Stat 461 HW4 due 11/19 Roshan Podural

1) a) $\mathbb{E}[r_t] = \mathbb{E}[t + a_t + .25\, a_{t-1}] = t$

$r_t = t + a_t + .25\, a_{t-1}$

$r_{t-1} = (t-1) + a_{t-1} + .25\, a_{t-2}$

$r_{t-2} = (t-2) + a_{t-2} + .25\, a_{t-3}$

$\mathrm{Cov}(r_t, r_{t-1}) = .25\, \sigma_a^2$

$\mathrm{Cov}(r_t, r_{t-2}) = 0$? ...

Weakly stationery b/c constant mean & γ_k depends on k.

b) $\mathbb{E}[w_t] = \mathbb{E}[r_t - r_{t-1}] = t - t = 0$

$\mathrm{Cov}(w_t, w_{t-1}) = -.75\,\sigma_a^2 + (.75)(.25)\,\sigma_a^2 = \gamma_1$

$\mathrm{Cov}(w_t, w_{t-2}) = .25\,\sigma_a^2 = \gamma_2$

$\mathrm{Cov}(w_t, w_{t-3}) = 0 = \gamma_3$

$w_t = [t + a_t + .25\, a_{t-1}] - [(t-1) + a_{t-1} + .25\, a_{t-2}] = 1 + a_t - .75\, a_{t-1} - .25\, a_{t-2}$

$w_{t-1} = \quad = 1 + a_{t-1} - .75\, a_{t-2} - .25\, a_{t-3}$

$w_{t-2} = \quad = 1 - a_{t-2} - .75\, a_{t-3} - .25\, a_{t-4}$

$w_{t-3} = \quad = 1 - a_{t-3} - .75\, a_{t-4} - .25\, a_{t-5}$

Weakly stationary b/c constant mean & γ_k depends on k.

2) a) $\mathbb{E}[r_t] = \mu = \phi_0 + \phi_1 \mathbb{E}[r_{t-1}] = \phi_0 + \phi_1 \mu \implies \mu - \phi_1\mu = \phi_0 \implies 1 - \phi_1 = \frac{\phi_0}{\mu}$

$= \dfrac{\phi_0}{1-\phi_1}$ also $\phi_0 = (1-\phi_1)\mu$

$r_t - \mu = \phi_1 (r_{t-1} - \mu) + a_t$

many $\vdots$ substitutions

$r_t - \mu = a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \dots = \sum_{i=0}^{\infty} \phi_1^i a_{t-i}$

$\psi_i = \phi_1^i$

$r_t = \mu + \sum_{i=1}^{\infty} \psi_i a_{t-i}$ ✓

b) from above (and $\mathbb{E}[a_t] = 0$), we know $\mu = \mathbb{E}[r_t] = \dfrac{\phi_0}{1-\phi_1}$

$\mathbb{E}[(r_t - \mu) a_{t+1}] = 0$ b/c a_t's are indep.

$\mathrm{Cov}(r_{t-1}, a_t) = \mathbb{E}[(r_t - \mu) a_t] = 0$

$$\gamma_0 = Var(r_t) = \phi_1^2 \, Var(r_{t-1}) + \sigma_a^2 \implies Var(r_t) = \phi_1^2 \, Var(r_t) + \sigma_a^2$$

$$= \frac{\sigma_a^2}{1-\phi_1^2} \quad (\phi_1^2 > 1)$$

c) $\mathbb{E}[a_t(r_t-\mu)] = \mathbb{E}[\phi_1 a_t (r_{t-1}-\mu)] + \mathbb{E}[a_t^2] = \mathbb{E}[a_t^2] = \sigma_a^2$

$$\gamma_k = \begin{cases} \phi_1 \gamma_1 + \sigma_a^2 & \text{if } k=0 \\ \phi_1 \gamma_{k-1} & \text{if } k>0 \end{cases} \quad \text{and } \gamma_k = \gamma_{k-1}$$

$\rho_k = \phi_1 \rho_{k-1}$ for $k \geq 0$

$\rho_k = \phi_1^k$ (b/c $\rho_0 = 1$)

3) a) $r_t = \phi_0 + \phi_1 r_{t-1} + a_t$ $\quad (\phi_0 = 0, \; \phi_1 = .8, \; a_t \sim N(0,1), \; t=1,\dots,T)$

$\rho_k = \phi_1^k$ so $\rho_1 = \phi_1 = .8$ & $\rho_5 = \phi_1^5 = .32768$ (and $\rho_1 = \phi_1^0 = 1$)

b) See R code

c) $\hat{\rho}_1 \xrightarrow{\text{appx.}} N\left(.8, \frac{.36}{T}\right)$ and $\hat{\rho}_5 \xrightarrow{\text{appx.}} N\left(.32768, \frac{1}{T} \cdot 4.555\right)$

d)

e) See R code; the distributions are close to theoretical (mean is off, and std. dev. is too high, but this makes sense for 10 rounds of T or n = 1000)

4) a) For MA(2) model, $r_{h+l} = \mu + a_{h+l} - \theta_1 a_{h+l-1} - \theta_2 a_{h+l-2}$

$r_t = 0.08 + a_t - 0.3 a_{t-1} + 0.12 a_{t-2}$

$\mu = .08$, $\theta_1 = .3$, $\theta_2 = -.12$; Since $a_t \sim t_4$ $Var(a_t) = \frac{4}{4-2} = 2 = \sigma_a$

$Var(r_t) = (1+\theta_1^2 + \theta_2^2)\sigma_a^2 = (1.1044) 2^2 = 4.4176$

b) $\hat{r}_n(1) = \mu - \theta_1 a_n - \theta_2 a_{n-1}$ $\qquad e_n(1) = r_{n+1} - \hat{r}_n(1) = a_{n+1}$ $\qquad a_{100} = -1.36$

$\hat{r}_{100}(1) = .08 - .3(-1.36) + .12(.34)$ $\qquad e_{100}(1) = a_{101}$ $\qquad a_{99} = 0.34$

$= \underline{.5288}$

c) $\hat{r}_n(2) = \mu - \theta_2 a_n$ $\qquad e_n(2) = r_{n+2} - \hat{r}_n(2)$

$\hat{r}_{100}(2) = .08 + .12 a_{100}$ $\qquad = \mu + a_{n+2} - \theta_1 a_{n+1} - \theta_2 a_n - \mu + \theta_2 a_n$

$= .08 + .12(-1.36)$ $\quad = a_{n+2} - \theta_1 a_{n+1}$

$= \underline{-0.08}$ $\quad e_{100}(2) = \underline{a_{102} - .3 a_{101}}$ $\quad \mathbb{E}[e_{100}(2)] = 0$ & $Var(e_{100}(2)) = 2 + (.09)2$

$= 2.18$

d) With MA(2), for 10-step ahead

$\hat{r}_{100}(10) = \mu$ (b/c $10 > 2$)

$e_n(10) = r_{n+10} - \hat{r}_n(10)$

$= \mu + a_{n+10} - \theta_1 a_{n+9} - \theta_2 a_{n+8} - \mu$

$= a_{n+10} - \theta_1 a_{n+9} - \theta_2 a_{n+8}$

$e_{100}(10) = a_{110} - .3 a_{109} + .12 a_{108}$ $\quad \mathbb{E}[e_{100}(100)] = 0$ & $Var(e_{100}(100)) = 2 + (.09)2 + (.0144)2$

$= 2.2088$

e) We always predict $\mu (= .08)$, so a 95% C.I.

of the error of the prediction is: $(-.109, .087)$

See R code

Stat 461 HW4

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3b

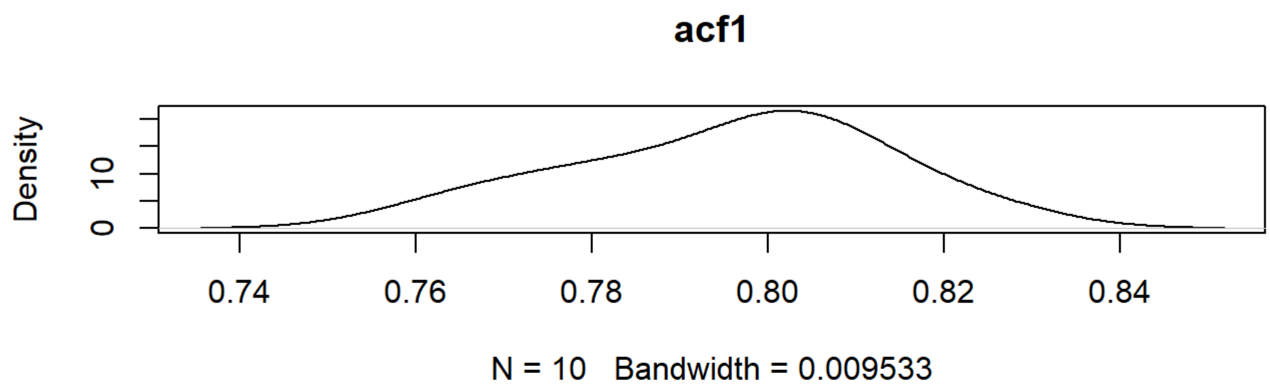
```
set.seed(1234)
T = 1000
N = 10000
Y = matrix(, nrow=1000, ncol=N/T)
for (i in 1:(N/T)) {
  Y[,i] = arima.sim(model=list(ar=c(0.8)), n=T)
}
print(acf(Y[,1], lag=5, main="", plot=FALSE)[c(1,5)])
```

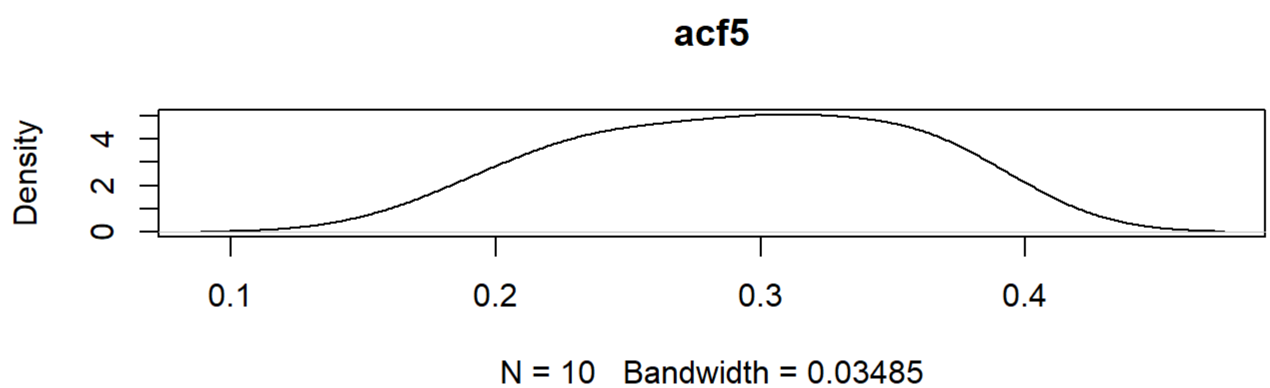
```
##
## Autocorrelations of series 'Y[, 1]', by lag
##
##     1     5
## 0.801 0.298
```

```
my_acfs = matrix(, nrow=2, ncol=N/T)
for (i in 1:(N/T)) {
  my_acfs[1,i] = acf(Y[,i], lag=5, main="", plot=FALSE)[1]$acf
  my_acfs[2,i] = acf(Y[,i], lag=5, main="", plot=FALSE)[5]$acf
}
```

3e

```
acf1 = my_acfs[1,]
acf2 = my_acfs[2,]
par(mfrow=c(2,1))
plot(density(acf1), main="acf1")
plot(density(acf2), main="acf5")
```






```
paste("acf1; mean:", mean(acf1), "sd:", sd(acf1))
```

```
## [1] "acf1; mean: 0.795480109270835 sd: 0.0177323385836998"
```

```
paste("acf5; mean:", mean(acf2), "sd:", sd(acf2))
```

```
## [1] "acf5; mean: 0.292728570789667 sd: 0.0613727922762759"
```

4e

```
ten_predict_error <- function(a110, a109, a108) {
  return(a110 - (.3 * a109) + (.12 * a108))
}

n_samples = 1000

a110 = rt(n_samples, 4)
a109 = rt(n_samples, 4)
a108 = rt(n_samples, 4)

predictions = ten_predict_error(a110, a109, a108)

a = mean(predictions)
s = sd(predictions)

error = qnorm(.975)*s/sqrt(n_samples)
paste0("95% C.I. of 10-step ahead prediction error(n=", n_samples,"): (", a-error, ", ", a+erro
r, ")")
```

```
## [1] "95% C.I. of 10-step ahead prediction error(n=1000): (-0.109011459873612, 0.0866523929473
549)"
```