THE CARATHEODORY CONSTRUCTION OF MEASURES

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Abstract. In this paper, I will cover the Caratheodory construction of measures.

1. Introduction

In the year 1901, Henri Lebesgue, a French mathematician, introduced the construction of measure, known as the Lebesgue measure, from an initial notion of the size of a very restricted class of subsets in $\mathbf{R}$, the intervals. Using this notion, he was able to define an outer measure on these sets and proceeded from there to obtain a measure. Even though the Lebesgue measure was shown to have many useful applications in measure and integration theory, it was restrictive by the fact that it could only assign measures to certain subsets of $\mathbf{R}$ in the Lebesgue σ-algebra.

In this paper, we will generalize the notion of an outer measure on a general set X and use this to obtain a measure. In particular, we will formally define what an outer measure on X is, and proceed from there to obtain a measure using Caratheodory's theorem. Following this, we will introduce what a premeasure is and show how a measure can be obtained using Caratheodory's extension theorem. Finally, we will finish the paper off by looking at measures on metric spaces. In addition to understanding this paper, we assume that the reader understands the basic concepts of measure theory(Lebesgue measure on the real line, properties of a measure, σ-algebra, algebra, etc...) and metric spaces.

2. Outer Measure

Definition 2.1. An outer measure on a set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies three conditions:

(1) $\mu^*(\emptyset) = 0$
(2) If A and B are subsets of X such that $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$
(3) If $A_j \subset X$ for each $j \in I$, where I is a countable index set, then $\mu^*(\bigcup_{j \in I} A_j) \leq \sum_{j \in I} \mu^*(A_j)$

Recall that $\mathcal{P}(X)$ is the power set of X, i.e., the collection of all subsets of X.

Parallel to constructing the outer measure for intervals, here is a way to construct outer measures for $S \subset X$. Let Ω be the family of subsets of X, such that $\emptyset \in \Omega$ and $X = \bigcup_{j \geq 1} X_j$, for some countable collection $X_j \in \Omega$. Define a function $\psi : \Omega \to [0, \infty]$ such that $\psi(\emptyset) = 0$. We define the outer measure of S in the following way

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Definition 2.2. The outer measure of any $S \subset X$ is defined by

$$\mu^*(S) = \inf\{\sum_{j\geq 1} \psi(E_j) : E_j \in \Omega,\ S \subset \bigcup_{j\geq 1} E_j\} \qquad (2.1)$$

Here $\{E_j\}$ is a countable cover of S by sets in Ω

Proposition 2.3. *If $\{A_j : j \geq 1\}$ is a countable family of subsets of A, where $A \subset X$, then*

$$\mu^*(\bigcup_j X_j) \leq \sum_j \mu^*(X_j) \qquad (2.2)$$

Proof. Fix $\epsilon > 0$. Each A_j has a countable cover $\{E_{j,k} : k \geq 1\}$, by elements of Ω, such that $\mu^*(A_j) \geq \sum_{k\geq 1} \psi(E_{jk}) - \frac{\epsilon}{2^j}$. Then $\bigcup_{j\geq 1} A_j \subset \bigcup_{j\geq 1} \bigcup_{k\geq 1} E_{j,k}$ so that

$$\mu^*(\bigcup_{j\geq 1} A_j) \leq \sum_{j\geq 1}\sum_{k\geq 1} \psi(E_{jk}) \leq \sum_{j\geq 1} \mu^*(E_j) + \sum_{j\geq 1} \frac{\epsilon}{2^j} = \sum_{j\geq 1} \mu^*(E_j) + \epsilon \qquad (2.3)$$

Since (2.3) holds for any arbitrary $\epsilon > 0$, it must hold for all $\epsilon > 0$. Letting $\epsilon \to 0$ gives the inequality (2.2). $\square$

Proposition 2.4. *Under the hypotheses given in Definition 2.1, μ^*, defined by* (2.1)*, is an outer measure.*

Proof. In order to show that μ^*, defined by (2.1), is an outer measure, all that needs to be done is to check the hypotheses given in Definition 2.1. (1) follows from $\psi(\emptyset) = 0$ and (3) follows from Proposition 2.3. All that remains to be shown is (2). This follows from the fact that, when $A \subset B$, any countable cover of B by elements of Ω is also a cover of A. Since μ^* satisfies all three hypotheses of Definition 2.1, it is indeed an outer measure. $\square$

Example 2.5. Let $E_j = I_j$, $\psi(E_j) = \ell(I_j)$ for $j \geq 1$, where I_j denote intervals and $\ell(I_j)$ denotes the length of the interval I_j, and S be a subset of an interval $[a, b]$ such that $a, b \in \mathbf{R}$. Then

$$\mu^*(S) = \inf\{\sum_{j\geq 1} \ell(I_j) : S \subset \bigcup_{j\geq 1} I_j\} \qquad (2.4)$$

is an outer measure by Proposition 2.4. In fact, it is the outer measure for intervals.

Example 2.6. For $\alpha \geq 0$, define

$$m^*_\alpha(E) = \lim_{\delta\to 0^+} \mathcal{H}^\delta_\alpha(E) \qquad (2.5)$$

$$\mathcal{H}^\delta_\alpha(E) = \inf\{\sum_{j\geq 1} \ell(\mathcal{Q}_j) : E \subset \bigcup_{j\geq 1} \mathcal{Q}_j,\quad \ell(\mathcal{Q}_j) \leq \delta\} \qquad (2.6)$$

where $\mathcal{Q}_j$ are cubes of length of side length $\ell(\mathcal{Q}_j)$ and $E \subset \mathbf{R^n}$ for $\delta > 0$. That is, $\mathcal{Q}_\ell$ is of the form $\mathcal{Q}_\ell(x) = \{y \in \mathbf{R^n} : |x_i - y_i| < \frac{\ell}{2}, i = 1, 2, ..., n\}$. Since $\mathcal{Q}_j = E_j$, $\ell(\mathcal{Q}_j) = \psi(E_j)$, and $S = E$, it follows by Proposition 2.4 that (2.5) is an outer measure.

Now that we have constructed an outer measure for subsets of X, we turn our attention to a specific collection of subsets of X. That is, the class of μ^*-measurable subsets of X, which we define next.

Definition 2.7. We say a set $A \subset X$ is μ^*-measurable provided

$$\mu^*(A) = \mu^*(Y \cap A) + \mu^*(Y \cap A^c) \qquad (2.7)$$

for all $Y \subset X$. Let $\mathcal{M}$ denote the class of μ^*-measurable subsets of X. Write $A \in \mathcal{M}$ if A is μ^*-measurable.

From $\mathcal{M}$, we will show that this specific collection of subsets of X is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a measure. These results are part of the Carathedory theorem, which are vital for constructing measures as we will see in the following section.

3. Caratheodory Theorem

Before the Caratheodory Theorem is proved, we first need to prove the two following propositions.

Proposition 3.1. *The class of $\mathcal{M}$ of μ^*-measurable subsets of X is an algebra, i.e., $\mathcal{M}$ is closed under complements and finite unions.*

Proof. Clearly, $\emptyset \in \mathcal{M}$. If $A \in \mathcal{M}$, then, for all $Y \subset X$, $Y \cap A^c = Y \setminus A$ and $Y \setminus A^c = Y \cap A$, so $\mathcal{M}$ is closed under complements.

Next, we want to show that (2.7) holds when $A = A_1 \cup A_2 \cup ... \cup A_N$ for all integers N such that $N \geq 2$. That is, $\mathcal{M}$ is closed under unions. In order to do this, it is sufficient to consider the case when $N = 2$. Suppose that $A_1, A_2 \in \mathcal{M}$. We want to show that (2.7) holds with $A = A_1 \cup A_2$. Since μ^* is subadditive, it follows that

$$\mu^*(Y) \leq \mu^*(Y \cap (A_1 \cup A_2)) + \mu^*(Y \cap (A_1 \cup A_2)^c) \qquad (3.1)$$

for all $Y \subset X$. All that remains to be shown is

$$\mu^*(Y) \geq \mu^*(Y \cap (A_1 \cup A_2)) + \mu^*(Y \cap (A_1 \cup A_2)^c) \qquad (3.2)$$

for all $Y \subset X$. Note that $A_1 \cup A_2 = A_1 \cup (A_2 \cap A_1^c)$ is a disjoint union. With that being stated, it follows from this that the right hand side of (3.2) is

$$\begin{aligned}
&\leq \mu^*(Y \cap A_1) + \mu^*(Y \cap A_2 \cap A_1^c) + \mu^*(Y \cap A_1^c \cap A_2^c) \\
&= \mu^*(Y \cap A_1) + \mu^*(Y \cap A_1^c) = \mu^*(Y)
\end{aligned}$$

where the last two identities use $A_2 \in \mathcal{M}$ and $A_1 \in \mathcal{M}$, respectively. This establishes (3.2) for all $Y \subset X$. Since (3.1) and (3.2) hold for all $Y \subset X$, it is established that $A_1 \cup A_2 \in \mathcal{M}$. Furthermore, since $\mathcal{M}$ is closed under unions and complements, it follows by definition that $\mathcal{M}$ is an algebra. $\square$

Proposition 3.2. *Suppose that $A_j \in \mathcal{M}$ are disjoint for $j \geq 1$. Then*

$$\mu^*(\bigcup_{j \geq 1} A_j) = \sum_{j \geq 1} \mu^*(A_j) \qquad (3.3)$$

Proof. In order to prove (3.3), it is necessary to show that

$$\mu^*(\bigcup_{j=1}^{N} A_j) = \sum_{j=1}^{N} \mu^*(A_j) \qquad (3.4)$$

for all integers $N \geq 2$. This is done by using the regular form of mathematical induction on N. The base case $N = 2$ is considered in the basis.

Basis:($N = 2$): Suppose that $A_1, A_2 \in \mathcal{M}$ disjoint. Set $Y = A_1 \cup A_2$ and $A_1 = A$. Then

$$\mu^*(A_1 \cup A_2) = \mu^*(A_1 \cap (A_1 \cup A_2)) + \mu^*((A_1 \cup A_2) \cap A_1^c) = \mu^*(A_1) + \mu^*(A_2)$$

as required in this case.
Inductive Claim: Suppose that k is an arbitrary integer such that $k \geq 2$. It is sufficient to use the inductive hypothesis to prove the inductive claim.
Inductive Hypothesis: Suppose that $A_j \in \mathcal{M}$ disjoint for $j \geq 1$. Then

$$\mu^*(\bigcup_{j=1}^{k} A_j) = \sum_{j=1}^{k} \mu^*(A_j)$$

Inductive Claim: Suppose that $A_j \in \mathcal{M}$ disjoint for $j \geq 1$. Then

$$\mu^*(\bigcup_{j=1}^{k+1} A_j) = \sum_{j=1}^{k+1} \mu^*(A_j)$$

To prove the inductive claim, start by assuming that $A_1, A_2, ..., A_{k+1} \in \mathcal{M}$ are disjoint. Then

$$\mu^*(\bigcup_{j=1}^{k+1} A_j) = \mu^* \underbrace{(\bigcup_{j=1}^{k} A_j \cup A_{k+1})}_{\text{disjoint}} = \mu^*(\bigcup_{j=1}^{k} A_j) + \mu^*(A_{k+1}) \qquad (3.5)$$

By inductive hypothesis, it follows that

$$(3.5) = \sum_{j=1}^{k} \mu^*(A_j) + \mu^*(A_{k+1}) = \sum_{j=1}^{k+1} \mu^*(A_j)$$

which establishes the inductive claim. By the base case, the inductive hypothesis, and the inductive claim, (3.4) is established for all $N \geq 2$. Taking $N \to \infty$ and using monotonicity of μ^* yields

$$\mu^*(\bigcup_{j\geq 1} A_j) \geq \sum_{j \geq 1} \mu^*(A_j)$$

and since the reverse inequality follows by Proposition 2.3, we have (3.3) for $A_j \in \mathcal{M}$ disjoint as desired. □

Theorem 3.3. *(Caratheodory Theorem) If μ^* is an outer measure on X, then the class $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a measure.*

Proof. By Proposition 3.1, it is known that $\mathcal{M}$ is closed under complements. That is, if $A \subseteq X$ then $X \setminus A \in \mathcal{M}$. This implies $X \in \mathcal{M}$ and $\emptyset \in \mathcal{M}$. By Proposition 3.2, we have additivity. All that remains to be shown is if $A_j \in \mathcal{M}$ for $j \geq 1$ is a countable disjoint family, then $A = \bigcup_{j \geq 1} A_j$ is μ^*-measurable.
Let $B_n = \bigcup_{j \leq n} A_j$. The measurability of A_n implies

$$\mu^*(Y \cap B_n) = \mu^*(Y \cap B_n \cap A_n) + \mu^*(Y \cap B_n \cap A_n^c) = \mu^*(Y \cap A_n) + \mu^*(Y \cap B_{n-1})$$

for all $Y \subset X$. Furthermore,

$$\mu^*(Y \cap A_n) + \mu^*(Y \cap B_{n-1}) = \mu^*(Y \cap A_n) + \mu^*(\underbrace{Y \cap \bigcup_{j \leq n-1} A_j}_{\text{disjoint}})$$

$$= \mu^*(Y \cap A_n) + \mu^*(Y \cap A_{n-1}) + ... + \mu^*(Y \cap A_1) = \sum_{j=1}^{n} \mu^*(Y \cap A_j)$$

Hence, it is established that

$$\mu^*(Y \cap B_n) = \sum_{j=1}^{n} \mu^*(Y \cap A_j) \qquad (3.6)$$

Since $\mathcal{M}$ is known to be an algebra by Proposition 3.1, $B_n \in \mathcal{M}$ for all $n \in \mathbf{N}$ and, using (3.6), we have

$$\mu^*(Y) = \mu^*(Y \cap B_n) + \mu^*(Y \cap B_n^c) \geq \sum_{j=1}^{n} \mu^*(Y \cap A_j) + \mu^*(Y \cap A^c) \qquad (3.7)$$

the last inequality holding because $B_n \subset A$. Now, taking $n \to \infty$ gives

$$\mu^*(Y) \geq \sum_{j \geq 1} \mu^*(Y \cap A_j) + \mu^*(Y \cap A^c) \geq \mu^*(\bigcup_{j \geq 1}(Y \cap A_j)) + \mu^*(Y \cap A^c) \qquad (3.8)$$

Since $\bigcup_{j \geq 1}(Y \cap A_j) = Y \cap \bigcup_{j \geq 1} A_j = Y \cap A$, it follows that

$$(3.8) = \mu^*(Y \cap A) + \mu^*(Y \cap A^c) \geq \mu^*(Y)$$

Therefore, $A \in \mathcal{M}$ as desired which establishes that $\mathcal{M}$ is closed under countable unions. This not only establishes that $\mathcal{M}$ is a σ-algebra, but it also establishes that the restriction of μ^* to $\mathcal{M}$ is a measure. This completes the proof of the Caratheodory Theorem. □

Example 3.4. Let $\mathcal{H}_\alpha \subset \mathcal{P}(\mathbf{R^n})$ be the collection of all sets A in $\mathbf{R^n}$ such that

$$m_\alpha^*(Y) = m_\alpha^*(Y \cap A) + m_\alpha^*(Y \cap A^c)$$

for all $Y \subset \mathbf{R^n}$, where m_α^* is defined by (2.5). By Theorem 3.3, $\mathcal{H}_\alpha$ is a σ-algebra and the restriction of $m_\alpha = m_\alpha^* \restriction_{\mathcal{H}_\alpha}$ is a measure.

4. Premeasures and Caratheodory's Extension Theorem

While the construction for (2.1) is very general, extra structure is needed to relate $\mu^*(S)$ to $\psi(S)$ when $S \in \Omega$. The following is a convenient setting.

Definition 4.1. Let $\mathcal{A}$ be an algebra of subsets of X. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a premeasure if it satisfies the following two conditions:

1. $\mu_0(\emptyset) = 0$
2. If $S_j \in \mathcal{A}$ countable, disjoint, $\bigcup_j S_j = S \in \mathcal{A}$, then $\mu_0(S) = \sum_j \mu_0(S_j)$

Example 4.2. Suppose $X = [a, b]$ for $a, b \in \mathbf{R}$, and let $\mathcal{A}$ consist of finite unions of intervals(open, closed, or half-open) in $[a, b]$. If $S = \bigcup_{k=1}^{N} J_k$ is a disjoint union of intervals, take $\mu_0(S) = \sum_{k=1}^{N} \ell(J_k)$, so μ_0 is the restriction of Lebesgue measure to $\mathcal{A}$.

Remarks 4.3. If μ_0 is a premeasure on $\mathcal{A}$, it induces an outer measure on X via the construction (2.1), i.e.,

$$\mu^*(E) = \inf\{\sum_{j\geq 1} \mu_0(A_j) : A_j \in \mathcal{A},\ E \subset \bigcup_{j\geq 1} A_j\} \qquad (4.1)$$

Proposition 4.4. *If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by* (4.1)*, then*

$$S \in \mathcal{A} \implies \mu^*(S) = \mu_0(S) \qquad (4.2)$$

and every set in $\mathcal{A}$ is μ^*-measurable

Proof. To prove (4.2), first note that $\mu^*(S) \leq \mu_0(S)$ for $S \in \mathcal{A}$ since S covers itself. Suppose that $S \in \mathcal{A}$ and $S \subset \bigcup_{j\geq 1} A_j$, where $A_j \in \mathcal{A}$. Then let $B_n = S \cap \left(A_n \setminus \bigcup_{j<n} A_j\right)$, so that $\bigcup_{n\geq 1} B_n = S$, and therefore, $\{B_n : n \geq 1\}$ is a disjoint family of members of $\mathcal{A}$ that covers S. By the definition of premeasure,

$$\mu_0(S) = \sum_{j\geq 1} \mu_0(B_j) \leq \sum_{j\geq 1} \mu_0(A_j) \qquad (4.3)$$

It follows that $\mu_0(S) \leq \mu^*(S)$. Since $\mu_0(S) \geq \mu^*(S)$ and $\mu_0(S) \leq \mu^*(S)$, we have (4.2).

All that remains to be proven is that every set in $\mathcal{A}$ is μ^*-measurable. To prove this, if $Y \subset X$ and $\epsilon > 0$, there is a sequence $\{B_j : j \geq 1\} \subset \mathcal{A}$ with $Y \subset \bigcup_{j\geq 1} B_j$ and $\sum_{j\geq 1} \mu_0(B_j) \leq \mu^*(Y) + \epsilon$. Since μ_0 is additive on $\mathcal{A}$,

$$\mu^*(Y) + \epsilon \geq \sum_{j\geq 1} \mu_0(B_j) = \sum_{j\geq 1} \mu_0((A \cap B_j) \cup (A^c \cap B_j))$$

$$= \sum_{j\geq 1} \mu_0(A \cap B_j) + \sum_{j\geq 1} \mu_0(A^c \cap B_j) \geq \mu^*(Y \cap A) + \mu^*(Y \cap A^c)$$

the latter inequality holding, since $\{B_j \cap A : j \geq 1\}$ is a cover of $Y \cap A$ by elements of $\mathcal{A}$. Taking $\epsilon \to 0$, we obtain (2.5), so any $A \in \mathcal{A}$ is μ^*-measurable. $\square$

When Proposition 4.4. is combine with Theorem 3.3, we obtain an extension of premeasure μ_0 to a measure. The following theorem, known as Caratheodory's extension theorem, establishes this result.

Theorem 4.5. *Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M} = \sigma(\mathcal{A})$ the σ-algebra generated by $\mathcal{A}$. Then there exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0, namely $\mu^* \restriction_{\mathcal{M}}$ where μ^* is given by* (4.3)

Proof. By Theorem 3.3, $\mu^* \restriction_{\mathcal{M}}$ is a measure, which we write as $\mu = \mu^* \restriction_{\mathcal{M}}$. Since the class of μ^*-measurable subsets of X is a σ-algebra containing $\mathcal{A}$, this measure restricted to $\mathcal{A}$ gives $\mu_0 = \mu \restriction_{\mathcal{A}}$ by Proposition 4.4. It follows from this that (4.1) holds for every set in $\mathcal{A}$. This completes the proof of this theorem. $\square$

Example 4.6. If Theorem 4.5 is applied to Example 4.2, then it yields the fact that finite unions of intervals in $I = [a, b]$ are Lebesgue measurable. Hence, so are sets in the σ-algebra $\sigma(\mathcal{A})$ generated by this algebra of subsets of I, In particular, all open sets in I are measurable, since they are countable unions of open intervals.

Next, we examine the extent to which the extension μ of μ_0 to $\sigma(\mathcal{A})$ is unique.

Proposition 4.7. *Let $\mathcal{A}$ be an algebra of subsets of X, generating the σ-algebra $\mathcal{M} = \sigma(\mathcal{A})$. Let μ_0 be a premeasure on $\mathcal{A}$, and let μ be the measure on $\mathcal{M}$ given by Theorem 4.5, extending μ_0. If v is another measure on $\mathcal{M}$ which agrees with μ_0, then for all $S \in \mathcal{M}$, $v(S) \leq \mu(S)$, and*

$$\mu(S) < \infty \implies \mu(S) = v(S) \qquad (4.4)$$

Furthermore, if there is a countable family A_j such that

$$X = \bigcup_{j \geq 1} A_j \quad A_j \in \mathcal{M}, \quad \mu(A_j) < \infty \qquad (4.5)$$

then $\mu(S) = v(S)$ for all $S \in \mathcal{M}$.

Proof. Suppose μ is a measure on $\mathcal{M}$ given by Theorem 4.5, and v is also a measure on $\mathcal{M} = \sigma(\mathcal{A})$, which also agrees with μ_0 on $\sigma(\mathcal{A})$. If $S \in \mathcal{M}$ and $S \subset \bigcup_{j \geq 1} A_j$, $A_j \in \mathcal{A}$, then

$$v(S) \leq \sum_{j \geq 1} v(A_j) = \sum_{j \geq 1} \mu_0(A_j) \qquad (4.6)$$

so, by the construction (4.1) of μ^*,

$$v(S) \leq \mu(S), \quad \textit{for all } S \in \sigma(\mathcal{A}) = \mathcal{M} \qquad (4.7)$$

To establish the reverse inequality, the following claim is needed.
Claim: If $B_j \in \mathcal{A}$ such that $B_j \subset B_{j+1}$ for each $j \in \mathbf{N}$, then $v(B) = u(B)$ where $B = \bigcup_{j \geq 1} B_j$.

Proof. Suppose $B_j \in \mathcal{A}$ as described above for each $j \in \mathbf{N}$. Then $B_j \nearrow B$ which implies that $v(B) = \lim_{n \to \infty} v(B_n) = \lim_{n \to \infty} u(B_n) = u(B)$. Therefore, $v(B) = u(B)$ as desired to complete the proof of this claim. □

Suppose now that $S \in \mathcal{M}$ and $\mu(S) < \infty$. Fix $\epsilon > 0$. Then we can choose the cover $\{A_j : j \geq 1\}$ in (4.1) such that $\sum_{j \geq 1} \mu(A_j) < \mu(S) + \epsilon$. Set $B_n = \bigcup_{j \leq n} A_j$ and $B = \bigcup_{j \geq 1} A_j$. Since each $A_j \in \mathcal{A}$, $B_n \in \mathcal{A}$ for each $n \geq 1$ so that $B_n \nearrow B$. Hence,

$$\mu(\bigcup_{j \leq n} A_j) = \mu(B_n) \leq \lim_{n \to \infty} \mu(B_n) = \mu(B) = \mu(\bigcup_{j \geq 1} A_j) \leq \sum_{j \geq 1} \mu(A_j) < \mu(S) + \epsilon$$

which gives $\mu(B) < \mu(S) + \epsilon$. It follows by this inequality and (4.7) that $v(B \setminus S) < \epsilon$. Meanwhile, by the claim, $v(B) = \mu(B)$, so

$$\mu(S) \leq \mu(B) = v(B) = v((S \cup (B \setminus S)) = v(S) + v(B \setminus S) \leq v(S) + \epsilon$$

Since $\epsilon > 0$ was arbitrary, the last inequality holds for all $\epsilon > 0$. Taking $\epsilon \to 0$ gives $u(S) \leq v(S)$. Since $u(S) \leq v(S)$ and $u(S) \geq v(S)$, (4.4) is established. All that remains to be shown is (4.5).

If (4.5) holds, we can assume the A_j are disjoint. If $S \in \mathcal{M}$, we have the disjoint union $S = \bigcup_{j \geq 1} S_j$, $S_j = S \cap A_j$. By (4.4), $\mu(S_j) = v(S_j)$, and then $\mu(S) = v(S)$ follows by countable additivity. This completes the proof. □

Considering that a subset of $\mathbf{R}$, in the Lesbesgue σ-algebra, is said to be Lebesgue measurable provided that the inner measure for intervals coincides with the outer measure for intervals, we can look at measurability for subsets of X in $\mathcal{M}$ in an identical way. The following proposition establishes this notion. However, before this proposition can be established, the following lemma is needed.

Lemma 4.8. *Let μ^* arise from a premeasure on an algebra $\mathcal{A}$. Let $\mathcal{A}_\sigma$ consist of countable unions of sets in $\mathcal{A}$. If $S \subset X$, then*

$$\mu^*(S) = \inf\{\mu^*(E) : E \in \mathcal{A}_\sigma, S \subset E\} \qquad (4.8)$$

Proof. Suppose μ^* arises from a premeasure on an algebra $\mathcal{A}$. If we let A_σ consist of countable unions of sets in $\mathcal{A}$, then $A_\sigma \subset \mathcal{A}$. Note that

$$\mu^*(S) = \inf\{\sum_{j\geq 1} \mu_0(A_j) : A_j \in \mathcal{A}, S \subset \bigcup_{j\geq 1} A_j\} \qquad (4.9)$$

If we define $\mathcal{C} = \{E \in A_\sigma : S \subset E\}$, then by (4.9) there exists the cover $\{A_j : j \geq 1\}$ so that $\sum_{j\geq 1} \mu_0(A_j) \leq \mu_0(E)$ for all $E \in \mathcal{C}$. In fact, $S = \bigcup_{j\geq 1} A_j$ so $\bigcup_{j\geq 1} A_j \in \mathcal{C}$ Since $\mathcal{C} \subset \mathcal{A}$, it follows by Proposition 4.4 that

$$\inf\{\sum_{j\geq 1} \mu_0(A_j) : A_j \in \mathcal{A}, S \subset \bigcup_{j\geq 1} A_j\} = \inf\{\mu^*(E) : E \in \mathcal{C}\} = \inf\{\mu^*(E) : E \in \mathcal{A}_\sigma, S \subset E\}$$

Therefore, we have arrived at the right hand side of (4.8) as desired. This completes the proof of this lemma. $\square$

Proposition 4.9. *Suppose an algebra $\mathcal{A}$ of subsets of X is given and a premeasure μ_0 on $\mathcal{A}$, with associated outer measure μ^*, defined by* (4.1). *Assume $Z \subset X$ is μ^*-measurable and $\mu^*(Z) < \infty$. Then a set $S \subset Z$ is μ^*-measurable if and only if*

$$\mu^*(S) + \mu^*(Z \setminus S) = \mu^*(Z) \qquad (4.10)$$

Proof. To begin the proof, suppose that (4.10) holds for any $Z \subset X$ such that Z is μ^*-measurable, $\mu^*(Z) < \infty$, and $S \subset Z$. We need to show that S is μ^*-measurable. That is, we need to show that

$$\mu^*(Y) = \mu^*(Y \cap S) + \mu^*(Y \cap S^c) \qquad (4.11)$$

for all $Y \subset X$. It is enough to show that the left hand side of (4.11) is greater than or equal to the right hand side of (4.11), since we know that the reverse inequality will follow from subadditivity of outer measures. In order to show this, we first write $Y = Y_0 \cup Y_1$, a disjoint union, where $Y_0 = Y \cap Z$ and $Y_1 = Y \cap Z^c$. Then using the measurability of Z, we have

$$\mu^*(Y) = \mu^*(Y \cap Z) + \mu^*(Y \cap Z^c) = \mu^*(Y_0) + \mu^*(Y_1) \qquad (4.12)$$

and

$$\mu^*(Y \cap S^c) = \mu^*(Y \cap S^c \cap Z) + \mu^*(Y \cap S^c \cap Z^c) = \mu^*(Y_0 \cap S^c) + \mu^*(Y_1) \qquad (4.13)$$

Furthermore, by (4.12) and (4.13), it follows that

$$\mu^*(Y \cap S) = \mu^*(Y) - \mu^*(Y \cap S^c) = \mu^*(Y_0 \cap S) \qquad (4.14)$$

Consequently, it suffices to prove for $Y = Y_0$, i.e., for $Y \subset Z$. In order to do this, fix $\epsilon > 0$. By Lemma 4.8, there exists $\hat{A} \in \mathcal{A}_\sigma$ such that $\hat{A} \supset Y$ and $\mu^*(\hat{A}) \leq \mu^*(Y) + \epsilon$. Set $A = \hat{A} \cap Z$. Then $A \in \mathcal{M}$($\mathcal{M}$ is a σ-algebra of μ^*-measurable sets), $A \supset Y$ (if $Y \subset Z$), and $\mu^*(A) \leq \mu^*(Y) + \epsilon$.
Claim: $\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c)$

Proof. Since $A = \hat{A} \cap Z \subset Z$ and $\mu^*(Z) < \infty$, it follows by measurability of A that

$$\mu^*(Z) = \mu^*(A) + \mu^*(Z \cap A^c) \qquad (4.15)$$

Now, the left side of (4.15) is equal to $\mu^*(S) + \mu^*(Z \setminus S)$, by hypothesis. By subaddivitiy of outer measure,

$$\begin{aligned}\mu^*(A) + \mu^*(Z \cap A^c) &= \mu^*((A \cap S) \cup (A \cap S^c)) + \mu^*(((Z \setminus A) \cap S) \cup ((Z \setminus A) \cap S^c)) \\ &\leq \mu^*(A \cap S) + \mu^*(A \cap S^c) + \mu^*((Z \setminus A) \cap S) + \mu^*((Z \setminus A) \cap S^c) \\ &= \mu^*(S \cap A) + \mu^*((Z \setminus S) \cap A) + \mu^*(S \setminus A) + \mu^*((Z \setminus S) \setminus A) \\ &= \mu^*(S) + \mu^*(Z \setminus S)\end{aligned}$$

the last identity following by grouping together the odd terms and the even terms on the second equality line and using measurability of A. Since this identity is equal to the left side of (4.15), the inequality in the first line must be equality. That inequality arose from the sum of two inequalities, and so both of them must be equalities. One of them is

$$\mu^*(A) + \mu^*(Z \cap A^c) = \mu^*(A \cap S) + \mu^*(A \cap S^c) + \mu^*(Z \cap A^c)$$

which establishes the above claim. □

By the claim, it follows that

$$\mu^*(Y) + \epsilon \geq \mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c) \geq \mu^*(Y \cap S) + \mu^*(Y \cap S^c) \qquad (4.17)$$

Since ϵ was arbitrary, (4.17) holds for all $\epsilon > 0$. In the limit $\epsilon \to 0$, we get

$$\mu^*(Y) \geq \mu^*(Y \cap S) + \mu^*(Y \cap S^c) \qquad (4.18)$$

Therefore, we have established (4.11) as required.

Conversely, suppose that $S \subset Z$ is measurable. We want to show that (4.10) holds. By measurability, (4.11) holds for all $Y \subset X$. Take $Y = Z$ and it now follows that

$$\mu^*(Z) = \mu^*(Z \cap S) + \mu^*(Z \cap S^c) = \mu^*(Z \cap S) + \mu^*(Z \setminus S)$$

which establishes (4.10). This completes the proof of this proposition. □

To finish this paper off, we will next describe an important class of outer measures on metric spaces, for which all open sets and closed sets can be shown to be μ^*-measurable. Before we do, we first state the definition of a metric outer measure. All of this is done in the following section.

5. Measures on Metric Spaces

Definition 5.1. Let X be a metric space, with distance function $d(x, y)$. An outer measure μ^* is called a metric outer measure provided

$$\rho(S_1, S_2) = \inf\{d(x_1, x_2) : x_j \in S_j\} > 0$$

$$\implies \mu^*(S_1 \cap S_2) = \mu^*(S_1) + \mu^*(S_2) \qquad (5.1)$$

for $j \in \{1, 2\}$

Proposition 5.2. *If μ^* is a metric outer measure on a metric space X, then every closed subset of X is μ^*-measurable.*

Proof. Suppose μ^* is a metric outer measure on a metric space X. We must show that if $F \subset X$ is closed and $Y \subset X$ satisfies $\mu^*(Y) < \infty$, then

$$\mu^*(Y) \geq \mu^*(Y \cap F) + \mu^*(Y \setminus F) \qquad (5.2)$$

In order to do this, let $B_n = \{x \in Y \setminus F : \rho(x, F) \geq \frac{1}{n}\}$. Note that $B_n \subset B_{n+1}$ for all $n \geq 1$, and so $B_n \nearrow Y \setminus F$. Also $\rho(B_n, F) \geq \frac{1}{n}$, so, by (5.1),

$$\mu^*(Y \cap F) + \mu^*(B_n) = \mu^*((Y \cap F) \cup B_n) \leq \mu^*(Y)$$

Hence, it will suffice to show that $\mu^*(B_n) \to \mu^*(Y \setminus F)$ as $n \to \infty$. We do this by first letting $C_n = B_{n+1} \setminus B_n$. Note that if $|j - k| \geq 2$, then $\rho(C_j, C_k) > 0$. Hence, for any N, one obtains inductively (using (5.1)) that

$$\sum_{j=1}^{N} \mu^*(C_{2j}) = \mu^*(\bigcup_{j=1}^{N} C_{2j}) \leq \mu^*(Y)$$

$$\sum_{j=1}^{N} \mu^*(C_{2j+1}) = \mu^*(\bigcup_{j=1}^{N} C_{2j+1}) \leq \mu^*(Y)$$

and consequently $\sum_{j \geq 1} \mu^*(C_j) < \infty$. Now countable subadditivity implies

$$\mu(Y \setminus F) = \mu(B_n \cup \bigcup_{j \geq 1} C_j) \leq \mu^*(B_n) + \sum_{j \geq n} \mu^*(C_j)$$

so as $n \to \infty$, the last sum tends to zero, and we obtain

$$\mu^*(Y \setminus F) \leq \lim_{n \to \infty} \sup \mu^*(B_n) \leq \lim_{n \to \infty} \inf \mu^*(B_n) \leq \mu^*(Y \setminus F)$$

the last inequality by monotonicity. This implies (5.2). On the other hand, the reverse side of the inequality follows from subadditiity of outer measures. Since we have both sides of inequalities, it follows that every closed subset of X is μ^*-measurable. □

Example 5.3. Lebesgue outer measure is a metric outer measure on $I = [a, b]$.

6. Conclusion:

In this paper, we have looked at different ways of obtaining measures using Caratheodory's theorem and Caratheodory's extension theorem. In particular, we have started the paper off by defining formally what an outer measure is, and then proceeded from there to obtain a measure for the class of μ^*-measurable subsets of X using Caratheodory's theorem. From there, we noted that this way of constructing measures was too general and lacked structure. As a result, we introduced pre-measures and then related them to outer measures using Caratheodory's extension theorem from which we were able to obtain measures. Despite the fact that this paper mainly looked at the general construction of measures, there were a couple measures mentioned in this paper. In particular, the Hausdorff measure and the Lebesgue measure. Even though though only a few measures were mentioned in this paper, the Caratheodory construction of measures extends to a wide array of measures which are used in many various fields of mathematics.

References

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