Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

January 2017 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Some of the problems may cover multiple pages. Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

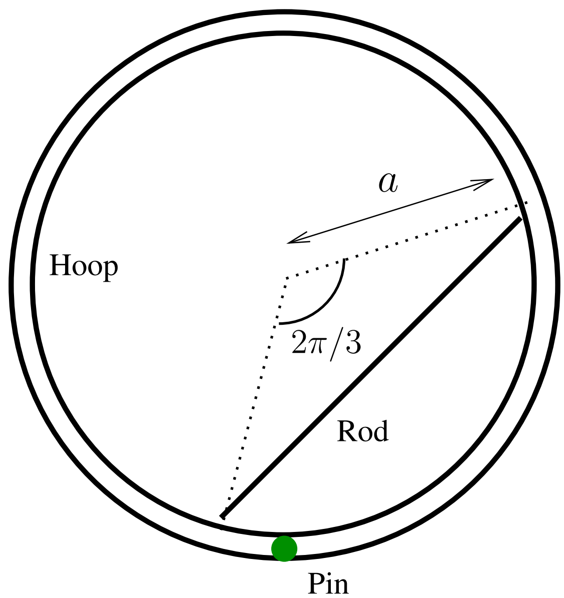
Write your ID number (not your name!) on the exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

A rod in a ring

A rod slides without friction in a hoop of inner radius a experiencing earths gravitational acceleration, g. The hoop is fixed to the floor by a small pin that does not influence the motion. The rod subtends an angle of 120^o (or $2\pi/3$ radians) as shown below.



(a) (9 points) Write down the Lagrangian of the system and solve for the frequency of small oscillations.

(b) (3 points) For small oscillations with a maximum angle of θ_o, what is the maximum force on the pin? Draw a sketch (or sketches) showing the position of the rod and the direction of the force on the pin when the magnitude of the force is maximal.

(c) (6 points) Now assume that the maximum oscillation angle θ_o is small, but large enough that the first non-linear corrections become important. Determine the period of oscillation including the first dependence on θ_0. (Hint: note the identity $\cos(3x) = 4\cos(x)^3 - 3\cos(x)$.)

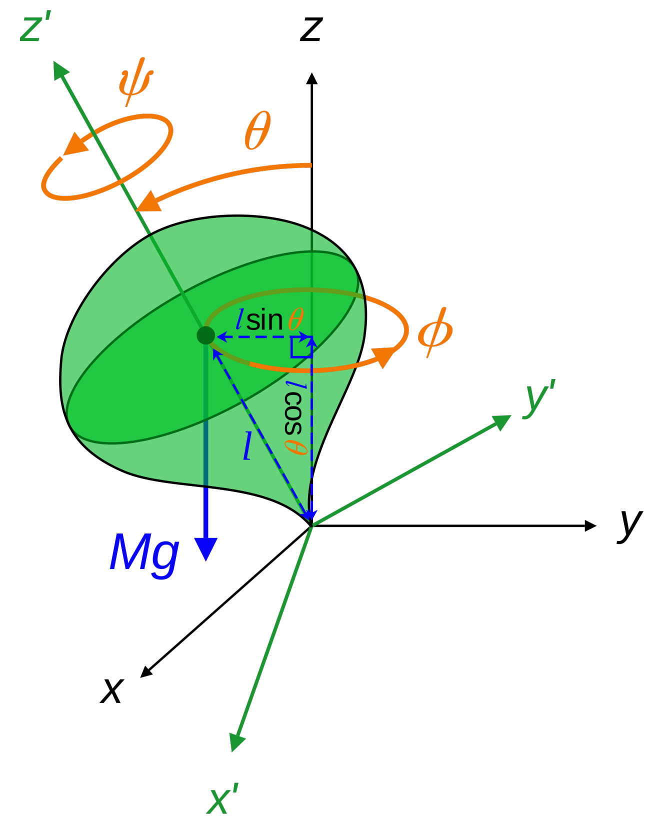
(d) (2 points) For the non-linear oscillations described in the last item, the motion is periodic with period τ but it is not sinusoidal. The angle as a function of time can be expanded as a Fourier series

$$\frac{\theta(t)}{\theta_o} = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n(t/\tau)} \tag{1}$$

Qualitatively sketch the power spectrum (i.e. $|c_n|^2$ as a function of n) for oscillations in the linear and weakly non-linear regimes.

Classical Mechanics 2

Stability of a symmetric top



The Lagrangian of a symmetric top rotating around its fixed lowest point on the axis of symmetry can be expressed in terms of the Euler's angles ϕ, θ, ψ as

$$\mathcal{L} = \frac{I}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_\parallel}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta\,,$$

where M is the mass of the top, $I = I_\perp + Ml^2$, with l being the distance between the rotation center and the center of mass above it; $I_\parallel$, $I_\perp$ – the two principal moments of inertia of the symmetric top, for the rotation, respectively, along the axis of symmetry and an axis orthogonal to it, and the axis z used in the definition of the Euler's angles points vertically up.

(a) (6 points) Find expressions for the two integrals of motion (besides the energy E) and identify the physical nature of these integrals. Briefly (in one sentence for each) state explicit physical reason for their conservation.

(b) (6 points) With the two integrals of motion from part (a), the problem reduces effectively to a problem with one degree of freedom. Derive an expression for the energy of this one-dimensional problem using the integrals of motion.

(c) (8 points) Assume that the top rotates around its axis with the angular velocity Ω. Using the result of (b), find the condition on Ω which ensures that a rotating top in the vertical position (i.e. $\theta \simeq 0$) is stable.

Classical Mechanics 3

Time as a canonical variable

In special relativity, space and time are treated on equal footing. In this problem we reformulate classical mechanics in this way. We parametrize both the space coordinates q^i of a point particle, but also the time t, as functions of a new time variable θ, so $q^i = q^i(\theta)$ and $t = t(\theta)$. Thus θ parametrizes the space-time path $x^\mu(\theta) = (ct(\theta), q^i(\theta))$ of the point particle.

(a) (2 points) If $L(q^i, \dot{q}^i, t)$ is the Lagrangian of a system with coordinates $q^i = q^i(t)$, show that the Lagrangian L_θ for the corresponding system with θ as the time variable is given by

$$L_\theta(q(\theta), t(\theta), q'(\theta), t'(\theta)) = t' L(q, q'/t', t)$$

where $t'(\theta) = dt/d\theta$ and $q'(\theta) = dq/d\theta$.

(b) (2 points) Show that the momentum conjugate to t in the new formulation is given by $p_t = -H$ where $H = H(p, q, t)$ is the ordinary Hamiltonian. Show that the momentum conjugate to q is unchanged in the new formulation. So now, phase space is $4 + 4$ instead of $3 + 3$ dimensional.

(c) (2 points) Derive the two Euler-Lagrange equations of motion for L_θ. Show that they are equivalent to the equation of motion for L.

(d) Consider a nonrelativistic point particle in ordinary classical mechanics with Lagrangian $L = T - V$ and a potential V that does not explicitly depend on time t.

   (i) (3 points) Derive the Noether charge for time translational invariance.

   (ii) (3 points) Now consider the corresponding Lagrangian L_θ. Is there a corresponding Noether charge for θ-translational invariance?

(e) Now consider the Lagrangian for a relativistic point particle with mass m and electric charge e, coupled to ordinary electromagnetic fields $A_\mu(q, t)$.

   (i) (4 points) Construct the Hamiltonian H of ordinary classical mechanics with $A_\mu(q(t), t)$.

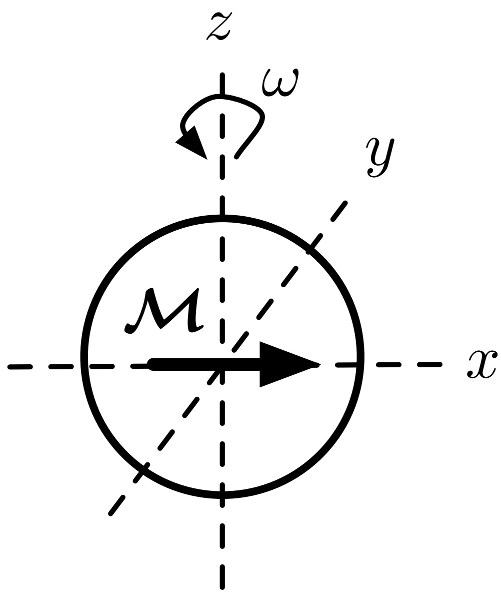
   (ii) (4 points) Construct the Hamiltonian H_θ with $A_\mu(x(\theta))$ with $x^\mu = (ct(\theta), q(\theta))$.

Electromagnetism 1

Magnetic field on the surface of a star

A star (roughly modeled on the Crab Pulsar) has mass M of $1.4\,M_{\odot}$ (or $1.4 \times 2 \times 10^{30}$ kg) and radius R of 10 km. It rotates (non-relativistically) with a period $\tau = 2\pi/\omega$ of 33 milli-seconds. The period slowly decreases due to the emission of electromgnetic radiation. The change in period per time is, $|\dot{\tau}| = 4.0 \times 10^{-13}$.

Model the star as a uniformly magnetized sphere spinning around the z-axis, with a magnetization $\boldsymbol{\mathcal{M}}$ lying in the $x-y$ plane.



(a) (7 points) Determine the total magnetic dipole moment of the star $m_o = \mathcal{M}\,\frac{4}{3}\pi R^3$ in terms of $\dot{\tau}$.

(b) (6 points) Determine the magnitude of the magnetic field at the north pole of the star. Check that your expression is dimensionally correct, and make a rough order of magnitude estimate for the magnetic field in Tesla.

Parts (c) and (d) are indpendent of parts (a) and (b); refer to the solution of part (a) as m_o.

(c) (5 points) Determine the electric field at the north pole of the star as a function of time. (Neglect the slow decrease of the rotational period with time.) Hint: what is the vector potential of the star?

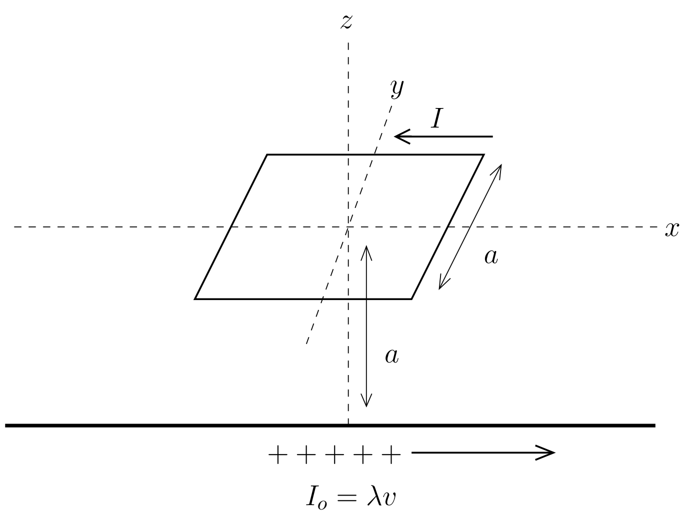
(d) (2 points) Numerically estimate the ratio of the energy density in the magnetic field to the energy density in the electric field at the north pole of the star.

Electromagnetism 2

Torques in Relativity

After the Michelson-Morley experiment of 1887, there was another experiment to measure the velocity of the earth through the aether: the Trouton-Noble experiment of 1903. It seemed to show that Maxwell's theory of electromagnetism is inconsistent. In this problem we will study a simplified version of this experiment and show how special relativity removes this inconsistency.

A neutral square loop of wire with sides of length a carries a current I. The square lies flat in the xy plane and is centered at the origin. Directly below the square is an infinite line of positive charge with charge per length λ_o. The line is parallel to the x-axis, but is displaced by a distance a below the origin in the negative z direction (see below). The positive charges in the infinite line move to the right with velocity v, producing a net current $I_o = \lambda_o v$. The neutral square can rotate around the x-axis



(a) (4 points) Calculate the net torque on the square due to the line of charge.

(b) (6 points) An observer moving to the right along the x-axis with velocity v measures a charge density in the square loop. Determine the charge per length in all four legs of the square loop. Make a sketch illustrating the distribution of charges in each leg.

(c) (3 points) Qualitatively explain the origin of the net torque according the right moving observer of part (b).

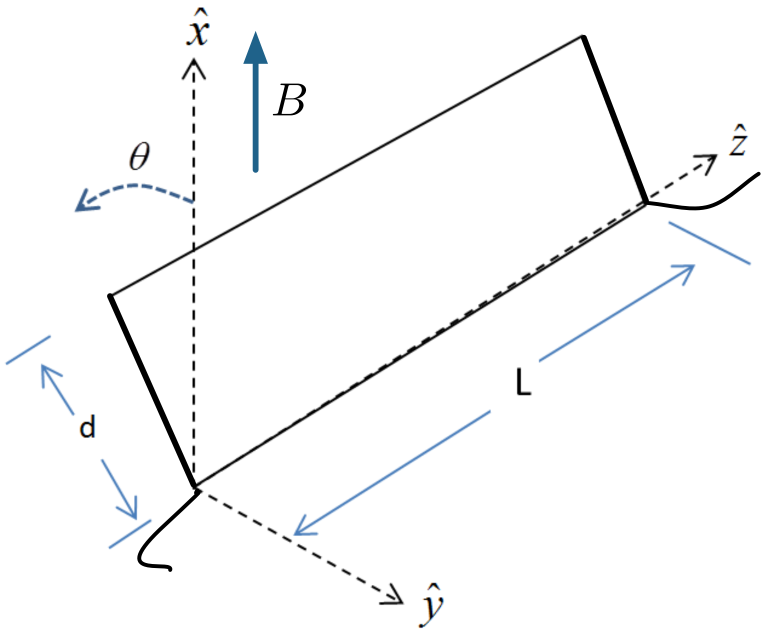
(d) (3 points) Determine the torque according to the right moving observer of part (b).

(e) (4 points) Compare the torques computed in parts (a) and (d). Are they equal? How does the Lorentz force per volume (i.e. $\boldsymbol{f} = \rho \boldsymbol{E} + \frac{\boldsymbol{J}}{c} \times \boldsymbol{B}$) transform under Lorentz transformation. Transform the Lorentz forces per volume in part (a) to explain results of part (d).

Electromagnetism 3

A conducting plate in a magnetic field

A thin rectangular nonmagnetic metal plate with dimensions $L \times d \times t$ has mass density ρ, and conductivity σ. The bottom edge of the plate is held fixed, but the plate is free to rotate around the z-axis (see below). Charge can flow on and off the plate through two leads of negligble resistance at either end of the plate. The plate sits in a constant magnetic field directed along the x-axis ($\mathbf{B} = B\,\hat{\boldsymbol{x}}$), and experiences the Earth's gravitational pull ($\mathbf{g} = -g\,\hat{\boldsymbol{x}}$) as shown below. For the analysis below assume that the length L is very large, so that the plate is essentially infinitely long. Also assume that the thickness t is very small, so that the plate is essentially two dimensional.



(a) (5 points) Assume plate rotates in a counter-clockwise fashion around the z axis (increasing θ as shown in the figure): (i) sketch how the current flows in the plate. Explain your reasoning. (ii) sketch a free body diagram showing the magnetic and gravitational forces on the plate.

(b) (8 points) Find an equation for the plate's angle $\theta(t)$ as a function of time.

(c) (3 points) Specialize this equation to the case $\theta \ll 1$ and find the small angle solution for $\theta(t)$, given the initial conditions $\theta(0) = \theta_0 \ll 1$ and $\dot{\theta}(0) = 0$.

(d) (4 points) For the typical values given below, show that the angle increases approximately as $e^{\gamma t}$ after an initial transient. Calculate the time constant $1/\gamma$ in seconds. Assume that $d = 1\,\mathrm{m}$, $\rho = 10^4\,\mathrm{kg/m^3}$, $\sigma = 10^8\,(\Omega\,\mathrm{m})^{-1}$, $B = 1\,\mathrm{T}$, and $g = 10\,\mathrm{m/s^2}$, and $\epsilon_o = 8.85\,\times 10^{-12}\,\mathrm{Nm^2/C^2}$

Quantum Mechanics 1

Quantum mechanics of a charged rod

A rod of length l and uniform mass distribution rotates around its center in the xy-plane. The rod has mass M. Two charge $+Q$ and $-Q$ are fixed at the end of the rod.

(a) (5 points) Describe this system quantum mechanically by finding its eigenfunctions and eigenvalues. Is the spectrum degenerate and why?

(b) (5 points) A constant weak electric field $\mathbf{E} = E\hat{\mathbf{x}}$ lying in the plane of rotation is applied. Find the new energies and eigenfunctions.

(c) (5 points) For a constant but strong electric field $\mathbf{E} = E\hat{\mathbf{x}}$, find an approximate wave function and energy for the ground state.

(d) (5 points) If at time $t = 0$ a weak time dependent electric field is applied $\mathbf{E}(t) = 2\cos(\omega t)\mathbf{E}$, find the probability for the transition from the ground state to any excited state.

Quantum Mechanics 2

Approximations of an anharmonic oscillator

The purpose of this problem is to compare various estimates for the ground state energy of the anharmonic oscillator. For that, consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x^4 \equiv H_\omega + \lambda x^4 \tag{13}$$

a. (3 points) For a weakly anharmonic oscillator use dimensional and physical reasoning to estimate (i) the typical size of the ground state wave function and (ii) the shift in the ground state energy due to the anharmonic term. Explain your reasoning.
b. (3 points) Use first order perturbation theory to determine the ground state energy of the weakly anharmonic oscillator.
c. (6 points) If Ψ_ω denotes the ground state wavefunction of the harmonic oscillator Hamiltonian H_ω (see eq. (13)), use Ψ_Ω as a variational ansatz to determine the optimal variational frequency $\Omega(m, \omega, \lambda)$ and corresponding variational energy $\mathcal{E}(m, \omega, \lambda)$ for the ground state of the anharmonic Hamiltonian. Compare to the results in **b.**
d. (3 points) Write the classical equation of motion corresponding to the anharmonic potential in (13), and approximately solve the equation with a single harmonic ansatz, $x(t) = A\cos(\Omega t)$, i.e. determine the oscillation frequency Ω as a function of A. For what value of A does classical oscillation frequency equal the variational frequency Ω in **c.**? Compare your result to the estimate of part **a**.
e. (5 points) Use the WKB approximation to evaluate the energy of the ground state of the anharmonic oscillator to first order in λ. How does your result compare to the results in **b, c** and why?

Quantum Mechanics 3

Cyclic quantum evolution and Berry phases

Consider a Hamiltonian $\hat{H}'(\lambda(t))$ that depends on time through a slowly varying parameter $\lambda(t)$. (For example H' could describe the spin Hamiltonian of a neutron in a time dependent magnetic field.) The eigenstates $|\epsilon_n(\lambda)\rangle$ and eigenvalues $\epsilon_n(\lambda)$ of H' are all slow functions of time through the parameter $\lambda(t)$. Any state of the system $|\Psi(t)\rangle$ can be expanded in terms of the instantaneous eigenstates of $\hat{H}'$, i.e. $|\Psi(t)\rangle = \sum \varphi_n(t)\,|\epsilon_n(\lambda)\rangle$.

(a) (6 points) Assume that the system starts in $|\epsilon_0(\lambda)\rangle$ and subsequently evolves in a single eigenstate, i.e. $|\Psi(t)\rangle = \varphi(t)\ |\epsilon_0(\lambda)\rangle$. The parameter $\lambda(t)$ is cyclically varied over a time T so that $\lambda(T) = \lambda(0)$:

   (i) Determine the final state $|\Psi(T)\rangle$ of the system given its initial condition $|\Psi(0)\rangle$.

   (ii) Show that $|\Psi(T)\rangle$ contains a nonvanishing phase ϕ (known as the *Berry phase*) that is independent of the period T. Express the Berry phase in terms of the *Berry connection*, $A \equiv i\hbar\langle\epsilon|\frac{d}{d\lambda}|\epsilon\rangle$.

Now consider a quantum mechanical particle. A time independent Hamiltonian $\hat{H}'(\mathbf{r})$ describes the internal degrees of freedom of the particle (e.g. its spin state). Now $\hat{H}'$ varies slowly in space, but does not vary in time. Its eigenstates $|\epsilon_n(\mathbf{r})\rangle$ and eigenvalues $\epsilon_n(\mathbf{r})$ also vary slowly in space, but not in time.

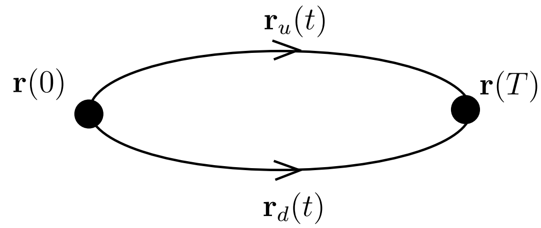
(b) (6 points) Assume that the full wave function in coordinate and "spin" space takes the form $|\Psi(\mathbf{r},t)\rangle = \varphi(\mathbf{r},t) \otimes |\epsilon_0(\mathbf{r})\rangle$ at all times. Analyze the time evolution of $|\Psi(\mathbf{r},t)\rangle$ under the full Hamiltonian $\hat{H} = \hat{\mathbf{p}}^2/2m + \hat{H}'$ and show that the Schrödinger equation for $\varphi(\boldsymbol{r},t)$ involves the effective Hamiltonian

$$\hat{H}_\varphi = \frac{1}{2m}\left[\hat{\mathbf{p}} - \mathbf{A}(\mathbf{r})\right]^2 + V(\mathbf{r}) + \epsilon_0(\mathbf{r}),$$

where $V(\boldsymbol{r}) = \sum_{n\neq 0} |\langle\epsilon_n|\boldsymbol{\nabla}\epsilon_0\rangle|^2$ is a scalar function of no further interest, and $\boldsymbol{A}(\mathbf{r}) \equiv i\hbar\langle\epsilon_0(\mathbf{r})|\nabla\epsilon_0(\mathbf{r})\rangle$ is the Berry connection in this case.

The Hamiltonian $\hat{H}_\varphi$ is analogous to the motion of a charged particle in a magnetic field. The "magnetic field" in this case is known as the *Berry curvature*, $\mathbf{B} = \nabla \times \mathbf{A}$. Now consider a wave packet in the set-up of part (*b*). The packet is localized at a central position $\mathbf{r}(t)$, which slowly changes in time. At $\mathbf{r}(0)$ the wave packet is split into two, $\varphi_\mathrm{u}(\mathbf{r}_u(t))$ and $\varphi_d(\mathbf{r}_d(t))$, and the two waves propagate along opposites sides of the contour (up and down as shown below) to finally interfere at $\mathbf{r}(T)$.

(c) (2 points) Use the results of part (*a*) (with $\boldsymbol{r}(t)$ as the adiabatic parameters) to determine the interference $\varphi_u^*\varphi_d$ at $\mathbf{r}(T)$ in terms of the "magnetic flux" through the contour and the wave functions φ_u and φ_d at $\mathbf{r}(0)$.



(d) (3 points) The eigenfunctions $|\epsilon_n(\mathbf{r})\rangle$ of H' are defined only up to an overall $\mathbf{r}$-dependent phase. If the kets are rotated by a function $\chi(\mathbf{r})$, $|\epsilon_n(\mathbf{r})\rangle \to e^{i\chi(\mathbf{r})}\,|\epsilon_n(\mathbf{r})\rangle$, how does the Berry connection $\mathbf{A}(\mathbf{r})$ and interference of part (c) change? Explain.

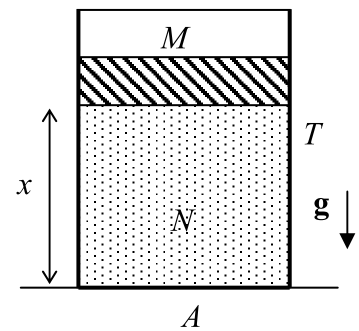
The analysis in parts (a) and (c) was for a single non-degenerate eigenstate $|\epsilon_0(\mathbf{r}(t))\rangle$ with a slowly varying parameter $\mathbf{r}(t)$. Now repeat the analysis of parts (a) and (c) but assume that at each point there are two degenerate eigenstates, $|\epsilon_0(\mathbf{r}(t))\rangle$ and $|\epsilon_1(\mathbf{r}(t))\rangle$, depending on the parameter $\boldsymbol{r}(t)$. Assume that the full wave function lies in their span, $|\Psi(t)\rangle = \varphi_0(t)\,|\epsilon_0(\mathbf{r}(t))\rangle + \varphi_1(t)\,|\epsilon_1(\mathbf{r}(t))\rangle$.

(e) (3 points) Derive an equation of motion and effective Hamiltonian for the pair of functions $(\varphi_0(t), \varphi_1(t))$ involving an appropriate Berry connection, $\mathbf{A}_{ab}(\boldsymbol{r}(t)) = i\hbar\langle\epsilon_a|\nabla\epsilon_b\rangle$ (where $a, b{=}0, 1$). Is adiabatic evolution possible? Explain.

Note that the Berry connection does not commute in this case. This is called non-abelian behavior.

Statistical Mechanics 1

Heavy piston



A vertically positioned cylinder, whose walls are kept at constant temperature T, is closed with a very heavy piston of mass M (see Fig. on the right), and contains $N >> 1$ molecules of an ideal gas. Neglecting the external pressure, and the friction between the piston and the cylinder's walls, calculate:

A (1 point) the equilibrium position x_0 of the piston;

B (2 points) the frequency of small oscillations of the piston near x_0;

C (3 points) the r.m.s. value δx of the piston's thermal fluctuations.

D (4 points) Now let the piston, moving with velocity u, also experience a drag force with the statistical average $\langle F \rangle = -\eta u$. Use this relation and the fluctuation-dissipation theorem to re-derive the answer to question C.

E (6 points) The drag force, mentioned in question D, may arise due to the molecules reflecting from the moving piston. Assuming that the velocity u of the piston is much lower than the typical molecular velocity, use the elementary kinetics and statistics of the ideal gas to calculate the drag coefficient η.

F (2 points) Formulate quantitatively the conditions of validity of your results. In particular, what is the condition that the oscillation process is isothermal?

Statistical Mechanics 2

Gas condensation

A closed container of volume V with a classical gas of $N >> 1$ indistinguishable particles. The inner surfaces of the container's walls have $N_S >> 1$ similar traps (potential wells of small size). Each trap can hold only one particle, in one of g_S degenerate states; energy $\Delta > 0$ is required to free the particle from the trap.

A (3 points). Assuming that the chemical potential μ of the system is known, calculate the number N_g of particles in the gas phase (i.e. in the volume of the container). What condition should be imposed on N_g for the gas to behave classically?

B (4 points). Again assuming that the chemical potential is known, calculate the probability of each trap to be filled, and the full number of filled traps (i.e. of condensed particles), as functions of μ.

C (4 points). Use the results obtained in A and B to derive the equation for the chemical potential of the system in equilibrium.

D (4 points). Solve the equation analytically in the limit of $N/N_S >> 1$, and analyze the solution; in particular, calculate the gas pressure.

E (4 points). Solve the equation for the chemical potential in the opposite limit, $N/N_S << 1$, and calculate the gas pressure in this case.

F (1 point). Summarizing your results, spell out the conditions at which the gas pressure is significantly affected by particle condensation in the traps.

Statistical Mechanics 3

The 1D Potts model

In the so-called Potts model, a uniform 1D chain of N classical spins (in the absence of an external magnetic field) is described by the following interaction Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\eta_i,\eta_j} \text{ , with } J > 0 \text{ ,} \tag{1}$$

where J is a positive coupling constant, η_i is the classical spin variable at the site i, describing the spin state, which may take integer values in the set $\{1, ..., q\}$, $\delta_{a,b}$ is the Kronecker delta symbol; and the summation is over all pairs of adjacent spins. Consider the model with $q = 3$ in thermal equilibrium at temperature T.

Do parts parts (a), (b), and (c) for finite N, and then take the limit $N \to \infty$ for parts (d), (e), and (f). For parts (c), (d), (e), and (f), you should give an explicit closed-form expression, not an abstract expression involving a summation.

(a) (1 pts.) Write the general expression for the statistical sum (= partition function) Z of the system.

(b) (2 pts.) Assuming periodic boundary conditions, express Z via the appropriate transfer matrix.

(c) (4 pts.) Use this expression to calculate Z. (Check your work as all subsequent parts depend on this result.)

Take $N \to \infty$ for the remainder of the problem:

(d) (3 pts.) Calculate the free energy per site, F, and the average energy per site, E.

(e) (3 pts.) Calculate the specific heat capacity per site, C, and the entropy per site, S.

(f) (5 pts.) Calculate the values of E, C, and S in the limits $T \to 0$ and $T \to \infty$. Physically and quantitatively explain your results for E and S in both limits.

(g) (2 pts.) Does this system have a symmetry-breaking phase transition at finite temperature? Prove your answer.