this contradicts the fact that G is connected. So precisely two of the sets $V_{i,j,p}$ must be empty. They cannot have Hamming distance 2 because they are then on a common 4-cycle of the cube, and the cut having that 4-cycle on the left side has at least $2k$ edges going to the right side because G is k-edge-connected. But there are precisely k such edges. So the two empty $V_{i,j,p}$ have Hamming distance 3 which implies that G is a blow-up of a 6-cycle. It is now easy to complete the proof. ∎

COROLLARY 3.3. *Let a, b, c, d be four distinct vertices in a k-edge-connected graph G. Then some two of a, b, c, d are not separated by the other two by a k-cut. In other words, G does not contain three k-cuts such that one separates a, b from c, d, and another separates a, c from b, d, and the third separates a, d from b, c.*

Proof. If Corollary 3.3 were false, there would be three cuts S_1, S_2, S_3 such that any two of a, b, c, d are separated from the other two by one of S_1, S_2, S_3. Hence S_1, S_2, S_3 are pairwise crossing. By Lemma 3.2, G is a blow up of a 6-cycle. As each of a, b, c, d is separated from some other by one of S_1, S_2, S_3, it follows that a, b, c, d are in four distinct blow-up sets. No three consecutive blow-up sets can contain one of a, b, c, d because then the fourth would be separated from the three by one of S_1, S_2, S_3. So the two blow-up sets containing none of a, b, c, d are diametrically opposite. Now it is easy to obtain a contradiction. ∎

Note that the results so far in this section are for graphs that may have multiple edges. In the following we focus on simple graphs. We shall also use the following observation.

LEMMA 3.4. *Let S be a k-cut with left side $|A|$ in a k-edge-connected simple graph. If $|A| \geq 2$, then $|A| \geq k$ with equality only if A induces a complete subgraph. If some vertex in A has degree at least $k+1$, then $|A| \geq k+1$. If all vertices in A have degree at least $k+1$, then $|A| \geq k+2$.*

Proof. Consider first the case where all vertices on the left side have degree at least $k+1$.

Suppose (reductio ad absurdum) that G has q vertices on the left side, where $q \leq k+1$. Each vertex on the left side is adjacent to at most $q-1$ vertices on the left side and is therefore incident with at least $k+1-(q-1)$ edges in the cut S. Hence S contains at least $q(k-q+2) > k$ edges, a contradiction.

The cases where $|A| \geq 2$ or some vertex on the left side has degree at least $k+1$ are proved by a similar argument. ∎

4. Edge-connectivity augmenting matchings.

THEOREM 4.1. *Let G be a simple k-edge-connected graph, where $k \geq 1$. Assume $\overline{G}$ has a matching covering all vertices of degree k in G. Then $\overline{G}$ has a matching M such that $G \cup M$ is $(k+1)$-edge-connected.*

Proof. Let n be the number of vertices of G. By the assumption of the theorem, there is a matching M' in $\overline{G}$ such that $G \cup M'$ has minimum degree at least $k+1$. We may assume that M' is maximal with this property. Now we extend M' to a matching M'' with $\lfloor n/2 \rfloor$ edges such that all edges in $M'' \setminus M'$ are in G, and we select M', M'' such that $G \cup M'$ has as few k-cuts as possible. We claim that $G \cup M'$ is $(k+1)$-edge-connected. For suppose (reductio ad absurdum) that $G \cup M'$ has a k-cut with sides A, B. Then each edge in M' joins two vertices in the same side since otherwise, $G \cup M'$ would have more that k edges between A and B. The maximality of M' implies that the edges in $M'' \setminus M'$ can be chosen at random, and hence we may assume that all edges of $M'' \setminus M'$, except possibly one, join two vertices on the same side.

If $e \in M''$ joins two vertices a, b in A, and $e' \in M''$ joins two vertices c, d in B, then we may assume that G has at least one edge joining one of a, b with one of c, d. For otherwise, we replace in M'' the edges e, e' with either ac, bd or ad, bc. One of these two replacements results in a new M' such that the new $G \cup M'$ has fewer k-cuts (contradicting the minimality property of M', M''), because in the new $G \cup M'$ there are $k+2$ edges between A, B, and if each of the replacements result in a new k-cut, we would obtain a contradiction to Corollary 3.3.

We have chosen M'' such that M'' has $\lfloor |A|/2 \rfloor = \alpha$ edges joining vertices in A and $\lfloor |A|/2 \rfloor = \beta$ edges joining vertices in B. We have also proved that for each of the α edges, say e, and each of the β edges, say e', there is an edge e'' in G joining an end of e with an end of e'. The edge e'' is in the k-cut consisting of the edges in G between A and B. By Lemma 3.4, $|A| \geq k+2$ and $|B| \geq k+2$ implying that $\alpha\beta > k$ unless $k = 1$ and $|A| = |B| = 3$. This is a contradiction when $k > 1$. We leave the case $k = 1$ for the reader. $\square$

The proof in Theorem 4.1 can be turned into a polynomially bounded algorithm since a graph has at most $n(n-1)/2$ minimum cuts, as proved in [2]. Actually, we do not need that upper bound, since, after each replacement of a pair of edges by another pair, there are fewer pairs of edges in M'' that are separated by a k-cut. So, the number of iterations is less than $n^2/4$.

If G is a k-edge-connected graph with an even number of vertices, then it is an easy consequence of Mader's lifting theorem [5] that one can add a perfect matching to G so that the resulting graph is $(k+1)$-edge-connected. (For, if we first add a new vertex w joined by precisely one edge to each vertex of G, then the resulting graph G' is $(k+1)$-edge-connected. Now Mader's lifting theorem implies that we can lift successively all edges incident with w such that the resulting graph G'' is also $(k+1)$-edge-connected. Note that G'' is obtained from G by adding a perfect matching.)

Frank [3] found a matching M which is also a smallest edge-connectivity augmenting set. Bang-Jensen and Jordan [1] proved that, if $|M| > 3k^4/2$ and G is simple, then another edge-connectivity augmenting matching of the same cardinality can be found in polynomial time such that the new matching introduces no double edges when added to G. We show that (a slightly improved version of) this can also be obtained with our method. We formulate the result so that it also includes the case where all G-degrees are $> k$, as we later consider the case where all degrees are precisely k.

THEOREM 4.2. *Let G be a simple k-edge-connected graph. Assume that a smallest edge-connectivity augmenting matching allowing double edges has precisely q edges. If $q \geq 2k$ or if all G-degrees are $\geq k+1$ (or both), then $\overline{G}$ has a matching M_0 with at most q edges such that $G \cup M_0$ is $(k+1)$-edge-connected and simple. M_0 can be found in polynomial time.*

Proof. Let M be a smallest edge-connectivity augmenting matching allowing double edges such that the number of double edges in $G \cup M$ is minimum. If $G \cup M$ has no double edges we are done. So assume uv is an edge in both G and M.

Consider first the case where each of u, v has G-degree k. Since M has at least $2k$ edges, M has an edge $u'v'$ which is not incident with any neighbor of u or v. Now we replace in M the two edges $uv, u'v'$ by uu', vv' or uv', vu'. Both replacements result in a matching M' such that $G \cup M$ has fewer double edges, and one of the two replacements results in a graph with no new k-cuts, by Lemma 3.1. (For otherwise, the two new k-cuts S_1, S_2, say, would be crossing, and, with the notation of Lemma 3.1, u, v would have Hamming distance 2, and u', v' would have Hamming distance 2. But then the edge e joins vertices of Hamming distance 2 which contradicts Lemma 3.1.) This contradicts the assumption that the number of double edges in $G \cup M$ is minimum. So, we may assume that u has G-degree at least $k+1$.

Let e be the edge joining u, v in M. As $M-e$ is not edge- connectivity augmenting, $(G \cup M) - e$ has a k-cut S with sides A, B such that $u \in A$. We choose S such that $|A|$ is minimum. (For the algorithmic part of Theorem

Bang-Jensen and Jordan [1] proved that, if we need to add at least $3k^4/2$ edges in order to augment a simple graph G to a k-edge-connected graph (allowing multiple edges), then the same number of edges suffice also if we do not allow multiple edges. With our method, the bound $3k^4/2$ can be reduced to $2k^2$. The details are given in [4].

The complete bipartite graph with an odd number k of vertices in each bipartite class needs $k+1$ additional edges to become a $(k+1)$-edge-connected simple graph, but only k additional edges to become a $(k+1)$-edge-connected graph with one double edge. So the bound $2k$ in Theorem

It is easy to give examples that the matching we find in Theorem 4.1 is not a minimum edge-connectivity augmenting set. Indeed, Tibor Jordan [1] has proved that the problem of finding a minimum edge-connectivity augmenting set is NP-hard. This changes dramatically when we consider a simple k-edge-connected graph G such that $\overline{G}$ has no matching covering all vertices of degree k in G. Before that we consider the simpler problem of augmenting the minimum degree.

5. Smallest edge sets covering all vertices of maximum degree. In this section we show how to find a smallest set of edges whose addition to G increases the minimum degree.

PROPOSITION 5.1. *Let G be a simple graph of maximum degree Δ. Then a smallest edge set E_Δ covering all vertices of degree Δ has size $|V(G_\Delta)| - m(G_\Delta)$ where G_Δ is the subgraph of G induced by the vertices of degree Δ. Moreover, E_Δ can be chosen such that it is the union of pairwise disjoint paths in G_Δ of length 1 or 2.*

Proof. The minimality of E_Δ implies that E_Δ contains no path or cycle with three edges. So E_Δ is a collection of pairwise disjoint stars. Let M be a largest matching in $G_\Delta \cap E_\Delta$. Then $|M| \leq m(G_\Delta)$. If v is one of the $|V(G_\Delta)| - 2|M|$ vertices in $V(G_\Delta)$ not covered by M, then v is incident with precisely one edge of E_Δ. (For, the maximality of M implies that v cannot be the center of a star in E_Δ with at least two edges.) If $u \neq v$, and u is a vertex in $V(G_\Delta)$, and u is not covered by M, then the edge in E_Δ incident with u is distinct from the edge in E_Δ incident with v. Hence $|E_\Delta| \geq |M| + (|V(G_\Delta)| - 2|M|) = |V(G_\Delta)| - |M| \geq |V(G_\Delta)| - m(G_\Delta)$.

To prove the opposite inequality, let M_1 be a maximum matching of G_Δ. Then the set S of vertices in G_Δ not covered by M_1 form an independent set. By Hall's theorem, G has a matching M_2 with $|S|$ edges covering all vertices of S. No two edges in M_2 cover the endvertices of an edge in M_1, by the maximality of M_1. So $M_1 \cup M_2$ is the edge set of a graph with $|V(G_\Delta)| - m(G_\Delta)$ edges consisting of pairwise disjoint paths in G_Δ of length 1 or 2. ∎

6. Edge-connectivity augmenting path systems. We now prove our main augmentation result.

THEOREM 6.1. *Let G be a simple k-edge-connected graph. Assume $\overline{G}$ has no matching covering all vertices of degree k in G. Then the smallest set of edges in $\overline{G}$ whose addition to G increases the edge-connectivity has the same cardinality as the smallest set of edges in $\overline{G}$ whose addition to G increases the minimum degree.*

Proof. By assumption, G has minimum degree k (since otherwise, the empty set is a matching covering all vertices of degree k). An edge-augmentation which increases the edge-connectivity also increases the minimum degree. Now let P be a smallest set of edges whose addition to G increases the minimum degree. The graph with vertex set $V(G)$ and edge set P is also (with a slight abuse of notation) called P. By Proposition 5.1, we may assume that P is a path system consisting of paths of length 1 or 2. Among all those path systems, we choose one such that the number of paths of length 1 in the system is maximum. Subject to that we choose P such that the number of k-cuts in $G \cup P$ is minimum. We claim that $G \cup P$ is $(k+1)$-edge-connected. Suppose therefore (reductio ad absurdum) that $G \cup P$ has a k-cut with sides A, B. The assumption of Theorem 6.1 implies that P has at least one 2-path $u_1u_2u_3$. We may assume that it is in A. Let x, respectively y, respectively z denote the number of connected components in $P[B]$ with one, respectively two, respectively three vertices. Then B has $x + 2y + 3z$ vertices. We now count edges from B to A. Each of the x isolated vertices in $P[B]$ is joined (in G) to both of u_1, u_3 by the maximality of the number of 1-paths in P. (For, if the vertex u is a component in $P[B]$ and u is not joined in G to u_3 say, then we replace in P the edge u_2u_3 by uu_3 and obtain a contradiction to the maximality property of P.) If $u_1'u_2'u_3'$ is a 2-path in $P[B]$, then each of u_1', u_3' is joined to each of u_1, u_3. (For, if u_1' is not joined to u_1, say, then we may replace the two edges $u_1u_2, u_1'u_2'$ by the edge u_1u_1' and thereby obtain a contradiction to the minimality property of P.)

If uv is any edge in $P[A]$, and $u'v'$ is any of the y edges in $P[B]$, then G has an edge joining one of u', v' to one of u, v. For otherwise, we replace (in P) $uv, u'v'$ by either uu', vv' or uv', vu' and obtain a contradiction to the minimality of k-cuts by Corollary 3.3.

It follows that the number of edges in G from B to $u_1u_2u_3$ is at least $2x + 4z + y$. Since S has precisely k edges, we cannot have $y = 0$ because this would imply that $k = |S| \geq 2x + 4z + y = 2x + 4z > x + 3z = |B| \geq k$, a contradiction. So $y > 0$. Moreover, we claim that, for at least one of the y edges, say $u'v'$, in $P[B]$, there is at most one edge in G from u', v' to u_1, u_2, u_3. If this claim were false, then an earlier count yields $k = |S| \geq 2x + 4z + 2y \geq x + 2y + 3z = |B| \geq k$ which is a contradiction unless $x = z = 0$, and $k = |S| = 2y = |B|$. By Lemma 3.4, B induces a complete graph which, however is not possible because the edges in $P[B]$ are in the complement of G. So we reach in any case a contradiction which proves the claim that $P[B]$ has an edge $u'v'$ such that there is at most one edge in G from u', v' to u_1, u_2, u_3. We have earlier proved that G has an edge from u', v' to u_1, u_2 and also an edge from u', v' to u_2, u_3. As these two edges are identical, the notation can be chosen such that $u_1'u_2$ is the unique edge from u', v' to u_1, u_2, u_3. We now replace in P the two edges $u_1u_2, u'v'$ by u_1u', u_2v'. The minimality of the number of k-cuts implies that G has a k-cut S' with u_1, u' on the left side and u_2, v' on the right side. In particular, the cuts S, S' cross, and with the notation of Lemma 3.1 (where S, S' play the role of S_1, S_2), the edge $u_1'u_2$ joins sets of Hamming distance 2 (because u_1', u_2 are separated by each of S, S'), a contradiction to Lemma 3.1.

7. Applications to regular graphs and an extremal result. Theorems 4.1 and

6.1 combine nicely (together with Proposition 5.1) for regular graphs:

THEOREM 7.1. *If G is a simple, k-regular, k-edge-connected graph, then the smallest number of edges in $\overline{G}$ whose addition to G results in a $(k+1)$-edge-connected simple graph equals $|V(G)| - m(\overline{G})$.*

Proof. If $\overline{G}$ has no perfect matching, then, by Theorem 6.1, the smallest set of edges in $\overline{G}$ whose addition to G increases the edge-connectivity has the same cardinality as the smallest set of edges in $\overline{G}$ whose addition to G increases the minimum degree. By Proposition 5.1, that number is $|V(G)| - m(\overline{G})$. On the other hand, if $\overline{G}$ has a perfect matching, then Theorem 4.1 implies that the perfect matching can be chosen such that its addition also increases the edge-connectivity. As the number of edges in a perfect matching is $|V(G)|/2 = |V(G)| - m(\overline{G})$, the proof is complete

Theorem 6.1 also implies immediately the first part of the following:

THEOREM 7.2. *If G is a simple graph with n vertices, then the smallest number of edges in $\overline{G}$ whose addition to G results in a $(k+1)$-edge-connected simple graph is at most $2n/3$. Equality holds if and only if $\overline{G}$ is the disjoint union of 3-cycles.*

Proof. If $\overline{G}$ has a perfect matching, then Theorem 4.1 gives the upper bound $n/2 < 2n/3$. If $\overline{G}$ has no perfect matching, then by Theorem 6.1 and Proposition 5.1, the upper bound $2n/3$ follows from the fact that this is the upper bound for the number of edges in a path system of 1-paths and 2-paths in the subgraph G_Δ induced by the vertices of maximum degree Δ. If G is not regular, then that upper bound can be replaced by $2(n-1)/3$. So, in order to prove the last part it only remains to prove the following: Every regular simple graph (except a graph with no edges) has a matching with more than $n/3$ edges unless the graph is the union of a collection of 3-cycles. This is trivial for 2-regular graphs and hence easy for $2k$-regular graphs. For k-regular graphs (where k is odd) it is a straightforward (though slightly tedious) application of Tutte's 1-factor theorem. ∎

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