STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

January 2015 (in 4 separate parts: CM, EM, QM, SM)

General Instructions (for each part):
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

Problem on Lagrangian and Hamiltonian mechanics

Consider the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} \frac{\dot{x}^2 + \dot{y}^2 + 2(x\dot{y} - y\dot{x})}{x^2 + y^2}$$

a. (6 pts.) Compute the Hamiltonian $H(x, y, p_x, p_y)$. The final form can be written as

$$\frac{1}{2} f(x, y)[(p_x - A_x(x, y))^2 + (p_y - A_y(x, y))^2]$$

for some f, A_x, A_y. Find the vector potential $\vec{A}$ and then compute the corresponding magnetic field. (Hint: recall that $\vec{B} = \vec{\nabla} \times \vec{A}$).

b. (4 pts.) Prove that the Lagrangian $L(x, y, \dot{x}, \dot{y})$ is invariant under two symmetries: rotations and scale transformations.

c. (4 pts.) Derive the conserved quantities for both symmetries.

d. (6 pts.) Rewrite the Lagrangian in polar coordinates, compute the Euler-Lagrange equations and solve them. Can you give a physical interpretation of the system?

Classical Mechanics 2

A particle in an attractive central potential

Consider the motion of a particle of mass m in an attractive central potential of the form

$$V(r) = \alpha\, r^k\,, \tag{1}$$

where k and α are real constants of the same sign (both positive or both negative).

a. (1 pt.) Write down the Lagrangian using polar coordinates (r, φ).

b. (3 pts.) Using conservation of the angular momentum, reduce the problem of determining the radial motion to an effective one-dimensional problem.

c. (4 pts.) Determine the radius and period of the circular orbits.

d. (2 pts.) For which values of k is the circular orbit *stable*?

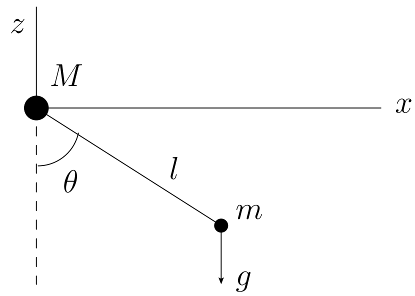
e. (5 pts.) Assuming that the circular orbit is stable, consider a small perturbation around it. Find the period of the small oscillations. In the approximation of small oscillations, for which values of k will the orbit *close*?

f. (5 pts.) Go back to the full $2d$ problem for r and φ (the polar coordinates in the plane of the orbit). Eliminate the time dependence and write a differential equation for the orbit.

Classical Mechanics 3

The Sliding Pendulum

Consider a sliding pendulum, consisting of a mass M which can move without friction along a horizontal bar, and which is connected by a massless rod of length l to another mass m.

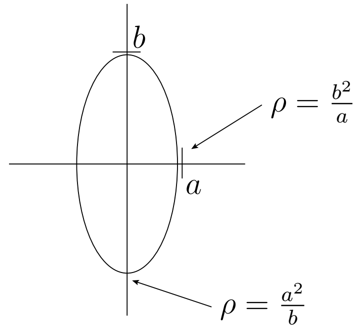


a. (4 pts.) Derive the angular frequency of small oscillations in the $x-z$ plane.

b. (4 pts.) Next consider the sliding conical (also called spherical) pendulum, for which the mass M can move without friction in the horizontal $x-y$ plane. Consider circular motion of both masses for a fixed, not necessarily small, angle θ. What is the angular frequency as a function of θ? Check your result for small θ.

c. (6 pts.) Now consider the inverted sliding pendulum. As it starts falling from rest at $\theta = \pi$, there comes a point where the tension T in the rod becomes zero. Find the value of θ when this happens, assuming for simplicity that M is infinitely large.

d. (6 pts.) What is the tension at $\theta = \frac{\pi}{2}$ and $\theta = 0$ if M and m are arbitrary?
Hint: For an ellipse given by $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, the curvature at the extremum with $x = a$ has a radius $\rho = \frac{b^2}{a}$, and at $y = -b$ one has $\rho = \frac{a^2}{b}$.



Electromagnetism 1

Fields of a non-relativistic particle

A charge particle of charge q moves non-relativistically with trajectory $\boldsymbol{R}(t)$:

(a) (6 pts.) Show that two of the four Maxwell equations are satisfied by expressing the fields $\boldsymbol{E}, \boldsymbol{B}$ in terms of the scalar and vector potentials, $A^{\mu} = (\varphi, \boldsymbol{A})$. Use the remaining Maxwell equations to derive the equations for the scalar and vector potentials in the Lorentz gauge.

(b) (8 pts.) Recall that the Green function of the wave equation is*

$$G(t - t_o, \boldsymbol{r} - \boldsymbol{r}_o) = \frac{\theta(t - t_o)}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|}\delta\left(t - t_o - \frac{|\boldsymbol{r} - \boldsymbol{r}_o|}{c}\right) . \tag{2}$$

Use this Green function to derive the potentials φ and $\boldsymbol{A}$ that are appropriate in the far field and the non-relativistic limit. Explicitly explain how the non-relativistic and far-field approximations are used at various points in the derivation to arrive at the final result.

(c) (4 pts.) If the particle is speeding up along the z axis

$$\boldsymbol{R}(t) = \left(v_o t + \frac{1}{2}a t^2\right)\hat{\boldsymbol{z}} ,$$

determine the electric field in the far field as measured on the x-axis. What is the polarization of the radiated field when measured on this axis?

(d) (2 pts.) Assuming the motion as in part (c), determine the power radiated per solid angle in the $\hat{\boldsymbol{x}}$ direction.

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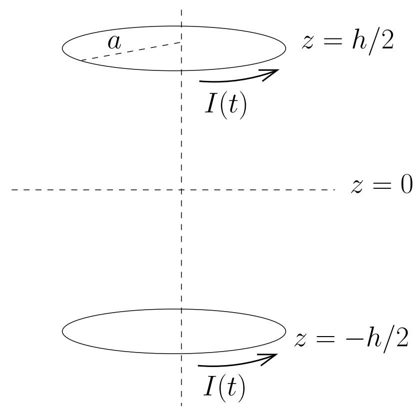
* The Green function satisfies

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) G(t, \boldsymbol{r}) = \delta(t)\delta^3(\boldsymbol{r}) \tag{1}$$

Electromagnetism 2

Electric and magnetic fields of a solenoid

Two identical circular coils of wire of radius a are separated by a length h. The coils each carry a slowly varying sinusoidal current $I(t) = I_o \cos(\omega t)$. The axis of the coils is aligned with the $z-$axis and the geometry is centered at $z = 0$ (see below).



a. (5 pts.) At lowest order in the frequency, a magnetostatic approximation is valid. Using this approximation, show that close to the axis, and near $z = 0$, the Taylor series for the axial and radial components of the slightly *off-axis* magnetic field take the approximate form:

$$B_z \simeq \sigma_o + \frac{1}{2}\sigma_2(z^2 - \frac{\rho^2}{2}) + \dots \qquad B_\rho \simeq -\frac{1}{2}\sigma_2 z\rho + \dots , \tag{1}$$

where σ_0 and σ_2 are determined by the Taylor series of the *on-axis* magnetic field

$$B_z(z) \simeq \sigma_0 + \frac{1}{2}\sigma_2 z^2 + \dots . \tag{2}$$

Here $\rho = \sqrt{x^2 + y^2}$.

b. (5 pts.) Using the magnetostatic approximation, determine the magnetic field in the z direction close to the axis of the solenoid, and near $z = 0$, to quadratic order in z and ρ. Describe the magnetic field when $h = a$.

c. (5 pts.) Determine the electric field close to the axis of the solenoid *at* $z = 0$ to the lowest non-trivial order in the frequency and ρ.

d. (5 pts.) Briefly answer the following:

1. (3pts.) For parts b. and c., give an estimate for the size of finite-frequency corrections.

2. (2pts.) For part b., estimate at what large z the magnetostatic approximation breaks down when computing the magnetic field on the z-axis.

Electromagnetism 3

Waves in metals

Consider an ohmic metal with high (but not infinite) conductivity σ and magnetic permeability[†] $\mu = 1$.

a. (6 pts.) Show that for harmonic time dependence, and high conductivity[‡] $\sigma \gg \omega$, that damped wave like solutions propagating in z-direction in the metal take the approximate form:

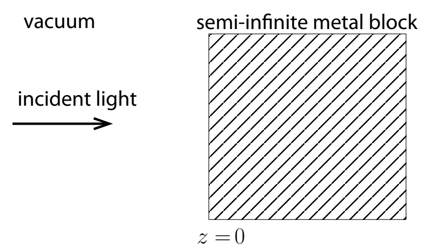
$$\boldsymbol{H}(t, z) = \boldsymbol{H}_c e^{-i\omega t + ik_c z} \tag{1}$$

where[§]

$$k_c = \frac{1+i}{\sqrt{2}} \frac{\sqrt{\sigma\omega}}{c} \tag{2}$$

b. (4 pts.) The electric field obeys a similar equation, $\boldsymbol{E}(t, z) = \boldsymbol{E}_c e^{-i\omega t + ik_c z}$. Use the Maxwell equations to express the amplitude of the electric field $\boldsymbol{E}_c$ in terms of the magnetic field $\boldsymbol{H}_c$.

c. (4 pts.) Now consider a linearly polarized plane wave in vacuum of frequency ω, which is normally incident upon a semi-infinite metal block with *infinite* conductivity as shown below.



When the metal has infinite conductivity, the amplitude of the reflected equals equals the amplitude of the incident wave, but the polarization of the reflected wave is inverted. Explain this familiar fact using the appropriate boundary conditions.

d. (6 pts.) Now consider the same reflection problem as in part 3, but this time the metal has a large (but finite) conductivity σ. Determine the electric and magnetic fields in the metal to leading order in ω/σ. The amplitude of the incident wave is E_o.

† In SI units this reads $\mu = \mu_o$

‡ In SI units this condition reads $(\sigma/\epsilon_o) \gg \omega$

§ This is written in Heaviside-Lorentz units. In SI units $k_c = (1+i)/\sqrt{2}\,\sqrt{\omega(\sigma/\epsilon_o)}\,/c$, while in Gaussian units, $k_c = (1+i)/\sqrt{2}\,\sqrt{4\pi\sigma\omega}\,/c$.

Quantum Mechanics 1

A particle in a perturbed harmonic potential

A particle of mass m in two dimensions is confined by an isotropic harmonic oscillator potential of frequency ω, while subject to a weak and anisotropic perturbation of strength $\alpha \ll 1$. The total Hamiltonian describing the motion of this particle is

$$H = H_0 + V = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + \alpha\, m\omega^2\, xy \tag{1}$$

a. (2 pts.) What are the energies and degeneracies of the three lowest-lying unperturbed states?

b. (5 pts.) Use perturbation theory to correct the energies to first order in α.

c. (5 pts.) Find the exact spectrum of H.

d. (4 pts.) Check that the perturbative results in part b. are recovered.

e. (4 pts.) Assume that 2 identical electrons are subject to the same anisotropic Hamiltonian (1). Write down the explicit wave-functions and degeneracies of the 2 lowest energy states.

Quantum Mechanics 2

Scattering of a particle from two static delta functions

Consider a one-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential barrier $U(x)$ composed of two static delta-functions

$$U(x) = \beta_1 \left(\delta(x) + \delta(x - a)\right) \tag{1}$$

a. (3 pts.) Can the particle tunnel through this barrier without reflection? Explain your answer.

b. (5 pts.) If so, at what value(s) of the kinetic energy does this happen?

Now consider a three-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential $U(\vec{r})$ composed of two static delta-functions

$$U(\vec{r}) = \beta_3 \left(\delta^{(3)}(\vec{r} - \vec{r}_1) + \delta^{(3)}(\vec{r} - \vec{r}_2)\right) \tag{2}$$

Suppose that for each of the delta-functions in (2) the S-wave scattering length $a < 0$.

c. (5 pts.) Write explicitly the S-wave bound wave-function near each of the centers. Check that each does not support a bound state for $a < 0$.

d. (7 pts.) Can the potential $U(\vec{r})$ with two centers support a bound state?
If so, under what conditions? Explain your answer.

Hint: For a hard core potential of radius R, the S-wave scattering length a is defined as $d\ln\chi(R)/dR = -1/a$, with $\chi(r)$ the S-wave reduced wave-function.

Quantum Mechanics 3

Harmonic oscillator subject to a transient external force

Consider a one-dimensional quantum-mechanical harmonic oscillator with mass m and resonance frequency ω. The oscillator initially (at $t \to -\infty$) is in its ground state. It is then subjected to a transient classical force $F(t)$, with $F(t \to \pm\infty) \to 0$.

a. (6 pts.) Write down the Hamiltonian $\hat{H}$ of the forced oscillator described above in terms of the usual ladder operators $\hat{a}$ and $\hat{a}^\dagger$, and solve their equations of motion in the Heisenberg picture. Show that the Hamiltonian for $t \to \pm\infty$ takes the form $\hat{H} = \hbar\omega(\hat{a}^\dagger_{\pm\infty}\hat{a}_{\pm\infty} + 1/2)$, where $\hat{a}^\dagger_{\infty} = \hat{a}^\dagger_{-\infty} - \alpha^\star$ and $\hat{a}_{\infty} = \hat{a}_{-\infty} + \alpha$, and determine the complex term α.

b. (6 pts.) At $t \to \pm\infty$, the ladder operators act on the states $|n_{\pm\infty}\rangle = (1/\sqrt{n!})(\hat{a}^\dagger_{\pm\infty})^n|0_{\pm\infty}\rangle$, where $|0_{\pm\infty}\rangle$ denote the vacuum with respect to $\hat{a}_{\pm\infty}$ and $\hat{a}^\dagger_{\pm\infty}$.

   Determine the probabilities $|c_n|^2$ that the oscillator has undergone a transition from the initial ground state to the n-th excited state at the end of the time evolution.

c. (3 pts.) What is the expectation value of the energy at the end of the time evolution?

d. (5 pts) Now assume that the force has a Gaussian profile, $F(t) = F_0 \exp(-t^2/(2\sigma_t^2))$, with amplitude $F_0 = \eta\hbar\omega/a_{ho}$, where η is a dimensionless parameter, and $a_{ho} = \sqrt{\hbar/m\omega}$ is the harmonic-oscillator length.

   For short pulses with $\sigma_t\omega \ll 1$, determine the maximum pulse strength η for which less than 1% of the population gets lost from the ground state. Show explicitly that in the limit $\sigma_t\omega \gg 1$, losses can be suppressed for any given value of η. Give a physical interpretation of these results.

Statistical Mechanics 1

Heat capacity and heat conductance in a ballistic system

Consider a one-dimensional system of free massless bosons with one polarization, and the dispersion relation $E_k = \hbar v|k|$, where v is the particle velocity, k – wavevector, E_k – energy. The particles are not interacting either among themselves or with external scattering potentials. If the system is in equilibrium at temperature T,

a. (6 pts.) calculate the specific heat capacity C.

b. (6 pts.) calculate the heat conductance G_{th}.

c. (8 pts.) repeat the calculations in (a) and (b) for massive fermions, for which $E_k = \hbar^2 k^2/2m$, and the chemical potential μ is far above the bottom of the energy spectrum: $\mu \gg k_B T$. (Consider one spin direction for the fermions.)

Hints: The heat conductance for the ballistic systems is defined, as usual, as the ratio of the energy flux flowing through a system over the infinitesimal temperature difference across the system that drives this flux. To evaluate the integrals needed to obtain the final numerical constants, you might find useful the following formulas:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

Statistical Mechanics 2

Ising model in an external magnetic field

Consider the Ising model of N spins $\sigma_i = \pm 1$ in an external magnetic field h. Within the mean field approximation, its Hamiltonian can be written as

$$\mathcal{H}_{\mathcal{MF}} = \frac{1}{2} N J m^2 - (Jm + h) \sum_i \sigma_i \,, \tag{1}$$

where the co-ordination number of the lattice has been absorbed into the coupling constant J, and m is the magnetization.

The magnetization, specific heat and magnetic susceptibility are defined, respectively, as

$$m = \frac{\partial f}{\partial h}, \quad C = \frac{\partial U}{\partial T}, \quad \chi = \frac{\partial m}{\partial h} \,,$$

where T is the temperature, U the internal energy, and f the free energy per site.

a. (6 pts.) Derive the (mean field) partition function following from eq. (1) and hence calculate the free energy of the system.

b. (8 pts.) Derive the magnetization, and graphically solve it for $h = 0$. Discuss the physical nature of the various solutions as a function of the temperature T. Identify a critical temperature T_c in terms of the system parameters, and discuss its physical meaning.

c. (6 pts.) Derive the expression for the dependence of the magnetization, the specific heat and the magnetic susceptibility on the quantity

$$t = \frac{T - T_c}{T_c} \,,$$

where T_c is the critical temperature, and thereby determine the mean-field critical exponents $\alpha_c, \beta_c, \gamma_c$ which are defined through the relations

$$m \sim |T - T_c|^{\beta_c} \,, \quad C \sim |T_c - T|^{-\alpha_c} \,, \quad \chi = \frac{\partial m}{\partial h} \,|_{h=0} \sim |T_c - T|^{-\gamma_c} \,.$$

For the calculation of the specific heat, note that, within the mean field approximation near the critical point, the internal energy is $U \propto Jm^2$.

Statistical Mechanics 3

Vapor pressure

a. (2 pts.) Write down the condition for thermodynamic equilibrium between the liquid and gas phases along a liquid-gas coexistence curve.

b. (14 pts.) Using your solution to part (a) and taking into account the entropy change at a liquid-gas phase transition, derive the relation for the vapor pressure along a liquid-gas coexistence curve.

c. (4 pts.) Calculate the vapor pressure of water at 27° C. Some relevant physical constants include the gas constant, $R = 8.315$ J/mol/K and the latent heat of vaporization of water, $L = 2.26 \times 10^3$ J/g or equivalently, $L = 4.06 \times 10^4$ J/mol.)