# Comprehensive Examination

## Department of Physics and Astronomy

## Stony Brook University

**January 2016 (in 4 separate parts: CM, EM, QM, SM)**

**General Instructions:**
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages.
Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

# Classical Mechanics 1

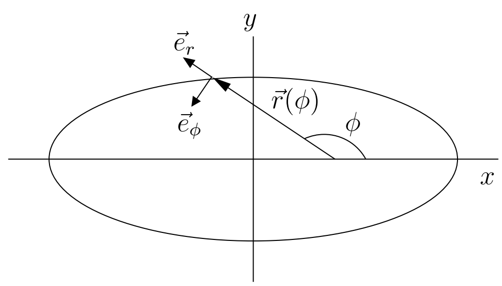
## Orbits of planets

As you know, the orbits $\vec{r}(\phi)$ of planets are ellipses, but, as you will see, the orbits $\vec{v}(\phi)$ in velocity space are circles! If one adds a perturbation $\frac{\beta}{r^2}$ to the $\frac{1}{r}$ potential, the ellipses start to precess, and the circles become a kind of epicycles. In this problem we prove these statements, and construct exact solutions for the potential

$$V(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2} \qquad (\alpha > 0,\ \beta \geq 0)\,. \tag{1}$$

Consider a point particle with mass $m$ and negative energy $E = -|E|$ in this potential.

a) (2 points) First prove that $\vec{e}_r = -\frac{d\vec{e}_\phi}{d\phi}$, where $\vec{e}_r$ is the unit vector along the radius and $\vec{e}_\phi$ the unit vector orthogonal to $\vec{e}_r$ in the direction of increasing $\phi$ (see the figure)




Set $\beta = 0$. Then prove that

$$\frac{d\vec{v}}{d\phi} = \gamma \frac{d\vec{e}_\phi}{d\phi}\,. \tag{2}$$

What is the constant $\gamma$?

b) (4 points) It follows from (2) that $\vec{v}(t) = \vec{w} + \gamma \vec{e}_\phi(t)$ with constant $\vec{w}$. By taking the scalar product of this equation with $\vec{e}_\phi$, show that one obtains elliptical orbits given by $r(\phi)$ (see definition below*). *Hint:* Express $v_\phi \equiv \vec{v}(t) \cdot \vec{e}_\phi(t)$ in terms of polar coordinates, and choose a coordinate system such that $\vec{w}$ lies along the positive $y$-axis.

*An ellipse $(x/a)^2 + (y/b)^2 = 1$ can be parametrized in polar coordinates relative to the focus by $r(\phi)$

$$r(\phi) = \frac{b^2}{a + c\cos\phi}\,,$$

where $c = \sqrt{a^2 - b^2}$ is the distance between the $(x, y)$ origin and the focus. The angle $\phi$ and radius $r(\phi)$ are indicated in the figure.

c) (4 points) On the other hand, show that $\vec{v}(\phi)$ describes circles. Draw pictures of these elliptical and circular orbits and locate in these pictures the angle $\phi$.

d) (2 points) Now consider the case that $\beta > 0$. Derive the relation

$$\tfrac{1}{2}m\dot{r}^2 + \left(\tfrac{A}{r} - B\right)^2 = \boldsymbol{E}^2\,. \tag{3}$$

What are $A$, $B$ and $\boldsymbol{E}$? Set $\sqrt{\frac{m}{2}}\,\dot{r} = \boldsymbol{E}\sin f(t)\,;\ \left(\frac{A}{r} - B\right) = \boldsymbol{E}\cos f(t)$. Is this always possible?

e) (4 points) Show that $\frac{\dot{f}(t)}{\dot{\phi}} = \frac{df}{d\phi} = \omega = \text{constant}$. Show that the orbits $r(\phi)$ are now ellipses with precession.

f) (4 points) Find the equation which generalizes (2) to the case when $\beta$ is nonvanishing. How would you solve this equation?

# Classical Mechanics 2

## A bead on a hoop

A bead of mass $m$ is constrained to move (without friction) on a hoop of radius $R$. The hoop rotates with constant angular velocity $\omega$ around the vertical axis. The bead is subjected to the force of gravity at the surface of the Earth.

a) Write down the Lagrangian for the system and the Lagrangian equations of motion. [4pts]

b) Find any constants of motion that may exist. Construct the Hamiltonian. Is it equal to the energy in the fixed (i.e. non-rotating) frame? Is the fixed-frame energy conserved? [2pts]

c) Find the critical angular velocity $\Omega$ below which the bottom of the hoop is a position of *stable* equilibrium. Find the stable equilibrium positions for both $\omega < \Omega$ and $\omega > \Omega$. [7pts]

d) Calculate the frequencies of small oscillations around the positions of stable equilibrium. [7pts]

# Classical Mechanics 3

## Magnetic mirrors

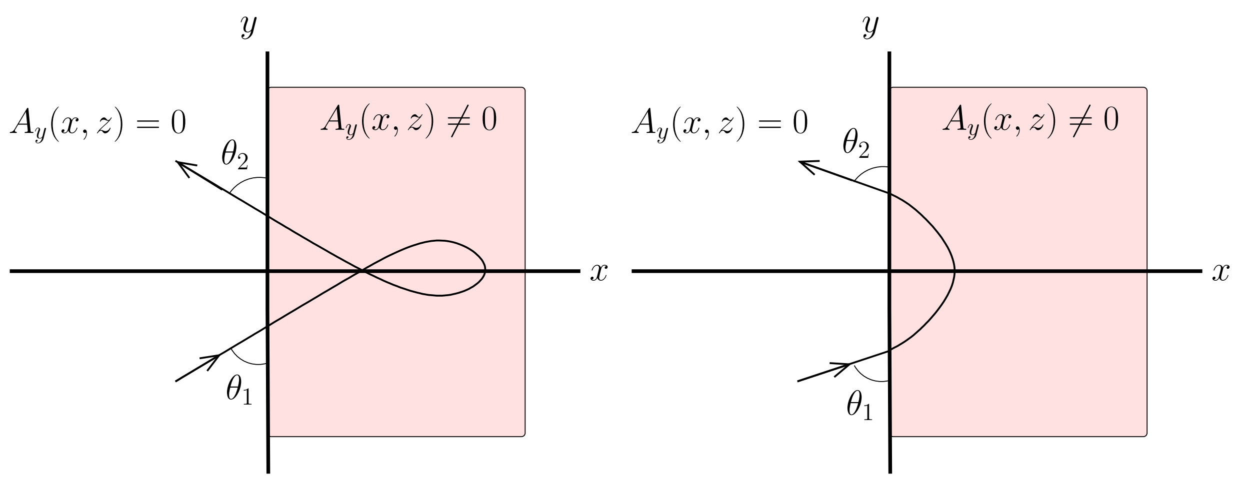



Fig. 1: Planar view of the system. The $z$-axis is perpendicular to the plane with $\hat{e}_z = \hat{e}_x \times \hat{e}_y$.

A relativistic electron (charge $e = -|e|$, rest mass $m$) with mechanical momentum

$$\vec{p} = p_0 \left(\hat{e}_x \cdot \sin\theta_1 + \hat{e}_y \cdot \cos\theta_1\right) \tag{1}$$

propagates from free space with zero magnetic field (and zero vector potential) at $x < 0$ into a time-independent magnetic field (see Fig. 1). The magnetic field has no $y$-component and is due to a vector potential which has only a $y$-component:

$$\vec{A} = \hat{e}_y A_y(x,z)\,; \quad \vec{B} = \vec{\nabla} \times \vec{A}\,;$$
$$\vec{B} = \hat{e}_x B_x(x,z) + \hat{e}_z B_z(x,z)\,.$$

At $z = 0$ the magnetic field is perpendicular to the $x - y$ plane.

a) (2 points) Show that a trajectory of an electron located at $z = 0$ with its momentum in $x - y$ plane (as in equation (1)) will stay in $x - y$ plane.

b) (5 points) Construct the relativistic Hamiltonian of the system and the canonical momentum of the particle. Is the Lagrangian, or the action, or the Hamiltonian Lorentz invariant? Exlain. (Hint: Recall that the Lagrangian of a point particle in an external electromagnetic potential is

$$L = -mc^2\sqrt{1 - \vec{v}^2/c^2} - e\,\varphi(\vec{x}(t)) + \frac{e}{c}\,\vec{v} \cdot \vec{A}(\vec{x}(t)) \tag{2}$$

where $\vec{v}$ is the particle velocity, $\varphi$ is the electrostatic potential, and $\vec{A}$ is the vector potential.)

c) (4 points) Obtain two integrals of the motion for the problem described above. Using these two integrals of the motion, derive an effective 1D Hamiltonian for motion in the $x$-direction of the following form:

$$H^* = \frac{p_x^2}{2m^*} + U(x)\,.$$

Find an expression for $m^*$ and for $U(x)$ in terms of the integrals of the motion and $A_y(x)$.

Hints: (1) One of these invariants is generic for any motion in a magnetic field, while the other is specific to this system's translation symmetry. (2) Write the equations of motion for $x$-components using the full Hamiltonian and substitute the two invariants into these equations. Compare these equations with those from the effective Hamiltonian to define $m^*$ and $U(x)$.

d) (5 points) Using the two integrals of the motion show that this system is indeed a "mirror" for trajectories in the $x-y$ plane, namely, an electron with initial momentum (1) is reflected such that angle of the incoming and outgoing electron in the figure are equal:

$$\theta_2 = \theta_1.$$

e) (5 points) Again, for a trajectory in $x - y$ plane, find an equation for the depth of penetration for the cases illustrated in Fig. 1(a) and Fig. 1(b). Solve this equation for the field of a quadrupole with field gradient $G$:

$$\vec{B} = G\left(\hat{e}_x z - \hat{e}_z x\right); \quad x > 0\,.$$

Which signs of $G$ correspond to the trajectories in Fig. 1(a) and Fig. 1(b)?
(Denote the charge of the electron by $e$, where $e = -|e|$.)

# Electromagnetism 1

## Radiation from a relativistic electron

Consider a relativistic electron (of charge $e$) traveling with an initial speed $v_o$ along the $z$-axis. At time $t = 0$ it slows down to a stop over a time $\tau$ while moving along the $z$-axis

$$v(t) = v_o \left(1 - \frac{t}{\tau}\right) , \qquad 0 \le t \le \tau \, . \tag{1}$$

Recall that the electric field in the far field radiated from a point charge following a trajectory with position $\boldsymbol{x}(t)$, and velocity $\boldsymbol{v}(T) = \boldsymbol{x}'(t)$ is

$$\boldsymbol{E}_{\rm rad}(t, \boldsymbol{r}) = \frac{e}{4\pi c^2} \left[ \frac{\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}}{R\,(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \right]_{\rm ret} , \tag{2}$$

where all quantities in square brackets are evaluated at the *retarded time*, $T(t, \boldsymbol{r})$ (which you will define below). The other symbols are defined as $\boldsymbol{n} \equiv (\boldsymbol{r} - \boldsymbol{x}(T))/|\boldsymbol{r} - \boldsymbol{x}(T)|$, $R \equiv |\boldsymbol{r} - \boldsymbol{x}(T)|$, and $\boldsymbol{\beta} = \boldsymbol{v}/c$.

(a) (3 points) Define the retarded time and compute the derivatives $\partial T/\partial t$ and $\partial T/\partial \boldsymbol{r}^i$

(b) (3 points) The radiation field $\boldsymbol{E}_{\rm rad}$ is derived from the *Liénard-Wiechert* potentials

$$\varphi(t, \boldsymbol{r}) = \frac{e}{4\pi} \left[ \frac{1}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} , \tag{3}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi c} \left[ \frac{\boldsymbol{v}}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} . \tag{4}$$

Using far field approximations, show that the Lorenz gauge condition is satisfied by these potentials.

(c) (6 points) For the decelerating electron described above, compute:

(i) the energy radiated per solid angle per *retarded time*.

(ii) the energy radiated per solid angle *per time*.

Describe in what physical situations you would be interested in (i) and (ii) respectively. Use no more than two sentences to describe each case.

(d) (4 points) Now consider a relativistic electron with initial energy of 1 GeV.

Examining your results of part (c), you should find that at $t = 0$ the radiation is initially emitted (predominantly) at a characteristic angle. Give an order of magnitude estimate for this angle. Explain your estimate by pointing to specific terms in your formulas from part (c).

(e) (4 points) Determine the total energy per solid angle emitted as the electron decelerates to a stop.

# Electromagnetism 2

## Induction and the energy in static magnetic fields

Consider a closed circuit of wire formed into a circular coil of $n$ turns with radius $a$, resistance $R$, and self-inductance $L$. The coil rotates around the $z$-axis in a uniform magnetic field $H$ directed along the $x$-axis (see below).

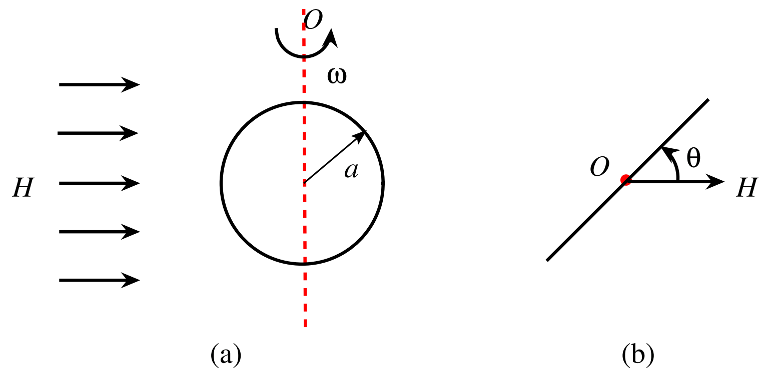



Figure 1: (a) side view; (b) top view.

a) (6 points) Find the current in the coil as a function of $\theta$ for rotation at a constant angular velocity $\omega$. Here $\theta(t) = \omega t$ is the angle between the plane of the coil and $H$ (the x-axis).

b) (4 points) Find the externally applied torque that is needed to maintain the coil's uniform rotation.

c) Because of the time-dependent currents induced in the coil, electromagnetic waves are radiated. Briefly answer the following questions:

(i) (2 points) What is the frequency of the radiation? Explain.

(ii) (2 points) What is the the polarization of the radiated waves propagating along the positive $z$-axis? Explain.

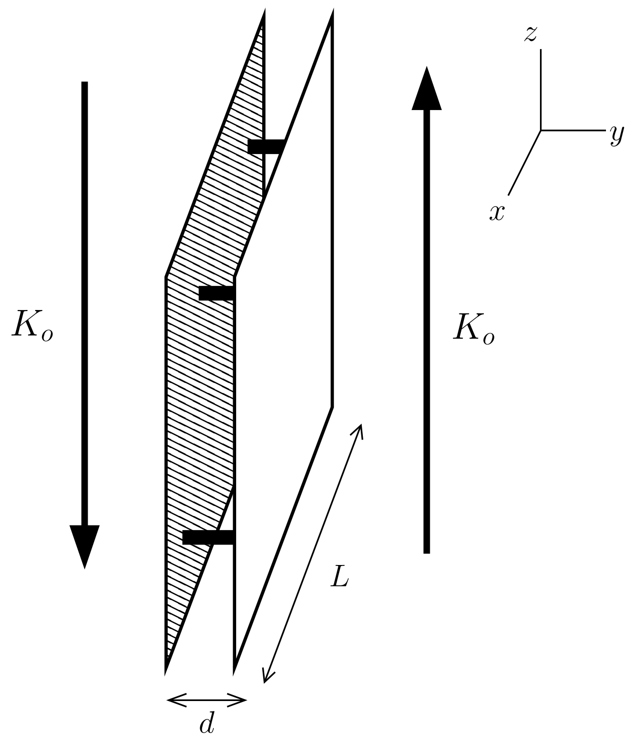
d) (6 points) Compute the total power radiated by the rotating coil of wire.

Note: in all parts you should assume that all transient effects have died away.

# Electromagnetism 3

## Two current sheets under Lorentz boosts

Consider two large square sheets of conducting material (with sides of length $L$ separated by a distance $d$, $d \ll L$) each carrying a uniform surface current of magnitude $K_o$. (The total current in each sheet is $I_o = K_o L$.) The current flows up the right sheet and returns down the left sheet. The mass of the sheets is negligible. The sheets are mechanically supported by four electrically neutral columns of mass $M_{\text{col}}$ and cross sectional area $A_{\text{col}}$ (three shown). Neglect all fringing fields.




(a) (3 points) Write down the electromagnetic stress tensor $\Theta^{\mu\nu}_{\text{em}}$ covariantly in terms of $F^{\mu\nu}$ and compute all non-vanishing components of $F^{\mu\nu}$ and $\Theta^{\mu\nu}_{\text{em}}$ both in between and outside of the two sheets.

(b) (1 point) Compute the total rest energy of the system (or $M_{\text{tot}}c^2$) including the contribution from the electromagnetic energy.

(c) (3 points) Determine the electromagnetic force per area on the current sheets (magnitude and direction) and the components of the mechanical stress tensor in the columns, $\Theta^{00}_{\text{mech}}$ and $\Theta^{yy}_{\text{mech}}$ (use the coordinates system in the figure). You can assume that the stress is constant across the cross sectional area of the columns.

(d) (6 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/c$ up the $z$-axis.

   (i) Determine the electric and magnetic fields (magnitudes and directions) using a Lorentz transformation. Check that the direction of the Poynting vector measured by this observer is consistent with physical intuition.

   (ii) Determine the charge and current densities in the sheets according to this observer. Are your charges and currents consistent with the fields computed in the first part of (d)? Explain.

(e) (7 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/z$ to the *right* along the $y$-axis (use the coordinate system shown in the figure).

   (i) Determine the total mechanical energy in the columns according to this observer.

   (ii) Determine the total electromagnetic energy according to this observer.

   (iii) Determine the total energy of this configuration. Is your result for the total energy consistent with part (b)? Explain.

**Comment:** There is of course stress in the sheets. But, since it does not have a $yy$ component the stress in the sheets can be neglected in this problem.

# Quantum Mechanics 1

## Interaction of two nucleons

The Schrödinger equation for the interaction of two nucleons can be reduced to the form:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2}+\left[V(r)+\frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u(r)=Eu(r) \tag{1}$$

where $u(r)=r\psi(r)$, $r$ is a separation between the proton and the neutron, and the $m$ is the reduced mass of the neutron-proton system. Use: $M_p = 938$ MeV/$c^2$, $M_n = 939$ MeV/$c^2$, $\hbar c = 197$ eV$\cdot$ nm, 1 barn $= 10^{-28}\mathrm{m}^2$.

a) (5 points) The deuteron is a bound state of a proton and a neutron which are primarily in an orbital $s$-wave with total angular momentum $J = 1$ and total spin $S = 1$. The deuteron potential can be approximated as a three-dimensional, spherically symmetric, square-well:

$$V(r)=\begin{cases}-V_0 & \text{for } r<R\\ 0 & \text{for } r\geq R\end{cases} \tag{2}$$

Given the (small!) deuteron binding energy $E \simeq -2\,\mathrm{MeV}$ and the potential range $R \simeq 2$ fm ($\sim$diameter of a deuteron), find an equation for the depth of the potential $V_o$. By analyzing this equation, show that the depth $V_o$ must be greater than about $25\,\mathrm{MeV}$. (Hint: $\tan\theta$ changes sign at $\pi/2$.)

An exact calculation of the potential depth in part ($a$) yields $V_0 \simeq 35$ MeV.

b) (10 points) In low energy neutron-proton scattering (incident neutron energy $E_o \leq$ 10 keV), one can use the same potential (Eq. 2) as for the deuteron state in part a). Find the wave function $u(r)$ and determine a formula for the neutron-proton scattering cross section. Which partial wave dominates the cross section? Hint: The wave function at large distances ($r \to \infty$) can be represented as a superposition of an incident wave and a scattered wave:

$$\psi(r,\theta,\phi)\sim e^{ikz}+f(\theta,\phi)\frac{e^{ikr}}{r}, \qquad \text{where } f(\theta,\phi)=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta),$$

with $k \equiv \sqrt{(2mE/\hbar^2)}$, and the normalization of the Legendre polynomials is:

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta=\frac{2}{2\ell+1}\delta_{\ell\ell'}.$$

c) (5 points) At low energies the experimental value of the unpolarized neutron-proton cross section is $\sigma \simeq 20$ barns (Fig. 1), while an analysis of part b) yields a cross section of 4.5 barns. What is the explanation for the discrepancy between the calculated value of the cross section in part b) and the experimental value? Make a specific prediction for a proton-neutron cross section in different spin channels which can be checked experimentally. Hint: Use the fact that that the deuteron has total spin $S = 1$.

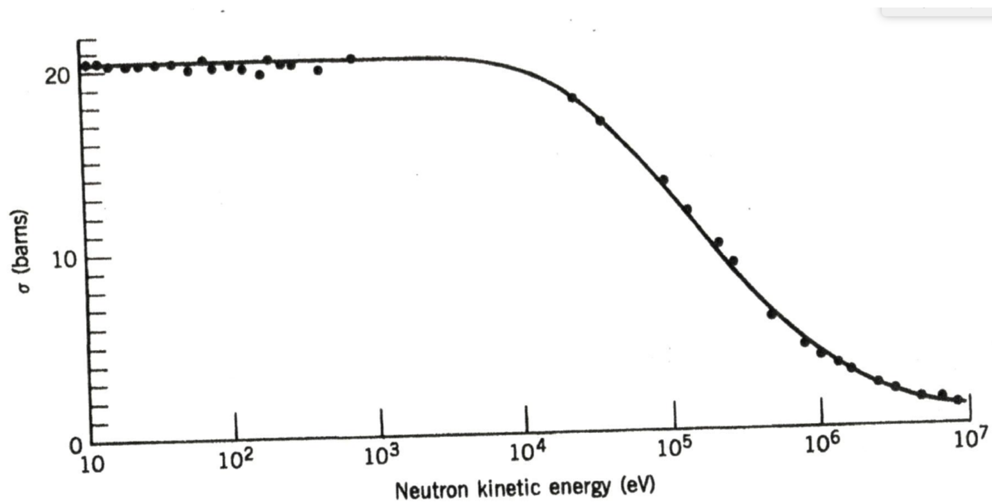



Figure 1: The neutron-proton scattering at low energy. Data taken from R. K. Adair, Rev. Mod. Phys. **22**, 249 (1950) and T. L. Houk, Phys. Rev. C **3**, 1886 (1970).

# Quantum Mechanics 2

## An oscillator in an electric field

A particle of mass $m$ and electric charge $q$ moves in 1-dimension under the effects of a a harmonic potential and a homogeneous electrostatic field $\mathcal{E}$. The Hamiltonian for the system is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - q\mathcal{E}x = H_0 - q\mathcal{E}x \tag{1}$$

**1. (4 points)** Show that $H$ can be written as $H = e^{-A}H_0 e^{A} + B$ by explicitly determining the two operators $A, B$. Use this to show that the spectrum of $H$ follows from that of $H_0$ by a shift operator. Use this observation to solve the eigenvalue problem.
**2. (4 points)** Express $A$ in terms of the creation $a$ and annihilation $a^\dagger$ operator of $H_0$. Use this to evaluate the probability to find the system in the ground state of $H$ at time $t$ if at time $t = 0$ it is in the ground state of $H_0$.
**3. (4 points)** What is the probability for the system to start at $t = 0$ in the ground state of $H_0$ and remain in this state at time $t$? For what time this probability is 1?
**4. (4 points)** Repeat **3** but now for the system to be found in the first excited state of $H_0$. Comment physically on the similarities and differences between **3** and **4**.
**5. (4 points)** Express the dipole moment $d = qx$ in terms of $a, a^\dagger$. Use this to calculate the mean value of the dipole moment $d = qx$ at time $t$, assuming that at $t = 0$ the system is again in the ground state of $H_0$.

# Quantum Mechanics 3

## Approximations for a quartic potential

Consider a particle with mass $m$ in a one-dimensional quartic potential $V = \beta x^4$ where $\beta$ is a positive constant.

(a) (2 points) Use dimensional analysis to determine how the eigenstate energies depend on $\beta$. (Hint: write the Schrödinger equation in terms of dimensionless variables

$$\left(-\frac{1}{2}\frac{d^2}{d\bar{x}^2} + \bar{x}^4\right)\psi = \epsilon\psi\,,$$

where $\bar{x}$ is a suitably rescaled coordinate.)

(b) (4 points) Calculate the eigenstate energies $E_n$ with $n = 0, 1, 2, \cdots$ in the WKB approximation. Compare the WKB spectrum of this quartic anharmonic oscillator with the spectrum of the harmonic oscillator and the particle in a box.

(c) (2 points) For which values of $n$ is the WKB method most accurate?

(d) (6 points) Approximate the energy $E_0$ of the ground state of the $\beta x^4$ anharmonic oscillator by applying the variational method with Gaussian wave function $\psi_0 = Ce^{-x^2/\lambda^2}$ where $\lambda$ is a real variable parameter.

(e) (4 points) Do the results obtained in part (d) satisfy the virial theorem? Explain. Do the variational method and/or the WKB method provide upper and/or lower bounds on the ground state energy?

(f) (2 points) Write down a wave function that can be used for the variational method to obtain an approximate value of the energy $E_1$ of the first excited state of the quartic anharmonic oscillator.

You may use the following integral:

$$\int_0^1 \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,. \qquad (1)$$

Here the gamma function satisfies the recursion relation $\Gamma(z+1) = z\Gamma(z)$, with representative values:

$$\Gamma(1/4) = 3.62561 \tag{2}$$
$$\Gamma(1/2) = \sqrt{\pi} \tag{3}$$
$$\Gamma(3/4) = 1.22542 \tag{4}$$
$$\Gamma(1) = 1 \tag{5}$$

You may also use the following results for Gaussian integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} x^n = \begin{cases} \sqrt{\frac{\pi}{a}} & \text{for } n = 0 \\ \sqrt{\frac{\pi}{a}}\left(\frac{1}{2a}\right) & \text{for } n = 2 \\ \sqrt{\frac{\pi}{a}}\left(\frac{3}{4a^2}\right) & \text{for } n = 4 \end{cases} \tag{6}$$

## Statistical Mechanics 1

### Ultra-relativistic electron gas

Consider an ideal 3D gas of $N$ ultra-relativistic electrons with energies $\varepsilon = pc$ (where **p** is electron's momentum, and $c$ is the speed of light), confined to volume $V$.

(a) (3 points) For the gas in equilibrium at zero temperature, calculate its chemical potential $\mu$ (i.e. the Fermi energy $\varepsilon_F$) and the total energy $E_0$, and express $E_0$ in terms of $N$ and $\varepsilon_F$.

(b) (6 points). Now consider the gas in equilibrium at a low temperature $T << \varepsilon_F/k_B$. In the first nonvanishing approximation in $T$, calculate the chemical potential, and express your result in terms of $\varepsilon_F$ and $T$.

(c) (4 points) For the same conditions as in Task 2, calculate the specific heat (i.e. the heat capacity per particle) of the gas, and express it in terms of $\varepsilon_F$ and $T$.

(d) (4 points) Obtain general expressions for the grand thermodynamic potential of the gas and its pressure, and express them via the total energy of the gas and its volume. Compare the result with that for an ideal gas of non-relativistic particles.

(e) (3 points) Express the gas pressure at $T = 0$ in terms of $N$ and $V$.

*Hint*: You may find the following *Sommerfeld expansion* useful:

$$\int_0^\infty F(\varepsilon) f(\varepsilon) d\varepsilon = \int_0^\mu F(\varepsilon) d\varepsilon + \frac{\pi^2}{6}\left(k_B T^2\right) F'(\mu) + O\left(\frac{k_B T}{\mu}\right)^4,$$

where

$$f(\varepsilon) \equiv \frac{1}{\exp\{(\varepsilon - \mu)/k_B T\} + 1},$$

is the Fermi-Dirac distribution, $F(\varepsilon)$ is any differentiable function, growing slower than $1/f(\varepsilon)$ at $\varepsilon \to \infty$, and $F'(\varepsilon)$ is its derivative.

# Statistical Mechanics 2

## Magnetic refrigeration

An external magnetic field **B** is applied to a set of $N$ non-interacting spin-½ particles with gyromagnetic ratio $\gamma$, and fixed spatial positions. For the thermal equilibrium at temperature $T$, calculate:

(a) (3 points) the average energy and heat capacity,

(b) (3 points) the average magnetic moment of the system and the variance of its fluctuations, and

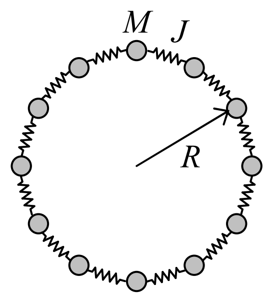
(c) (4 points) the entropy per spin.

(d) (5 points). Sketch the temperature dependence of the entropy, for two substantially different field magnitudes, and discuss (qualitatively) what would happen with the entropy and the internal energy of the system if it is first thermally isolated from the environment, and then the applied field is turned off.

(e) (5 points). Suggest a way to use this system as a refrigerator, assuming that its thermal contacts with hot and cold heat baths, and the applied magnetic field, may be controlled at will.

## Statistical Mechanics 3

### 1D vibrational modes




Consider a system of $N >> 1$ similar particles of mass $M$, equally spaced on a circle of radius $R$, and constrained to move only around the circle. Nearest neighbor particles are connected by springs with equal spring constants $J$ – see Fig. on the right.

(a) (3 points). Write down the Lagrangian function of the system, and the equation of angular motion of an arbitrary particle, assuming that spring deformations are relatively small.

(b) (2 points). Prove that if the particles are numbered sequentially, the equation of motion of the $n^{\text{th}}$ particle is satisfied by the following function:

$$\varphi_n(t) = \mathrm{Re}\sum_{j=0}^{N-1} c_j(t)\exp\left(i\frac{2\pi jn}{N}\right),$$

where $\varphi_n(t)$ is the angular displacement of the particle, and derive the differential equation obeyed by functions $c_j(t)$. What simple physical system obeys the similar differential equation?

(c) (3 points). For a single one-dimensional harmonic oscillator of frequency $\omega$ that may be comparable with $k_B T/\hbar$, write down the statistical sum ("partition function"), and calculate its heat capacity $C(T)$ in thermal equilibrium at temperature $T$. Analyze the low-temperature and high-temperature limits of the function $C(T)$.

(d) (5 points). Returning to the system of $N$ particles on a circle (see Fig. above), calculate its heat capacity in the intermediate temperature range

$$\frac{\hbar\omega_0}{N} << k_B T << \hbar\omega_D, \qquad \text{where } \omega_D \equiv \left(\frac{J}{M}\right)^{1/2}.$$

(e) (4 points). Now suppose that $M$ is so large that there is a broad range of lower temperatures:

$$\frac{\hbar^2 N}{MR^2} << k_B T << \frac{\hbar\omega_D}{N}.$$

What is the heat capacity of the system in this range?

(f) (3 points). Estimate the temperature at which the heat capacity becomes exponentially small, if the particles are distinguishable. How does the answer change if they are indistinguishable bosons?

*Hint*: You may use the following integral: $\displaystyle\int_0^\infty \frac{x\,dx}{e^x - 1} = \frac{\pi^2}{6}$.