Option Valuation

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Options

Option: a contract provides the holder with the right to trade a specified quantity of an unerlying asset at a fixed price (called a strike price or an exercise price) at or before the expiration date of the option.

stock

- Right, Not obligation.
- Two types of options:
  - Call option: to buy.
  - Put option: to sell.
- Two common options:
  - European option: can exercise only at expiration date.
  - American option: can exercise any time before or at expiration date.
- Two sides of every option contract:
  - Long position: the investor who has bought or own the option.
  - Short position: the investor who has sold or written the option.

Call Option: Example

Example: suppose that an investor owns a European call option to buy 100 shares of stock A at \$25 per share expired in six months. Suppose that the options costs \$3 per share. The risk-free interest rate is 8% per annum.

$$\text{premium payment} = \$3 \times 100 = \$300.$$

1. Suppose that the stock is selling for \$30 per share six months later. The owner will exercise the option and the net profit in dollar value at the time of purchase is

$$\exp\left(\frac{-0.08}{2}\right) \times \overset{(30\times 100 - 25\times 100)}{\$500} - \$300 = \$180.40.$$

At the exercise date the dollar value of the net profit is

$$\$500 - \exp\left(\frac{0.08}{2}\right) \times \$300 = \$187.76.$$

Call Option: Example

2. If the stock price falls below the exercise price at the maturity, a call option is worthless and the owner will not exercise the call option.
3. Suppose the stock is selling for \$27 per share six months later. The owner will still exercise the option and the net profit in dollar value at the time of purchase is

$$\exp\left(\frac{-0.08}{2}\right) \times \$200 - \$300 = -\$107.84.$$

At the exercise date the dollar value of the net profit is

$$\$200 - \exp\left(\frac{0.08}{2}\right) \times \$300 = -\$112.24.$$

(The negative sign indicates a loss.)

Call Option: Example

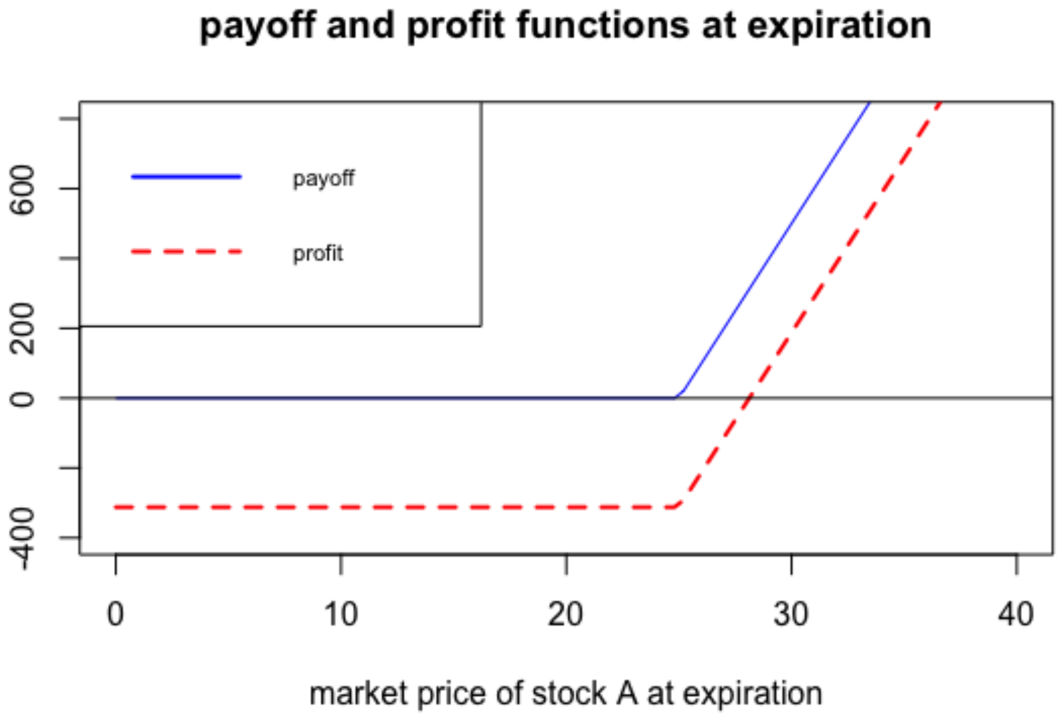


Figure: Payoff (solid line) and profit (dashed line) for the owner (long position) of a European call option on 100 shares of stock A at expiration. The negative sign in profit function indicates a loss.

Put Option: Example

Example: suppose that an investor owns a European put option to sell 100 shares of stock B at \$40 per share expired in three months. Suppose that the options costs \$1 per share. The risk-free interest rate is 6% per annum. Premium payment=$\$1 \times 100 = \$100$.

1. Suppose that the stock has a market price of \$38 per share three months later. The owner will exercise the option and the net profit in dollar value at the time of purchase is

   $$\exp\left(\frac{-0.06}{4}\right) \times \overset{(40\cdot 100 - 38\cdot 100)}{\$200} - \$100 = \$97.20.$$

   At the exercise date the dollar value of the net profit is

   $$\$200 - \exp\left(\frac{0.06}{4}\right) \times \$100 = \$98.49.$$

2. If the stock price rises above the exercise price at the maturity, a put option is worthless and the owner will not exercise the put option.

Put Option: Example

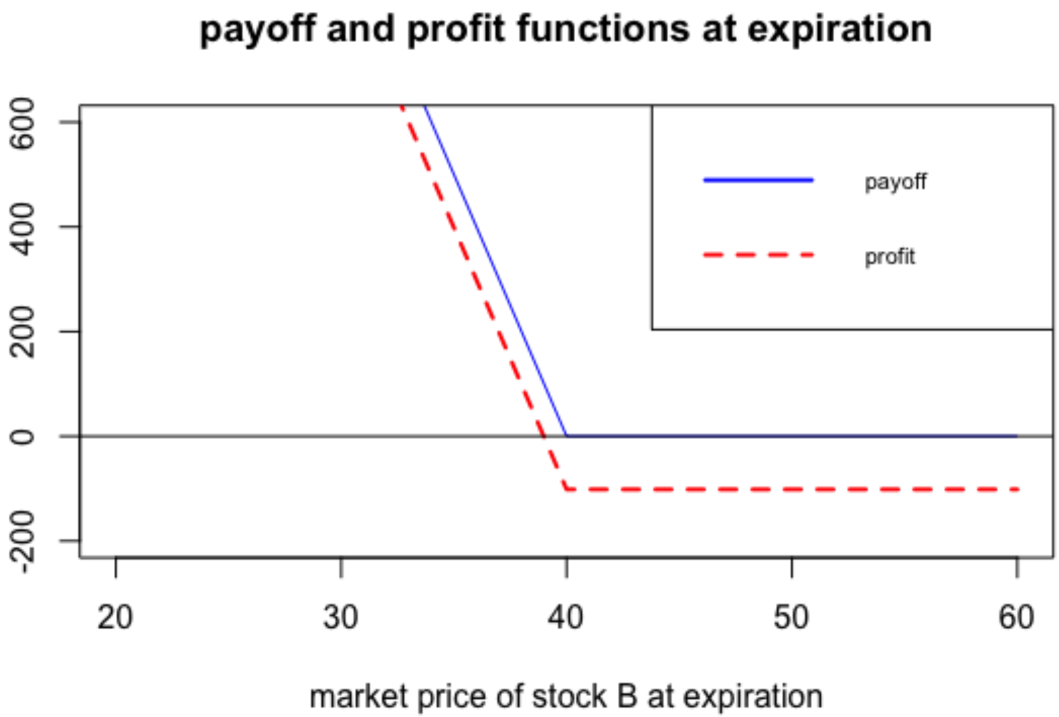


Figure: Payoff (solid line) and profit (dashed line) for the owner (long position) of a European put option on 100 shares of stock B at expiration. The negative sign in profit function indicates a loss.

Options

- For a European call option, the pay-off at the maturity is

$$\max(S_T - K, 0) = (S_T - K)_+.$$

The holder only exercises his/her option if $S_T > K$ (buys the stock via exercising the option and sells the stock on the market).

- For a European put option, the pay-off at the maturity is

$$\max(K - S_T, 0) = (K - S_T)_+.$$

The holder only exercises his/her option if $S_T < K$ (buys the stock from the market and sells it via option).

Arbitrage

Arbitrage: ”free lunch” in plain English.

- Arbitrage means locking in a guaranteed risk-free profit by trading in the market without investing capital.
- Arbitrage often involves with simultaneously entering transactions in two or more markets.

Example: Suppose that a stock is traded on the New York Stock Exchange at \$88 per share and on the London Stock Exchange at £50 per share when exchange rate is \$1.80 per pound.

An arbitrage opportunity occurs: simultaneously purchase 100 shares of the stock in the New York Stock Exchange and sell them in the London Stock Exchange and obtain a risk-free profit of

$$(\$1.80 \times £50 - \$88) \times 100 = \$200.$$

$(1.8 \cdot 50) \cdot 100 - 88 \cdot 100 = 20$

General Assumption: NO ARBITRAGE

Major arbitrage opportunities occur rarely and if occur, they usually last for a short period of time. It is a wide belief that without taking a risk, one can not make a profit on free competitive markets.

General assumption: NO ARBITRAGE.

- **Law of one price**: two financial instruments have the same price if their payoffs are exactly the same.
- One can price an option by finding a portfolio with a known price and having exactly the same payoffs as the option. By the law of one price, the price of the option must be equal to the known price of the portfolio.
- Pricing by the law of one price is also called **arbitrage pricing**.

Put-Call Parity: $C + Ke^{-rT} = P + S_0$

Two portfolios:

- Portfolio A: one European call option with strike price K plus cash amount $= Ke^{-rT}$;
- Portfolio B: one European put option with strike price K plus one share of the underlying asset.

Then at the expiration time T, the two portfolios each has value

- Portfolio A:

$$\max(S_T - K, 0) + e^{rT} \times Ke^{-rT} = \max(S_T, K).$$

- Portfolio B:

$$\max(K - S_T, 0) + S_T = \max(K, S_T).$$

Put-Call Parity: $C + Ke^{-rT} = P + S_0$

Because the options are European, they cannot be exercised before T, they must also have the same relative value today.

C – Price of call option @ current time
P – " " put " " must be for same stock

Put-call Parity:

$$C + Ke^{-rT} = P + S_0$$

where C and P are the prices of European call and put options, respectively.

- This relationship is often used to compute put option price from call option price, or vice versa.
- If put-call parity does not hold, arbitrageurs can make a sure profit by shorting the securities in the expensive portfolio and buying the securities in the cheaper one.

Arbitrage when Put-Call Parity does not Hold

Case 1: if

$$C + Ke^{-rT} < P + S_0,$$

(Put-Call Parity Theorem)

then short both the put and the stock and use the proceeds ($= P + S_0$) to buy the call and invest the difference ($P + S_0 - C$) at the risk free rate.

At expiration, one of the two options is in the money and one is out. In either case, the arbitrageur ends up buying one share at a price of K, leaving him/her with a profit of

$$(P + S_0 - C)e^{rT} - K.$$

Put option tends to be associated with stock

Call is associated with cash

Case 2: if

$$C + Ke^{-rT} > P + S_0,$$

then arbitrage by shorting the call, borrowing $P + S_0 - C$, and buying the put and the stock. The investor will end up selling one share for K, earning a profit of $K - (P + S_0 - C)e^{rT}$.

Short - sell stocks you don't have (say you borrow stocks from a broker)
and pay the broker back later (by buying stock [hopefully if went down])

You can either short the put and the stock (and buy the call) or
short the call (and buy the put and the stock)
Short the put/call if the put/call side is bigger

$S_0 = 31 \quad K = 30$

Arbitrage when Put-Call Parity does not Hold

Example: consider a European call option on stock C with strike price $K = \$30$ and has 3 months until expiration. The current price of stock C is $S_0 = \$31$ and the risk-free interest rate is $r = 10\%$ per annum. The price of this call option is $C = \$3$. On the other hand, the price of a European put option on stock C with same strike price and the expiration date is $P = \$2.25$. In this case,

$$C + Ke^{-rT} = \$3 + \$30e^{-0.1\times 0.25} = \$32.26$$

call side is cheaper, so we buy call

$$P + S_0 = \$2.25 + \$31 = \$33.25$$

put side is more expensive so we short the put & stock

the put-call parity does not hold and portfolio B is overpriced relative to portfolio A.

Arbitrage strategy: buy the call option, short the put option and one share of the underlying stock.

Arbitrage when Put-Call Parity does not Hold

Arbitrage strategy: buy the call option, short the put option and one share of the underlying stock.

buy low
sell high

Action now:

Buy call for \$3 −3
Short put to realize \$2.25 +2.25
Short the stock to realize \$31 +31
Invest \$30.25 for 3 months 30.25

Action in 3 months:

If $S_T > 30$	If $S_T < 30$
Receive \$31.02 from investment	Receive \$31.02 from investment
Exercise call to buy stock for \$30	Put exercised: buy stock for \$30
Net profit =\$1.02	Net profit =\$1.02

One-Step Binomial Tree

Example: suppose that a stock has price $50 at current time and its price at the end of six months will be either $45 or $55. Consider a European call option with maturity six months and strike price $53. The risk-free interest rate is 6% per annum.

(binomial) two choices for stock prices

Beginning Value | **End of Month Value**

Stock price=$55; payoff=$55-$53=$2.

Stock price=$50

Stock price=$45; payoff=$0.

One-Step Binomial Tree

Question: what should be the price of the call option today?

Approach: create a risk free portfolio:

portfolios = \$9 at $T = .5$ (6 months)

$\Delta = .2$

- long position in $\triangle$ shares of the stock;
- short position in one call option.

Investment **Payoff**

\$9 at current time

$= e^{-\frac{.06}{2}} \cdot 9$ at previous time $(T=0)$

$\triangle$ share of stock-call option

$\$55\triangle - \$2 \;\approx \$9$

$\triangle$ share of stock-call option

$\$50\triangle -$ call option

$\triangle$ share of stock-call option

$\$45\triangle - 0 = \$45\triangle = \$9$

(or 2 shares 10 call options)

Riskless portfolio requires $\$55\triangle - \$2 = \$45\triangle$, that is, $\triangle = 0.2$.

One-Step Binomial Tree

The portfolio has a fixed value

$$\$55 \times 0.2 - \$2 = \$45 \times 0.2 = \$9$$

at the maturity of the option regardless of the state of the stock. With no arbitrage assumption, the riskless portfolio will earn risk-free interest rate. Portfolio value at the time of the purchase is

$$\exp(-0.06 \times 0.5) \times \$9 = \$8.73.$$

Denote the fair price of call option at current time by C_0, the portfolio at the time of purchase has dollar value

(if not = this price, there is arbitrage oportunity)

$$\$50 \times 0.2 - C_0 = \$8.73 \qquad \boxed{C_0 = \$1.27.\ \text{Fair Price}}$$

Arbitrage opportunity exists when the option is not trading at the fair value \$1.27.

- With option value more than \$1.27, the portfolio would earn interest at rate higher than the risk-free rate.
- While with option value less than \$1.27, shorting the portfolio would be mean borrowing money at interest rate lower than the

Arbitrage when Option is not Trading at its Fair Value

Example: suppose that the option can be bought or sold at \$2 per share. Arbitrage opportunity exists. $(> 1.27 = C_0)$

Since this option costs more than C_0, we should sell/short (buy low, sell high).

Action now:

Sell one call option for \$2

Borrow \$8

Buy 0.2 shares of stock

Action in 6 months:

If $S_T = 55$

(Since strike price of call is less than $S_T=55$, person will exercise option)

Buy 0.8 shares of stock $\$55 \times 0.8$

Sell 1 share of stock through call \$53

Pay back $\exp(0.06 \times 0.5) \times \$8$

If $S_T = 45$ (less than call option strike prices, so no exercise)

Sell 0.2 shares of stock $\$45 \times 0.2$

No action by onwer of the call

Pay back $\exp(0.06 \times 0.5) \times \$8$

Net profit at expiration:

$$\$53 - 0.8 \times \$55 - \exp(0.06 \times 0.5) \times \$8 = \$0.76$$

$$\$45 \times 0.2 - \exp(0.06 \times 0.5) \times \$8 = \$0.76$$

Arbitrage when Option is not Trading at its Fair Value

Example: suppose that the option can be bought or sold at \$1 per share. Arbitrage opportunity exists. < 1.27

Option is cheap, so we buy it.

Action now:

Buy one call option for \$1

Sell 0.2 shares of stock to gain \$10

Deposit \$9

Action in 6 months:

If $S_T = 55$	If $S_T = 45$
Sell 0.8 shares of stock $\$55 \times 0.8$	Buy 0.2 shares of stock $\$45 \times 0.2$
Buy 1 share of stock through call \$53	No action by onwer of the call
Collect $\exp(0.06 \times 0.5) \times \$9$	Collect $\exp(0.06 \times 0.5) \times \$9$

Net profit at expiration:

$$0.8 \times \$55 - \$53 + \exp(0.06 \times 0.5) \times \$9 = \$0.27$$

$$-0.2 \times \$45 + \exp(0.06 \times 0.5) \times \$9 = \$0.27$$

Arbitrage

Arbitrage is possible when one of three conditions is met:

(1) The same asset does not trade at the same price on all markets ("the law of one price").

(2) Two assets with identical cash flows do not trade at the same price.

(3) An asset with a known price in the future does not today trade at its future price discounted at the risk-free interest rate.

One-Step Binomial Tree: General Set Up

General assumptions:

- Non-dividend-paying stock.
- Current price: S_0.
- Price at expiration could move up to S_0u or move down to S_0d.
- Risk-free interest rate: r.

Consider a European call option with maturity T and strike price K. Further let C_u and C_d be the option values at expiration. Then

$$
\begin{aligned}
C_u =& \max(S_0u - K, 0);\\
C_d =& \max(S_0d - K, 0).
\end{aligned}
$$

Further assume $0 < d < 1 \leq \underbrace{\exp(rT) < u}_{\text{risk-free interest rate should be lower than any investment that has risk}}$ to make the problem realistic.

(should be rewarded for taking on risk)

One-step Binomial Tree: General Set Up

Approach: create riskless portfolio:

- long position in $\triangle_0$ shares of stock;
- short position in one call option.

Riskless portfolio requires

$$S_0 u \triangle_0 - C_u = S_0 d \triangle_0 - C_d \quad \Longrightarrow \quad \triangle_0 = \frac{C_u - C_d}{S_0(u-d)}.$$

With above given $\triangle_0$, the portfolio has value at expiration of

$$S_0 u \triangle_0 - C_u = S_0 d \triangle_0 - C_d = \frac{C_u d - C_d u}{u-d}.$$

The portfolio is risk-free and must earn the risk-free interest rate. Therefore, the dollar value of the portfolio at current is

$$S_0 \triangle_0 - C_0 \quad = \exp(-rT)\frac{C_u d - C_d u}{u-d}.$$

One-step Binomial Tree: General Set Up

The cost of setting up the portfolio at the present time is $S_0 \triangle_0$ for stock shares minus C_0 for shorting one option, that is,

$$S_0 \triangle_0 - C_0.$$

Equating the two portfolio values

$$S_0 \triangle_0 - C_0 = \exp(-rT) \frac{C_u d - C_d u}{u - d}$$

and we arrive at,

$$C_0 = \exp(-rT) \left[q C_u + (1-q) C_d \right], \tag{1}$$

where

(weight for weighted avg.)

$$q = \frac{\exp(rT) - d}{u - d}. \quad q \in (0,1) \tag{2}$$

Example: $S_0 = 50$, $u = 1.1$, $d = 0.9$, $T = 0.5$, $K = 53$, $r = 0.06$, $C_d = 0$ and $C_u = 2$. From the formula in above, we have

$$q = \frac{\exp(rT) - d}{u - d} = \frac{\exp(0.06 \times 0.5) - 0.9}{1.1 - 0.9} = 0.6523$$

and

$$\begin{aligned} C_0 =& \exp(-rT)\left[qC_u + (1-q)C_d\right] \\ =& \exp(-0.06 \times 0.5) \times (0.6523 \times 2 + (1 - 0.6523) \times 0) \\ =& \$1.27. \end{aligned}$$

Interpretations:

1. Under the assumption that $0 < d < 1 \leq \exp(rT) < u$, q defined by (2) is between 0 and 1 and can be interpretated as a probability that the stock price moves up from S_0 to $S_0 u$ at expiration date.
2. The expression $qC_u + (1-q)C_d$ is the expected payoff of the option at maturity with respect to the probability q, and C_0 given by (1) is the discounted expected payoff of the option.
3. The fabraticated probability q derived from no arbitrage condition is different from the probability that the stock price actually moves up or down in the real world, which is called physical probability.
4. The option price C_0 does not depend on the physical probability or the expected stock return actually occurred in the real world.
5. q: risk-neutral probability, or the pricing measure, or the risk-neutral measure.

Risk-Neutral Valuation

Given S_0, the expected stock price at time T with respect to the risk neutral probability q can be computed as

$$\begin{aligned} E_q(S_T|S_0) =& qS_0u + (1-q)S_0d \\ =& qS_0(u-d) + S_0d \\ =& \frac{\exp(rT) - d}{u-d} \cdot S_0(u-d) + S_0d \\ =& S_0\exp(rT). \end{aligned}$$

That is, in the risk-neutral world, the stock price grows up on average at the risk-free rate.

Define discounted price $S_i^* = \exp(-ri)S_i$, $i = 0, T$. Then

$$E_q(S_T^*|S_0^*) = S_0^*.$$

That is, with respect to risk-neutral probability, discounted price S_i^* is a martingale.

Risk-Neutral Measure=Martingale Measure

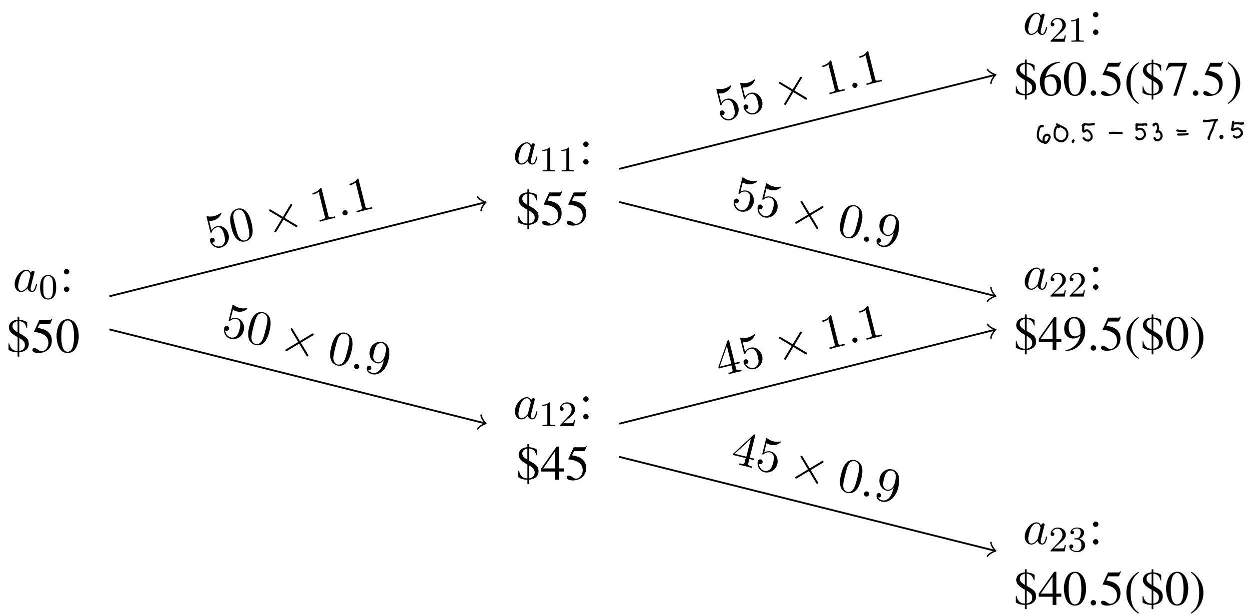
General principle of option pricing and risk-neutral valuation:

Risk neutral preferences are neither risk-averse nor risk seeking. A risk neutral party's decisions are not affected by the degree of uncertainty in a set of outcomes, so a risk neutral party is indifferent between choices with equal expected payoffs even if one choice is riskier.

In the risk-neutral world, every investor is risk-neutral, meaning being indifferent to risk. No compensation for risk is required and stock prices are expected to rise at the risk-free interest rate. When pricing an option, we can assume with complete impunity a risk-neutral world by setting q for the probability of stock price moving up, and the option price is the discounted expected payoff of the option at expiration in the risk-neutral world.

Multi-Step Binomial Tree

Example: consider a stock with \$50 as currect price and two time steps with six months in length for each time step. At each two time steps, the stock price may move up by 10% or down by 10%. The risk-free interest rate is 6%. Consider a European call option with maturity one year and strike price \$53.



Approach: break the two-step binomial tree into three one-step trees and repeatedly apply the risk-neutral valuation established for one-step binomial tree to each of the three one-step trees backward.

- For node a_{11}, the risk neutral probability is

$$q = \frac{\exp(0.06/2) - 0.9}{1.1 - 0.9} = 0.6523,$$

q is the same for all nodes b/c there is always u=1.1 and d=.9 .

and the option price is

$$\exp(-0.06/2) \times (0.6523 \times \$7.5 + 0.3477 \times \$0) = \$4.7477.$$

- For node a_{12}, option price is \$0. $[\exp(-.06/2) \times (.6523 \times 0 + .3477 \times 0)] = 0$
- For initial node a_0, option price is

$$\exp(-0.06/2) \times (0.6523 \times \$4.7477 + 0.3477 \times \$0) = \$3.0054.$$

Multi-Step Binomial Tree: General Set Up

General set up for n steps Binomial Tree:

- Time index $0, 1, 2, \cdots$ denote present time, one-step ahead, two-step ahead, etc.
- τ: time length between consecutive steps
- S_0: initial price
- At each step, price can only move up by a factor u or down by a factor d.

Let S_i, $i = 0, 1, 2, \cdots, n$ be the price of the stock at the end of the ith step. Then

$$S_i = S_0 u^j d^{i-j},$$

where $j = 0, 1, \cdots, i$ equal to the number of times during the past i steps where the price moves up along the tree.

Multi-Step Binomial Tree: General Set Up

Let

(tau)

$$q = \frac{\exp(r\tau) - d}{u - d} \qquad \tau = \frac{T}{n}$$

be the probability of moving up and $1 - q$ be the probability of moving down at each step. Denote by Q the risk-neutral probability on the whole tree. Define the discounted stock price

$$S_i^* = \exp(-ri\tau) S_i,$$

then with respect to the risk-neutral probability Q,

$$\begin{aligned}
E_Q\left[S_{i+1}^* | S_i^*, \cdots, S_0^*\right] =& \exp\left[-r(i+1)\tau\right]\left[qS_i u + (1-q)S_i d\right] \\
=& \exp\left[-r(i+1)\tau\right]\left[qS_i(u-d) + S_i d\right] \\
=& \exp\left[-r(i+1)\tau\right]\left[\frac{\exp(r\tau) - d}{u-d} S_i(u-d) + S_i d\right] \\
=& \exp\left[-ri\tau\right] S_i = S_i^*,
\end{aligned}$$

that is, S_i^* is a martingale.

Multi-Step Binomial Tree

Risk-neutral valuation: the option price at time 0 is the discounted expected payoff with respect to risk-neutral probability.

For a European call option with strike price K and expiration $T = n\tau$, the option price at present time is

$$C_0 = \exp(-rT) E_Q \left[(S_n - K)_+ | S_0 \right].$$

Since

$j = 0, \cdots, n$

$$S_n = S_0 u^j d^{n-j}$$

j is how many steps up

and follows a binomial distribution under the risk-neutral probability Q

$$Q[S_n = S_0 u^j d^{n-j}] = \binom{n}{j} q^j (1-q)^{n-j},$$

payoff at maturity $= (S_n - K)_+$

the option price has expression,

$$C_0 = \exp(-rT) \sum_{j=0}^{n} (S_0 u^j d^{n-j} - K)_+ \cdot \binom{n}{j} q^j (1-q)^{n-j}.$$

not addition

$$C_0 = \exp(-rT)\sum_{j=0}^{n}\left(S_0 u^j d^{n-j} - K\right)_+ \cdot \binom{n}{j} q^j (1-q)^{n-j}$$

$S_0 = 50$, $K = 53$, $r = 6\%$, $u = 1.1$, $d = .9$, $n = 2$, $T = 1$, $\tau = \frac{T}{n} = .5$

$$q = \frac{\exp(r\tau) - d}{u - d}$$

$$C_0 = \exp(-.06)\sum_{j=0}^{2}\left(50 \cdot (1.1)^j (.9)^{2-j}\right)_+ \cdot \binom{2}{j} q^j (1-q)^{2-j}$$

Binomial tree $\xrightarrow{n \to \infty}$ Geometric BM / BS Model

$$S_i\left(uq + d(1-q)\right) = e^{r/n} S_i$$

$$uq + d(1-q) = e^{r/n}$$

$$u^2 q + d^2(1-q) - \left(uq + d(1-q)\right)^2 = \frac{\sigma^2}{n}$$

Geometric Brownian Motion

1. **Binomial tree:**
   - Pro: simple
   - Con: unrealistic

   $S_n = S_0 u^j d^{n-j}$ $\quad \binom{n}{j} q^j (1-q)^{n-j}$ (probability)

2. **Solution:** increase the number of steps in a binomial model so that we could divide the life of the option into many steps with small time length between adjacent steps.

3. **Result:** binomial model approaches to the black-scholes model and the stock price is governed by **geometric Brownian Motion**.

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t \qquad S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

$$\left(\ln(S_t) = \ln(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma W_t\right) \sim N(\ \ , \sigma^2 t)$$

Standard Brownian Motion: Wiener Process

Wiener Process: let $\triangle W_t = W_{t+\triangle t} - W_t$, then a continuous time stochastic process $\{W_t, t \in \mathcal{R}_+\}$ is a Wiener process if it satisfies

$\triangle W_t \sim N(0, \triangle t)$

1. $\triangle W_t = \epsilon\sqrt{\triangle t}$ where ϵ is a standard normal random variable;
2. $\triangle W_t$ is independent of W_j for all $j \leq t$.

$W_t \sim N(0,t)$

$W_u - W_t \sim N(0, u-t)$

- The first condition $\triangle W_t \sim N(0, \triangle t)$ implies that $W_T - W_0 \sim N(0, T)$. This says that the variance of a Wiener process increases linearly with the length of time interval.

- The second condition is a Markov property saying that conditional on the present value W_t, any past information of the process W_j, with $j < t$, is irrelevant to the future W_{t+l} with $l > 0$. From this property, it is easily seen that for any two nonoverlapping time intervals $\triangle_1$ and $\triangle_2$, the increments $W_{t_1+\triangle_1} - W_{t_1}$ and $W_{t_2+\triangle_2} - W_{t_2}$ are independent.

Geometric Brownian Motion

General set up: consider a call option with unit expiration $[0,1]$. For $i=0,1,\cdots,n$, let $t_i = i/n$. Divide $[0,1]$ into n steps with subintervals $[t_{i-1},t_i]$, $i=1,\cdots,n$. The length for each subinterval is $1/n$. The stock has initial price S_0 and moves up by a factor u or down by a factor d where

$$u = \exp\left(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}\right) \qquad \text{and} \qquad d = \exp\left(\frac{\mu}{n} - \frac{\sigma}{\sqrt{n}}\right),$$

μ is a real number and σ is a positive number. Later we will see that μ and σ correspond to the drift and volatility of the stock return. u and d are chosen so that

$$q = \frac{e^{r/n} - d}{u-d}$$

$$qu + (1-q)d = \boxed{\exp\left(\frac{\mu}{n}\right)} \quad - S_i$$

$$[qu^2 + (1-q)d^2] - [qu + (1-q)d]^2 = \boxed{\frac{\sigma^2}{n}} \quad - S_i$$

Geometric Brownian Motion

Let $S^n_{t_i}$ be the stock price at the end of i-th step and X be the number of steps price going up, Then

$$\begin{aligned}
S^n_{t_i} &= S_0 u^X d^{i-X} = S_0 u^X d^i d^{-X} \\
&= S_0 \left(\frac{u}{d}\right)^X d^i \\
&= S_0 \left[\frac{\exp(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}})}{\exp(\frac{\mu}{n} - \frac{\sigma}{\sqrt{n}})}\right]^X \left[\exp(\frac{\mu}{n} - \frac{\sigma}{\sqrt{n}})\right]^i \\
&= S_0 \exp\left(2\frac{\sigma}{\sqrt{n}}X\right) \exp\left(\frac{\mu}{n}i - \frac{\sigma}{\sqrt{n}}i\right) \\
&= S_0 \exp\left[\frac{\mu}{n}i + \frac{\sigma}{\sqrt{n}}(2X - i)\right].
\end{aligned}$$

$S_{t_i} \to \frac{i}{n}$

Let W_j be a random variable such that $W_j = 1$ if the price go up and $W_j = 0$ if the price go down for $j = 1, 2, \cdots, i$. $X = \sum_{j=1}^{i} W_i$

Moreover, let Y_j be a random variable such that $Y_j = 1$ if the price go up and $Y_j = -1$ if the price go down for $j = 1, 2, \cdots, i$. $Y_j = 2W_j - 1$

For each j, we have $Y_j = 2W_j - 1$.

$$2X - i = 2\sum_{j=1}^{i} W_j - i = 2\sum_{j=1}^{i} W_j - \sum_{j=1}^{i} 1 = \sum_{j=1}^{i}(2W_j - 1) = \sum_{j=1}^{i} Y_j.$$

It follows that

$$S_{t_i}^n = S_0 \exp\left(\frac{\mu}{n} i + \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{i} Y_j\right) = S_0 \exp\left(\mu t_i + \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{i} Y_j\right).$$

If we let $Y = \sum_{j=1}^{i} Y_j$, then

$$S_{t_i}^n = S_0 \exp\left(\mu t_i + \frac{\sigma}{\sqrt{n}} Y\right).$$

where Y is a random walk and $S_{t_i}^n$ is a geometric random walk.

Denote by Q the risk-neutral probability. Under Q, the stock price moves up with probability q and down with probability $1-q$ at each step, where

$$
\begin{aligned}
q =& Q(Y_j = 1) = \frac{\exp(r\tau) - d}{u - d} = \frac{\exp(r/n) - d}{u - d} \\
=& \frac{\exp(r/n) - \exp(\frac{\mu}{n} - \frac{\sigma}{\sqrt{n}})}{\exp(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}) - \exp(\frac{\mu}{n} - \frac{\sigma}{\sqrt{n}})} = \frac{\exp(\frac{r}{n} - \frac{\mu}{n}) - \exp(-\frac{\sigma}{\sqrt{n}})}{\exp(\frac{\sigma}{\sqrt{n}}) - \exp(-\frac{\sigma}{\sqrt{n}})} \\
\approx& \frac{1 + \frac{r}{n} - \frac{\mu}{n} - 1 + \frac{\sigma}{\sqrt{n}} - \frac{1}{2}\frac{\sigma^2}{n}}{1 + \frac{\sigma}{\sqrt{n}} + \frac{1}{2}\frac{\sigma^2}{n} - 1 + \frac{\sigma}{\sqrt{n}} - \frac{1}{2}\frac{\sigma^2}{n}} = \frac{\frac{r}{n} - \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} - \frac{1}{2}\frac{\sigma^2}{n}}{2\frac{\sigma}{\sqrt{n}}} \\
=& \frac{1}{2}\left(1 - \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma\sqrt{n}}\right), \quad = \frac{1}{2} - \frac{1}{2}\,\frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma\sqrt{n}}
\end{aligned}
$$

where the approximation is by $\exp(x) \approx 1 + x + \frac{x^2}{2}$ and we ignore terms with higher order than $1/n$.

Geometric Brownian Motion

Under Q, $E_Q[Y_j] = 2q-1$ and $\mathrm{Var}_Q[Y_j] = 4q(1-q)$, and

W_j follows binomial distribution

$$E_Q\left[\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{i} Y_j\right] = \frac{\sigma}{\sqrt{n}} E_Q\left[\sum_{j=1}^{i} Y_j\right] = \frac{\sigma i(2q-1)}{\sqrt{n}}$$

$$\approx \frac{\sigma i\left(1 - \frac{\mu - r + \sigma^2/2}{\sigma\sqrt{n}} - 1\right)}{\sqrt{n}} = \frac{i(-\mu + r - \sigma^2/2)}{n} = t_i(r - \mu - \sigma^2/2)$$

$$\mathrm{Var}_Q\left[\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{i} Y_j\right] = \frac{\sigma^2}{n}\mathrm{Var}_Q\left[\sum_{j=1}^{i} Y_j\right] = \frac{\sigma^2 i 4q(1-q)}{n} \approx t_i\sigma^2$$

given that $q(1-q) \approx 1/4$

$$\left(\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{i} Y_j\right) \sim N\left(t_i\left(r-\mu-\frac{\sigma^2}{2}\right),\ t_i\sigma^2\right)$$

Geometric Brownian Motion

We can conclude that under probability measure Q, as $n \to \infty$,

$\left\{ \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{i} Y_j, i = 1, \cdots, n \right\}$ as a stochastic process on $[0, 1]$

converges in distribution to

$W_t \sim N(0, t)$

(seen on prev. slide)

$$\left[\left(r - \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t, \right] \sim N\left(t_i\left(r-\mu-\frac{\sigma^2}{2}\right), t_i\sigma^2\right)$$

where $t \in [0, 1]$ and W_t is a standard Brownian motion. It follows that under probability measure Q,

this is under the risk-neutral assumption, not physical world

$$S_{t_i}^n = S_0 \exp\left\{ \mu t_i + \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{i} Y_j \right\}$$

as a stochastic process converges to S_t in distribution where

$$S_t = S_0 \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}$$

for $t \in [0, 1]$, which is the geometric Brownian motion.

Geometric Brownian Motion

A stochastic process $\{S_t, t \in \mathcal{R}_+\}$ is said to follow a Geometric Brownian Motion if it satisfies the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where W_t is a Wiener process, and drift μ and volatility σ are constants.

- Drift μ is used to model deterministic trends;
- Volatility σ is used to model a set of unpredictable events occurring during this motion.

As a continuous-time model, the stochastic differential equation can be interpreted as a limit of the discrete model with $\triangle \longrightarrow 0$,

$$\frac{S_{t+\triangle} - S_t}{S_t} = \mu \triangle + \sigma \sqrt{\triangle} \epsilon_t,$$

where ϵ_t are standard normal random variables. The discrete model relates to stock return over time period $\triangle$ with mean rate μ of return and volatility σ.

The Black-Scholes Formula

Consider a European call option with strike price K and maturity T. Again r is the constant risk-free interest rate. At the expiration date T, the option has payoff $(S_T - K)_+$ and its price at present time must be

$$C_0 = \exp(-rT) E_Q[(S_T - K)_+],$$

where Q is the risk-neutral probability, and E_Q stands for the expectation taken with respect to Q.

Suppose that a stock price process S_t, $t \in [0, T]$, follows geometric Brownian motion. In the risk-neutral world, the stock price has expression

$$S_T = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}.$$

Let $X = \log S_T - \log S_0$, then $S_T = S_0 e^X$. Under Q,

$$X \sim N\left((r - \frac{\sigma^2}{2})T, \sigma^2 T\right).$$

Direct computation shows

$$\begin{aligned}
E_Q[(S_T - K)_+] =& E_Q[(S_T - K)\mathbf{1}_{S_T \geq K}] \\
=& E_Q[S_T \mathbf{1}_{S_T \geq K}] - K E_Q[\mathbf{1}_{S_T \geq K}] \\
=& E_Q[S_0 e^X \mathbf{1}_{S_0 e^X \geq K}] - K Q[S_0 e^X \geq K] \\
=& S_0 E_Q[e^X \mathbf{1}_{X \geq ln(K/S_0)}] - K Q[X \geq ln(K/S_0)].
\end{aligned}$$

Normality of X under Q leads to

$$\begin{aligned}
Q[X \geq ln(K/S_0)] =& 1 - \Phi\left(\frac{ln(K/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=& \Phi\left(\frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

where Φ is the cdf of a standard Normal distribution.

On the other hand,

$$
\begin{aligned}
E_Q[e^X \mathbf{1}_{X \geq ln(K/S_0)}] = & \int_{ln(K/S_0)}^{\infty} e^x \frac{1}{\sigma\sqrt{T}} \phi\left(\frac{x-(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) dx \\
= & e^{rT} \int_{ln(K/S_0)}^{\infty} \frac{1}{\sigma\sqrt{T}} \phi\left(\frac{x-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) dx \\
= & e^{rT} \left[1 - \Phi\left(\frac{ln(K/S_0)-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)\right] \\
= & e^{rT} \Phi\left(\frac{ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

where the second equality above uses the fact that

$$
e^x \phi\left(\frac{x-(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) = e^{rT} \phi\left(\frac{x-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right).
$$

$$e^x \Phi\left(\frac{x-(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) = e^{rT}\,\Phi\left(\frac{x-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

$$e^x \cdot \exp\left(-\frac{[x-(r-\sigma^2/2)T]^2}{2\sigma^2 T}\right)$$

$$x - \frac{x^2+[(r-\sigma^2/2)T]^2 - 2x(r-\frac{\sigma^2}{2})Tx}{2\sigma^2 T} \quad = -2xT\left(r-\frac{\sigma^2}{2}+\sigma^2\right) = -2xT\left(r+\frac{\sigma^2}{2}\right)$$

$$\frac{[(r-\sigma^2/2)T]^2}{2\sigma^2T} = \frac{(r^2+(\sigma^2/2)^2-2r\frac{\sigma^2}{2})T^2}{2\sigma^2T} = \frac{[r^2-(\sigma^2/2)^2+2r\frac{\sigma^2}{2}-4r\frac{\sigma^2}{2}]T^2}{2\sigma^2T} = \frac{[(r+\frac{\sigma^2}{2})T]^2}{2\sigma^2T} + \frac{-4r\frac{\sigma^2}{2}T^2}{2\sigma^2T} = \frac{[(r+\frac{\sigma^2}{2})T]^2}{2\sigma^2T} - rT$$

Proof: $e^x\phi\left(\frac{x-(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) = e^{rT}\phi\left(\frac{x-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)$

Let $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ be the pdf of standard normal

$$e^x\phi\left(\frac{x-(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

$$= \exp(x)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{[x-(r-\sigma^2/2)T]^2}{2\sigma^2T}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{-2\sigma^2Tx + x^2 - 2x(r-\sigma^2/2)T + (r-\sigma^2/2)^2T^2}{2\sigma^2T}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2-2(r+\sigma^2/2)Tx + (r-\sigma^2/2)^2T^2}{2\sigma^2T}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2-2(r+\sigma^2/2)Tx + (r-\sigma^2/2)^2T^2 - 2r\sigma^2T^2}{2\sigma^2T}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{[x-(r+\sigma^2/2)T]^2}{2\sigma^2T} + rT\right)$$

$$= \exp(rT)\frac{1}{\sqrt{2\pi}}\left(-\frac{[x-(r+\sigma^2/2)T]^2}{2\sigma^2T}\right.$$

$$= \exp(rT)\,\phi\left(\frac{x-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

$$= e^{rT}\cdot\phi\left(\frac{x-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

Put-Call Parity suggests that $P_0 = C_0 + K\exp(-rT) - S_0$

$$= S_0\Phi(d_1) - K\exp(-rT)\Phi(d_2) + K\exp(-rT) - S_0$$

$$= K\exp(-rT)[1-\Phi(d_2)] - S_0[1-\Phi(d_1)]$$

$$= K\exp(-rT)\Phi(-d_2) - S_0\Phi(-d_1)$$

Now

$$
\begin{aligned}
& E_Q[(S_T - K)_+] \\
=& S_0 e^{rT} \Phi\left(\frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

and the option price at current time follows to be

$$
\begin{aligned}
C_0 =& e^{-rT} E_Q[(S_T - K)_+] \\
=& S_0 \Phi\left(\frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=& S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)
\end{aligned}
$$

where

$$
d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.
$$

Black-Scholes Formula Option price at current time for a

- European call option

$$C_0 = \exp(-rT)E_Q[(S_T - K)_+] = S_0\Phi(d_1) - K\exp(-rT)\Phi(d_2)$$

- European put option

$$P_0 = \exp(-rT)E_Q[(K - S_T)_+] = K\exp(-rT)\Phi(-d_2) - S_0\Phi(-d_1)$$

where

$$d_1 = \frac{log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Example: suppose that the current price of Intel stock is \$80 per share with volatility $\sigma = 20\%$ per annum. The risk free interest $r = 8\%$ per annum. Consider a call option with maturity $T = 3$ months.

- Strike price: $K = 90$.

$$d_1 = \frac{ln(80/90) + (0.08 + 0.2^2/2) \times 0.25}{0.2\sqrt{0.25}} = -0.9278,$$

$$d_2 = \frac{ln(80/90) + (0.08 - 0.2^2/2) \times 0.25}{0.2\sqrt{0.25}} = -1.0278,$$

$$\Phi(-0.9278) = 0.1767 \qquad \Phi(-1.0278) = 0.1520,$$

$$C_0 = 80 \times 0.1767 - 90e^{-0.08 \times 0.25} \times 0.1520 = 0.73.$$

The Black-Scholes Formula

- Strike price: $K = 85$.

$$d_1 = \frac{ln(80/85) + (0.08 + 0.2^2/2) \times 0.25}{0.2\sqrt{0.25}} = -0.3562,$$

$$d_2 = \frac{ln(80/85) + (0.08 - 0.2^2/2) \times 0.25}{0.2\sqrt{0.25}} = -0.4562,$$

$$\Phi(-0.3562) = 0.3608 \qquad \Phi(-0.4562) = 0.3241,$$

$$C_0 = 80 \times 0.3608 - 85e^{-0.08\times 0.25} \times 0.3241 = 1.86.$$

$C_0 = S_0 \Phi(d_1) - Ke^{-rT} \cdot \Phi(d_2)$

$P_0 = Ke^{-rT} \cdot \Phi(-d_2) - S_0 \Phi(-d_1)$

Marginal Effects

data: $r_1, r_2, \cdots, r_n$

Variance σ^2

$N(m, \sigma^2 \Delta)$

From the formulas, the price of a call or put option depends on five variables:

- **the current stock price S_0,** $\uparrow S_0 \Longrightarrow \uparrow C_0$?
- **the strike price K,** $\uparrow K \Longrightarrow \downarrow C_0$?
- **the time to expiration T measured in years,** $\uparrow T \Longrightarrow \uparrow C_0$? more complicated than this
- **the volatility σ per annum,** $\uparrow \sigma \Longrightarrow \uparrow C_0$? more complicated than this
- **the interest rate r per annum.** $\uparrow r \Longrightarrow \uparrow C_0$?

see slide 54, 56

Marginal Effects

Current stock price S_0: C_0 is positively related to S_0.

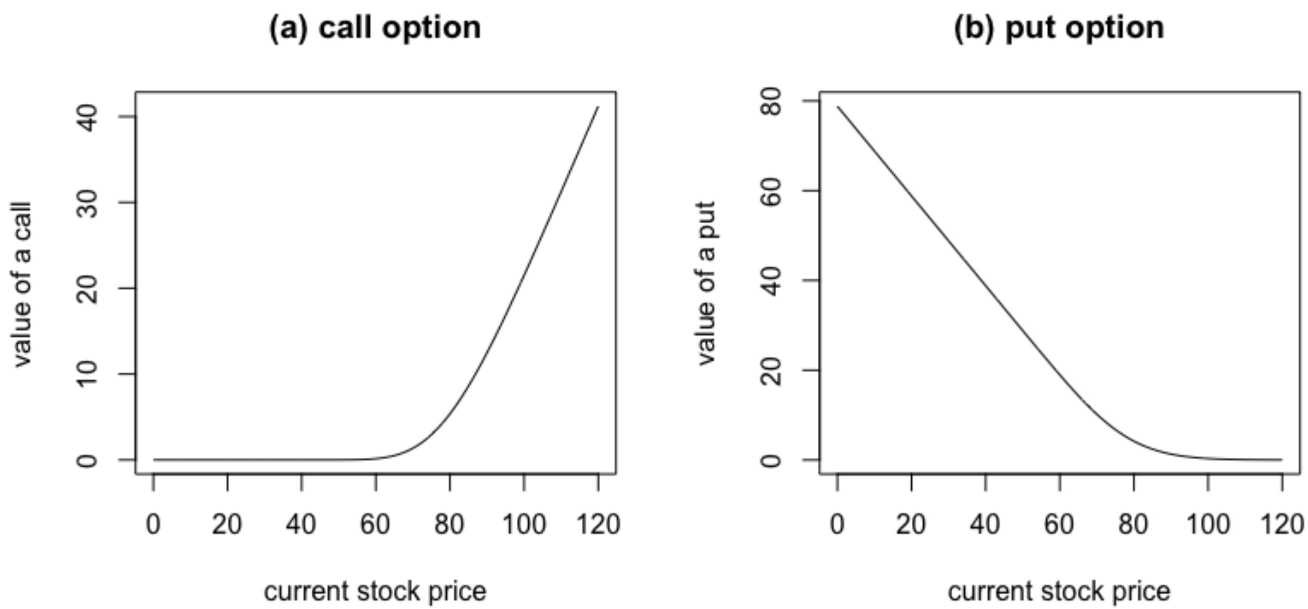


Figure: Marginal effects of the current stock price on the price of an option with $K = 80$, $T = 0.25$, $\sigma = 0.25$ and $r = 0.06$: (a) call option and (b) put opion.

Marginal Effects

Strike price K: C_0 is negatively related to K.

Time to expiration T: C_0 is related to T in a complicated manner. We consider the limiting results here by rewriting d_1 and d_2 as

$$d_1 = \frac{log(S_0/K)}{\sigma\sqrt{T}} + \frac{(r + \sigma^2/2)\sqrt{T}}{\sigma}$$

and

$$d_2 = \frac{log(S_0/K)}{\sigma\sqrt{T}} + \frac{(r - \sigma^2/2)\sqrt{T}}{\sigma}$$

If $S_0 < K$, then $C_0 \to 0$ as $T \to 0$.
If $S_0 > K$, then $C_0 \to S_0 - K$ as $T \to 0$ and $C_0 \to S_0$ as $T \to \infty$.

Marginal Effects

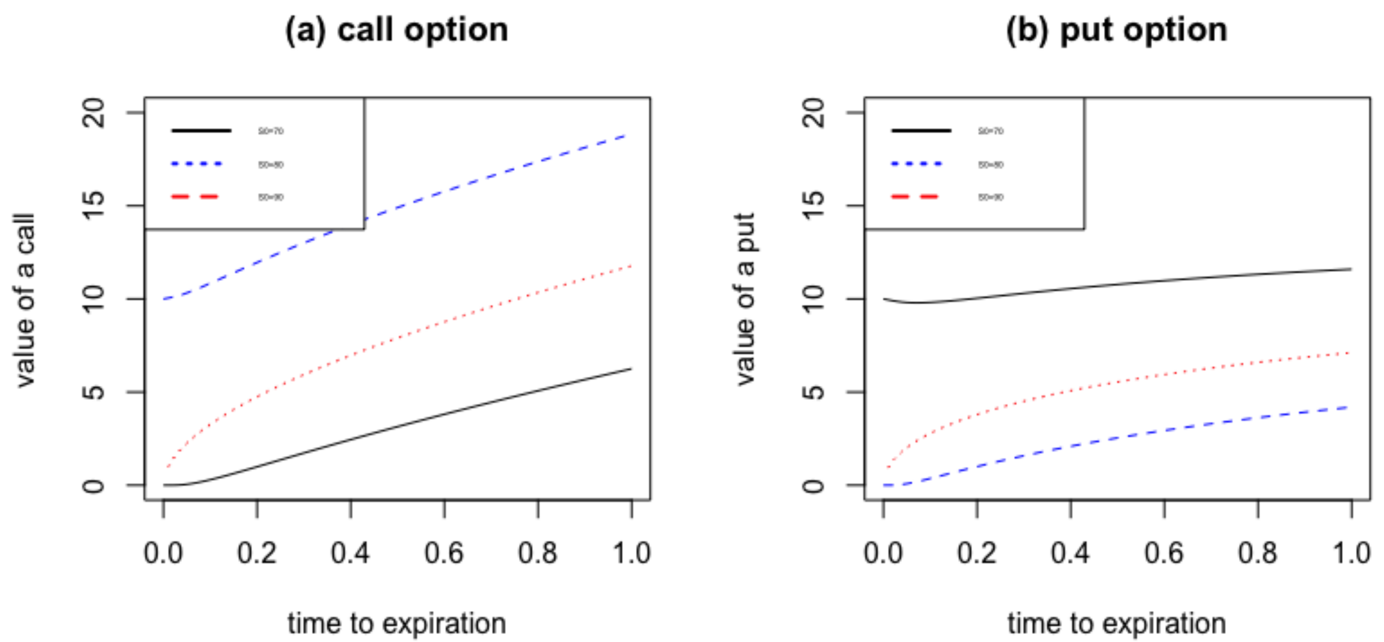


Figure: Marginal effects of time to expiration on the price of an option with $K = 80$, $\sigma = 0.3$ and $r = 0.06$: (a) call option and (b) put opion. The solid, dotted, and dashed lines are for the current stock price $S_0 = 70, 80$, and 90, respectively.

Marginal Effects

Volatility σ: complicated manner. We also consider the limiting results here, note that

$$d_1 = \frac{log(S_0/K) + rT}{\sigma\sqrt{T}} + \frac{\sigma}{2}\sqrt{T}$$

and

$$d_2 = \frac{log(S_0/K) + rT}{\sigma\sqrt{T}} - \frac{\sigma}{2}\sqrt{T}$$

If $ln(S_0/K) + rT < 0$, then $C_0 \to 0$ as $\sigma \to 0$.
If $ln(S_0/K) + rT \geq 0$, then $C_0 \to S_0 - Ke^{-rT}$ as $\sigma \to 0$ and $C_0 \to S_0$ as $\sigma \to \infty$

Interest rate r: C_0 is positively related to r such that $C_0 \to S_0$ as $r \to \infty$.

Marginal Effects

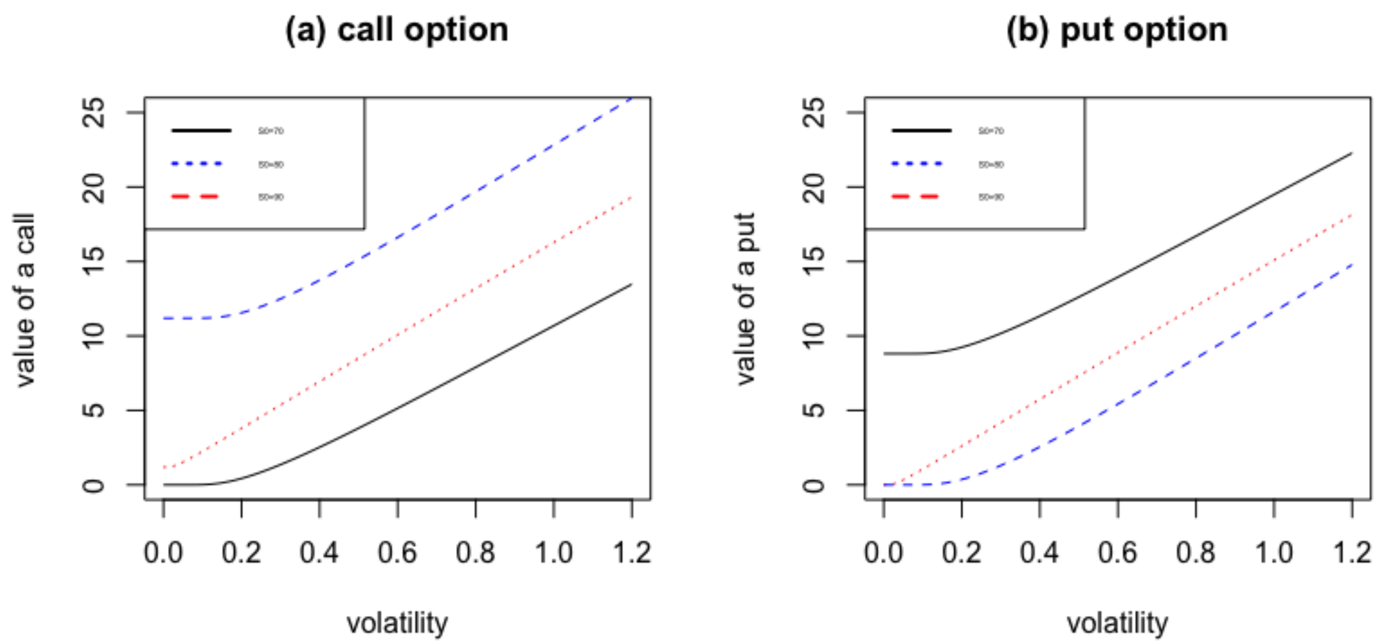


Figure: Marginal effects of stock volatility on the price of an option with $K = 80$, $T = 0.25$, $\sigma = 0.3$ and $r = 0.06$: (a) call option and (b) put opion. The solid, dotted, and dashed lines are for the current stock price $S_0 = 70$, 80, and 90, respectively.

Implied Volatility via the Black-Scholes Formula

Often, BS Model doesn't really fit data

Useful for educational purposes

Implied volatility: from the observed prices of a European call option, volatility can be deduced from the Black-Scholes formula.

- This approach is often criticized for using a specific model, which is based on some assumptions that might not hold in practice.
- Experience shows that implied volatility of an asset return tends to be larger than that obtained by using a GARCH type of volatility model.
- Although there lacks explicit formula for the inverse function of the Black-Scholes formula as a function of σ, given a numerical value c, root-finding algorithms such as Newton-Raphson and bisection are very effective in solving the equation and obtaing the value for the implied volatility.

Realized volatility: estimate daily volatility based on intraday data of the stock.

- Let r_t be the daily log return of an asset, suppose that there are n equally spaced intradaily log returns available such that $r_t = \sum_{i=1}^{n} r_{t,i}$. The quantity

$$RV_t = \sum_{i=1}^{n} r_{t,i}^2$$

is called the realized volatility of r_t.
- For heavily traded assets in the US, a time interval of 3-15 minutes is often used.
- Stock volatility consists of intraday volatility and overnight volatility. However, the high-frequency intraday returns contain only very limited information about the overnight volatility.