Roshan Poduval Stat 461 Midterm Roshan Poduval 10/21/2020

1) a)

$$\begin{tikzcd}[column sep=small]
 & & S_{2,0} = 121 \\
 & S_{1,0} = 110 \arrow[ur] \arrow[dr] & \\
S_0 = 100 \arrow[ur] \arrow[dr] & & S_{2,1} = 99 \\
 & S_{1,1} = 90 \arrow[ur] \arrow[dr] & \\
 & & S_{2,2} = 81
\end{tikzcd}$$

$121 - 90 = 31$

$99 - 90 = 9$

$(81-90)_+ = 0$

b) $q = \dfrac{\exp(rT) - d}{u - d} = \dfrac{\exp(.05/2) - .9}{1.1 - .9} = .6265$

so prob of move down is $1 - .6265 = \underline{.3734}$

c) At $S_{1,0}$, the option price is $\exp(-.05/2) \cdot (.6265 \cdot 31 + .3734 \cdot 9) = 22.22$

" " $S_{1,1}$ " " $\exp(-.05/2) \cdot (.6265 \cdot 9 + 0) = 5.50$

At S_0 " " $\exp(-.05/2) \cdot (.6265 \cdot 22.22 + .3734 \cdot 5.5) = \underline{\$15.58}$

d)

$$\begin{tikzcd}[column sep=small]
 & S_{1,0} = 110 \\
S_0 = 100 \arrow[ur] \arrow[dr] & \\
 & S_{1,1} = 90
\end{tikzcd}$$

$C_0 = 3.06$

$(110 - 105)_+ = 5$

$(90 - 105)_+ = 0$

$110\Delta - 5 = 90\Delta \Rightarrow \Delta = .25$

Action now:

- Sell one call for \$4
- Buy .25 shares for \$25
- Borrow \$21

Action in 6 months:

If $S_t = 110$

- buy .75 shares for \$82.5
- sell 1 share thru call for \$105

If $S_t = 90$

- sell .25 shares for \$22.5
- no action by call owner

• Pay back $\exp(.05/2)\cdot 21$ • Pay back $\exp(.05/2)\cdot 21$

$$\left.\begin{aligned} -82.5 + 105 - \exp(.05/2)\cdot 21 &= .97 \\ 22.5 - \exp(.05/2)\cdot 21 &= .97 \end{aligned}\right\} \text{free lunch}$$

2) a) $q = \dfrac{1+r/n - d}{u-d} = \dfrac{1+r/n - (1+r/n - \sigma/\sqrt{n})}{(1+r/n+\sigma/\sqrt{n}) - (1+r/n - \sigma/\sqrt{n})} = \dfrac{1+r/n - 1 - r/n + \sigma/\sqrt{n}}{2\sigma/\sqrt{n}} = \dfrac{\sigma/\sqrt{n}}{2\sigma/\sqrt{n}} = \dfrac{1}{2}$

b) $E_Q[S^*_{k+1} \mid S^*_k, \dots, S^*_0] = (1+r/n)^{-(k+1)} (q S_k u + (1-q) S_k d)$

$(1+r/n)^{-(k+1)} (q S_k (u-d) + S_k d)$

$(1+r/n)^{-(k+1)} (\frac{1}{2} S_k (u-d) + S_k d)$

$(1+r/n)^{-k} \cdot S_k = S^*_k$

So, S^*_k is a martingale

c) For binomial distribution $Bin(n,q)$, this is simulating taking n trials, each with success probability q, and counting the number of successes. For our situation, each trial is a step (up or down) of the stock price, and the probability of a step up is q, and the number of steps up is X.

X follows the exact description of $Bin(n,q)$.

d) $C_0 = (1+r/n)^{-n} E_Q[(S_n - K)_+ \mid S_0]$ $\quad S_n = S_0 u^j d^{n-j}$ $\quad Q[S_n = S_0 u^j d^{n-j}] = \binom{n}{j} q^j (1-q)^{n-j}$

$C_0 = (1+r/n)^{-n} \sum_{j=0}^{n} \left[((S_0 u^j d^{n-j}) - K)_+ \cdot \binom{n}{j} q^j (1-q)^{n-j} \right]$

e) $S^n_{t_i} = S_0 u^x d^{i-x} = S_0 \left(\frac{u}{d}\right)^x d^i = S_0 \left[\frac{\exp(\mu/n + \sigma/\sqrt{n})}{\exp(\mu/n - \sigma/\sqrt{n})}\right]^x \left[\exp(\mu/n - \sigma/\sqrt{n})\right]^i = S_0 \exp\left(2\frac{\sigma}{\sqrt{n}} x\right) \exp\left(\frac{\mu i}{n} - \frac{\sigma i}{\sqrt{n}}\right)$

$= S_0 \exp\left(\mu i/n + \sigma/\sqrt{n}\,(2x - i)\right)$

Let W_j be a RV such that $W_j = 1$ if the stock price goes up and $W_j = 0$ if the stock goes down.

$\sum_{j=1}^{i} W_i = X \qquad Y_j = 2W_j - 1$

$2X - i = 2\sum_{j=1}^{i} W_j - i = 2\sum_{j=1}^{i} W_j - \sum_{j=1}^{i} 1 = \sum_{j=1}^{i} (2W_j - 1) = \sum_{j=1}^{i} Y_j$

So, $S^n_{t_i} = S_0 \exp\left(\mu i/n + \sigma/\sqrt{n}\,(2x-i)\right) = S_0 \exp\left(\mu t_i + \sigma/\sqrt{n} \sum_{j=1}^{i} Y_j\right) \qquad Y = \sum_{j=1}^{i} Y_j$

$= S_0 \exp\left(\mu t_i + \sigma/\sqrt{n}\, Y\right)$

Under prob. measure Q, as $n \to \infty$ $\sigma/\sqrt{n}\, Y$ converges in distribution to

$$\left(r - \mu - \sigma^2/2\right)t + \sigma W_t \qquad (\mu = 0)$$

So, $S_n = S_0 \exp\left(\sigma W_t + [r - \sigma^2/2]t\right)$ ✓

f) We know that stock price is martingale, and since $r = 0$, there is no interest (\$100 now is \$100 at any time, no need to discount). Because of this, to know S_1, we can take any other stock price S_t and "discount back" with $r = 0$.

So $E_Q[S_1 | S_t] = S_t$.