Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

January 2019

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

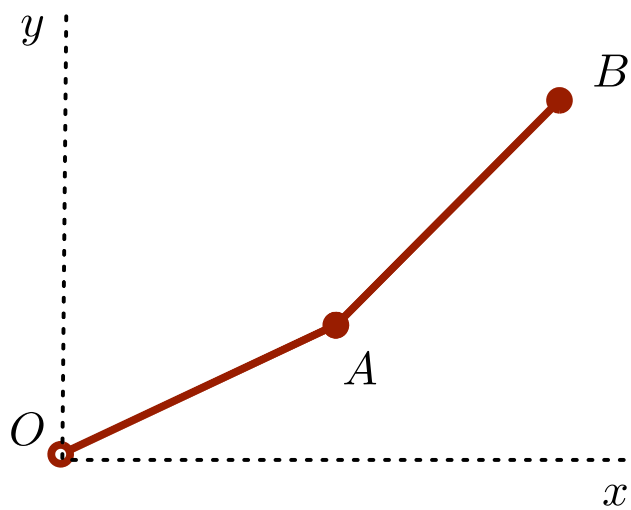
Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

Linked rods in a plane

Two massless rods lie in the $x - y$ plane (neglect gravity). Each rod has length ℓ, and each rod has mass M at one end (points A and B below). The massless end of the first rod is hinged to a fixed pivot O, while its massive end is hinged at the point A to massless end of the second rod as shown below. The hinges at O and A are constructed so that the second rod can swing past past the first without obstruction.



(a) (7 points) Write down the Lagrangian of the system. Determine and interpret any integrals of motion.

(b) At time $t = 0$ both rods are aligned along the x axis. The mass at A moves with velocity v_0, while the mass at B moves with velocity $2v_0 + \Delta v$, with $\Delta v \ll v_0$.

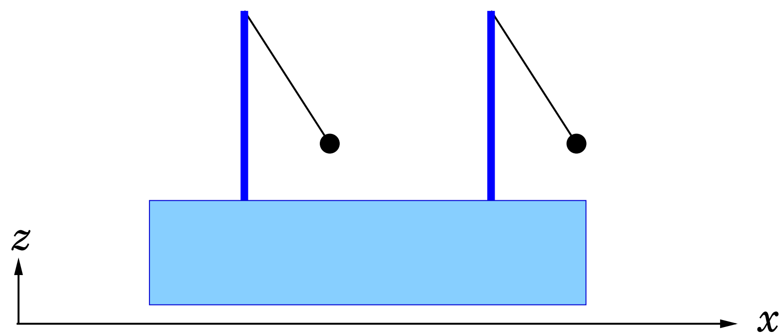
  i (3 points) Qualitatively describe the subsequent motion of the system.

  ii (10 points) Determine the angles of the masses relative to the x axis as functions of time.

Classical Mechanics 2

Coupled pendulums

Two identical pendulums each have a point mass m suspended (in a uniform gravitational field) on a massless rigid rod of length ℓ hanging from a frictionless pivot. The pendulums are constrained to swing in the $x - z$ plane, and are mounted on a block of mass $M = 2m$ which is free to slide, without friction, in the x-direction.



(a) (10 points) Construct a Lagrangian of the system assuming small oscillations: (i) find the resulting normal modes and frequencies, (ii) interpret any zero modes, and (iii) qualitatively sketch the oscillation pattern for each mode.

(b) (6 points) Now the base (the block with mass $2m$) is pushed in the x-direction by an external force $F(t) = P_0\delta(t)$. Determine thee subsequent motion of the system, assuming that the impulse P_0 is so small that the subsequent oscillations may be treated in a harmonic approximation.

(c) (4 points) Now consider a time dependent force pushing the base in the x-direction:

$$F(t) \equiv F_0 e^{-|t|/\tau}\,, \tag{33}$$

where $\tau > 0$ is some characteristic time scale. Assuming that the system is at rest for $t \to -\infty$, determine the total work done on the system by the time dependent force as $t \to +\infty$.

Classical Mechanics 3

Scattering between two particles

An incoming particle (particle A) of mass m and velocity $\vec{v}$ and collides with a free particle (particle B) also of mass m, which is initially at rest in the laboratory frame. The two particles interact with a repulsive $1/r^2$ potential

$$U(r) = \frac{h}{r^2}\,, \quad h > 0\,,$$

where $\vec{r} = \vec{r}_A - \vec{r}_B$ is the relative coordinate of the two particles.

Find the angular distribution (the differential cross-section) of the deflected particles, $d\sigma(\theta)/d\theta$, where θ is the angle between $\vec{v}$ and particle A's velocity after the collision. Assume that the process is repeated many times with random impact parameter b, distributed so that the number of incident particles per transverse area is constant. In steps:

(a) (3 points) Write down the Lagrangian of the two particles. Introduce the center-of-mass and relative coordinates of the system, and show that the equation of motion of the relative coordinate $\vec{r}$ is that of a single (effective) particle in the potential $U(r)$. Express the energy E and angular momentum ℓ of the relative motion with respect to the origin of $\vec{r}$ in terms of the "laboratory" frame quantities m, v and b.

(b) (6 points) Find the trajectory $r(\phi)$, where the ϕ is the polar angle, of the effective particle with energy E and angular momentum ℓ moving in the potential $U(r)$. Follow the convention that $\phi = 0$ corresponds to the distance of closest approach as shown below.

(c) (6 points) Find the differential cross-section $d\sigma(\chi)/d\chi$ in the center-of-mass frame, where χ is the scattering angle in this frame.

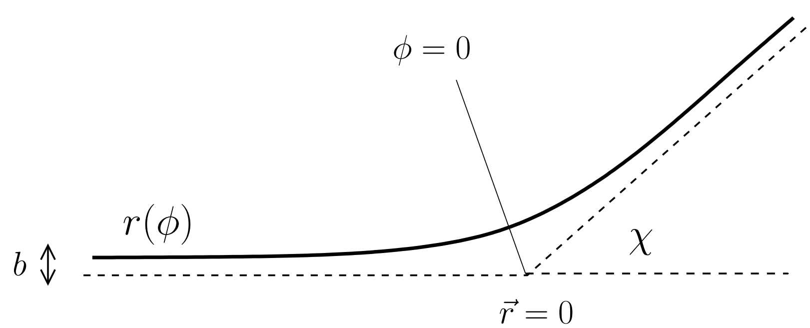


Figure 1: The scattering angle χ in the center-of-mass frame.

(d) (5 points) Convert $d\sigma(\chi)/d\chi$ into the initial "laboratory" frame to find $d\sigma(\theta)/d\theta$.

Electromagnetism 1

Radiation from an undulator

Consider an ultra-relativistic electron of charge q, mass m, and velocity v, propagating in the z direction with $\gamma \equiv 1/\sqrt{1 - v^2/c^2} \gg 1$. In the lab frame $\mathcal{F}_0$, the electron experiences a weak external sinusoidal magnetic field directed along the x axis:

$$\boldsymbol{B}(z) = B_0 \sin(kz)\, \hat{\boldsymbol{x}} \,. \tag{1}$$

The electron is only slightly deflected from its straight line motion as it propagates in the magnetic field.

In this problem we will compute the average energy radiated per unit time by the undulating electron in two ways. In parts (a), (b), (c) we will work in a frame $\mathcal{F}$ moving (with the electron) at constant speed v in the z direction relative to the lab. In part (d) we will work directly in the lab frame $\mathcal{F}_0$.

(a) (4 points) Explicitly determine the external electromagnetic field in the moving frame $\mathcal{F}$ by making a Lorentz transformation. Compute the instantaneous Poynting vector in $\mathcal{F}$ (both magnitude and direction). Show that the transformed fields are equivalent to a plane wave. What is the wavelength and amplitude associated with these fields?

(b) (6 points) Determine the average energy radiated per unit time by the electron in frame $\mathcal{F}$.

(c) (4 points) Show that the energy radiated per unit time by an accelerating charged particle is invariant under Lorentz boosts in the z direction. What then is the average energy radiated per unit time in the lab frame $\mathcal{F}_0$ by the undulating electron?

Hint: Boost the radiated energy and momentum in a time interval Δt from the instaneous rest frame of the accelerating particle to a frame moving in the z direction.

(d) (6 points) The relativistic Larmor formula for the total energy W radiated per retarded time T is

$$\frac{dW}{dT} = \frac{2}{3}\left(\frac{q^2}{4\pi c^3}\right)\left(\gamma^6 \boldsymbol{a}_\parallel^2 + \gamma^4 \boldsymbol{a}_\perp^2\right) \,. \tag{2}$$

Here $\boldsymbol{a}_\parallel$ is the acceleration of the electron parallel to its velocity, and $\boldsymbol{a}_\perp$ is the acceleration perpendicular to its velocity.

Working in the lab frame, use Eq. (2) with the appropriate kinematic approximations to determine the average energy lost per time by the relativistic electron. Compare the result to that of part (c).

Electromagnetism 2

A current loop and a sphere

A circular loop of radius R, lying flat in the xy-plane with center at the origin, carries a uniform current I. A sphere of radius $a \ll R$ and permeability μ is placed at a height $h \gg a$ on the z axis above the xy-plane.

a. (6 points) Determine the magnetic field $\vec{\mathbf{B}}$ inside and outside the sphere.

b. (4 points) Determine the force between the sphere and the ring. Give a qualitative explanation for the direction of the force for both paramagnetic ($\mu > \mu_0$) and diamagnetic ($\mu < \mu_0$) materials.

c. (5 points) Determine the angular distribution of the force per area on the surface of the sphere to leading order in a/R.

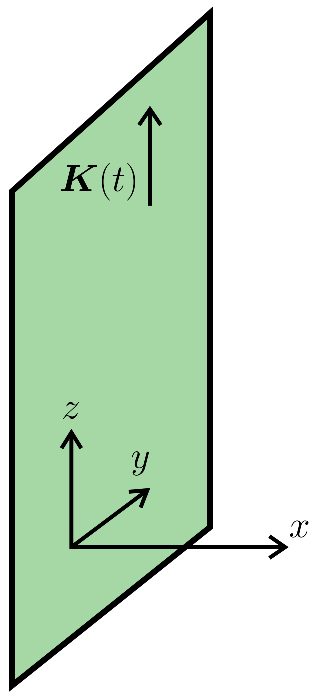
d. (3 points) Integrate the force per area in (c) to find the total force and compare the result to (b). Do they agree? Why or why not? Explain.

e. (2 points) Now place the sphere in the center of the ring at zero height, $h = 0$. Is this configuration stable or unstable configuration? Explain. Consider both paramagnetic and diamagnetic materials.

Electromagnetism 3

A current sheet in an ohmic medium

Consider an infinite sheet lying in the zy plane carrying surface current[1], $\boldsymbol{K} = K_0\, e^{-i\omega t}\, \hat{\boldsymbol{z}}$. The current is driven by an external source which sustains the current's amplitude and frequency.



(a) (4 points) First consider the this current-carrying sheet in vacuum. Determine the magnetic and electric fields to lowest (non-trivial) order in the frequency. Sketch the amplitude of the electric field as a function of x for both positive and negative values of x. Do your results for the electromagnetic fields hold everywhere in space? Explain.

(b) (8 points) Now place the same current sheet into an ohmic medium of conductivity σ with[2] $\sigma \gg \omega$. Determine the (real) electric and magnetic fields everywhere in space. Sketch the amplitude of the electric field as a function of x for both positive and negative values of x.

Show how your results for the fields follow from the Maxwell equations and their boundary conditions. Assume that[3] $\epsilon = \mu = 1$.

(c) (8 points) Determine the total energy dissipated per time by the induced electric fields in the ohmic medium. Show that it equals the work done per time by the external source maintaining the surface current.

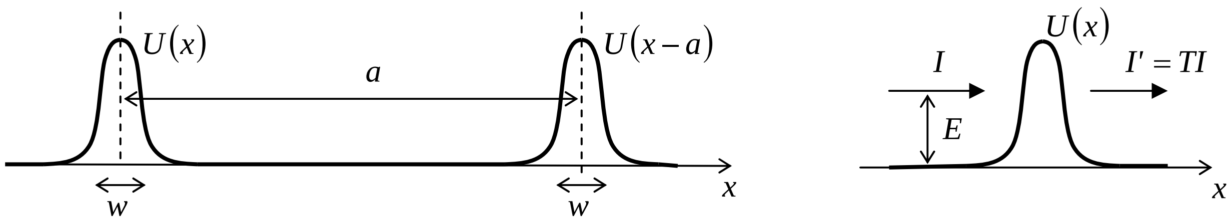
---

[1] The units of K_0 are amps/meter in SI units.
[2] In SI units this reads $\sigma/\epsilon_0 \gg \omega$.
[3] In SI units this reads $\epsilon = \epsilon_0$ and $\mu = \mu_0$.

Quantum Mechanics 1

Resonant tunneling and state metastability

Your goal in this problem is to study and relate two basic quantum properties of a one-dimensional non-relativistic particle of mass m, moving in the system of two similar, symmetric potential barriers $U(x)$ of width scale w, separated by a much larger distance $a >> w$ – see the left panel in the figure below. Each barrier has a tunneling transparency $T \leq 1$ (which is some smooth function of the particle's energy E), defined as shown on the right panel of the figure, where I and I' are the probability currents of the incident and transferred monochromatic de Broglie waves.



A (4 points). Calculate the (similarly defined) transparency T' of the two-barrier system for a monochromatic de Broglie wave incident from afar, as a function of a, T, and the particle's energy E.

B (2 points). Sketch the calculated transparency as a function of a, and prove that its largest value, reached at certain resonance values a_{res}, equals 1, and hence may be larger than T. Give a physical interpretation of this fact.

C (4 points). For the case of a very low barrier transparency, $T << 1$, and the distance a close to one of its resonance values a_{res}, calculate the energy width δE of the transparency resonance. (For certainty, use the standard "FWHM" definition of the width as the energy interval within that the transparency is larger than ½.)

D (7 points). Now consider the situation when the particle is initially placed between the barriers (again with a very low transparency $T << 1$), in its lowest-energy state, which is metastable due to the tunneling through the barriers. Calculate the law of the time decay of the probability to find the particle between the barriers, in the lowest nonvanishing approximation in T.

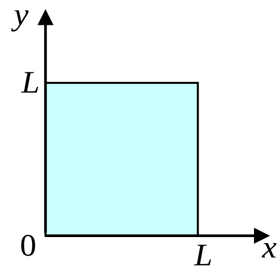
E (3 points). Compare the characteristic time τ of the metastable state's decay to the energy width δE calculated in Task C, and comment. Explain why such "energy-time uncertainty relation" is less general than the canonical uncertainty relations - such as those between Cartesian components of the generalized coordinates and the corresponding momenta.

Hint: Working on Task D, it may be helpful to represent the standing de Broglie wave describing the metastable state as a sum of two waves traveling in opposite directions.

Quantum Mechanics 2

Orbital motion perturbations

A spinless point particle of mass m is confined to a square-shape two-dimensional region of size $L \times L$:



A (2 points). Write down the stationary wave functions and the corresponding eigenenergies of the particle.

B (5 points). Now consider there is a "point defect" inside the region, whose potential can be modeled as

$$U = \lambda L^2 \delta(\vec{r} - \vec{r}_0),$$

where $\vec{r}_0 = \{x_0, y_0\}$ is the location of the defect. Treating the potential as a perturbation, calculate the first-order corrections to the energies of the ground state and the first excited state. Find out the locations of the defect (inside the square), at which the 1st excited state remains degenerate.

C (7 points). Now consider a moving "defect", which oscillates along the x-direction at the center of the square:

$$\vec{r}_0 = \left\{ \frac{L}{2} + l \sin \omega t, \frac{L}{2} \right\},$$

where $l \ll L$. Find out the selection rule for the particle excitation from the ground state to an arbitrary excited state. Calculate the time-dependent probability for the charge to be in the first excited state, for $\omega \ll \omega_{fi}$, where $\hbar\omega_{fi}$ is the energy difference between the initial and final states, using the time-dependent perturbation theory.

D (6 points). At $\omega \ll \omega_{fi}$, the time evolution of the system obeys the adiabatic theorem. Use this fact to calculate the time-dependent transition probability from the ground state to the first excited state. Compare your result with what you got in Task C.

Quantum Mechanics 3

Two fermions on a ring

Consider two spin-1/2 fermions of mass m, with coordinates x_1 and x_2, confined to move on a circle of circumference L and interacting through a spin-dependent potential

$$V = -u\delta(x_1 - x_2)\vec{s}_1 \cdot \vec{s}_2 \,, \quad u > 0,$$

where $\vec{s} = \{s_x, s_y, s_z\}$ is the operator of the spin 1/2 (in units of $\hbar$), so that the Hamiltonian H of the two-electron system (in the standard notations) is:

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + V \,.$$

The circle is threaded by an infinitely-long solenoid which carriers a magnetic flux Φ, so that the electron momenta are

$$p_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA \,,$$

where A is the vector potential produced by the magnetic flux, which can be taken to be constant along the circle, and e is the electron's charge.

(a) (4 points) What is the relation between A and Φ? Use this relation to write down the single-particle eigenstates $\psi_n(x)$ and eigenenergies E_n of one electron on the circle in terms of ϕ and other parameters in the problem.

(b) (3 points) Derive the boundary conditions for the orbital part $\psi(x_1, x_2)$ of the two-electron wavefunction at $x_1 = x_2$, if electron spins are in the triplet state.

(c) (4 points) Assuming that $|\Phi/(h/e)| < \frac{1}{2}$, find the ground-state energy $E_0^{(t)}$ and the orbital part $\psi_0^{(t)}(x_1, x_2)$ of the corresponding two-electron state.

(d) (4 points) Derive the same boundary conditions as in part (b) for electron spins in the singlet state.

(e) (5 points) For $\Phi = 0$, determine the ground-state energy $E_0^{(s)}$ and the orbital part $\psi_0^{(s)}(x_1, x_2)$ of the ground-state two-electron wavefunction in the singlet spin state in the limit of strong potential $u \to \infty$.

Statistical Mechanics 1

Ideal gas in gravitational potential

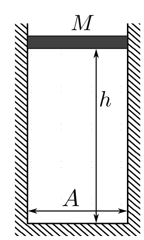
Consider an ideal gas of N indistinguishable molecules of mass m in a cylindrical volume $V = Ah$ with base area A and height h.

(a) (5 points) Calculate the canonical partition function of the ideal gas $Z^g(T, V)$, including the effect of gravity. You may find useful the integral $\int_0^\infty dt\, t^2\, e^{-t^2} = \sqrt{\pi}/4$.

(b) (4 points) Using the partition function, calculate the internal energy of the system. Then, determine the heat capacity in two limits, $T \gg mgh$ and $T \ll mgh$. You may find the Stirling approximation $N! \approx (N/e)^N$ helpful.

(c) (2 points) Write down the condition for when the effect of gravity is negligible compared to the thermal energy. What is the internal energy in that case? Neglecting internal degrees of freedom, estimate at what temperature the thermal energy for oxygen molecules O_2 (^{16}O) at 1m height is of the order of the potential energy.

Now assume that the top of the cylinder is a piston of mass M that can move vertically (change the height h), and that the gravity effects on the gas itself are *negligible*, so that the gas volume can change but the pressure is constant.



(d) (2 points) Write down the Hamiltonian of the combined piston + gas system H^{g+p}, and express the potential energy of the piston in terms of the pressure and the volume of the gas.

(e) (4 points) Find the canonical partition function Z^{g+p} of the system with the piston. One way to do this is to use the result for the partition function $Z^g(T, V)$ from part (a) (now neglecting the potential energy of the gas). First, show that Z^{g+p} is proportional to its Laplace transform

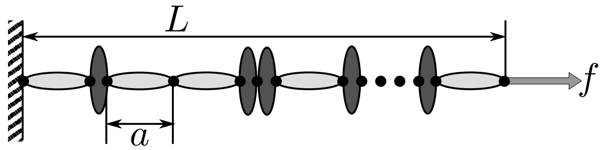
$$Z^{g+p}(T, P) = B \int_0^\infty dV e^{-\alpha V} Z^g(T, V) \tag{82}$$

for some B and α that you have to determine. Then compute Z^{g+p} as above (you may use the integral $\int_0^\infty dx\, x^N\, e^{-bx} = \frac{N!}{b^{N+1}}$).

(f) (3 points) Compute the variance of the piston height h and express it in terms of particle number N, piston mass M, and temperature T.
Hint: one way to do this is to use the pressure-dependent partition function $Z^{g+p}(T, P)$.

Statistical Mechanics 2

Thermodynamics of a polymer molecule



Consider a single polymer molecule that consists of $N \gg 1$ connected elementary identical links. These links may be either "folded" (zero length) or extended (to length a) in the same direction, and have the same intrinsic energy in both states. One end of the molecule is fixed, and tension f may be applied to the other end.

(a) (3 points) Find the average length L_0 of the molecule when no tension is applied to the molecule ($f = 0$). How does L_0 depend on temperature? What is the variance of the length $\langle(\delta L)^2\rangle$ (assuming constant temperature)?

Hint: Throughout the problem, assume that the change in the length of the molecule is small, $\Delta L \ll L_0$.

(b) (4 points) The molecule is "stretched" to fixed length $L = L_0 + \Delta L$. How many microscopic states of the molecule correspond to this state, and what is the corresponding entropy? You may use the Stirling approximation $\ln n! = (n \ln n - n)$.

(c) (3 points) Calculate the tension f required to stretch the molecule in part(b) to length $L = L_0 + \Delta L$, and its elasticity k_T at constant temperature T.

(d) (3 points) Now assume that the heat capacity of the unstretched molecule $C(L_0) \equiv C_0$ is independent of temperature. Calculate the heat capacity C_f for a molecule stretched by a constant force f.

(e) (3 points) What is the adiabatic elasticity k_S if the molecule?

(f) (4 points) Suggest a design for a heat engine based on the temperature dependence of the elasticity $k_T(T)$ and calculate its efficiency. Draw the engine's operating cycle in the (L, T) plane and show the direction of the cycle.

Statistical Mechanics 3

The thermoelectric effect

A (3 points). For a gas of similar non-relativistic particles, write:

- the Liouville theorem,
- the Boltzmann transport equation in its general form, and
- the Boltzmann equation in the relaxation-time approximation.[1]

B (2 points). Spell out the last equation for a free, isotropic Fermi gas of particles with electric charge q, in the presence of a uniform, time-independent external electric field $\mathbf{E}$.

C (4 points). Solve the obtained equation in the linear approximation in the weak applied field $\mathbf{E}$, and use the result to express the densities of the electric current ($\mathbf{j}_e$) and of the heat flow ($\mathbf{j}_h$) as integrals over the single-particle energy ε.

Hint 1: The heat flow density in a gas with the single-particle probability distribution $w(\mathbf{r}, \mathbf{p})$ may be calculated as

$$\mathbf{j}_h = \int (\varepsilon - \mu) \mathbf{v}\, w d^3 p ,$$

where μ its chemical potential, and $\mathbf{v}$ is the particle's velocity.

D (2 points). Give a physical interpretation of the formula given in *Hint 1*.

E (2 points). Use the first result of Task C to obtain an explicit expression for the Ohmic conductivity σ (defined as the coefficient in the differential form $\mathbf{j}_e = \sigma\mathbf{E}$ of the Ohm law) via the gas density n, particle's charge q and mass m, and the relaxation time τ, for arbitrary temperature.

F (3 points). For a degenerate Fermi gas, use the second result of Task C and the Sommerfeld expansion formula to calculate the so-called *Peltier coefficient* in the linear relation $\mathbf{j}_h = \Pi \mathbf{j}_e$.

Hint 2: The Sommerfeld expansion may be represented in several forms; for this problem, the most useful of them is

$$\int_0^\infty \varphi(\varepsilon)\left(-\frac{\partial f(\varepsilon)}{\partial \varepsilon}\right) d\varepsilon \approx \varphi(\mu) + \frac{\pi^2}{6} T^2 \frac{d^2 \varphi(\varepsilon)}{d\varepsilon^2}\Big|_{\varepsilon=\mu}, \qquad \text{at } T \ll \mu ,$$

where $\varphi(\varepsilon)$ is a smooth function equal to zero at $\varepsilon = 0$, and $f(\varepsilon)$ is the Fermi-Dirac distribution.

G (4 points). Suggest a simple method to measure the coefficient Π experimentally.

Hint 3: Think about a loop made of two different conducting materials, one of them with a known Π.

---

[1] The approximation is sometimes called the BGK model.