Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2016 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages.
Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

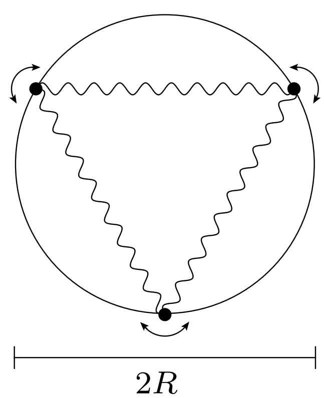
Classical Mechanics 1

Three beads

Consider three beads of mass m, connected by three identical springs with spring constant k and unstretched equilibrium length l_0. In equilibrium this system has the shape of an equilateral triangle whose sides have length l_0. We are going to study the small oscillations of this system.

1. (3 points) First consider the motion of this system in a horizontal plane. All three beads can freely move in the plane, only restricted by the springs attached to them. The motion is assumed to be without friction. (i) How many normal modes does this system have? (ii) How many zero modes are there (by zero mode we mean a normal mode with frequency $\omega = 0$)? (iii) How many nonzero modes are there, and is there degeneracy (by degeneracy we mean that two or more normal modes have the same frequency)? (iv) Sketch the motion of the normal modes with nonvanishing frequencies. Do *not* calculate the values of these frquencies.

2. (2 points) Next consider the case that the beads can only move along a fixed ring of radius R in the plane. To fit the triangle on the ring, one may need to stretch or compress the springs, depending on the values of l_0 and R. Consider an arbitrary value of l_0. Again the motion of the beads along the ring is assumed to be frictionless. We



   ask again the same questions: (i) How many modes are there? (ii) How many zero modes are there? (iii) How many nonzero modes are there, and is there degeneracy?

3. (1 points) Depending on the value of $\frac{l_0}{R}$, the triangle on the ring may be stable or unstable. Can you guess a case when it is stable?

4. (6 points) Determine the value of $\frac{l_0}{R}$ when the system changes from stable to unstable.

5. (3+5 points) Expand the potential to second order in small deviations. Compute the frequencies of small oscillations in the stable regime.

Classical Mechanics 2

The rocking of a half-cylinder

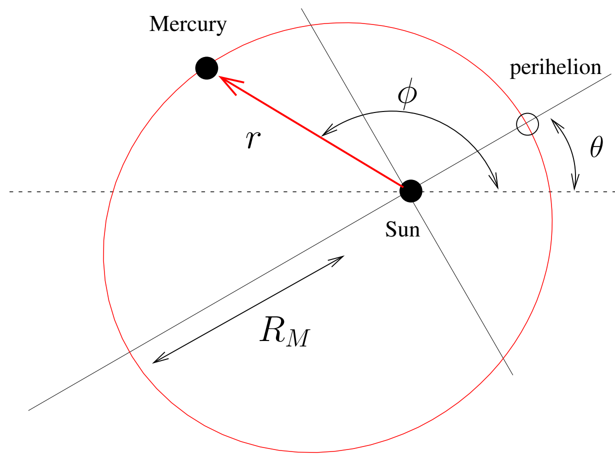
Consider a uniform half-cylinder of mass m and radius a rocking without slipping on a horizontal plane.

1. (2 points) Determine the position of the center of mass of the half-cylinder and the moment of inertia I_{cm} around the center of mass.
2. (4 points) Derive the Lagrangian L in terms of the angle θ between the flat surface of the half-cylinder and the horizontal.
3. (4 points) Write down the Lagrange equation of motion and find the frequency ω of small oscillations around the equilibrium.
4. (5 points) Determine the force $\vec{F}(\theta)$ produced by the plane on the half-cylinder at the line of contact as a function of the angle θ of the half-cylinder in the regime of small oscillations.
5. (5 points) Now consider large oscillations. Give physical arguments to predict whether the normal force exerted by the plane is larger or smaller than the gravitational force in the following two cases: when the angle θ is maximal, and when $\theta = 0$.

Classical Mechanics 3

The precession of Mercury due to Jupiter

Recall that the trajectory of Mercury $r(\phi)$ is an ellipse with the sun at one focus as shown below. The perihelion (defined as the distance of closest approach – see below) is rotated relative to the x-axis by an angle θ. The semi-major axis is denoted R_M. The eccentricity of Mercury is small, $\epsilon = 0.2$, although it is the most eccentric of the Sun's planets.



Due to perturbations from the other planets, the angle of the perihelion θ changes (it precesses) as function of time. The precession rate due to the plantes is small. The contribution of Jupiter to the precession rate is of order 150 arcsec/century, or (since the orbital period of Mercury is 88 days) approximately 1.78×10^{-6} rad/turn.

The goal of this problem is to estimate Jupiter's contribution to the precession rate[1]. Specifically, we will model Jupiter as a ring of mass M_J at the orbital radius of Jupiter R_J (not shown), and compute how the ring perturbs Mercury's orbit and causes the perihelion of Mercury to precess. Jupiter's orbital radius is significantly larger than Mercury's, $R_J \simeq 10\,R_M$, and its mass is significantly smaller than the sun's, $M_J \simeq 0.95 \times 10^{-3}\,M_\odot$.

1. (4 points) Show that for $R_J \gg R_M$ the Lagrangian of Mercury interacting with the sun of mass $M_\odot$, and a ring of mass M_J and radius R_J is approximately

$$L \simeq \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GmM_\odot}{r} + \alpha r^2\,, \tag{1}$$

   and determine the coefficient α.

[1] Famously, general relativity also perturbs the classical orbit and contributes 43 arcsecs/century to the total precession rate. This "anomalous" precession of Mercury was measured in the nineteenth century by Le Verrier and finally explained by Einstein in 1915. The total precession rate due to all the planets is approximately 550 arcsec/century.

2. (2 points) By introducing a dimensionless radius $\underline{r} = r/R_M$ (the radius in units of Mercury's semi-major axis) and other dimensionless variables show that a dimensionless Lagrangian for the system is

$$\underline{L} \simeq \frac{1}{2}\left(\frac{d\underline{r}}{d\underline{t}}\right)^2 + \frac{1}{2}\underline{r}^2\left(\frac{d\phi}{d\underline{t}}\right)^2 + \frac{1}{\underline{r}} + \underline{\alpha}\,\underline{r}^2\,, \tag{2}$$

where the dimensionless constant $\underline{\alpha}$ is of order

$$\frac{M_J}{M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 0.4 \times 10^{-6}\,. \tag{3}$$

To lighten the notation, stop underlining the variables in what follows.

3. (2 points) Determine the Hamiltonian of the dimensionless system, and use the Hamiltonian to determine the equations of motion.

4. (6 points) For $\alpha = 0$ determine the trajectory of Mercury $r(\phi)$, in terms of the energy and the angular momentum.

   For convenience, we note the elementary integral

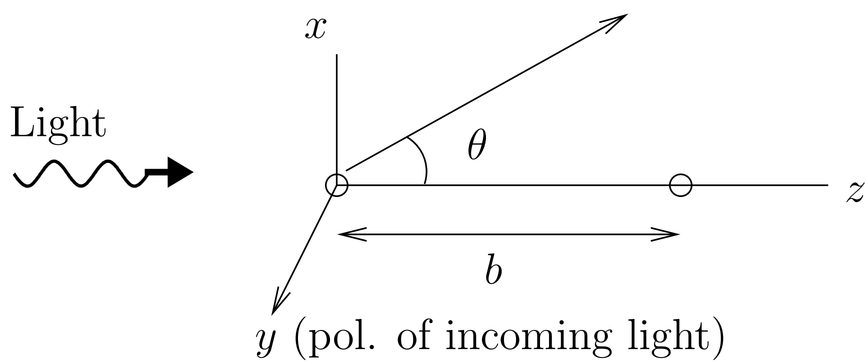
$$\int^x \frac{dy}{\sqrt{1-y^2}} = \sin^{-1}(x)\,. \tag{4}$$

5. (6 points) Determine the change in the orbital period and precession of the perihelion of Mercury to first order in α. You may treat the eccentricity of Mercury as small so that the orbit is approximately circular. Evaluate the precession rate numerically in radians per turn and compare to the experimental result of 1.78×10^{-6}rad/turn.

Electromagnetism 1

Scattering at different scales

Consider the scattering of an electromagnetic plane wave of wavenumber k and frequency ω propagating in the z direction. The incident light is linearly polarized in the y direction, $\boldsymbol{E}(t,\boldsymbol{r}) = \hat{\boldsymbol{y}} E_0 e^{ikz-i\omega t}$ (out of the page in the diagram below). The light is scattered by two small dielectric spheres of radius a separated by a distance b with $b \gg a$. The first sphere is centered at the origin, while the second sphere is located on the z axis with $z = b$. The two spheres have dielectric constant $\epsilon = 1 + \chi$ with $\chi \ll 1$.



(a) (5 points) Consider the scattering of long wavelength light $kb \ll 1$. Determine the total cross section of the two spheres to leading order in kb. *Hint:* in the long wavelength limit the two spheres are polarized by the same (approximately) uniform incoming field.

- How does the cross section of the two spheres compare to the cross section of a single sphere in the long wavelength limit?

(b) (5 points) Remaining in the long wavelength limit $kb \ll 1$, determine the electric field as a function of time at the specific point, $\boldsymbol{r} = (x, y, z) = (2b, 0, 0)$, along the x axis. *Hint:* Is this point in the near or far field? Explain.

(c) (5 points) Now consider the scattering of shorter wavelength light with $kb \sim 1$ but still $ka \ll 1$. Determine the differential cross section $d\sigma/d\Omega$ of the two spheres for light scattered at an angle θ in the z, x plane (see diagram above).

- Sketch the differential cross section $d\sigma/d\Omega$ for scattering at $\theta = \pi/2$ (along the x axis) as a function of k for $kb = 0 \ldots 8\pi$.

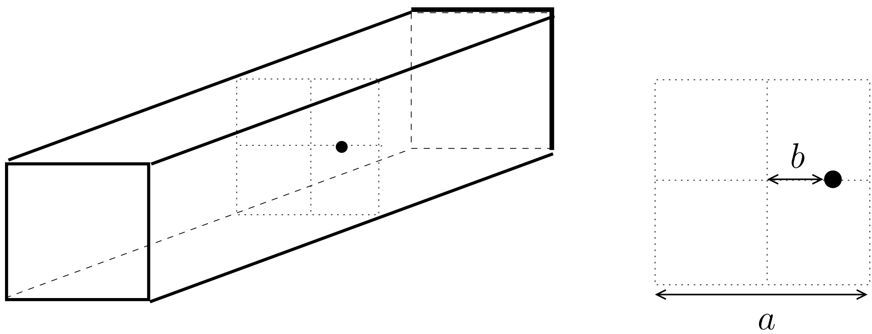
(d) (5 points) Now instead of a plane wave of light, consider the scattering of a wave packet with mean wavenumber $\bar{k}$ and bandwidth Δk, with $\Delta k/k \simeq 1/10$. The differential cross section is the energy scattered per solid angle divided by the total energy in the wave packet.

- Qualitatively sketch the differential cross section $d\sigma/d\Omega$ for scattering at $\theta = \pi/2$ as a function of $\bar{k}$, and contrast this sketch with the $\Delta k = 0$ limit drawn in the second part of (c). At large k how does the cross section for the two spheres compare to the cross section for one sphere?

Electromagnetism 2

A charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.



1. (2 points) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x z)\,\Phi(k_y y)\,e^{\pm\kappa_z z} \qquad \text{where} \qquad \Phi(u) = \left\{ \cos(u) \text{ or } \sin(u) \right\} \tag{1}$$

   for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

2. (4 points) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential.

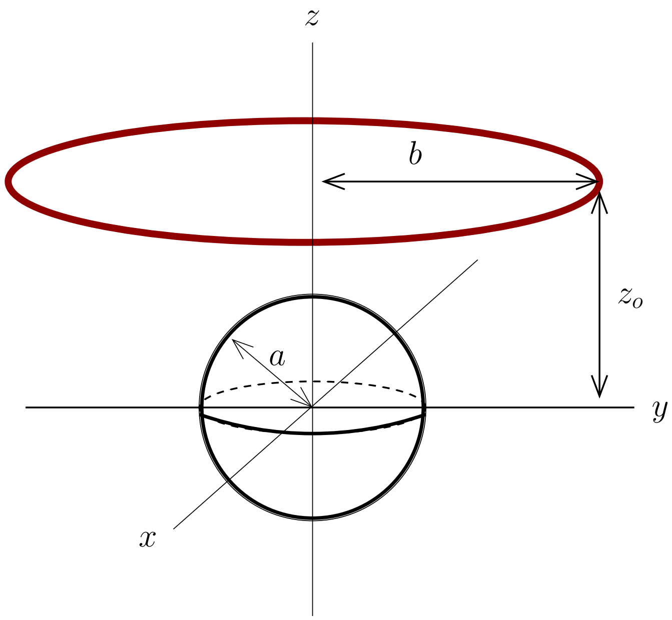
3. (7 points) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (1) to determine the potential from the point charge described in part (2).

4. (7 points) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$.

Electromagnetism 3

A ring and a sphere in a magnetic field

A sphere of radius a with magnetic permeability μ is placed in an external slowly varying homogeneous magnetic field (not shown), $\boldsymbol{B}_{\rm ext}(t) = B_o(t)\,\hat{\boldsymbol{z}} = \mathcal{B}\cos(\omega t)\hat{\boldsymbol{z}}$. Placed above the sphere at height z_o is an ohmic ring of radius b and resistance $\mathcal{R}$. The center of the ring coincides with the z-axis and the plane of the ring is perpendicular to the z-axis (see below).



(a) (6 points) The induced magnetic moment of the sphere is proporitonal to the external field

$$\boldsymbol{m} = \alpha_B \boldsymbol{B}_{\rm ext}\,. \tag{1}$$

Determine the polarizability, α_B. Neglect the fields from the currents induced in the ring.

(Hint: recall that for a permeable sphere in a constant external magnetic field, the magnetic field outside the sphere is that of an induced magnetic dipole plus the external field, while the magnetic field inside the sphere is constant, $\boldsymbol{B}_{\rm in} = B_{\rm in}\,\hat{\boldsymbol{z}}$. Determine α_B and $B_{\rm in}$ from the appropriate boundary conditions at the surface of the sphere.)

(b) (6 points) Determine the current induced in the ring.

(c) (2 points) Under what conditions can the induced magnetic fields from the ring be neglected in part (a)? Estimate.

(d) (6 points) Determine the force on the ring.

Quantum Mechanics 1

A particle in crossed $\boldsymbol{E}$ and $\boldsymbol{B}$ fields

Consider a particle of mass m and charge e moving in a 2-dimensional xy-plane in the presence of both a scalar potential $V(\vec{r})$ and a magnetic potential $\vec{A} = Bx\,\hat{y}$. For notational simplicity assume $e = \hbar = c = 1$.

a. (5 points) For $V(\vec{r}) = -ax$, identify the conserved momentum, and derive the energy spectrum of the system. If we were to trade $ax \to ay$, is there a conserved momentum?
b. (8 points) Define the following localized state for the potential $V(\vec{r}) = -ax$

$$|\Psi(0)\rangle = \int_{-\infty}^{+\infty} dk\, \Psi(k)\ |k\rangle_0\,,$$

where $|k\rangle_0$ is the lowest energy eigenstate with momentum k, and $\Psi(k)$ is any normalized momentum space wavefunction. Write down an expression for the time-evolved state, $|\Psi(t)\rangle$, and calculate its mean velocity along the conserved momentum direction. Give a physical interpretation of your result.
c. (7 points) Now consider instead, the hard wall potential

$$V(\vec{r}) = \begin{cases} 0 & x > 0 \\ \infty & x \leq 0 \end{cases} \tag{1}$$

with the same magnetic potential. Find the lowest energy of 3 indistinguishable particles, each at rest, that carry in addition a spin $\frac{1}{2}$ with a g-factor g, and write the corresponding wavefunction.

Quantum Mechanics 2

BEC of Lithium-7

At about the same time that Ketterle, Wiemann, and Cornell formed Bose-Einstein Condensates (BEC) of rubidium-87 and sodium-23 atoms (and later won the Nobel prize), Randy Hulet's lab in Texas was trying to form a BEC of lithium-7 ($^7_3$Li) atoms in a harmonic trap. We will try to model Hulet's system with a mean-field approach, using the Gross-Pitaevskii equation — essentially a nonlinear generalization of Schrödinger's equation:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{Ng}{2}|\psi|^2$$

where H acts on the wave function ψ. Here ψ is a single particle wave function. At low temperatures, all the N bosons should be in the same state, and the contribution of the interactions to the energy should scale as the local density of the wave function $|\psi|^2$, giving rise to the third term in our Hamiltonian. One very significant difference between these experiments is that lithium-7 atoms attract ($g < 0$) while rubidium-87 and sodium-23 repel ($g > 0$).

a. **(2 points)** Explain how lithium-7 is a boson.

b. **(7 points)** Using a Gaussian trial wave function $\psi = c\, e^{-r^2/2a^2}$, calculate the expectation value of the energy $E(a) = \langle H \rangle$ as a function of a. [Hint: The expectation value has the form $E(a) = A/a^2 + Ba^2 + C/a^3$ for some constants A, B, and C which you need to determine.]

c. **(5 points)** In the limit in which the interaction energy is large compared to the kinetic energy, minimize $E(a)$ as a function of a for the repulsive case $g > 0$. How do a_{min} and $\langle H \rangle_{\text{min}}$ scale with N?

d. **(6 points)** The coupling, g, is proportional to the scattering length, ℓ, for the bosons

$$g = \frac{4\pi\hbar^2\ell}{m} \ .$$

Assume the trap has a frequency $\omega = 2\pi \times 145$ Hz and a scattering length $\ell = -1.5$ nm for lithium-7. What is the maximum number of lithium-7 atoms that can be placed in the trap?

Quantum Mechanics 3

Zitterbewegung and the Darwin Term

The phenomenon of Zitterbewegung ("quivering motion") for an electron was predicted by Schrödinger in 1928, and is a peculiar consequence of Dirac's 1928 relativistic quantum theory. In this problem we will use the Zitterbewegung to give a physical interpretation of the Darwin term, which is a relativistic correction to the non-relativistic Hamiltonian arising from an approximation of the Dirac equation.

(a) (5 points) For the electron in the hydrogen atom the Hamiltonian takes the following approximate form

$$H = \frac{p^2}{2m} + V(r) - \frac{p^4}{8m^3c^2} + \frac{\hbar^2}{4m^2c^2}\frac{1}{r}\frac{dV}{dr}\,\boldsymbol{\sigma}\cdot\boldsymbol{L} + \underbrace{\frac{\hbar^2}{8m^2c^2}\nabla^2 V(r)}_{\text{The Darwin term!}} \quad , \tag{1}$$

where $V(r) = -e^2/r$ is the Coulomb potential, and m is the electron mass. The last term is known as the Darwin term.

(i) Briefly state the origin of the the p^4 term and the $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ term (no long derivations).

(ii) Determine the energy shift δE from the Darwin term to the $1s$ state of hydrogen with wave function $\psi_{1s}(r) = e^{-r/a_o}/\sqrt{\pi a_0^3}$.

(iii) Evaluate the magnitude of the shift $\delta E/E_{1s}$ numerically.

(iv) What is the shift for the $2p$ state? Explain.

The origin of the Darwin term can be understood with the Dirac Hamiltonian

$$H = c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta\, mc^2\,. \tag{2}$$

Here $\boldsymbol{\alpha}$ and β are 4×4 Dirac matrices which (in the Schrödinger picture) are[4]

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{3}$$

where σ^k are the Pauli matrices and $\mathbb{I}$ is the unit matrix. In the Heisenberg representation these matrices are time-dependent operators, $\hat{\boldsymbol{\alpha}}(t)$ and $\hat{\beta}(t)$. They satisfy the anti-commutation relations

$$\left\{\alpha^i, \alpha^j\right\} = 2\,\delta^{ij}\cdot\mathbb{I}_{4\times4}, \qquad \{\beta,\beta\} = 2\cdot\mathbb{I}_{4\times4}, \qquad \{\alpha,\beta\} = 0. \tag{4}$$

(b) (2 points) Using the Heisenberg picture evaluate the velocity operator $\hat{v}^k = \mathrm{d}\hat{x}^k(t)/\mathrm{d}t$, and show that the eigenvalues of $\hat{v}^z$ are $\pm c$.

[4] These matrices are given in the Dirac representation.

(c) (5 points) Now evaluate $\mathrm{d}\hat{v}^k/\mathrm{d}t$ in the Heisenberg picture, and write the result in terms of $\hat{p}^k$ and $\hat{v}^k$ and H. Why are $\hat{p}^k$ and H time independent?

(i) First for $p^k \simeq 0$, determine $\hat{v}^k(t)$ and then $\hat{x}^k(t)$ by integrating the Heisenberg equations of motion.

(ii) Now for general p^k, determine $\hat{v}^k(t)$ and then $\hat{x}^k(t)$ by integrating the Heisenberg equations of motion.

You should find a term linear in time and an oscillatory term. Interpret the linear term. The oscillating motion in $\hat{x}^k(t)$ is known as Zitterbewegung or "quivering motion".

(d) (6 points) Now consider the following electron wave function with $\langle\vec{p}\rangle \simeq 0$ in a specific superposition of states at time $t = 0$

$$\psi(\vec{x}) = \frac{\psi_0(\vec{x})}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \tag{5}$$

Here $\psi_0(\vec{x})$ is a normalized Gaussian wave packet with a large spatial width, so that the uncertainty in $\vec{p}$ is negligibly small.

(i) Using the results of part (c), determine the mean position and velocity of the electron in the z direction, $\langle\hat{z}(t)\rangle$ and $\langle\hat{v}^z(t)\rangle$.

(ii) Numerically determine the amplitude of the oscillating term of $\langle\hat{z}(t)\rangle$ in meters. Compare the amplitude of the Zitterbewegung oscillatory motion to the Bohr radius a_o.

(iii) Using microscopy, could it be possible in the future to directly detect the oscillatory motion of the electron? Explain.

(e) (2 points) Show that the Darwin correction to the Hamiltonian can be qualitatively, and even quantitatively, explained as the average potential experienced by an electron undergoing Zitterbewegung oscillations around its equilibrium position $\vec{r}$.

Statistical Mechanics 1

Blackbody radiation and its fluctuations

This problem addresses properties of the spontaneous electromagnetic radiation at thermal equilibrium.

(a) [2 points] Calculate the probability for a one-dimensional quantum harmonic oscillator of eigenfrequency ω to be on its n^{th} energy level, in thermal equilibrium at temperature T.

(b) [3 points] Calculate the average energy, the free energy, and the entropy of the oscillator, and discuss their dependences on temperature.

(c) [4 points] Calculate the variance (dispersion) of fluctuations of oscillator's energy, and express it via the average energy and $\hbar\omega$.

(d) [2 points] Calculate the number of electromagnetic standing-wave modes in a large, closed free-space volume V, with frequencies within a narrow interval $[\omega, \omega + d\omega]$, where $d\omega$ is much smaller than ω, but still large enough to contain many modes. Briefly explain why each mode may be treated as a one-dimensional quantum harmonic oscillator.

(e) [6 points] Calculate the average total energy of the electromagnetic field in volume V (including all essential modes), and the variance of its fluctuations. Express the variance via the average energy and temperature, and find the dependence of the relative r.m.s. fluctuation of the energy on temperature T and volume V.

(f) [3 points] How large should volume V be for your results to be qualitatively valid? Evaluate the condition for room temperature.

Hint: You may like to use the following table integral: $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$.

Statistical Mechanics 2

Ising model on a triangle

The Ising model on a triangle is described by the energy:

$$E = -J(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - h(\sigma_1 + \sigma_2 + \sigma_3)\,.$$

Here J and h are known parameters: exchange energy and external magnetic field, respectively. The Ising spins $\sigma_{1,2,3}$ are the only degrees of freedom in the problem and they are taking values ± 1. Assume that the temperature of the system is T.

(a) [2 points] Compute the partition function of the model.

(b) [3 points] Compute the free energy and the entropy of the model.

(c) [3 points] Compute the specific heat at temperature T and $h = 0$. Plot qualitative dependence of the specific heat as a function of T.

(d) [4 points] Compute the magnetization $M = \langle\sigma\rangle \equiv \langle\sigma_1 + \sigma_2 + \sigma_3\rangle$ at given h and $T \ll J$. What is the behavior of the magnetic susceptibility $\chi = \frac{\partial M}{\partial h}\Big|_{h\to 0}$ at low temperature?

(e) [8 points] Find the fluctuation of magnetization $\langle(\sigma - M)^2\rangle$ at $T \ll J$.

Statistical Mechanics 3

Bose-Einstein condensation

This problem addresses the Bose-Einstein condensation (BEC) of the gas of $N >> 1$ indistinguishable, noninteracting bosons of mass m, in various confining potentials.

(a) [4 points] Calculate the critical temperature T_c of the condensation in a rectangular, hard-wall box of volume $V = a\times b\times c$, with all linear sizes of the same order. What is the exact value of the chemical potential μ at $T < T_c$?

(b) [3 points] Now one of the box sizes (say, c) is slowly reduced, while other two dimensions are increased to keep the volume V constant. Estimate the value c_0 at which T_c becomes substantially affected by the change.

(c) [4 points] Can the BEC take place at $c << c_0$? If yes, calculate the corresponding T_c. If not, provide a proof.

(d) [7 points] Now the same particles are placed into a soft, spherically-symmetric potential well, whose potential may be approximated as $U = m\omega^2r^2/2 \equiv m\omega^2(x^2 + y^2 + z^2)/2$. Can the BEC take place in this system? If yes, calculate the T_c.

(e) [2 points] Suggest (and justify by estimations) a simple way of experimental detection of the BEC in the case of a soft confining potential.

Hints: You may use the following table integrals,

$$\int_{a>0}^{\infty} \frac{dx}{e^x - 1} = \ln\frac{1}{1-e^{-a}}; \qquad \int_0^{\infty} \frac{x^{s-1}dx}{e^x - 1} = \Gamma(s)\zeta(s), \quad \text{for } s > 1,$$

and treat particular values of the gamma-function $\Gamma(s)$ and the Riemann zeta-function $\zeta(s)$ as known numbers. (For $s \sim 1$, they are of the order of 1 as well, for example, $\Gamma(3/2) = \pi^{1/2}/2$, $\zeta(3/2) \approx 2.612$; $\Gamma(2) = 1$, $\zeta(2) = \pi^2/6$, etc.)