Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2019

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

The adiabatic oscillator

Many physical systems involve motion at different time scales. We can treat such a system using a form of the WKB approximation. Consider an undamped harmonic oscillator with a slowly time dependent frequency $\omega(t)$, i.e. $|\dot{\omega}| \ll \omega^2$.

(a) (4 points) Write down the time-dependent Lagrangian for a particle of mass m moving along the x-axis in a harmonic potential with oscillation frequency $\omega(t)$.

Substitute the ansatz

$$x(t) = e^{a(t)+i\theta(t)} ,$$

into the Euler-Lagrange equations for $x(t)$, and without approximation determine the equations of motion for $a(t)$ and $\theta(t)$ by examining the real and imaginary parts.

(b) (4 points) Assuming that $\dot{a}^2$, $|\ddot{a}| \ll \omega^2$, determine the leading-order solution for the real $x(t)$ given the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. Use this result to show that $|\dot{a}| \ll \omega$ is equivalent to $|\dot{\omega}| \ll \omega^2$.

Hint: $a(t)$ can be expressed in terms of $\dot{\theta}(t)$ without approximation.

(c) (4 points) Show that the energy E is *not* constant in time to first order in $\dot{\omega}$. Comment on the result in light of Noether's theorem.

(d) (4 points) Show, however, that the local average $\overline{E(t)/\omega(t)}$ *is* constant in time to first order in $\dot{\omega}$. The time average is taken over a local oscillation period $T = 2\pi/\omega(t)$ over which $\omega(t)$ does not change by much. Comment on the result in light of adiabatic invariants.

An example of such a system might be a pendulum consisting of a heavy mass on a thin cable with a time-dependent length $\ell(t)$. Imagine that the length of the cable varies due to the variation in the temperature throughout the day as

$$\ell(t) = \ell_0(1 + \beta\cos(\Omega t)) ,$$

where $t = 0$ at noon and $\Omega = 2\pi/1\,\text{day}$. Assume that $\beta \ll 1$ and that the oscillation amplitude is small.

(e) (4 points) If the pendulum is at its maximum displacement ϕ_0 at noon, what are the amplitude and phase of the oscillations at 6:00 p.m.?

Classical Mechanics 2

Stellar Orbits

The gravitational potential of the Milky Way galaxy can be reasonably approximated by the axisymmetric form

$$\Phi(r, z) = \frac{1}{2} v_0^2 \ln \left(r^2 + \frac{z^2}{q^2} \right) , \tag{1}$$

where $r = \sqrt{x^2 + y^2}$ is the radial distance from the z-axis (i.e. we are using cylindrical coordinates r, ϕ, z), and v_0 and q are constants. In this problem we will study the orbits of stars in this potential.

(a) (2 points) For a star test particle of mass m, determine two constants of motion.

(b) (6 points) Find the location and period of stable circular orbits.

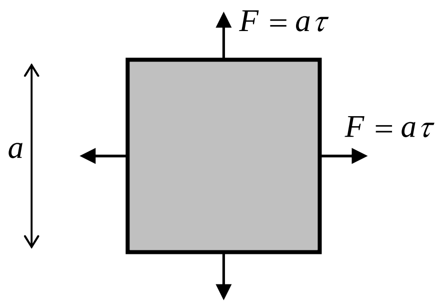
(c) (6 points) Now, consider orbits which are nearly circular $r(t) = r_{circ} + \delta r(t)$, $z = z_{circ} + \delta z(t)$. Determine the frequency of small oscillations in δr and δz. What do we mean by "small"? (That is, what are the criteria on δr and δz for this analysis to be valid?)

(d) (6 points) Our sun is at a distance of about 8 kpc (2.5×10^{17} km) from the center of the Milky Way galaxy, on an orbit that is approximately circular with a period of 225 million years. The sun undergoes vertical oscillations with an period of 87 million years, and radial oscillations with a period of 160 million years. Determine matter density profile. If you are uncertain of your results from (b) and (c) leave the result in terms of v_0 and q for partial credit. Newton's constant is $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}.

Classical Mechanics 3

Waves vs. oscillations

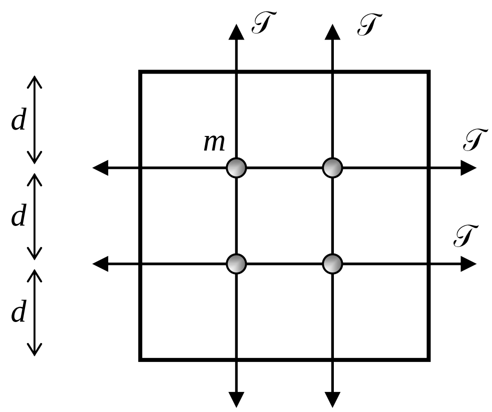
A uniform flexible membrane of area $a \times a$, with mass μ per unit area, is stretched on a thin plane frame, with an isotropic tension τ per unit length – see figure below.



A (6 points). Use any approach you like to derive the 2D wave equation describing small transverse displacements $z(x, y, t)$ of the membrane. (Here x and y are the coordinates in the plane's frame.)

B (3 points). Use the derived equation to calculate the frequency spectrum of standing waves in the system; sketch a few lowest wave modes.

Now consider a discrete-point analog of the system, with 4 particles of equal masses m, connected with light, flexible strings that are stretched with equal tensions $\mathscr{T}$ – see figure below. (Just as in the case of the membrane, the thin frame does not allow the string ends to deviate from its plane.)



C (4 points). Use any approach you like to derive the system of equations describing small transverse oscillations $z_{i,j}(t)$ of the particles.

D (5 points). Use the derived system to calculate the modes and frequencies of the oscillations. (*Hint*: You may take clues from the system's symmetry and from the solution of Part B.)

E (2 points). Compare the calculated frequencies with those of the membrane quantitatively, and comment.

Electromagnetism 1

A cylinder and a line charge

A line charge with linear charge density λ is placed at a distance R and parallel to a conducting cylinder of radius b ($R > b$), which is grounded.

a. 8 points Determine the electrostatic potential at any point and discuss its limits.

b. 8 points Determine the charge per area $\sigma(\theta)$ induced on the surface of the cylinder. For $R/b = 2, 4$, evaluate and sketch $\sigma(\theta)$ as function of the azimuthal angle θ.

c. 4 points Determine the force per unit length on the line charge. Explain its behavior as the line charge approaches the cylinder.

Electromagnetism 2

Radiation from a current sheet:

(a) (5 points) The retarded Green function of the $1+1$ dimensional wave equation dimensional wave equation is defined as the solution to

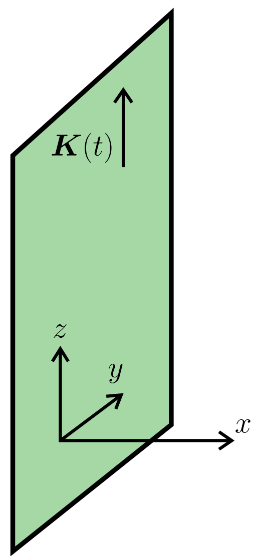
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) G(tx|t'x') = \delta(t-t')\delta(x-x')\,. \tag{1}$$

Show that the retarded Green function is

$$G(tx|t'x') = \frac{c}{2}\,\theta\left(t-t'+(x-x')/c\right)\,\theta(t-t'-(x-x')/c)\,. \tag{2}$$

Does $G(tx|t'x')$ satisfy the appropriate boundary conditions? Explain.

Now consider an infinite sheet spanning the yz plane, with uniform surface current, $\boldsymbol{K}(t) = K(t)\,\hat{\boldsymbol{z}}$.



(b) (5 points) Determine the gauge potentials $\varphi(t,x)$ and $\boldsymbol{A}(t,x)$ in the Lorenz gauge (without approximation) to the right and left of the sheet. Check that the Lorenz gauge condition is satisfied.

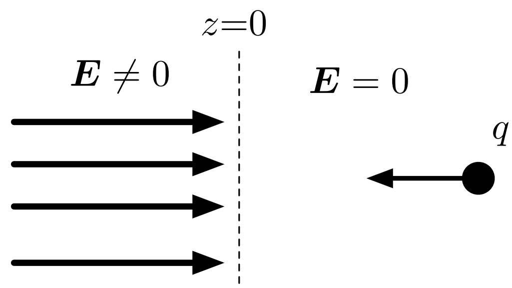
(c) (5 points) Determine the electric and magnetic fields, and the energy radiated per unit time and area by the current sheet.

(d) (5 points) The electric and magnetic fields close to the sheet can be calculated using the elementary methods of Ampere's and Faraday's laws. (i) For $x > 0$, use these methods to calculate the magnetic field $\boldsymbol{B}(t,x)$ and the electric field difference, $\Delta\boldsymbol{E} \equiv \boldsymbol{E}(t,x) - \boldsymbol{E}(t,0)$, close to the sheet. (ii) Show that the general results of (c) agree with these calculations in the appropriate limit. (iii) How close to the sheet does one need to be in order for the elementary methods to be applicable? Assume that the time-dependent currents are characterized by a time scale τ.

Electromagnetism 3

Radiation during linear acceleration

(a) (6 points) An ultra-relativistic ($\gamma \gg 1$) positively charged particle of charge q and mass m is traveling with velocity $v_0 \equiv c \tanh y_0$ in the negative z direction from positive infinity as shown below. At $z = 0$ the particle enters a semi-infinite region ($z < 0$) of homogeneous electric field directed in the positive z direction, $\boldsymbol{E} = E\,\hat{\boldsymbol{z}}$.



(i) Determine the particle's position $z(\tau)$ as a function of proper time τ when the particle is in the electric field.

(ii) Determine how long (in the laboratory frame) the particle remains in the electric field.

(b) (6 points) Determine the total energy radiated by the particle as it accelerates in the electric field.

(c) (2 points) At what angle(s) relative to the z-axis is the radiation peaked? Explain.

(d) To determine the radiation of low frequency photons, the particle's acceleration may be treated with an impulsive approximation

$$\boldsymbol{a}(t) = 2v_0\,\hat{\boldsymbol{z}}\,\delta(t)\,. \tag{1}$$

(i) (4 points) For what range of frequencies is the impulsive approximation valid? Explain.

(ii) (2 points) Use dimensional and physical reasoning to deduce the dependence on ω of the distribution of radiated photons per unit frequency $\mathrm{d}N/\mathrm{d}\omega$ (at low frequency).

Quantum Mechanics 1

An electron in E&M fields

An electron is confined to move within the $[x, y]$ plane. An external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ is applied. Neglect spin.

(a) (2 points) What are the stationary values (i.e. eigenvalues) of the electron's energy?

Hint: work in the Landau gauge where $\mathbf{A} = Bx\hat{\mathbf{y}}$.

(b) (2 points) What is the ground state wavefunction (ignoring normalization)?

(c) (2 points) Show that the ground state degeneracy scales with the area of the system and the magnetic field.

Now an electric field is added, $\mathbf{E} = \mathcal{E}\hat{\mathbf{x}}$. This corresponds to a new term in the Hamiltonian, $H_{\text{elec}} = -e\mathcal{E}x$.

(d) (3 points) What are the stationary values of the electron's energy now?

(e) (3 points) How does the electric field change the ground-state wavefunctions?

(f) (3 points) Derive an expression for the probability current density, $\langle \mathbf{j}(x, y) \rangle$, evaluated in these states.

(g) (2 points) Evaluate the current density of part (f) at the point (x, y) where the wave function squared $|\psi(x, y)|^2$ is maximal.

(h) (1 point) Give a physical interpretation of the last result.

Quantum Mechanics 2

Orbital and spin dynamics in magnetic field

A. In this part of the problem, we consider a spinless particle with mass m and electric charge q, moving in an external magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$.

A.1 (4 points). Prove that the expression $\hat{\mathbf{v}} \equiv (\hat{\mathbf{p}} - q\mathbf{A})/m$ is the legitimate operator of particle's velocity in the sense that $\hat{\mathbf{v}} = d\hat{\mathbf{r}}/dt$, where $\mathbf{r}$ is its radius-vector. Derive the commutation relations between the Cartesian components of the vector-operator $\hat{\mathbf{v}}$, in terms of the components of the vector $\mathbf{B}$.

A.2 (4 points). Prove that

$$m\frac{d\hat{\mathbf{v}}}{dt} = q\frac{\hat{\mathbf{v}}\times\mathbf{B} - \mathbf{B}\times\hat{\mathbf{v}}}{2}.$$

Discuss the relation between this equality and the classical expression for the Lorentz force.

B. In this part of the problem, we ignore the orbital motion of the particle, but take into account its spin-½, with a gyromagnetic ratio $\gamma \neq 0$.

One of the key components of the currently developed quantum information technology is the so-called *Hadamard gate* – a system that acts on a *qubit*, i.e. of a two-level quantum system, performing the following transformation of its basis states - in this field, traditionally called 0 and 1:

$$\hat{\mathscr{H}}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \qquad \hat{\mathscr{H}}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{1}$$

B.1 (8 points). Identifying the state 0 with the spin-½ fully polarized along the z-axis, and the state 1 with its opposite polarization, prove that the Hadamard transform may be implemented by applying to the particle a constant magnetic field $\mathbf{B}$ with the following Cartesian components:

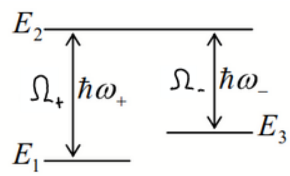
$$B_x = \frac{B}{\sqrt{2}}, \qquad B_y = 0, \qquad B_z = \frac{B}{\sqrt{2}},$$

for a certain time interval t, and calculate this interval.

B.2 (4 points). Calculate $\langle\mathbf{S}(t)\rangle$ for the time evolution of the spin vector-operator $\hat{\mathbf{S}}$ in this field for the initial state 0. Give a geometric description of this evolution, and mark the trajectories describing the particular transformations (1).

Quantum Mechanics 3

Dynamics of a driven 3-level atom



Consider a three-level atom with non-degenerate eigenstates $|\psi_j\rangle$ and energies E_j ($j = 1, 2, 3$) as shown in the figure (set $E_1 = 0$). The system is subject to weak resonant couplings $\hat{H}'_\pm(t) = \hbar\Omega_\pm(t)\cos\omega_\pm t$ between $|\psi_1\rangle$ and $|\psi_2\rangle$, and between $|\psi_3\rangle$ and $|\psi_2\rangle$, with $\omega_+ = (E_2 - E_1)/\hbar$ and $\omega_- = (E_2 - E_3)/\hbar$ respectively. The coupling amplitudes $\Omega_\pm(t)$ are adiabatically varying in time, and can be assumed to be real. Neglect any direct coupling between $|\psi_1\rangle$ and $|\psi_3\rangle$.

(a) **Equation of motion [5pts].** Use time-dependent perturbation theory to find the equations of motion for the amplitudes $a_j(t)$ in the evolution of $|\psi(t)\rangle = \sum_{j=1}^{3} a_j(t)|\psi_j\rangle$, and write these equations out in matrix form.

(b) **Rotating-wave Hamitonian [8pts].** (1) Re-write the equations of motion in terms of new amplitudes $b_1 = a_1$, $b_2 = a_2 \exp(i\omega_+ t)$, $b_3 = a_3 \exp(i[\omega_+ - \omega_-]t)$, neglecting any remaining oscillating terms. Why is this approximation justified? (2) Find the eigenenergies of the resulting effective Hamiltonian, and show that one of its three eigenstates, the so-called "dark state", has amplitudes $(\cos\theta, 0, -\sin\theta)$ in the new effective basis, where $\tan\theta = \Omega_+/\Omega_-$. Why does the term "dark state" make sense?

(c) **Sequential state transfer [3pts].** Suppose that you want to transfer the state of the system from $|\psi(0)\rangle = |\psi_1\rangle$ to $|\psi(\infty)\rangle = |\psi_3\rangle$. One way to do this is to first apply Ω_+ (with Ω_-=0) in order to perform a state rotation $|\psi_1\rangle \to |\psi_2\rangle$, and to then apply Ω_- (with Ω_+=0) for $|\psi_2\rangle \to |\psi_3\rangle$. Assuming that both $\Omega_\pm(t)$ are square pulses with amplitude Ω, solve the equation of motion for each state rotation and find the pulse duration τ necessary for making the transfer complete.

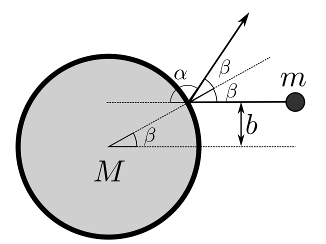
(d) **Adiabatic state transfer [4pts].** The existence of the dark state can be used to smoothly transfer the state of the system from $|\psi(0)\rangle = |\psi_1\rangle$ to $|\psi(\infty)\rangle = |\psi_3\rangle$ without going through $|\psi_2\rangle$. Explain why and how this can be done, and qualitatively describe suitable pulse profiles $\Omega_\pm(t)$. What can you say about the temporal ordering and overlap of the two pulses? Which of the two methods is more robust in view of possible fluctuations of the pulse parameters?

Statistical Mechanics 1

Brownian motion

The goal of this problem is to explore several aspects of the Brownian motion of a classical particle.

In this model, the particle is a hard sphere of a radius R and mass M, which is hit by molecules of an ideal classical gas at temperature T. The size r and mass m of the gas molecules are negligible compared to those of the particle. Assume also that all the molecular collisions with the particle are elastic and specular (i.e. without any angular momentum transfer).



First, assume that the particle is initially at rest.

(A) [4pt] Calculate the *hit rate*, i.e., the number of collisions of the molecules with the particle per unit time.

(B) [4pt] Calculate the *average energy transferred* to the particle in one collision.
Now, assume that the particle has initial velocity some $\mathbf{V}$ with respect to the gas, which is much slower than the r.m.s. velocity of the molecules.

(C) [5pt] Calculate the *drag coefficient* η defined by the relation

$$\langle \mathbf{F} \rangle = -\eta \mathbf{V}\,, \tag{1}$$

where $\mathbf{F}$ is the effective force resulting from molecular hits.

Hint: For the calculation, you may want to consider how the Maxwell's distribution is modified in the reference frame of the particle.

(D) [4pt] Use the Langevin equation describing the molecular hits (but no other forces)

$$M\dot{\mathbf{V}} = \mathbf{F} = \langle \mathbf{F} \rangle + \widetilde{\mathbf{F}} = -\eta\,\mathbf{V} + \widetilde{\mathbf{F}}\,, \quad \text{where } \langle \widetilde{\mathbf{F}} \rangle = \langle (\mathbf{F} - \langle \mathbf{F} \rangle) \rangle = 0\,, \tag{2}$$

and the results from (A,B,C) to calculate the *time dependence* of the average kinetic energy of the particle and its *equilibrium value.* Compare the latter to the equipartition theorem.

(E) [3pt] Use your results to estimate the molecular hit rate and the energy relaxation time for a 1-micron dust particle in air at room temperature.

Statistical Mechanics 2

2D and 3D gases in equilibrium

A closed volume V, with inner wall surface of area A, contains $N >> 1$ similar, non-relativistic particles of mass m each. Any of the particles may be either moving freely inside the volume, as a component of a 3D classical gas, with degeneracy g_V of each orbital state, or condense on the inner walls, where it can also move freely as a component of a 2D gas – also classical, with a generally different degeneracy g_A of each orbital state. The condensation of a particle releases energy Δ.

A (3 points). Using any statistical approach you like, calculate the average number N_3 of particles in the 3D gas, as a function of its chemical potential μ and temperature T.

B (5 points). Perform a similar calculation of the average number N_2 of particles condensed on the surface, taking into account the 2D character of their motion.

C (2 points). Assuming that N_2, $N_3 >> 1$, use the conditions of thermal and chemical equilibrium of the 3D and 2D phases, and the results obtained in tasks A and B, to derive a system of algebraic equations relating these numbers.

D (3 points). Solve the obtained system of equations to calculate the chemical potential μ of the system, the number of particles in the 3D gas, and its pressure P, as explicit functions of N and T.

E (4 points). Analyze the results in detail. In particular, simplify them for very low and very high values of the ratio $k_B T/\Delta$, and sketch the function $P(T)$ at fixed N and Δ, paying special attention to the temperature region where the ratio is of the order of 1.

F (3 points). Discuss the physics of your results. Does this model describe a phase transition between the 2D and 3D gases? (Justify your answer.) If not, suggest an example how the model may be modified to describe such a transition. (No quantitative analysis is required.)

Hint / reminder: $\int_{-\infty}^{+\infty} \exp\{-\xi^2\} d\xi = \pi^{1/2}$.

Statistical Mechanics 3

The 1D Potts model

In the so-called Potts model, a uniform 1D chain of N classical spins (in the absence of an external magnetic field) is described by the following interaction Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\eta_i, \eta_j} \text{ , with } J > 0 \text{ ,} \tag{1}$$

where J is a positive coupling constant, η_i is a classical spin variable at the site i, describing the spin state, which may take integer values in the set $\{1, ..., q\}$, δ_{ab} is the Kronecker delta symbol, and the summation is over all pairs of adjacent spins. Consider the model with $q = 3$ in thermal equilibrium at temperature T.

Do parts parts (a), (b), and (c) for finite N, and then take the limit $N \to \infty$ for parts (d), (e), and (f). For parts (c), (d), (e), and (f), you should give an explicit closed-form expression, not an abstract expression involving a summation.

(a) (1 pts.) Write the general expression for the statistical sum (partition function) Z of the system.

(b) (2 pts.) Assuming periodic boundary conditions, express Z via the appropriate transfer matrix.

(c) (4 pts.) Use this expression to calculate Z. (Check your result carefully as all subsequent parts depend on this result.)

Take $N \to \infty$ for the remainder of the problem:

(d) (3 pts.) Calculate the free energy per site, F, and the average energy per site, E.

(e) (3 pts.) Calculate the specific heat capacity per site, C, and the entropy per site, S.

(f) (5 pts.) Calculate the values of E, C, and S in the limits $T \to 0$ and $T \to \infty$. Physically and quantitatively explain your results for E and S in both limits.

(g) (2 pts.) Does this system have a symmetry-breaking phase transition at finite temperature? Prove your answer.