Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2018 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Some of the problems may cover multiple pages. Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

Write your ID number (not your name!) on *each* exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, if approved by the proctor, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

Periodic one-dimensional motion

Part I. Consider one-dimensional motion $x(t)$, as described by the Lagrangian

$$L = \frac{m}{2}\,\dot{x}^2 - g\,|x|\,.$$

(a) (3 points) First draw a qualitative picture of the motion. Then write down and integrate the equation of motion.

(b) (3 points) Find $T(E)$, the period of the motion at given total nonrelativistic energy E.

(c) (3 points) Find the truncated action per period, $\int_{t_0}^{t_0+T} p\,dx$.

Part II. Same as in Part I, but for the relativistic Lagrangian

$$L = -m\sqrt{1-\dot{x}^2} - g\,|x| \;\; \text{with } c = 1\,.$$

(a) (5 points) Integrate the equation of motion. Check that the nonrelativistic limit reduces to the solution in I(a).

(b) (2 points) Find $T(E)$, the period of the motion at given total relativistic energy E.

(c) (2 points) Evaluate again the truncated action per period.

(d) (2 points) Compare the relativistic and the nonrelativistic motion of a particle that is momentarily at rest at $x = x_0$.

Classical Mechanics 2

Discretizing canonical transformations

Many physical systems are described by Hamiltonians which give rise to equations of motion that cannot be solved analytically, but must be discretized and solved numerically. Discretizations which preserve the symmetries of the continuum theory are especially effective when numerically integrating the equations of motion for long times. In this problem, we will explore some of the techniques available to describe such systems.

Consider a one-dimensional classical system whose *finite* time evolution is described by a canonical transformation. Specifically, let

$$x \equiv x(t) \quad , \quad x' \equiv x(t') \quad , \quad p \equiv p(t) \quad , \quad p' \equiv p(t')$$

and consider a generating function $F_2(x, p')$. Then the evolution from (x, p) to (x', p') is obtained by solving the equations

$$p = \frac{\partial F_2}{\partial x} \quad , \quad x' = \frac{\partial F_2}{\partial p'} \tag{1}$$

(a) (5 points)

(i) Show that this evolution preserves volume in phase space (that is, prove Liouville's theorem for this case).

(ii) Next show that for

$$F_2 = xp' + \delta t\, H$$

as $\delta t \equiv t' - t \to 0$, the evolution equations reduce to Hamilton's equations of motion.

(b) (5 points) Suppose there is a conserved quantity $G(x, p)$. Noether's theorem states that this means the system has a symmetry.

(i) What are the transformations of x, p under this symmetry?

(ii) How does the Hamiltonian transform under this symmetry? Explain why this is equivalent to Noether's theorem.

(iii) Now compute the transformation of the Lagrangian expressed in the Hamiltonian form:

$$L(x, \dot{x}) = p\dot{x} - H(x, p)$$

Show that L transforms by a total derivative. You should use Hamilton's equations, but do NOT use the Euler-Lagrange equations.

(iv) A simple example of a conserved quantity is the Hamiltonian itself. For this example, what are the transformations of x, p and L? What is the physical interpretation of this symmetry?

(c) (5 points) For a Hamiltonian of the form $\frac{p^2}{2m} + U(x)$, show that the naive discretization of Newton's equations of motion (for δt small but finite)

$$p' = p - \frac{\partial U(x)}{\partial x}\delta t \quad , \qquad x' = x + \frac{p}{m}\delta t \tag{2}$$

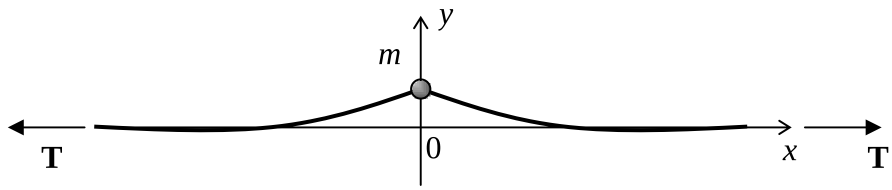
does NOT preserve volume in phase space. For a harmonic oscillator, will the volume shrink or grow? What does this say about the long time behavior of this approximation? Estimate the number of iterations before the error is of order one, in terms of the mass m of the particle, the spring constant k, and the finite interval δt.

(d) (5 points) What is the analogous discretization using canonical transformations? Find the right $F_2(x, p')$, and work out the equations corresponding to (2) in part (c). Why is this *guaranteed* to preserve volume in phase space? What does this say about the long time behavior of this approximation?

Classical Mechanics 3

Open system

A particle of mass *m* is attached to an infinite string, with mass μ per unit length, and stretched with tension *T*. The particle is confined to move along the *y* axis normal to the string (see the figure below), in an additional potential $U(y)$, not related to the string, with a minimum at $y = 0$.



A (5 points). Derive the system of differential equations and boundary conditions describing the dynamics of small deviations of this system from equilibrium.

B (5 points). Assuming that the waves on the string are excited only by the motion of the particle (rather than any external source), reduce this system to an ordinary differential equation for the small displacement of the particle, as a function of time. (Hence consider the case that for $x>0$ there are only right-moving waves, and for $x<0$ only left-moving waves.)

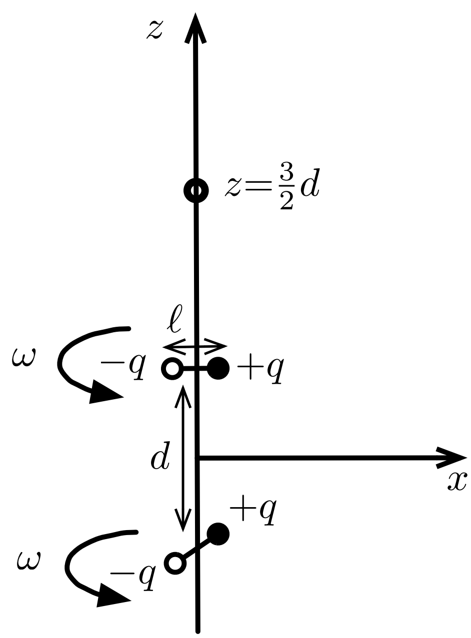
C (4 points). Solve this equation for the case $U(y) = 0$, assuming that the motion was initiated, from equilibrium, by a short external impulse *p* given to the particle. Calculate the final displacement of the particle. Sketch the resulting displacement $y(x, t)$ of some point of the string as a function of time.

D (4 points). For the case of a harmonic oscillator, when $U(y) = m\omega^2 y^2/2$, solve the equation of motion under the same assumptions as in C.

E (2 points). Would you describe this system as a Hamiltonian (energy-conserving) or dissipative?

Electromagnetism 1

Consider two identical rods of length ℓ with charges $+q$ and $-q$ pasted on their ends. The centers of the rods are located on the z-axis which is perpendicular to the length of the rods (see below). The two rods are separated by a distance $d \gg \ell$, with the top and bottom rods located at heights $z = \pm d/2$ respectively. The rods rotate around the z axis with the same frequency ω but are out of phase, and at time t=0 the bottom rod has an azimuthal angle of ϕ_0 while the top rod has $\phi = 0$.



(a) (8 points) First consider the limit $\omega d/c \ll 1$. Determine the (real) electric *and* magnetic fields as a function of time at a height $z = 3d/2$ on the z axis.

(b) (4 points) Next consider the limit $\omega d/c \gg 1$, but still with $\omega\ell/c \ll 1$. Determine the (real) electric *and* magnetic fields at a height $z = 3d/2$ on the z axis.

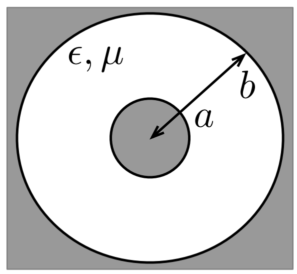
(c) (4 points) With the approximations of part (b), determine the phase ϕ_0 when the fields from the two dipoles add destructively. Explain your result physically.

(d) (4 points) With the approximations of part (b), determine the time averaged electromagnetic power passing through a small area A at a height $z = 3d/2$ on the z axis, with front face directed towards the origin.

Electromagnetism 2

A guided wave:

A simple coaxial cable consists of two cylindrical perfect conductors of infinite length as shown below. The inner and outer conducting cylinders have radii a and b respectively. The space between the conductors is filled with a dielectric with electric and magnetic permeablities ϵ and μ respectively. Assume that the currents are on the conducting surfaces and do not penetrate into the interior of the metal. This is an appropriate approximation at high frequency when the skin depth is small compared to the transverse dimensions.



(a) (2 points) (i) A static potenial difference is maintained between the inner and outer conductors. Determine the capacitance per unit length $\mathcal{C}$. (ii) A current runs down the cable on the surface of the inner conductor and returns on the surface of the outer conductor. Determine the inductance per unit length $\mathcal{L}$. (iii) Determine the product $\mathcal{LC}$ and find a pleasing result.

(b) (4 points) Now consider an electromagnetic wave propagating down the cable in the z direction. Assume that the electric and magnetic fields are perpendicular to z and take the form

$$\boldsymbol{E}(t, \boldsymbol{x}) = \boldsymbol{E}_\perp(\boldsymbol{x}_\perp)\, e^{ikz-i\omega t}\,, \qquad \boldsymbol{B}(t, \boldsymbol{x}) = \boldsymbol{B}_\perp(\boldsymbol{x}_\perp)\, e^{ikz-i\omega t}\,, \tag{1}$$

where $\boldsymbol{x}_\perp$ denotes the (x, y) coordinates. Show that $\boldsymbol{E}_\perp(\boldsymbol{x}_\perp)$ and $\boldsymbol{B}_\perp(\boldsymbol{x}_\perp)$ satisfy the equations of two-dimensional electro and magneto statics; determine the relation between $\boldsymbol{E}_\perp$ and $\boldsymbol{B}_\perp$ and the relation between ω and k.

The wave solutions in Eq. (1) are known as transverse electromagnetic (TEM) waves.

(c) (4 points) Consider the propagating TEM wave of part (c). Show that the current on the conducting surfaces satisfies a one dimensional wave equation.

(d) (5 points) Determine the power transmitted by the TEM wave. Express your result in terms of the amplitude of current wave, I_0, and the radii a and b of the cable.

(e) (5 points) Now assume the walls of the cylinders are not perfect conductors, but have a large but finite conductivity σ. First estimate, and then compute the power lost per length by the TEM wave of part (d). Assume that $kb \ll 1$.

Electromagnetism 3

The Poisson Integral

(a) (5 points) Consider a grounded cylinder of radius a which is infinite in length. A line of charge inside the cylinder has charge per length λ, and is displaced from the center of the cylinder by a distance ρ_0, with $\rho_0 < a$. Determine the potential $\varphi(\rho, \phi)$ at all points inside the cylinder[1].

Hint: consider an appropriate image line of charge at a distance a/ρ_0^2 from the center, and check that the appropriate boundary conditions are satisfied.

(b) (8 points) Now the line of charge is removed, but the surface cylinder is held at potential

$$V_0(\phi) = \begin{cases} V_0 & 0 < \phi < \pi \\ -V_0 & \pi \leqslant \phi \leqslant 2\pi \end{cases} \tag{49}$$

Express the potential inside the cylinder $\varphi(\rho, \phi)$ as a definite integral using the Green function of (a).

(c) (7 points) The potential $\varphi(\rho, \phi)$ of part (b) may also be expressed as a series expansion in the appropriate separated solutions. Determine this expansion and check that the first term in the series agrees with the results of (b) for $\rho \ll a$.

[1] Here $\rho = \sqrt{x^2 + y^2}$ and $\phi = \mathrm{atan}(y/x)$, with x, y measured from the center of the cylinder.

Quantum Mechanics 1

One dimensional "atoms" and "molecules" in external fields

The purpose of this problem is to study the ionization process of a one-dimensional "atom", and the excitation process of a one-dimensional "molecule" under the action of an external time-dependent interaction.

a. (5 points) Consider a one-dimensional atom composed of a spinless particle of charge e and mass m bound by a delta-function potential $V(x) = -g\,\delta(x)$. Find the normalized energy eigenstates of the bound atom. Find the normalized energy eigenstates of the unbound atom assumed as free particles in a box of size L. Derive their density of states as a function of their energy E_k where k is the wavenumber.

b. (5 points) A time-dependent electric field is applied on the atom,

$$E(t) = E_0\theta(t)\theta(\tau - t) \tag{1}$$

with $\theta(t)$ a step function. What is the probability $P(E_k)dE_k$ for the charge to be in an unbound state with energy between E_k and $E_k + dE_k$? Discuss the physical nature of this result for small and large k.

c. (5 points) Now, consider a one-dimensional molecule composed of two distinguishable spin-$\frac{1}{2}$ particles with charges $e_{1,2}$ and masses $m_{1,2}$ coupled harmonically:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}\omega^2(x_1 - x_2)^2 + g_S\,\vec{\sigma}_1 \cdot \vec{\sigma}_2\,. \tag{2}$$

What are the energy levels, wavefunctions and degeneracies of the molecule? Give the explicit form of the normalized wave functions of the two lowest energy levels of the molecule for $g_S/\hbar\omega \ll 1$.

d. (5 points) A time- and spin-dependent interaction is now applied on the molecule

$$V(t) = \left(V_1 + V_2\frac{(x_1 - x_2)}{L}\right)\sigma_{1x}\,\theta(t)\sin(\omega t)\,. \tag{3}$$

Here V_1 and V_2 are constants, x_1 and x_2 are the spatial cooridates of the particles, and σ_{1x} is the spin operator acting on the first particle. The molecule is intially in the ground state when $V(t)$ is applied. What is the transition probability to find the molecule in an excited state if $V_{1,2}$ are small. Comment on any selection rule.

Quantum Mechanics 2

A complex potential

In this problem we will explore the partial transparency of the nucleus to high energy eutrons, i.e. their partial absorption. For that, a complex potential language will be used. Consider a beam of non-relativistic neutrons of mass m and energy E moving along the z-axis towards a nuclear target. The incoming neutron will be treated as a plane wave with wave number k and the nucleus will be approximated by a constant complex potential $\mathbf{V} = -U - iW$. By analogy with optics, we define $n = v/c$ as the index of refraction, where v and c are the effective velocities of the neutron inside and outside of the nucleus respectively.

a. 4 points Derive an expression for n, and express its limit (by Taylor expansion) for high energy neutrons. Derive the intensity of the neutron beam along the z-direction.

b. 4 points Derive the equation for the neutron flow of probability and show that it is not conserved. What is the net rate loss of neutrons? Where do the neutrons go?

The nucleus can be thought of as a collection of N (spinless) nucleons, moving in an attractive well potential $-U(r)$ for $r \leqslant R$, with pair interactions $V(\vec{r}_1, \vec{r}_2)$ described by the following Hamiltonian in the second quantized form

$$\mathbf{H} = \sum_a e_a a_a^\dagger a_a + \frac{1}{2} \sum_{ab,cd} \langle ab|V|cd\rangle \; a_a^\dagger a_b^\dagger a_d a_c \,, \tag{16}$$

with $e_a \,|a\rangle = (\vec{p}^2/2m - U(r))\,|a\rangle$. $a_a^\dagger, a_a$ are creation and annihilation operators.

c. 6 points Write the commutation relations for $a, a^\dagger$ and use them to define the properly symmetrized and normalized ground state $|0_F\rangle$ of the nucleus in zeroth order in V. Use perturbation theory to correct the ground state energy to first order in V.

d. 6 points The neutron in parts **a-b** can be thought to be in an initial state $|\Psi_E\rangle = a_E^\dagger\,|0_F\rangle$ with energy E. Use Fermi Golden rule to express the transition rate between this state and the allowed final states $|\Psi_F\rangle = a_p^\dagger a_{p'}^\dagger a_h\,|0_F\rangle$. This rate can be used as a microscopic estimate for W. Explain.

Quantum Mechanics 3

Polarized spin

In this problem, your task is to derive some basic properties of particle's spin, starting exclusively from the commutation relations between the Hermitian Cartesian components $\hat{S}_j$ of its vector operator $\hat{\mathbf{S}}$:

$$\left[\hat{S}_j, \hat{S}_k\right] = i\hbar \hat{S}_l \varepsilon_{jkl}$$

(where each of indices j, k, and l may take values from 1 to 3, and ε_{jkl} is the Levi-Civita permutation symbol), plus the fact that the each of these component operators has a set of eigenstates, with eigenvalues separated by multiples of $\hbar$. Then you would use the derived properties of the operators to analyze properties of a polarized spin.

A (2 points). Calculate the following commutators: $\left[\hat{S}^2, \hat{S}_j\right]$, $\left[\hat{S}_+, \hat{S}_-\right]$ and $\left[\hat{S}_3, \hat{S}_\pm\right]$, and prove the operator relation $\hat{S}^2 = \hat{S}_-\hat{S}_+ + \hat{S}_3^2 + \hbar\hat{S}_3$, where $\hat{S}^2 \equiv \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_2^2$, and $\hat{S}_\pm \equiv \hat{S}_1 \pm i\hat{S}_2$.

B (5 points). Prove that for the common eigenstates of the operators $\hat{S}^2$ and $\hat{S}_3$, the eigenvalue of the former operator is $\hbar^2 s(s + 1)$, where s is an either integer or half-integer quantum number,[1] which sets the limits for the possible eigenvalues $S_3 = \hbar m$ of the latter operator: $-s \leq m \leq +s$.

C (3 points). In the basis of the these common eigenstates, calculate all matrix elements of the operators $\hat{S}_\pm$ and $\hat{S}_{1,2}$, and then the diagonal matrix elements of operators $\hat{S}_1^2$, $\hat{S}_2^2$, $\hat{S}_1\hat{S}_2$, and $\hat{S}_2\hat{S}_1$.

D (7 points). A free particle with spin s and a gyromagnetic ratio $\gamma \neq 0$ has been placed into an external constant magnetic field, and allowed to relax into its ground state, thus "polarizing" the spin. Use the results of the previous tasks to calculate the expectation value of its spin component along a direction at angle θ with that of the field, and the r.m.s. uncertainty of this value.

E (3 points). Specify your results for $\theta = \pi/2$, and give physical interpretations for their dependence on the spin s.

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[1] This constant s is of course what is called the particle's spin in the narrow sense of the word.

Statistical Mechanics 1

Dipole interaction

Two similar classical electric dipoles, of a fixed magnitude d, are separated by a fixed distance r.

A (4 points). Represent the dipole-dipole interaction energy as a function of the angular orientation of each dipole.

B (3 points). Assuming that each dipole may rotate, with negligible mechanical inertia, write a general expression for its statistical sum Z in thermal equilibrium. (The expression may include a specific definite integral.)

C (7 points). Obtain an explicit expression for Z in the high-temperature limit, and use it to calculate the average interaction energy E, heat capacity, and entropy of the system.

D (3 points). Give a brief physical interpretation of the results. In particular, compare the obtained dependence $E(r)$ with the long-range part of the van der Waals interaction[1] between electrically neutral atoms. What are the main handicaps of this simple model for the description of such interaction between real atoms?

E (3 points). Calculate E explicitly in the limit $T \to 0$, and briefly discuss the result.

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[1] This part is called the *London dispersion force*.

Statistical Mechanics 2

Independent spins in magnetic field

Consider a system of N quantum spins $s = 1/2$ in magnetic field B. The Hamiltonian of such system is

$$H = -\sum_{i=1}^{N} 2\mu B S_i^z \,. \tag{28}$$

Here μ is the magnetic moment of the particle, S_i^z take values $\pm 1/2$ and we assumed that the magnetic field is in the z-direction.

(a) (2 points) Show that the partition function of such a system at temperature T is given by

$$Z(T, N, B) = [2\cosh(\mu B/T)]^N \,.$$

(b) (3 points) Compute the average value of total magnetization defined as

$$M = \langle \hat{\mathcal{M}} \rangle \,,$$

as a function of T, N, B. Here the total magnetization operator is given by $\hat{\mathcal{M}} = 2\mu \sum_{i=1}^{N} \hat{S}_i^z$.

(c) (1 point) Calculate the magnetic susceptibility of the system as a function of T, N, B defined by

$$\chi = \left(\frac{\partial M}{\partial B}\right)_{N,T} \,.$$

(d) (4 points) Find the variance of the magnetization $(\delta M)^2$ as a function of temperature and magnetic field. The variance is defined as

$$(\delta M)^2 = \left\langle \left(\hat{\mathcal{M}} - \langle \hat{\mathcal{M}} \rangle\right)^2 \right\rangle = \langle \hat{\mathcal{M}}^2 \rangle - \langle \hat{\mathcal{M}} \rangle^2 \,.$$

How is the variance of the magnetization related to the magnetic susceptibility?

(e) (3 points) Explicitly determine the variance in the limit of high and low temperatures. Interpret these limiting cases physically (i.e. explain why they are almost obvious).

(f) (8 points) Find the form of the partition function for a system of N classical spins and repeat the calculations of (b), (c), (d), and (e) for *classical* spins. You can take the Hamiltonian of the system to be a classical analogue of (28), i.e.

$$H = -\sum_{i=1}^{N} \boldsymbol{\mu} \cdot \mathbf{B} \,. \tag{29}$$

assuming now that $\boldsymbol{\mu}$ is a classical magnetic moment vector with fixed absolute value $\mu^2 = \boldsymbol{\mu}^2$. Compare the results of part (e) for the classical and spin $\frac{1}{2}$ cases.

Statistical Mechanics 3

Bose-Einstein Condensation:

Consider a non-interacting non-relativistic Bose gas in a macroscopic three-dimensional box of volume V.

(a) (3 points) Write down the appropriate partition function and derive an equation that gives the occupation number as a function of energy at a given temperature and chemical potential.

(b) (5 points) Consider a gas composed of a finite number of particles. Make a sketch of the chemical potential as a function of temperature. Does it go to zero? If so, indicate whether this happens at $T \to 0$ or at some other temperature.

Explicitly determine the chemical potential as a function of temperature in the classical (or high temperature) limit, and also show this approximate result in your sketch.

(c) (3 points) Compute the critical temperature, T_c, above which practically all the particles are in excited states, but below which a significant number is in the ground state.

(d) (4 points) Does the pressure at low temperatures ($T < T_c$) depend on the particle density? If yes, how? If no, explain why. What is limiting behavior of the pressure of the ideal Bose gas as $T \to 0$?

Consider now what happens for an infinite system ($V \to \infty$ with N/V fixed) in different dimensions, 1d, 2d, and 3d:

(e) (4 points) Can a non-relativistic ideal Bose gas of a given number density of particles undergo Bose condensation in d = 1, 2, 3 dimensions? Explain.

Useful Mathematical Formulas

$$\int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n) \tag{30}$$

$$\Gamma(n+1) = n\Gamma(n);\ \Gamma(1/2) = \sqrt{\pi} \tag{31}$$

$$\int_0^\infty \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s) \tag{32}$$

$$\{\zeta(\tfrac{3}{2}), \zeta(\tfrac{4}{2}), \zeta(\tfrac{5}{2}), \zeta(\tfrac{6}{2}), \zeta(\tfrac{7}{2}), \zeta(\tfrac{8}{2}), \ldots\} = \{2.61, 1.64, 1.34, 1.20, 1.13, 1.08, \ldots\} \tag{33}$$