

Introduction to Machine Learning

Maximum Margin Methods

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Outline

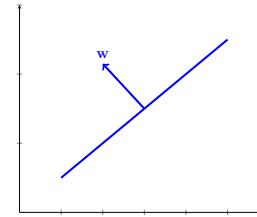
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1 Maximum Margin Classifiers

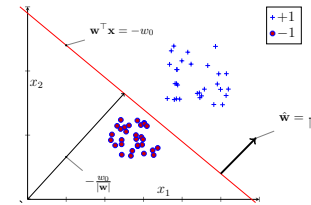
$$y = \mathbf{w}^\top \mathbf{x} + b$$

- Remember the Perceptron!
- If data is linearly separable
 - Perceptron training guarantees learning the decision boundary



- There can be other boundaries
 - Depends on initial value for \mathbf{w}

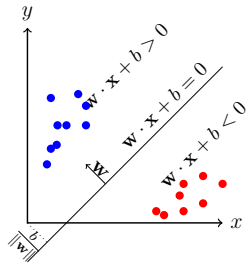
- **But what is the best boundary?**



1.1 Linear Classification via Hyperplanes

- Separates a D -dimensional space into two half-spaces
- Defined by $\mathbf{w} \in \mathbb{R}^D$
 - *Orthogonal* to the hyperplane
 - This \mathbf{w} goes through the origin
 - How do you check if a point lies “above” or “below” \mathbf{w} ?
 - What happens for points **on** \mathbf{w} ?

For a hyperplane that passes through the origin, a point \mathbf{x} will lie above the hyperplane if $\mathbf{w}^\top \mathbf{x} > 0$ and will lie below the plane if $\mathbf{w}^\top \mathbf{x} < 0$, otherwise. This can be further understood by understanding that $b f \mathbf{w}^\top \mathbf{x}$ is essentially equal to $|\mathbf{w}| |\mathbf{x}| \cos \theta$, where θ is the angle between \mathbf{w} and \mathbf{x} .

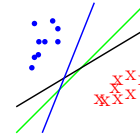


- Add a bias b
 - $b > 0$ - move along \mathbf{w}
 - $b < 0$ - move opposite to \mathbf{w}
- How to check if point lies above or below \mathbf{w} ?
 - If $\mathbf{w}^\top \mathbf{x} + b > 0$ then \mathbf{x} is *above*
 - Else, *below*
- Decision boundary represented by the hyperplane \mathbf{w}
- For binary classification, \mathbf{w} points **towards** the positive class

Decision Rule

$$y = \text{sign}(\mathbf{w}^\top \mathbf{x} + b)$$

- $\mathbf{w}^\top \mathbf{x} + b > 0 \Rightarrow y = +1$
- $\mathbf{w}^\top \mathbf{x} + b < 0 \Rightarrow y = -1$
- **Perceptron** can find a hyperplane that separates the data
 - ... if the data is linearly separable
- But there can be many choices!



- Find the one with best separability (largest margin)
- Gives better generalization performance
 1. Intuitive reason
 2. Theoretical foundations

1.2 Concept of Margin

- **Margin** is the distance between an example and the decision line
- Denoted by γ
- For a positive point:

$$\gamma = \frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|}$$

- For a negative point:

$$\gamma = -\frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|}$$

To understand the margin from a geometric perspective, consider the projection of the vector connecting the origin to a point \mathbf{x} on the decision line. Let the point be denoted as \mathbf{x}' . Obviously the vector \mathbf{r} connecting \mathbf{x}' and \mathbf{x} is given by:

$$\mathbf{r} = \gamma \hat{\mathbf{w}} = \gamma \frac{\mathbf{w}}{\|\mathbf{w}\|}$$

if \mathbf{x} lies on the positive side of \mathbf{w} . But the same vector can be computed as:

$$\mathbf{r} = \mathbf{x} - \mathbf{x}'$$

Equating above two gives us \mathbf{x}' as:

$$\mathbf{x}' = \mathbf{x} - \gamma \frac{\mathbf{w}}{\|\mathbf{w}\|}$$

Noting that, since \mathbf{x}' lies on the hyperplane and hence:

$$\mathbf{w}^\top \mathbf{x}' + b = 0$$

Substituting \mathbf{x}' from above:

$$\mathbf{w}^\top \mathbf{x} - \gamma \frac{\mathbf{w}^\top \mathbf{w}}{\|\mathbf{w}\|} + b = 0$$

Noting that $\frac{\mathbf{w}^\top \mathbf{w}}{\|\mathbf{w}\|} = \|\mathbf{w}\|$, we get γ as:

$$\gamma = \frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|} \quad (1)$$

Similar analysis can be done for points on the negative side of \mathbf{x} . In general, one can write the expression for the margin as:

$$\gamma = y \frac{\mathbf{w}^\top \mathbf{x} + b}{\|\mathbf{w}\|} \quad (2)$$

where $y \in \{-1, +1\}$.

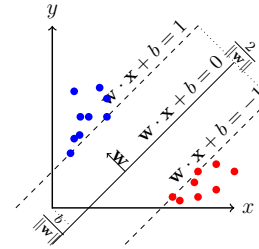
Functional Interpretation

- Margin **positive** if prediction is **correct**; **negative** if prediction is **incorrect**

From the figure one can note that the size of the margin is $\frac{2}{\|\mathbf{w}\|}$. We can show this as follows. Since the data is separable, we can get two parallel lines represented by $\mathbf{w}^\top \mathbf{x} + b = +1$ and $\mathbf{w}^\top \mathbf{x} + b = -1$. Using result from (1) and (2), the distance between the two lines is given by $2\gamma = \frac{2}{\|\mathbf{w}\|}$.

2 Support Vector Machines

- A hyperplane based classifier defined by \mathbf{w} and b



- Like perceptron
- Find hyperplane with *maximum separation margin* on the training data
- Assume that data is linearly separable (will relax this later)
 - Zero training error (loss)

SVM Prediction Rule

$$y = \text{sign}(\mathbf{w}^\top \mathbf{x} + b)$$

SVM Learning

- **Input:** Training data $\{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \dots, (\mathbf{x}_N, y_N)\}$
- **Objective:** Learn \mathbf{w} and b that maximizes the margin

2.1 SVM Learning

- SVM learning task as an optimization problem
- Find \mathbf{w} and b that gives zero training error
- Maximizes the margin ($= \frac{2}{\|\mathbf{w}\|}$)
- Same as minimizing $\|\mathbf{w}\|$

Optimization Formulation

$$\begin{aligned} & \underset{\mathbf{w}, b}{\text{minimize}} && \frac{\|\mathbf{w}\|^2}{2} \\ & \text{subject to} && y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1, \quad i = 1, \dots, n. \end{aligned}$$

- **Optimization** with N linear inequality constraint

A Different Interpretation of Margin

- What impact does the margin have on \mathbf{w} ?
- Large margin \Rightarrow Small $\|\mathbf{w}\|$
- Small $\|\mathbf{w}\| \Rightarrow$ regularized/simple solutions
- Simple solutions \Rightarrow Better generalizability (*Occam's Razor*)
- Computational Learning Theory provides a formal justification [1]

2.2 Solving SVM Optimization Problem

Optimization Formulation

$$\begin{aligned} & \underset{\mathbf{w}, b}{\text{minimize}} && \frac{\|\mathbf{w}\|^2}{2} \\ & \text{subject to} && y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1, \quad i = 1, \dots, n. \end{aligned}$$

- There is an quadratic objective function to minimize with N inequality constraints
- “Off-the-shelf” packages - quadprog (MATLAB), CVXOPT
- Is that the best way?

3 Constrained Optimization and Lagrange Multipliers

$$\underset{x, y}{\text{minimize}} \quad f(x, y) = x^2 + 2y^2 - 2$$

$$\begin{aligned} & \underset{x, y}{\text{minimize}} && f(x, y) = x^2 + 2y^2 - 2 \\ & \text{subject to} && h(x, y) = x + y - 1 = 0. \end{aligned}$$

- Tool for solving constrained optimization problems of differentiable functions

$$\begin{aligned} & \underset{x, y}{\text{minimize}} && f(x, y) = x^2 + 2y^2 - 2 \\ & \text{subject to} && h(x, y) : x + y - 1 = 0. \end{aligned}$$

- A Lagrangian multiplier (β) lets you combine the two equations into one

$$\underset{x, y, \beta}{\text{minimize}} \quad L(x, y, \beta) = f(x, y) + \beta h(x, y)$$

Solution 1. Writing the objective as Lagrangian.

$$L(x, y, \beta) = x^2 + 2y^2 - 2 + \beta(x + y - 1)$$

Setting the gradient to 0 with respect to x, y and β will give us the optimal values.

$$\frac{\partial L}{\partial x} = 2x + \beta = 0$$

$$\frac{\partial L}{\partial y} = 4y + \beta = 0$$

$$\frac{\partial L}{\partial \beta} = x + y - 1 = 0$$

Multiple Constraints

$$\begin{array}{ll} \underset{x,y,z}{\text{minimize}} & f(x,y,z) = x^2 + 4y^2 + 2z^2 + 6y + z \\ \text{subject to} & h_1(x,y,z) : x + z^2 - 1 = 0 \\ & h_2(x,y,z) : x^2 + y^2 - 1 = 0. \end{array}$$

$$L(x,y,z,\boldsymbol{\beta}) = f(x,y,z) + \sum_i \beta_i h_i(x,y,z)$$

Handling Inequality Constraints

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & f(x,y) = x^3 + y^2 \\ \text{subject to} & g(x) : x^2 - 1 \leq 0. \end{array}$$

- Inequality constraints are **transferred** as constraints on the Lagrangian, α

The Lagrangian in the above example becomes:

$$\begin{aligned} L(x,y,\alpha) &= f(x,y) + \alpha g(x,y) \\ &= x^3 + y^2 + \alpha(x^2 - 1) \end{aligned}$$

Solving for the gradient of the Lagrangian gives us:

$$\begin{aligned} \frac{\partial}{\partial x} L(x,y,\alpha) &= 3x^2 + 2\alpha x = 0 \\ \frac{\partial}{\partial y} L(x,y,\alpha) &= 2y = 0 \\ \frac{\partial}{\partial \alpha_1} L(x,y,\alpha) &= x^2 - 1 = 0 \end{aligned}$$

Furthermore we require that:

$$\alpha \geq 0$$

From above equations we get $y = 0$, $x = \pm 1$ and $\alpha = \pm \frac{3}{2}$. But since $\alpha \geq 0$, hence $\alpha = \frac{3}{2}$. This gives $x = 1$, $y = 0$, and $f = 1$.

Handling Both Types of Constraints

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & f(\mathbf{w}) \\ \text{subject to} & g_i(\mathbf{w}) \leq 0 \quad i = 1, \dots, k \\ \text{and} & h_i(\mathbf{w}) = 0 \quad i = 1, \dots, l. \end{array}$$

Generalized Lagrangian

$$L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = f(\mathbf{w}) + \sum_{i=1}^k \alpha_i g_i(\mathbf{w}) + \sum_{i=1}^l \beta_i h_i(\mathbf{w})$$

subject to, $\alpha_i \geq 0, \forall i$

Primal and Dual Formulations

Primal Optimization

- Let θ_P be defined as:

$$\theta_P(\mathbf{w}) = \max_{\boldsymbol{\alpha}, \boldsymbol{\beta} : \alpha_i \geq 0} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

- One can prove that the optimal value for the original constrained problem is same as:

$$p^* = \min_{\mathbf{w}} \theta_P(\mathbf{w}) = \min_{\mathbf{w}} \max_{\boldsymbol{\alpha}, \boldsymbol{\beta} : \alpha_i \geq 0} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

Consider

$$\begin{aligned} \theta_P(\mathbf{w}) &= \max_{\boldsymbol{\alpha}, \boldsymbol{\beta} : \alpha_i \geq 0} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\ &= \max_{\boldsymbol{\alpha}, \boldsymbol{\beta} : \alpha_i \geq 0} f(\mathbf{w}) + \sum_{i=1}^k \alpha_i g_i(\mathbf{w}) + \sum_{i=1}^l \beta_i h_i(\mathbf{w}) \end{aligned}$$

It is easy to show that if any constraints are not satisfied, i.e., if either $g_i(\mathbf{w}) > 0$ or $h_i(\mathbf{w}) \neq 0$, then $\theta_P(\mathbf{w}) = \infty$. Which means that:

$$\theta_P(\mathbf{w}) = \begin{cases} f(\mathbf{w}) & \text{if primal constraints are satisfied} \\ \infty & \text{otherwise,} \end{cases}$$

Primal and Dual Formulations (II)

Dual Optimization

- Consider θ_D , defined as:

$$\theta_D(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \min_{\mathbf{w}} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

- The **dual** optimization problem can be posed as:

$$d^* = \max_{\boldsymbol{\alpha}, \boldsymbol{\beta}: \alpha_i \geq 0} \theta_D(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \max_{\boldsymbol{\alpha}, \boldsymbol{\beta}: \alpha_i \geq 0} \min_{\mathbf{w}} L(\mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

$d^* == p^*$?

- Note that $d^* \leq p^*$
- “Max min” of a function is always less than or equal to “Min max”
- When will they be equal?
 - $f(\mathbf{w})$ is convex
 - Constraints are affine

Relation between primal and dual

- In general $d^* \leq p^*$, for SVM optimization the equality holds
- Certain conditions should be true
- Known as the **Karun-Kuhn-Tucker** conditions
- For $d^* = p^* = L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) &= 0 \\ \frac{\partial}{\partial \beta_i} L(\mathbf{w}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) &= 0, \quad i = 1, \dots, l \\ \alpha_i^* g_i(\mathbf{w}^*) &= 0, \quad i = 1, \dots, k \\ g_i(\mathbf{w}^*) &\leq 0, \quad i = 1, \dots, k \\ \alpha_i^* &\geq 0, \quad i = 1, \dots, k \end{aligned}$$

Optimization Formulation

$$\begin{aligned} &\underset{\mathbf{w}, b}{\text{minimize}} && \frac{\|\mathbf{w}\|^2}{2} \\ &\text{subject to} && y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1, \quad i = 1, \dots, n. \end{aligned}$$

A Toy Example

- $\mathbf{x} \in \mathbb{R}^2$
- Two training points:
 - $\mathbf{x}_1, y_1 = (1, 1), -1$
 - $\mathbf{x}_2, y_2 = (2, 2), +1$
- Find the best hyperplane $\mathbf{w} = (w_1, w_2)$

Optimization problem for the toy example

$$\begin{aligned} &\underset{\mathbf{w}}{\text{minimize}} && f(\mathbf{w}) = \frac{1}{2} \|\mathbf{w}\|^2 \\ &\text{subject to} && g_1(\mathbf{w}, b) = y_1(\mathbf{w}^\top \mathbf{x}_1 + b) - 1 \geq 0 \\ &&& g_2(\mathbf{w}, b) = y_2(\mathbf{w}^\top \mathbf{x}_2 + b) - 1 \geq 0. \end{aligned}$$

- Substituting actual values for \mathbf{x}_1, y_1 and \mathbf{x}_2, y_2 .

$$\begin{aligned} &\underset{\mathbf{w}}{\text{minimize}} && f(\mathbf{w}) = \frac{1}{2} \|\mathbf{w}\|^2 \\ &\text{subject to} && g_1(\mathbf{w}, b) = -(\mathbf{w}^\top \mathbf{x}_1 + b) - 1 \geq 0 \\ &&& g_2(\mathbf{w}, b) = (\mathbf{w}^\top \mathbf{x}_2 + b) - 1 \geq 0. \end{aligned}$$

The above problem can be also written as:

$$\begin{aligned} &\underset{w_1, w_2, b}{\text{minimize}} && f(w_1, w_2) = \frac{1}{2}(w_1^2 + w_2^2) \\ &\text{subject to} && g_1(w_1, w_2, b) = -(w_1 + w_2 + b) - 1 \geq 0 \\ &&& g_2(w_1, w_2, b) = (2w_1 + 2w_2 + b) - 1 \geq 0. \end{aligned}$$

To solve the toy optimization problem, we rewrite it in the Lagrangian form:

$$L(w_1, w_2, b, \alpha) = \frac{1}{2}(w_1^2 + w_2^2) + \alpha_1(w_1 + w_2 + b + 1) - \alpha_2(2w_1 + 2w_2 + b - 1)$$

Setting $\nabla L = 0$, we get:

$$\begin{aligned} \frac{\partial}{\partial w_1} L(w_1, w_2, b, \alpha) &= w_1 + \alpha_1 - 2\alpha_2 = 0 \\ \frac{\partial}{\partial w_2} L(w_1, w_2, b, \alpha) &= w_2 + \alpha_1 - 2\alpha_2 = 0 \\ \frac{\partial}{\partial b} L(w_1, w_2, b, \alpha) &= \alpha_1 - \alpha_2 = 0 \\ \frac{\partial}{\partial \alpha_1} L(w_1, w_2, b, \alpha) &= w_1 + w_2 + b + 1 = 0 \\ \frac{\partial}{\partial \alpha_2} L(w_1, w_2, b, \alpha) &= 2w_1 + 2w_2 + b - 1 = 0 \end{aligned}$$

Solving the above equations, we get, $w_1 = w_2 = 1$ and $b = -3$.

Back to SVM Optimization

Optimization Formulation

$$\begin{aligned} &\underset{\mathbf{w}, b}{\text{minimize}} \quad \frac{\|\mathbf{w}\|^2}{2} \\ &\text{subject to} \quad y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1, \quad i = 1, \dots, n. \end{aligned}$$

- Introducing [Lagrange Multipliers](#), $\alpha_i, i = 1, \dots, n$

Rewriting as a (primal) Lagrangian

$$\begin{aligned} &\underset{\mathbf{w}, b, \alpha}{\text{minimize}} \quad L_P(\mathbf{w}, b, \alpha) = \frac{\|\mathbf{w}\|^2}{2} + \sum_{i=1}^n \alpha_i \{1 - y_i(\mathbf{w}^\top \mathbf{x}_i + b)\} \\ &\text{subject to} \quad \alpha_i \geq 0 \quad i = 1, \dots, n. \end{aligned}$$

Solving the Lagrangian

- Set gradient of L_P to 0

$$\frac{\partial L_P}{\partial \mathbf{w}} = 0 \Rightarrow \mathbf{w} = \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i$$

$$\frac{\partial L_P}{\partial b} = 0 \Rightarrow \sum_{i=1}^n \alpha_i y_i = 0$$

- Substituting in L_P to get the dual L_D

Dual Lagrangian Formulation

$$\begin{aligned} &\underset{b, \alpha}{\text{maximize}} \quad L_D(\mathbf{w}, b, \alpha) = \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{m, n=1}^N \alpha_m \alpha_n y_m y_n (\mathbf{x}_m^\top \mathbf{x}_n) \\ &\text{subject to} \quad \sum_{i=1}^n \alpha_i y_i = 0, \alpha_i \geq 0 \quad i = 1, \dots, n. \end{aligned}$$

- Dual Lagrangian is a *quadratic programming problem* in α_i 's
 - Use “off-the-shelf” solvers
- Having found α_i 's

$$\mathbf{w} = \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i$$

- What will be the bias term b ?

Investigating Karun Kuhn Tucker Conditions

- For the primal and dual formulations
- We can optimize the dual formulation (as shown earlier)
- Solution should satisfy the **Karush-Kuhn-Tucker** (KKT) Conditions

3.1 Kahrn-Kuhn-Tucker Conditions

$$\frac{\partial}{\partial \mathbf{w}} L_P(\mathbf{w}, b, \alpha) = \mathbf{w} - \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i = 0 \quad (3)$$

$$\frac{\partial}{\partial b} L_P(\mathbf{w}, b, \alpha) = - \sum_{i=1}^n \alpha_i y_i = 0 \quad (4)$$

$$1 - y_i \{\mathbf{w}^\top \mathbf{x}_i + b\} \leq 0 \quad (5)$$

$$\alpha_i \geq 0 \quad (6)$$

$$\alpha_i (1 - y_i \{\mathbf{w}^\top \mathbf{x}_i + b\}) = 0 \quad (7)$$

- Use KKT condition #5

- For $\alpha_i > 0$

$$(y_i \{\mathbf{w}^\top \mathbf{x}_i + b\} - 1) = 0$$

- Which means that:

$$b = - \frac{\max_{n:y_i=-1} \mathbf{w}^\top \mathbf{x}_i + \min_{n:y_i=1} \mathbf{w}^\top \mathbf{x}_i}{2}$$

3.2 Support Vectors

Most α_i 's are 0

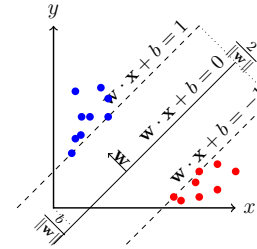
- KKT condition #5:

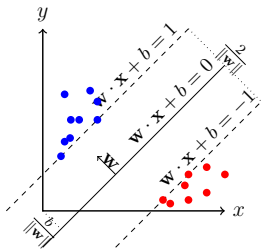
$$\alpha_i (1 - y_i \{\mathbf{w}^\top \mathbf{x}_i + b\}) = 0$$

- If \mathbf{x}_i **not** on margin

$$\begin{aligned} y_i \{\mathbf{w}^\top \mathbf{x}_i + b\} &> 1 \\ \Rightarrow \alpha_i &= 0 \end{aligned}$$

- $\alpha_i \neq 0$ only for \mathbf{x}_i on margin
- These are the **support vectors**
- Only need these for prediction





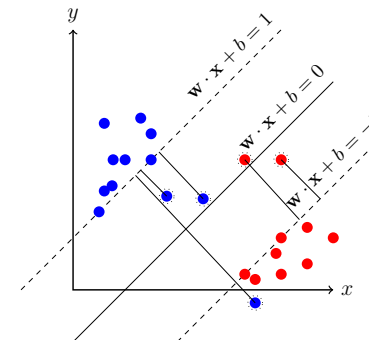
One can see from the prediction equation that:

$$y^* = \text{sign}\left(\sum_{i=1}^n \alpha_i y_i (\mathbf{x}_i^\top \mathbf{x}^*)\right)$$

In the summation, the entries for \mathbf{x}_i that do not lie on the margin will have no contribution to the sum because α_i for those \mathbf{x}_i 's will be 0. Hence we only need to the non-zero input examples to get the prediction.

What have we seen so far?

- For linearly separable data, SVM learns a weight vector \mathbf{w}
- Maximizes the margin
- SVM training is a **constrained optimization problem**
 - Each training example should lie outside the margin
 - N constraints
- Cannot go for zero training error
- Still learn a maximum margin hyperplane
 1. Allow some examples to be misclassified
 2. Allow some examples to fall **inside** the margin
- How do you set up the optimization for SVM training



Introducing Slack Variables

- **Separable Case:** To ensure zero training loss, constraint was

$$y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1 \quad \forall i = 1 \dots n$$

- **Non-separable Case:** Relax the constraint

$$y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1 - \xi_i \quad \forall i = 1 \dots n$$

- ξ_i is called **slack variable** ($\xi_i \geq 0$)
- For misclassification, $\xi_i > 1$

3.3 Optimization Constraints

- It is OK to have some misclassified training examples
 - Some ξ_i 's will be non-zero
- Minimize the number of such examples

$$\text{Minimize } \sum_{i=1}^n \xi_i$$

- Optimization Problem for Non-Separable Case

$$\begin{array}{ll} \underset{\mathbf{w}, b}{\text{minimize}} & f(\mathbf{w}, b) = \|\mathbf{w}\|^2 + C \sum_{i=1}^n \xi_i \\ \text{subject to} & y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1 - \xi_i, \xi_i \geq 0 \quad i = 1, \dots, n. \end{array}$$

- C controls the impact of margin and the margin error.
- What is the role of C ?
- Similar optimization procedure as for the separable case (QP for the dual)
- Weights have the same expression

$$\mathbf{w} = \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i$$

- Support vectors are slightly different
 1. Points on the margin ($\xi_i = 0$)
 2. Inside the margin but on the correct side ($0 < \xi_i < 1$)
 3. On the wrong side of the hyperplane ($\xi_i \geq 1$)

C dictates if we focus more on maximizing the margin or reducing the training error.

- Training time for SVM training is $O(n^3)$
- Many *faster* but approximate approaches exist
 - Approximate QP solvers
 - Online training
- SVMs can be extended in different ways
 1. Non-linear boundaries (**kernel trick**)
 2. Multi-class classification
 3. Probabilistic output
 4. Regression (Support Vector Regression)

References

References

- [1] V. Vapnik. *Statistical learning theory*. Wiley, 1998.