Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2015 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

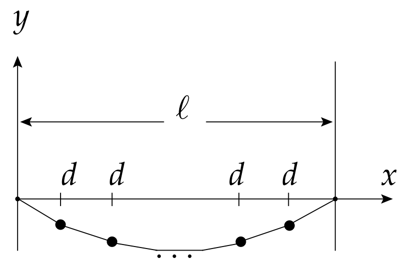
Each problem counts for 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages.
Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

Wave propagation in an (an)harmonically coupled system of N masses



Consider a massless string of length ℓ_0 with string constant κ, containing N masses, each with mass m. The string is stretched to a length $\ell > \ell_0$ and attached to two opposite walls, as in the figure. In equilibrium, the distance from each wall to the nearest mass is d, as is the distance between two neighbouring masses. The position of the masses is (x_k, y_k) with $k = 1, \cdots, N$, and the endpoints are fixed at $(x_0, y_0) = (0, 0)$ and $(x_{N+1}, y_{N+1}) = (\ell, 0)$. Neglect gravity. The potential is $V(\xi_k, y_k) = \frac{1}{2}\kappa \sum_{k=1}^{N+1} (\ell_k - a)^2$, where $a = \frac{\ell_0}{N+1}$ and ℓ_k is the length of the segment of the string between the k^{th} mass and the $(k-1)^{\text{th}}$ mass, and $\xi_k = x_k - kd$.

a) (2 points) Write down the Lagrangian L without any approximations. Then expand to **third** order in small deviations (ξ_k, y_k).

b) (6 points) Consider first harmonic motion (due to the terms in L which are quadratic in ξ_k and y_k). Write down the equation of motion for ξ_j and y_j. Expand x_j and y_j into normal modes. What are the integration constants? (Hint: Try the ansatz ξ_j proportional to $e^{ij\alpha}$, and similarly for y_j.)

c) (6 points) What are the frequencies ω as a function of j? Take the continuum limit $N \to \infty$, keeping the total mass $M = Nm$ fixed. Show that one obtains the wave equation. What are the dispersion relations for $\omega(k)$ as a function of the wave number k for transverse and longitudinal modes?

d) (6 points) Now consider the cubic anharmonic term in L. Obtain the wave equation for longitudinal waves and transverse waves with the contribution from this term. Describe the difference in the propagation of a longitudinal pulse and the propagation of a transverse pulse due to the anharmonic term.

Classical Mechanics 2

Point particle moving on a sphere

Consider a point particle with mass m, moving on a fixed sphere with radius R in the gravitational field of the Earth.

a) (3 points) Write down the Lagrangian in spherical coordinates (r, θ, ϕ). Write down explicit expressions for the two conserved quantities.

b) (6 points) Using the expressions in a), write the time t as a function of θ, and ϕ as a function of θ. You will find integrals which can not be evaluated in closed form. That is OK, just write them down.

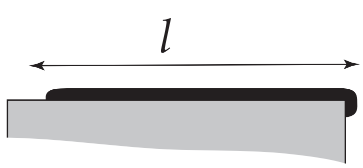
c) (4 points) If the particle moves on a horizontal circle with azimuthal angle θ, what is the angular velocity of this periodic orbit?

d) (7 points) If the particle has a z-component M_z of the orbital angular momentum, all its trajectories will lie between two horizontal circles on the sphere. Find an equation for the position of these circles. How many solutions does this equation have, and what can you say on physical grounds about these solutions?

Classical Mechanics 3

Rope sliding off a table

An ideal (flexible, uniform, frictionless, etc.) rope of the length l and mass M starts sliding off an ideal frictionless table as shown in the figure (the rope is initially at rest, the gravitational acceleration is g, the size of the piece of the rope initially hanging off the table is y_0).



a) (2 points) Introduce some generalized coordinate and write down the Lagrangian of the system.

b) (2 points) Derive the Euler-Lagrange equations of motion.

c) (2 points) Write down the energy of the system.

d) (6 points) Calculate the time T for the rope to slide half way off the table.

e) (8 points) Compute the horizontal component of the reaction force of the table at the moment when the rope is exactly half way off the table. *Hint:* calculate the rate of the change of the momentum of the rope.

Electromagnetism 1

Torque on a cylinder

The constitutive relation is a relation between the macroscopic electrical current density in a medium and the applied fields. Recall that for a normal isotropic conductor *at rest* in an electric ($\boldsymbol{E}$) and magnetic field ($\boldsymbol{B}$) the constitutive relation in a linear response approximation is known as Ohm's law: $\boldsymbol{J} = \sigma \boldsymbol{E}$.

a) (2 points) For most materials, a symmetry principle forbids a generalized Ohm's law in the rest frame of the material of the form:

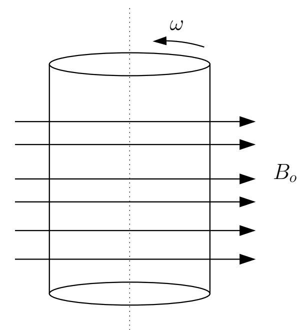
$$\boldsymbol{J} = \sigma \boldsymbol{E} + \sigma_B \boldsymbol{B} \,. \tag{1}$$

Explain.

b) (6 points) By making a Lorentz transformation for small velocities, deduce the familiar constitutive relation for a normal conductor *moving* non-relativistically with velocity $\boldsymbol{u}$ in an electric and magnetic field from the rest frame constitutive relation, Eq. ().

c) (4 points) Now consider a solid conducting cylinder of radius R and conductivity σ rotating rather slowly with constant angular velocity ω in a uniform magnetic field B_o perpendicular to the axis of the cylinder as shown below. Determine the current flowing in the cylinder.

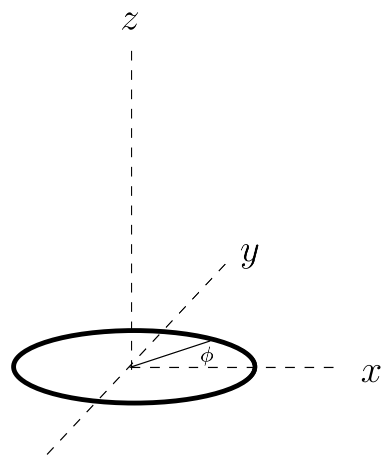
d) (8 points) Determine the torque required to maintain the cylinder's constant angular velocity. Assume that the skin depth is much larger than the radius of the cylinder.



Electromagnetism 2

Oscillating current on a ring

A current is driven through a ring of radius R in the xy plane (see below). Using complex notation, the current has a harmonic time dependence, $\boldsymbol{J}(t,\boldsymbol{r}) = e^{-i\omega t}\boldsymbol{J}(\boldsymbol{r})$, and the spatial dependence is $\boldsymbol{J}(\boldsymbol{r}) = I_0 \sin(\phi)\,\delta(\rho - R)\delta(z)\hat{\boldsymbol{\phi}}$.



a) (4 points) Sketch the current flow at time $t = 0$ and $t = \pi/\omega$, and determine the charge density $\rho(t,\boldsymbol{r})$. Show that it corresponds to an oscillating electric dipole, and determine the electric dipole moment.

b) In the long wavelength limit, and in the radiation zone, determine each of the following quantities in the xz plane at $y = 0$:

   (a) (6 points) The vector potential $\boldsymbol{A}(t,\boldsymbol{r})$ in the Lorentz gauge.

   (b) (4 points) The magnetic field $\boldsymbol{B}(t,\boldsymbol{r})$.

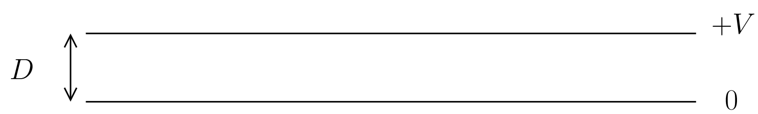
   (c) (4 points) The (time averaged) angular distribution of the radiated power, $dP/d\Omega$.

c) (2 points) What is the polarization of the radiated electric field when viewed along the z axis ?

Electromagnetism 3

Parameters of an electron tube

Consider an idealized electron tube (diode) consisting of infinite planar cathode and anode separated by a distance D in the z direction (see below). The cathode (at $z = 0$) may be regarded as an infinite supply of free electrons at rest. The anode (at $z = D$) is at potential $+V$ relative to the cathode. (V is sufficiently small that Newtonian physics applies.) The device is evacuated, so that only electrons are between the two electrodes. The current through such a device is determined by the flow of the charge of these electrons from the cathode ($z = 0$) to the anode ($z = D$).



a) (10 points) Use Poisson's equation, the equation of continuity, and the conservation of energy to derive a differential equation for the electric potential $\Phi(z)$ in steady state. Make sure you have the sign correct, and state the boundary conditions explicitly.

b) (6 points) Find $\Phi(z)$ and use it to determine the current density J as a function of the parameters of the problem and physical constants. *Hint:* Try a scaling solution of the form $\Phi(z) \propto z^{\beta}$.

c) (4 points) Put in numbers for a centimeter-sized device and an anode potential of 300 volts to *estimate* the impedance typical of electron tube circuits.

Quantum Mechanics 1

Spin-$\frac{1}{2}$ resonance and neutron interferometry

I. An electron of charge e and mass m_e is subject to a uniform magnetic field $B_0\hat{z}$ and has its spin along the positive z-axis. At $t = 0$ an additional time-dependent magnetic field is switched on in the transverse plane with

$$B_\perp(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y} \qquad (1)$$

a) (8 points) Write down the Schrödinger equation for this time-dependent problem and solve it.

b) (3 points) What is the probability in time to find the electron with its spin along the negative z-axis, and for what frequency is the spin flip maximum?

c) (3 points) Neutron spin flippers are based on this magnetic set-up. Denoting by t_n the time that a neutron is in the field, find the minimum value of t_n for a maximum spin flip to occur. Explicitly write down the neutron state at this time. The neutron magnetic moment is μ_n.

II. Now in a neutron interferometer, a neutron beam is split into two beams labeled (1) and (2). In beam (1) the neutrons are subject to the spin flipper with the time of flight adjusted for maximum spin flip. In beam (2) the neutrons acquire a spin-independent phase δ by a mechanism which we do not specify. There is no magnetic field acting on beam (2). Both beams are recombined and an interference pattern is observed. Let the initial neutron state at $t_i = 0$ be

$$\langle\vec{x}|\Psi_i\rangle = \frac{1}{\sqrt{2}}\left(\langle\vec{x}|\varphi_1\rangle|\uparrow\rangle + \langle\vec{x}|\varphi_2\rangle|\downarrow\rangle\right) \qquad (2)$$

Here $\langle\vec{x}|\varphi_{1,2}\rangle$ are fixed and normalized spatial states along route (1) and (2) respectively.

d) (6 points) Determine the final neutron state $\langle\vec{x}|\Psi_f\rangle$ as the neutron beam recombines in the interferometer and calculate the expectation value of the neutron spin $\vec{\mathbf{S}}_n$ in this final state.

Quantum Mechanics 2

Pairs of Hamiltonians in one dimension

Consider a particle on the x-axis with potential $U(x)$ such that $U(x)$ vanishes as $x \to \pm\infty$, and $U(x)$ is everywhere negative and nonsingular. Recall that the ground state for such a system is always a nondegenerate bound state.

a) (4 points) Define $V(x) = U(x) - E_0$ where E_0 is the ground state energy. Write the Hamiltonian in factorized form as $H = A^\dagger A + E_0$ where $A = c\frac{d}{dx} + W(x)$ and c is a constant. Determine c and $W(x)$. (Hint: Express $V(x)$ in terms of the ground state wave function $\phi_0(x)$ and try the logarithmic derivative of $\phi_0(x)$ for W.)

b) (4 points) Show that A annihilates $\phi_0(x)$. Show that $H_1 = A^\dagger A + E_0$ and $H_2 = AA^\dagger + E_0$ have the same non-vanishing eigenvalues $E_n > E_0$. Draw a picture of the eigenvalues of H_1 and H_2, both the discrete and the continuous ones. (Hint: Act with A on H_1.)

c) (6 points) Consider now two systems, one with Hamiltonian $H_1 = A^\dagger A + E_0$ and another with Hamiltonian $H_2 = AA^\dagger + E_0$. Let $A^\dagger A = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1$ and $AA^\dagger = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2$. Construct the $W(x)$ which gives $V_1 = a^2$, where a is a constant, and construct the corresponding $V_2(x)$. Plot $V_2(x)$ and $V_1(x)$ as functions of x. (Hint: The solution of the Riccati equation $\frac{d}{dx}y + y^2 = 1$ is given by $y(x) = \tanh x$.)

d) (6 points) If $A^\dagger A$ has a constant potential $V_1 = a^2$, the solutions for H_1 are plane waves. Prove that then the potential $V_2(x)$ of H_2 is also reflectionless. (A potential is called reflectionless if every incoming plane wave of the continuous spectrum is transmitted without reflection. In other words, there is total transmission.)

Quantum Mechanics 3

Shell model for atomic nuclei

Atomic nuclei can be described by the shell-model which consists of spin-1/2 protons and neutrons filling the states of a spherically symmetric potential $V(r)$. A simple approximation is $V(r) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$, with energy levels of the form

$$E_N = \left(N + \frac{3}{2}\right)\hbar\omega \tag{1}$$

a) (2 points) What is the degeneracy of E_N as a function of N (including spin)? As a check, list the single-particle states within the first 3 shells $N = 0, 1, 2$.

b) (6 points) We introduce a radial quantum number n and orbital quantum number l through $N = 2(n-1) + l$. Write the explicit expressions for the energies, parities and degeneracies (including spin) for the first 5 shells. Identify the magic numbers corresponding to closed shells by analogy with the noble gases.

c) (6 points) The spin-orbit interaction between the nucleons

$$V_{so} = a\,\vec{s}\cdot\vec{l} \quad , \quad a < 0 \tag{2}$$

splits the states within a given shell. Calculate the first-order energy shifts caused by V_{so} and the corresponding degeneracies for the first 5 shells.

d) (3 points) O^{17} consists of 8 protons and 9 neutrons. Identify its spectroscopic nl_j assignment (where $\vec{j} = \vec{l} + \vec{s}$), and find its energy shift and parity.

e) (3 points) Nuclear data indicate that the fourth magic number is 50, in disagreement with what you found in **b**. An intruder state from the 5th shell got into the 4th shell. What are the quantum numbers of the intruder state?

Statistical Mechanics 1

Thermodynamic properties of a ferroelectric crystral

Consider a ferroelectric system of N molecules in zero electric field $\mathcal{E} = 0$. Each molecule has two energy states available, of energy $-\frac{1}{2}\kappa$ and $+\frac{1}{2}\kappa$ respectively, where κ is a constant. If l molecules are in the excited state, then the energy of the crystal is given by

$$E^0_{N,l} = -\frac{1}{2}(N-l)\,\kappa + \frac{1}{2}\,l\,\kappa\,, \quad l = 0, 1, 2, ...N, \tag{1}$$

with degeneracy

$$g_{N,l} = 2\frac{N!}{(N-l)!\,l!}\,. \tag{2}$$

In the presence of a non-zero electric field $\mathcal{E}$ the degeneracies are partially split by the energy-level changes:

$$\Delta E_{N,l} = \pm\,\nu\,l\mathcal{E}, \tag{3}$$

where ν is the electric moment of the molecules that couples to the external electric field.

a) (3 points) What is the ground state energy per molecule $u_0(\mathcal{E})$ at $T = 0$, as a function of the electric field $\mathcal{E}$?

b) (2 points) Calculate the partition function $Q_N(T, \mathcal{E})$ and the free energy per molecule of this system *.

c) (7 points) Find the internal energy per molecule $u(T, \mathcal{E})$ at finite temperature T and electric field $\mathcal{E}$, and show that, in the limit $T \to 0$, $u(T, \mathcal{E})$ approaches $u_0(\mathcal{E})$. Find $u(T, \mathcal{E})$ as a function of $\mathcal{E}$ in the limit $T \to \infty$.

d) (8 points) Find the entropy per molecule, $s(T, \mathcal{E})$, for the two limits of the temperature ($T \to \infty$ and $T = 0$), paying special attention to the field values $\mathcal{E} = \pm\frac{\kappa}{\nu}$. Relate your results to the Third Law of Thermodynamics.

*Hint: $(a + b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k$

Statistical Mechanics 2

Two-dimensional, nonrelativistic Bose gas

A two-dimensional ideal, spinless and nonrelativistic Bose gas is maintained in an area A with finite temperature T and chemical potential μ, with $z = e^{\mu/k_B T}$.

a) (8 points) Calculate the grand partition function $\mathbf{Z}(z, A, T)$. Separate the zero momentum part.

b) (5 points) Calculate the average density of bosons $\mathbf{n}(z, A, T)$. Show that z must be less than 1 for any density.

c) (2 points) Can the two-dimensional idea gas Bose condense? Explain.

d) (5 points) What changes in these arguments in three dimensions, and why?[†]

[†]Hint: for nonzero momenta you may use the following expansion to do the integrals: $\ln(1/(1-x)) = \sum_{k=1}^{\infty} \frac{x^k}{k}$

Statistical Mechanics 3

Mean-field theory of an Ising-type model in a transverse field

Consider the system of spins-1/2 on a lattice with q nearest-neighbors for each site, characterized by the Hamiltonian

$$H = -J\big[\sum_{\langle i,j\rangle} \sigma_z^{(i)}\sigma_z^{(j)} + g\sum_i \sigma_x^{(i)}\big]\,,$$

in equilibrium at temperature T. Here $\sigma_z^{(i)}$, $\sigma_x^{(i)}$ are the Pauli matrices of the ith spin, and the sum in the first term is taken over the pairs of the nearest-neighbor spins. As usual, the dimensionality of the lattice does not affect the results in the mean-field approximation.

a) (6 points) Adopting the standard mean-field approach, introduce the effective single-spin Hamiltonian H_0 and calculate the thermal averages

$$m_z = \langle \sigma_z^{(i)} \rangle\,, \quad m_x = \langle \sigma_x^{(i)} \rangle\,.$$

b) (6 points) Imposing the relevant self-consistency condition, find the transition temperature T_c of the temperature-driven phase transition from the paramagnetic (vanishing m_z) to the ferromagnetic (non-vanishing m_z) state, and determine the range of the parameter g, when such a phase transition exists. What is the expression for T_c in the limit $q \to 0$?

c) (8 points) In the situation with vanishing temperature, $T = 0$, analyze m_z as a function of g from the same self-consistency condition. Determine the point g_c of a "quantum phase transition" into a ferromagnetic state, and find m_z and m_x as functions of the small difference $\delta g = g_c - g$.