Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2017 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Some of the problems may cover multiple pages. Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

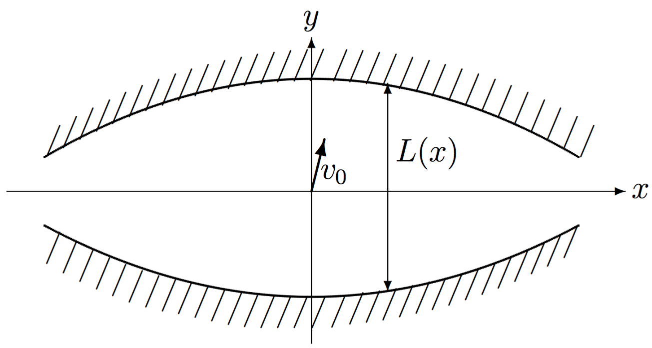
Write your ID number (not your name!) on the exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

A magnetic trap

Consider a non-relativistic particle of mass m moving freely between two fixed curved walls separated by a distance $L(x) = L_0(1 - x^2/a^2)$, where a is a constant with $a \gg L_0$. Assume that $v_x \ll v_y$ and that the particle crosses the midpoint with initial velocity $\vec{v}_0$, where $v_0^2 = v_{0x}^2 + v_{0y}^2$, as shown below. The initial angle is small, i.e. $\theta_0 \equiv \tan^{-1}(v_{0x}/v_{0y}) \ll 1$. There is no gravity.



(a) (6 points) Derive an approximate expression for the value of $x_{\max}$, the maximum distance the particle will reach in its motion in the x-direction. Describe what happens after that.

Hint: the adiabatic invariant, $J_y \equiv \oint p_y dy$, is approximately conserved during the motion.

Now consider a new problem without the walls of part (a). Consider a non-relativistic particle of mass m and charge q moving on a circular orbit in a plane perpendicular to a uniform static magnetic field $\vec{B} = B\hat{x}$ pointing in the $\hat{x}$-direction.

(b) (2 points) Write down an expression for the kinetic energy of the particle in terms of q, m, B, and the radius of the orbit.

(c) (6 points) Now imagine that the particle has a small velocity component in the x-direction, i.e. in the direction of the magnetic field. Moreover, imagine that the static magnetic field $\vec{B}$ acts in the x-direction, but has a tiny positive gradient in that direction, i.e. $\vec{B} = B(x)\,\hat{x}$. The Lagrangian of the system is

$$\mathcal{L} = \frac{m}{2}\left(v_\perp^2 + \dot{x}^2\right) + \frac{q}{2}Br^2\dot{\theta}\,,$$

where θ is the angle with respect to the y-axis in the yz-plane, r is the radius $r = \sqrt{y^2 + z^2}$, and $v_\perp^2 \equiv \dot{r}^2 + r^2\dot{\theta}^2$ is the square of the velocity perpendicular to $\vec{B}$. Assume $\dot{r}$ is small, so that $v_\perp^2 \simeq r^2\dot{\theta}^2$. Show that Br^2 is approximately constant along the particle's motion.

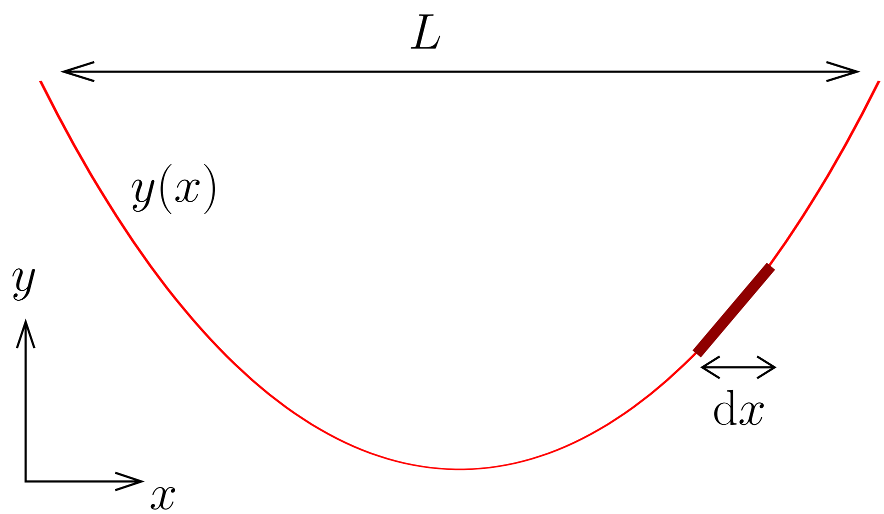
(d) (6 points) Argue now that in analogy to part (a), there is again a maximum value for x. What does the configuration of the magnetic field need to be in order to confine the particle in this “magnetic trap”.

Classical Mechanics 2

An engineered rope

Consider a flexible, but inextensible, rope with total mass m and an adjustable mass per length, $\lambda \equiv \mathrm{d}m/\mathrm{d}\ell$, where ℓ is the distance along the rope. The rope is to be hung between two points separated by a horizontal distance L.

The mass per length is adjusted so that λ is proportional to the tension in the rope when it is hung[1]. Thus for the rope shown below, $\lambda(x) = cT(x)$ with c a proportionality constant, and we have parametrized λ by the x coordinate, i.e. $\mathrm{d}m = \lambda\,\mathrm{d}\ell = \lambda(x)\sqrt{1 + y'(x)^2}\,\mathrm{d}x$. This problem determines the shape of hanging rope $y(x)$ and the necessary mass density.



(a) (8 points) Take a small segment of the rope extending from $x - \mathrm{d}x/2$ to $x + \mathrm{d}x/2$. Draw a well labeled free body diagram for this segment including the tension and gravity. Use the force balance equations in the x and y directions to determine a differential equation for the shape of the rope $y(x)$ with the specified mass density, $\lambda(x) = cT(x)$.

(b) (5 points) Determine $y(x)$. What is the maximum distance, $L_{\max}$, such a rope could traverse?

(c) (3 points) Keeping the mass density $\lambda(\ell)$ fixed, if a small weight of mass $m_o \ll m$ is now hung from the center of the rope, does the height of the center of mass of the rope increase or decrease under the additional load? Explain physically without detailed calculations.

(d) (4 points) Now an additional small mass m_o is hung, but λ is re-adjusted to maintain the constraint, $\lambda(x) = cT(x)$. Determine $y(x)$ with the additional load.

[1] Assuming that the maximum tension is proportional to the mass density, this engineered rope has an equal probability of breaking anywhere along its length, and not just at the points of greatest tension.

Possibly useful integrals:

$$\int \frac{\mathrm{d}u}{1+u^2} = \operatorname{atan}(u) + C$$

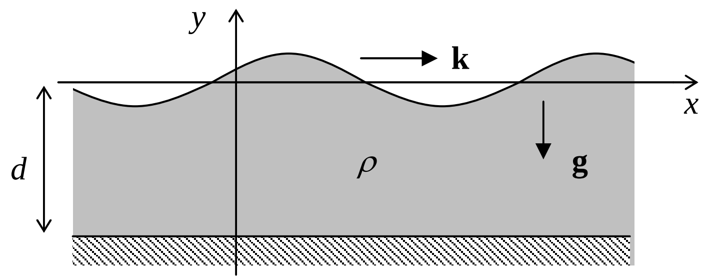
$$\int \frac{\mathrm{d}u}{\sqrt{e^u - 1}} = 2\operatorname{atan}(\sqrt{e^u - 1}) + C$$

$$\int \mathrm{d}u \sec^2(u) = \tan(u) + C$$

Classical Mechanics 3

“Gravity” Waves

The usual waves on a water surface[1] may be reasonably well described modeling the water as an ideal (viscosity-free), incompressible fluid of density ρ, placed in a constant gravity field $\mathbf{g}$. This problem addresses only sinusoidal waves, with all variables independent of one Cartesian coordinate - in the Fig. below, of z.



A (4 points). What general equations should be satisfied by an ideal fluid's density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$, where $\mathbf{r}$ is a fixed point in a lab reference frame (rather than the fluid particle's position)?

B (2 points). How may these equations be simplified:

- if the fluid is incompressible (ρ = const)?
- in the low-velocity limit?

C (4 points). Prove that the simplified equations are satisfied by the following expressions for the particle displacements:

$$q_x(\mathbf{r},t) = Ae^{ky}\cos(kx - \omega t), \qquad q_y(\mathbf{r},t) = Ae^{ky}\sin(kx - \omega t),$$

provided that the wave's amplitude A is small ($kA \ll 1$), and the fluid's depth d is large ($kd \gg 1$).

D (4 points). Calculate the waves’ dispersion relation $\omega(k)$, and find their phase and group velocities.

E (4 points). Modify these results (including the formulas for q_x and q_y) for a fluid of a finite depth $d \sim 1/k$ (but still $d \gg A$), and analyze the resulting dispersion relation.

F (2 points). Qualitatively, how would these dispersion relations be affected by surface tension?

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[1] Traditionally, they were called "gravity waves", but nowadays this term should probably be reserved for the recently observed "real" gravity waves in free space, described by the general relativity.

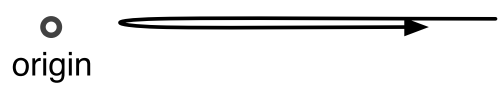
Electromagnetism 1

Radiation during a collision

A classical non-relativistic charged particle of charge q and mass m is incident upon a repulsive mechanical potential $U(r)$

$$U(r) = \frac{\mathcal{A}}{r^2}\,,$$

so that the force exerted on the particle is $\boldsymbol{F} = -\nabla U(r)$. The particle moves along the x-axis and strikes the central potential head on as shown below. The incident kinetic energy (i.e. the kinetic energy of the particle far from the origin) is K.

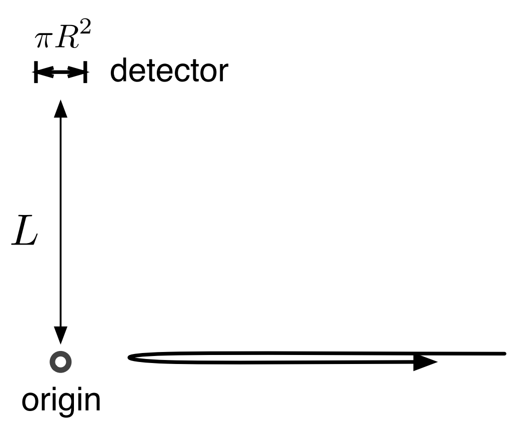


(a) (2 points) Determine the particle's classical trajectory $x(t)$. Adjust the integration constants so that the particle reaches its distance of closest approach at $t = 0$. Check that for late times $x(t)$ approaches $v_o\,t$ with the phyically correct value of v_o. Check that for small times $x(t)$ behaves as $x(t) \simeq x_o + \frac{1}{2}a_o t^2$ with the physically correct value of x_o.

(b) (4 points) Use dimensional reasoning and the Larmour formula to estimate the total energy lost to electromagnetic radiation during the collision. How does the energy lost scale with the incident energy?

(c) (2 points) Calculate quantitatively the energy lost to radiation during the collision processes. Some relevant integrals are given at the end of this problem.

Now consider a detector placed along the y-axis far from the origin as shown below. The front face of the detector has an area of πR^2, and the detector is placed at a distance L from the origin with $L \gg R$.



(d) (2 points) What is the direction of polarization of the observed light in the detector? Explain.

(e) (2 points) What is the typical frequency of the photons that are emitted at 90^o? Explain.

(f) (5 points) For the detector described above, determine the average number of photons received by the detector per unit frequency:

$$\frac{dN}{d\omega}\,. \tag{1}$$

Some relevant integrals are given at the end of the problem.

(g) (3 points) We have determined the photon radiation spectrum using classical electrodynamics. For what values of the parameters $\mathcal{A}$ and K is this approximation justified?

Useful integrals and formulas for EM1:

1. For positive integer n, we note the integrals

$$\int_{-\infty}^{\infty} du \frac{1}{(1+u^2)^n} = \pi\, c_n \tag{2}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \ldots \tag{3}$$

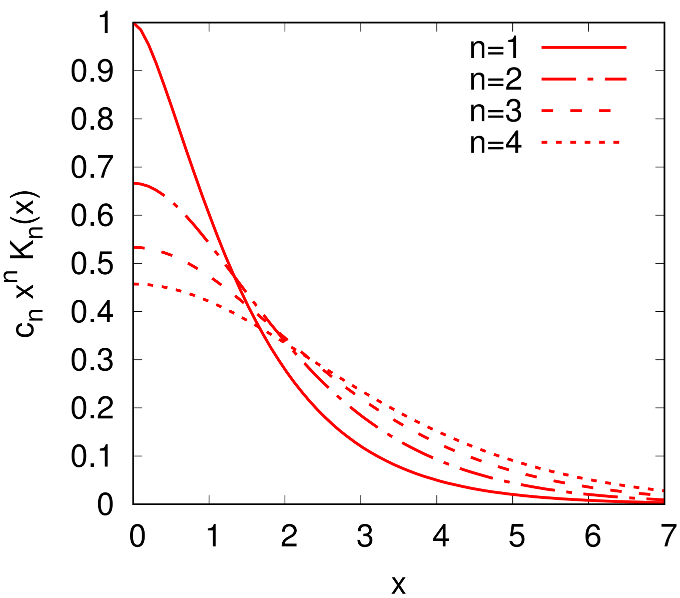
2. For positive integers n, we note the integrals

$$\int_0^{\infty} du \frac{\cos(xu)}{(u^2+1)^{n+\frac{1}{2}}} = c_n\, x^n K_n(x) \tag{4}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{3}, \frac{1}{15}, \frac{1}{105}, \ldots \tag{5}$$

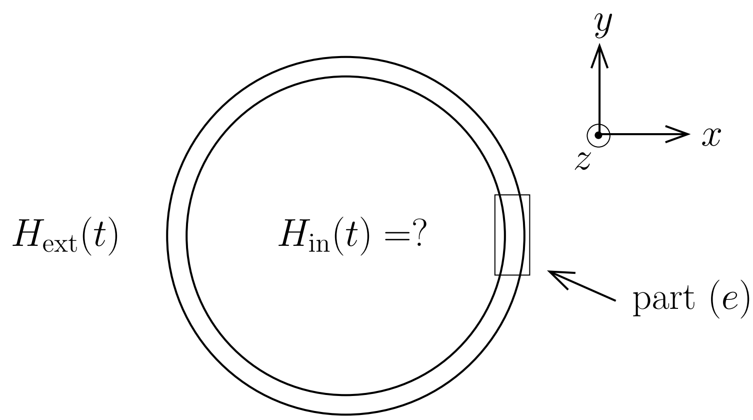
and $K_n(x)$ are the modified Bessel functions. The RHS of Eq. (4) is illustrated below.



Electromagnetism 2

A cylindrical shell in a magnetic field

Consider an infinitely long cylindrical ohmic shell of conductivity σ and radius a. The walls have thickness Δ, with $\Delta \ll a$. The shell is placed in a uniform, but time dependent, external magnetic field $H_{\text{ext}}(t)$, which is directed along the z-axis as shown below. The goal of this problem is to determine the magnetic field inside the cylinder. The thickness Δ is sufficiently small that the induced current density may be considered (spatially) constant inside the shell wall in parts (a)–(c).



(a) (1 point) For a specified surface current $\boldsymbol{K} = K(t)\,\hat{\boldsymbol{\phi}}$, how is the magnetic field inside the shell related to the external magnetic field.

(b) (5 points) Determine a differential equation for the evolution of the magnetic field inside the cylinder. Check that your equation is dimensionfully correct.

(c) (5 points) For a sinusoidal external field, $H_{\text{ext}}(t) = H_o e^{-i\omega t}$, determine the amplitude of the magnetic field's sinusoidal oscillations inside the cylinder. Make a graph of the ratio of the interior to exterior amplitudes as a function of frequency.

(d) (4 points) At higher frequency the induced current changes appreciably over the wall thickness Δ. Estimate the frequency where this (neglected) dynamics becomes important.

(e) (5 points) Determine the amplitude of magnetic field's sinusoidal oscillations inside the cylinder without assuming that the induced current is constant within the walls. Check that for small Δ you reproduce the results of part (c).

Hint: Magnify and analyze the highlighted region shown in the figure to relate the interior and exterior. Treat the walls of the cylinder as having infinite transverse (y and z) extent, so that all fields in the walls are functions x only.

Electromagnetism 3

Angular momentum in a wave packet

Consider a wave packet with a transverse profile $E_o(x, y)$ propagating in the z direction (see eq. (3) for a complete specification of $\boldsymbol{E}$ and $\boldsymbol{B}$). Although the precise form of $E_o(x, y)$ is not needed below, for definiteness you may assume that the wave packet has a Gaussian profile for

$$E_o(x, y) = \mathcal{A} e^{-\frac{x^2+y^2}{4\sigma^2}} , \tag{1}$$

and is infinitely broad in the z direction. The following integrals may be useful:

$$\int_{-\infty}^{\infty} \mathrm{d}u \; e^{-\alpha u^2} = \sqrt{\frac{\pi}{\alpha}} , \tag{2a}$$

$$\int_{-\infty}^{\infty} \mathrm{d}u \; e^{-\alpha u^2} e^{iku} = \sqrt{\pi} e^{-\frac{k^2}{4\alpha}} . \tag{2b}$$

(a) (2 points) When all derivatives of $E_o(x, y)$ are neglected, show that[3]

$$\boldsymbol{E}^{(0)}(t, \boldsymbol{r}) = E_o(x, y) \, e^{i(kz-\omega t)} \, \frac{(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})}{\sqrt{2}} , \tag{3a}$$

$$\boldsymbol{B}^{(0)}(t, \boldsymbol{r}) = \hat{\boldsymbol{z}} \times \boldsymbol{E}^{(0)} , \tag{3b}$$

is a solution to the Maxwell equations for $\omega = ck$.

(b) (3 points) Calculate the time averaged energy per length in the wave packet, $\langle U \rangle$.

(c) (5 points) When the derivatives of $E_o(x, y)$ are not neglected, Eq. (3) is not a solution to the Maxwell equations. Determine the corrections to $\boldsymbol{E}^{(0)}$ and $\boldsymbol{B}^{(0)}$ to first order in gradients for $k\sigma \gg 1$.

Hint: try a solution for $\boldsymbol{E}$ (and analogously for $\boldsymbol{B}$) of the form

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}^{(0)} + E^{(1)}(x, y) e^{i(kz-\omega t)} \hat{\boldsymbol{z}} , \tag{4}$$

and determine the correction $E^{(1)}(x, y)$ in terms of $E_o(x, y)$ and its derivatives.

(d) (4 points) Write the solution to part (c) as a linear superposition of the plane wave solutions to the Maxwell equations. First use the superposition to qualitatively explain the correction to the electric field (proportional to $\hat{\boldsymbol{z}}$), and then use the superposition to precisely reproduce this correction.

(e) (4 points) Calculate the z-component of the time averaged angular momentum per length in the wave packet, $\langle L^z \rangle$, to the lowest non-trivial order in $k\sigma$.

(f) (2 points) Determine the ratio $\langle L^z \rangle / \langle U \rangle$. Interpret the result using photons.

[3] This is Gaussian or Heaviside-Lorentz units. In SI units the magnetic field reads, $\boldsymbol{B}^{(0)} = \frac{1}{Z_0} \hat{\boldsymbol{z}} \times \boldsymbol{E}^{(0)}$ where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376\,\mathrm{Ohms}$ is the vacuum impedance.

Quantum Mechanics 1

A charged massive spinless particle in a magnetic field

Consider the spinless particle with charge e and mass m in a constant magnetic field B directed along the z-axis

1. (4 points) Write down the non-relativistic Hamiltonian describing this problem, and find the operator of particle velocity $\hat{\mathbf{v}}$.

2. (4 points) Establish the commutation relations for the spatial components of this operator $[\hat{v}_i, \hat{v}_j]$, and for $[\hat{v}_i, \hat{x}_j]$, where $\hat{x}_j$ is the coordinate operator. Explain physically what these commutators imply about measurements of the system.

3. (8 points) (i) Write down the Schrödinger equation describing the problem and find the energy spectrum. (ii) Using the Schrödinger equation from (i), write down the wave function of the lowest Landau level explicitly. (iii) Determine the degeneracy of this energy level for a system of area $A = L_x L_y$ perpendicular to the magnetic field.

4. (4 points) Evaluate the commutator of the angular momentum component $\hat{l}_z$ and velocity component $\hat{v}_z$. Provide a physical interpretation of the result.

Quantum Mechanics 2

Parity violation in atomic hydrogen

This problem is about the weak interactions, but its solution requires only non-relativistic quantum mechanics.

In atomic hydrogen, weak currents associated with the exchange of Z_0 bosons ($m_Z = 92.6$ GeV/c^2) give rise to an additional interaction between the electron and the proton of the form

$$H_w = \beta_w[\vec{s}\cdot\vec{p}\,\delta^3(\vec{r}) + \delta^3(\vec{r})\ \vec{s}\cdot\vec{p}] \quad \text{with } \beta_w \approx 1.4\times 10^{-8}\ \mathrm{m^4/Js} \tag{59}$$

Here, $\vec{s}$ and $\vec{p}$ are the spin and the momentum operators of the bound electron, and the proton is assumed to be fixed at $\vec{r} = 0$. The weak interaction leads to a modification of selection rules for electric-dipole transitions, and may thus e.g. be characterized via optical spectroscopy.

(a) (5 points) (i) Assuming that H_w has no effect, calculate the lifetime of *both* the $2p(m = 0)$ and the $2s$ states due to spontaneous decay with an electric-dipole approximation. (ii) Use your result to estimate the lifetimes of these states in seconds. [Hint: The decay rate, or Einstein A coefficient, is $A = \omega_0^3|D|^2/3\pi\epsilon_0\hbar c^3$, where D is the electric dipole moment and $\hbar\omega_0$ the energy difference. A list of hydrogen wave functions and useful integrals is given at the end of this problem.]

(b) (2 points) Estimate the spatial range of the weak interaction and compare it to the Bohr radius of the bound electron. In view of this result, how justified is the use of the δ functional in the expression for H_w?

(c) (9 points) (i) Using symmetry arguments, show that H_w violates parity. (ii) Give a rough estimate for the (leading-order) modification to the lifetime of an electron in $2s$ [Hint: the $2s_{1/2}$ and $2p_{1/2}$ states are separated in energy by the Lamb shift, $\Delta E \simeq 4\times 10^{-6}$ eV and mix; very crudely assume $\vec{s}\cdot\vec{p} \sim (\hbar/2)p_r$ and ignore angular dependencies]

(d) (4 points) Determine whether H_w can produce a permanent electric-dipole moment in the 2s state. If so, what would this imply for time-reversal symmetry?

QM2: Wave-functions and integrals

- The hydrogen wave functions read:

$$\psi_{1s}(r,\theta,\phi) \propto e^{-r/a_0} \tag{60a}$$

$$\psi_{2s}(r,\theta,\phi) \propto (2 - r/a_0)e^{-r/2a_0} \tag{60b}$$

$$\psi_{2p,m=\pm1}(r,\theta,\phi) \propto \frac{r}{a_0}e^{-r/2a_0}\sin\theta e^{\pm i\phi} \tag{60c}$$

$$\psi_{2p,m=0}(r,\theta,\phi) \propto \frac{r}{a_0}e^{-r/2a_0}\cos\theta \tag{60d}$$

- A relevant integral

$$\int_0^\infty \mathrm{d}x\, x^n e^{-x} = n! \tag{61}$$

Quantum Mechanics 3

Scattering from a spherical shell

Consider a particle of mass m and energy E scattering on a 3-dimensional and spherically symmetric shell potential $V(r)$ as described by the Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} + V(r) = \frac{\vec{p}^{\,2}}{2m} + \alpha\,\delta(r - r_0) \tag{1}$$

a. (5 points) Determine the stationary S-state wavefunction for $E > 0$ and the corresponding phase shift.

b. (5 points) In the long wavelength limit, give the form of the phase shift and the explicit form of the scattering length.

c. (5 points) How many bound states can exist for the lowest partial wave and how does their existence depend on α? A graphical proof is acceptable.

d. (5 points) What is the scattering length when a bound state appears at $E = 0$? Describe the behavior of the scattering length as a function of α, from repulsive to attractive and give a well labeled sketch of your description.

Statistical Mechanics 1

Semi-circular energy distribution

A system of identical, non-interacting, spin-½ particles has $N >> 1$ orbital single-particle eigenenergies ε, with the following semi-circular distribution function:

$$\rho(\varepsilon) = N\frac{2}{\pi}\left(\Lambda^2 - \varepsilon^2\right)^{1/2}, \qquad \text{for} -\Lambda \leq \varepsilon \leq +\Lambda\,, \tag{1}$$

where $\rho(\varepsilon)d\varepsilon$ is the number of different eigenenergies within a small interval $d\varepsilon$.

A (2 points). What number N_g of particles in the system provides the lowest value of its ground state energy E_g, and what is this value?

B (8 points). Now let us temperature be different from zero, but the number of particles still have the value N_g calculated in Task A. Derive an explicit expression for the free energy F of the system. (An integral that cannot be worked out analytically in the general case is acceptable.)

C (6 points). Simplify your result for F, and calculate the entropy S of the system, in the limit of low (but still non-zero) temperature.

D (4 points). Simplify your general result for F, and calculate S, in the opposite, high-energy limit, and interpret your result.

Statistical Mechanics 2

1D Boltzmann gases

Consider a monoatomic gas of identical non-relativistic particles with mass m. The particles are confined to move along a line segment of length L and they can pass through each other, so that the gas can be treated as ideal.

(a) (4 points). Calculate the chemical potential of such monoatomic gas in equilibrim at temperature T and density $n = N/L$, where $N \gg 1$ is the total number of particles.

(b) (6 points) Now assume that two atoms can form a bound state, a diatomic molecule with binding energy Δ. Calculate the density $n_{2\mathrm{at}} = N_{2\mathrm{at}}/L$ of such diatomic molecules in equilibrium with the monoatomic gas at temperature T and express it through the density $n_{1\mathrm{at}} = N_{1\mathrm{at}}/L$ of the latter. The total number of atoms $N_{1\mathrm{at}} + 2N_{2\mathrm{at}} = N$ is conserved.

(c) (5 points) Further, consider internal harmonic oscillations of the diatomic molecules from part (b) with frequency $\omega \ll \Delta/\hbar$. Calculate the chemical potential $\mu_{2\mathrm{at,osc}}$ of such diatomic gas assuming that $T \ll \Delta$.

(d) (5 points) Finally, find the density of oscillating diatomic molecules $n_{2\mathrm{at,osc}}$ in equilibrium with the monoatomic gas with density $n_{1\mathrm{at}}$ and temperature T. Compare this result to part (b) and discuss the difference qualitatively in the cases $T \ll \hbar\omega$ and $T \gg \hbar\omega$.

Statistical Mechanics 3

Molecular field

The molecular-field approach in the theory of continuous phase transitions,[1] first suggested in 1908 by P.-E. Weiss, is based on taking the random variable s (say, a component of a classical "spin" variable s) in the form

$$s = \eta + \tilde{s}, \qquad \text{with } \eta \equiv \langle s \rangle, \quad \text{and } |\tilde{s}| << \langle s \rangle .$$

A (6 points). Apply the molecular-field approach to the Ising model of ferromagnetic transitions,

$$E = -J \sum_{\{j,j'\}} s_j s_{j'} - h \sum_j s_j , \qquad \text{with } J > 0 \text{ and } s_j = \pm 1 ,$$

where $\{j, j'\}$ means the pairs of nearest neighbors, on an infinite, d-dimensional cubic lattice. In particular, derive a self-consistency equation for the order parameter η.

B (3 points). Use the self-consistency equation to calculate the critical temperature T_c of the phase transition, and sketch the magnetization curves $\eta(h)$ at $T < T_c$ and $T > T_c$. How close is it to the calculated T_c to the exact values (in the same model), for d = 1, 2, and 3? Briefly explain the sign of the difference.

C (9 points). Apply the same approach to the so-called classical Heisenberg model,

$$E = -J \sum_{\{j,j'\}} \mathbf{s}_j \cdot \mathbf{s}_{j'} - \sum_j \mathbf{h} \cdot \mathbf{s}_j , \qquad \text{with } J > 0 .$$

(Here, in contrast with the Ising model, the spin of each site is modeled with a classical 3D vector **s** of a fixed length $s = 1$.) Again, calculate the critical temperature and analyze the magnetization curves at low and high temperatures.

D (2 points). Compare the molecular-field results for T_c in these two models, and interpret their difference.

[1] Sometimes it is called the mean-field theory; however, this terminology may be misleading, because it invites confusion with the Landau-type mean-field theories (such as the Ginzburg-Landau or Gross-Pitaevskii equations), which are of a higher level of phenomenology, and in particular treat T_c as a given parameter.