Stat 461 HW3 Due 10/15 @ 2:30 Roshan Poduval
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1) $K = \$22$ (expires in 6 months)

$C = \$3$ $\quad C + Ke^{-rT} = 3 + 22 \cdot e^{-.1/2} = 23.927$

$P = \$4$ $\quad P + S_0 = 4 + 20 = 24$

$r = .1$ (10% per year) $\quad$ Call side is less than put side.

$S_0 = \$20$

Action Now:

- Buy call for \$3
- Short put for \$4
- Short stock for \$20
- Invest remaining (\$21) for 6 months

Action in 6 months:

- If $S_T > 22$
  - get \$22.0767 ≈ \$22.08 from investment
  - exercise call to buy stock for \$22

  net profit = \$0.08

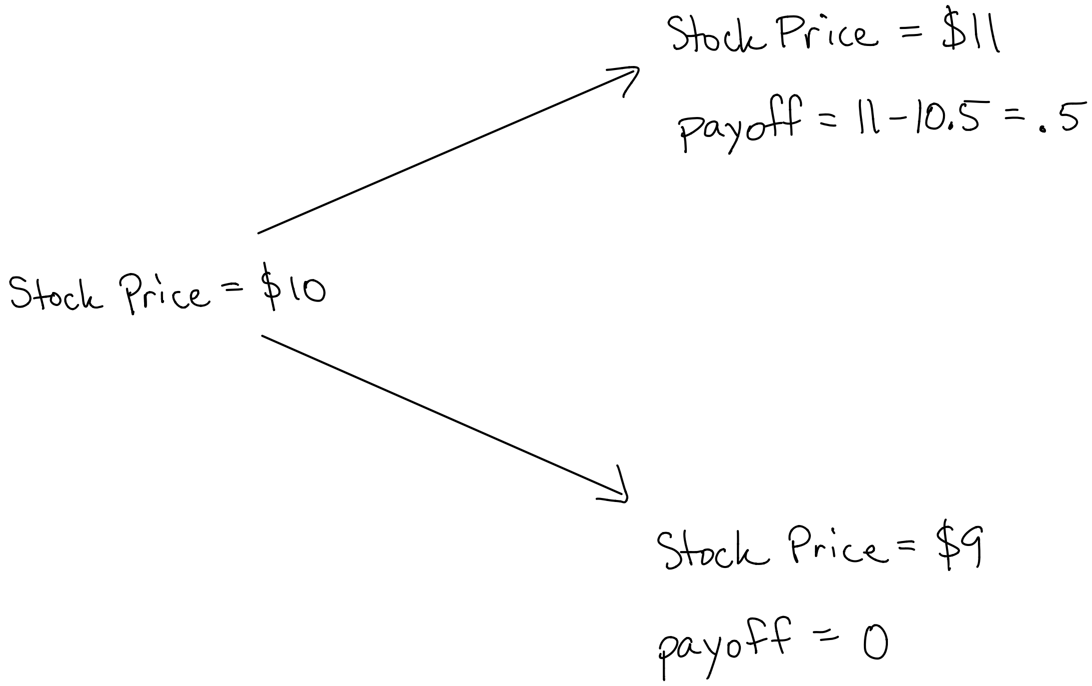
- If $S_T < 22$
  - get \$22.0767 ≈ \$22.08 from investment
  - holder of put (that we shorted) exercises to buy stock for \$22

  net profit = \$0.08

2)

After 3 Months

a)



$11\Delta - .5 = 9\Delta = 2.25 \rightarrow \Delta = .25$

Value of portfolio at time of purchase (no arbitrage assumption)

$e^{-.12/4} \cdot 2.25 = \$2.1835 \approx \$2.18$

$S_0 \times \Delta - C_0 = 2.18 \rightarrow 10 \times .25 - C_0 = 2.18 \rightarrow C_0 = \$0.32$

b) $.8 > .32$, so the option is overpriced

Action Now:

- Sell one call option for \$0.80
- Borrow \$1.70
- Buy .25 shares of stock (\$2.50)

Action in 3 Months:

- If $S_T = \$11$
  - Buy .75 shares of stock (\$8.25)
  - Sell 1 share of stock through call (\$10.5)
  - Pay back $e^{.12/4} \cdot 1.70$ ($= 1.75177 \approx \$1.75$)
- If $S_T = \$9$

- Sell .25 shares of stock (\$2.25)
- no action by owner of call
- pay back $e^{.12/4} \cdot 1.70$ ($= 1.75177 \approx \$1.75$)

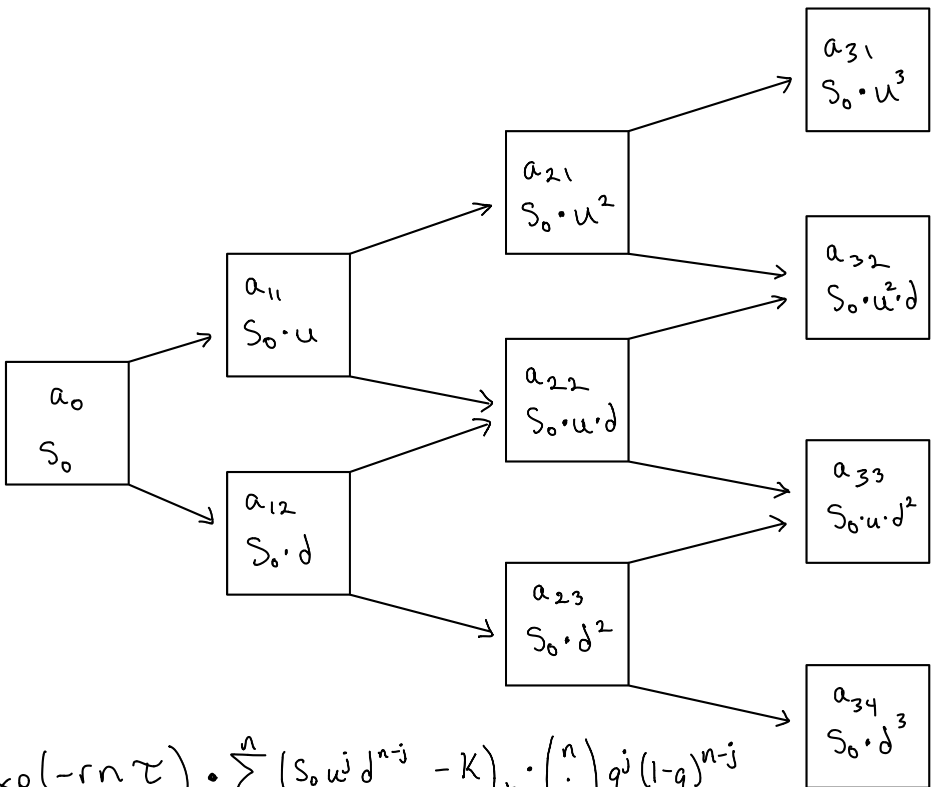
$$\left.\begin{array}{l} 10.5 - 8.25 - 1.75 = 0.5 \\ 2.25 - 1.75 = 0.5 \end{array}\right\} \text{net profit at expiration}$$

3) a) (risk-neutral probability) $q = \dfrac{\exp(r \cdot \tau) - d}{u - d}$

The probability of moving down at any given step is $(1-q)$

The prob. of moving down at every step (n steps) is $(1-q)^n$

b)



$$C_0 = \exp(-rn\tau) \cdot \sum_{j=1}^{n} (S_0 u^j d^{n-j} - K)_+ \cdot \binom{n}{j} q^j (1-q)^{n-j}$$

d) $S_9 = S_0 u^j d^{n-j}$ $\quad j = 5, \ S_0 = 100, \ u = 1.1, \ d = .9$

$= 100 \cdot (1.1)^5 (.9)^4$

$= 105.666 \xrightarrow[\text{back to 0}]{\text{discount from 9}} \$61.58$ $\qquad q = \dfrac{\exp(.06) - .9}{1.1 - .9} \cong .809$

P(stock price moves up 5 times in the first 9 steps) $= \binom{9}{5} q^5 (1-q)^4 = .057953$

4) a) $S_1 = S_0 \exp\left(\frac{\sigma}{\sqrt{n}} Y\right)$ in this case, Y is either -1 or 1

(stock price can go up or down once)

$S_2 = S_0 \exp\left(\frac{\sigma}{\sqrt{n}} Y\right)$ in this case, Y is $-2, 0, 2$

(stock price can either go up or down twice

or up once and down once)

$\vdots$

$S_k = S_0 \exp\left(\frac{\sigma}{\sqrt{n}} Y\right)$ $Y = -k, -k+2, -k+4, \cdots, k-4, k-2, k$

Y tells you how many times the stock went up/down

b) Given the values Y takes, $\frac{Y+k}{2} \in [0,k]$

This, we can think of as the number of times our stock goes up

$\text{Bin}(k,q)$ is a distribution that will give values of $[0,k]$

(# successes in k trials / # times stock goes up in k time increments)

and the probability of success/stock price increasing is q.

$$q = Q(Y_j = 1) = \frac{\exp(r\tau) - d}{u - d} = \frac{\exp(r/n) - d}{u - d} = \frac{\exp(r/n) - \exp(-\sigma/\sqrt{n})}{\exp(\sigma/\sqrt{n}) - \exp(-\sigma/\sqrt{n})} = \frac{0 - \exp(-\sigma/\sqrt{n})}{\exp(\sigma/\sqrt{n}) - \exp(-\sigma/\sqrt{n})}$$

$$\approx \frac{0 - 1 + \frac{\sigma}{\sqrt{n}} - \frac{\sigma 2}{n}}{\left(1 + \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{n}\right) - \left(1 - \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{n}\right)} = \frac{-1 + \frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{n}}{2\frac{\sigma}{\sqrt{n}}} = \frac{1}{2}\left(1 - \frac{\sigma}{2\sqrt{n}}\right)$$

5)

b) $P_0 = K\exp(-rT)\,\Phi(-d_2) - S_0\,\Phi(-d_1)$

6) a) b) See R code

c) $C_0 = S_0 \cdot \Phi(d_1) - K\exp(-rT) \cdot \Phi(d_2)$

$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = -.06$

$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$

$= -.16$

$= 60 \cdot \Phi(-.06) - 68 \cdot \exp(-.06) \cdot \Phi(-.16)$

$= -1.329 \approx -1.33$

d) $P_0 = K\exp(-rT) \cdot \Phi(-d_2) - S_0 \Phi(-d_1)$

$d_1 = -.06$

$d_2 = -.16$

$= 68 \cdot \exp(-.06) \cdot \Phi(.16) - 60 \cdot \Phi(.06)$

$= 1.329 \approx 1.33$

Stat 461 HW3

Roshan Poduval

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Question 3e

```
n = 10
tau = 1
r = .06
S0 = 100
u = 1.1
d = 0.9
K = 110

q = (exp(r) - d)/(u - d)

expectation = 0
for (j in 1:n) {
  price = max(((S0*(u^j)*(d^(n-j))) - K),0)
  probability = choose(n,j)*(q^j)*((1-q)^(n-j))
  expectation = expectation + (price*probability)
}

C0 = exp(-r*n*tau)*expectation
C0
```

```
## [1] 39.90502
```

Question 6

```
set.seed(1234)
mu = .06
sigma = 0.1
T = 1
S0 = 60
n = 10000
N = 500
dt = T/n
t = seq(0, T, by=dt)

R = mu*dt + sigma*rnorm(n*N, mean=0, sd=sqrt(dt))
R = matrix(R, n, N)

S = matrix(rep(0, n*N), n, N)
S = rbind(rep(S0, N), S)
for (j in 1:N) {
  for (i in 1:n) {
    S[i+1,j] = S[i,j]*R[i,j] + S[i,j]
  }
}
sum(S[10001,]>68)/N
```

```
## [1] 0.24
```

```
paste("95% Confidence Interval for stock price after 1 year: [", qnorm(.025)*sd(S[10001,])+mean
(S[10001,]), ",", qnorm(.975)*sd(S[10001,])+mean(S[10001,]), "] centered around", mean(S[10001
,]))
```

```
## [1] "95% Confidence Interval for stock price after 1 year: [ 50.6576374146889 , 76.2263573510
72 ] centered around 63.4419973828805"
```

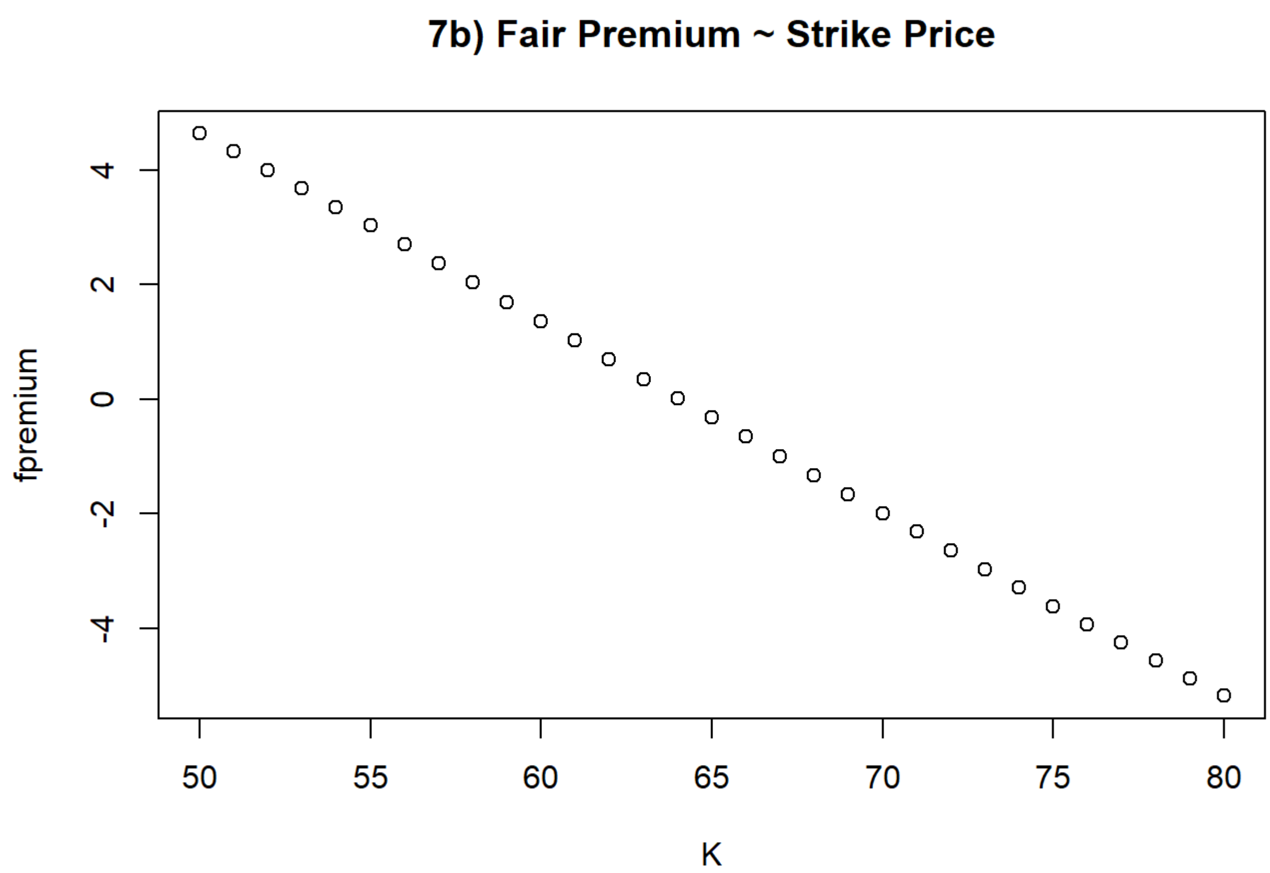
Question 7

```r
d1 <- function(S0, K, r, sigma, T) {
  return(log(S0/K) + ((r + ((sigma^2)/2))*T))/(sigma*sqrt(T))
}
d2 <- function(S0, K, r, sigma, T) {
  return(d1(S0, K, r, sigma, T) - (sigma*sqrt(T)))
}
C0 <- function(S0, K, r, sigma, T) {
  return((S0*dnorm(d1(S0, K, r, sigma, T))) - (K*exp(-r*T)*dnorm(d2(S0, K, r, sigma, T))))
}
P0 <- function(S0, K, r, sigma, T) {
  return((K*exp(-r*T)*dnorm(-d2(S0, K, r, sigma, T))) - (S0*dnorm(-d1(S0, K, r, sigma, T))))
}

S0 = 60
K = 68
r = .06
sigma = .1
T = 1
paste("7a) Call:", C0(S0, K, r, sigma, T), "Put:", P0(S0, K, r, sigma, T))
```

```
## [1] "7a) Call: -1.32941092022493 Put: 1.32941092022493"
```

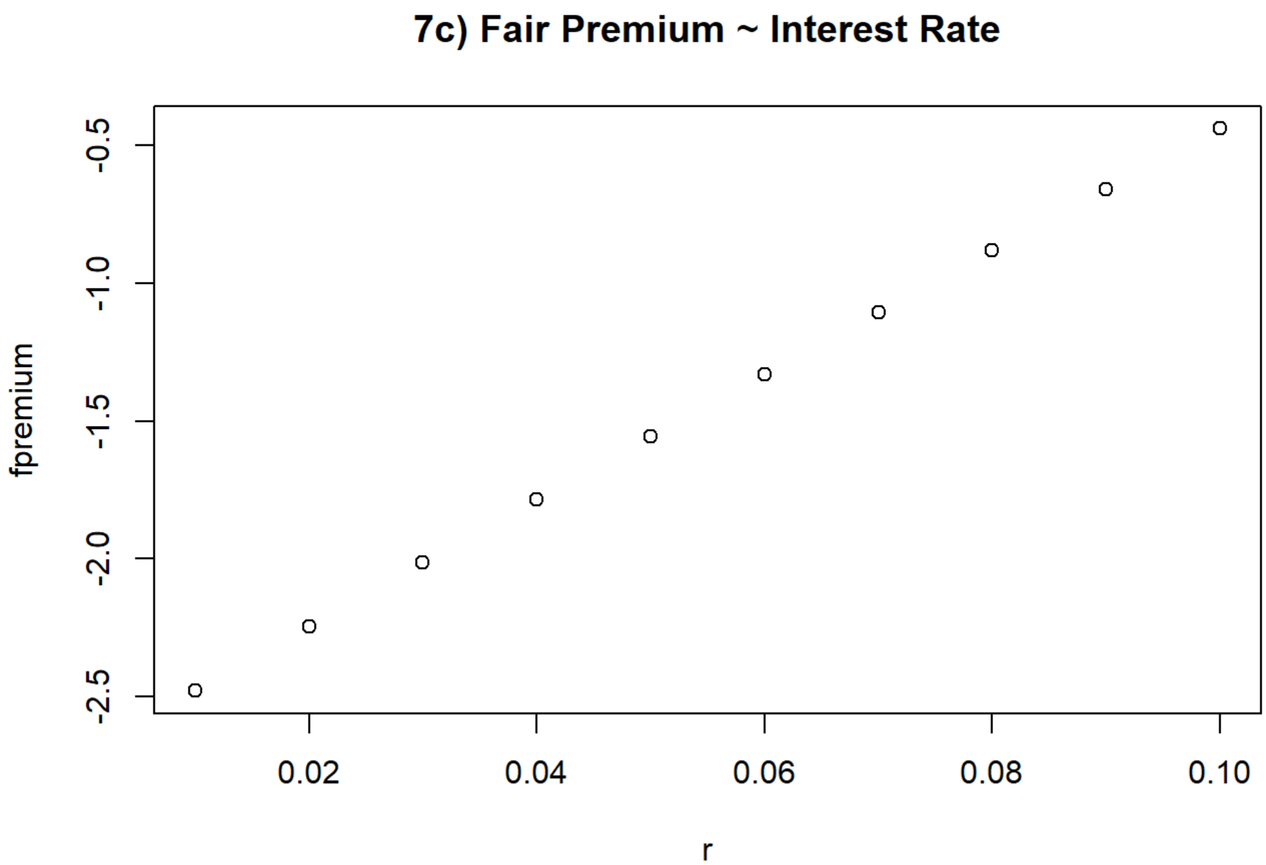
```r
S0 = 60
K = 50:80
r = .06
sigma = .1
T = 1
fpremium = C0(S0, K, r, sigma, T)
plot(K, fpremium, main="7b) Fair Premium ~ Strike Price")
```




```
paste("7b) As the strike price of the call goes up (the price we are allowed to buy the stock at
if the call option is exercised), the fair premium (price we are willing to pay) goes up. This m
akes sense because a call that allows you to buy at a lower price later has more value.")
```

```
## [1] "7b) As the strike price of the call goes up (the price we are allowed to buy the stock a
t if the call option is exercised), the fair premium (price we are willing to pay) goes up. This
makes sense because a call that allows you to buy at a lower price later has more value."
```

```
S0 = 60
K = 68
r = seq(.01, .1, .01)
sigma = .1
T = 1
fpremium = C0(S0, K, r, sigma, T)
plot(r, fpremium, main="7c) Fair Premium ~ Interest Rate")
```




```
paste("7c) As interest rate increases, the fair premium of the call also increases. This makes s
ense because if the (risk-free) interest rate is high, that means that stock price will likely a
lso grow relatively fast. If the stock price is going to grow, then it makes sense to buy a call
that secures a low price in the future for you to buy at. Conversely, if the interest rate is lo
w, it is likely that the stock price will not grow as much (compared to when interest rate is hi
gh). If the stock isn't going to get much more expensive to buy, there is less incentive to buy
 a call option that sets a price for you to buy at in the future (likely the call option will no
t be exercised).")
```

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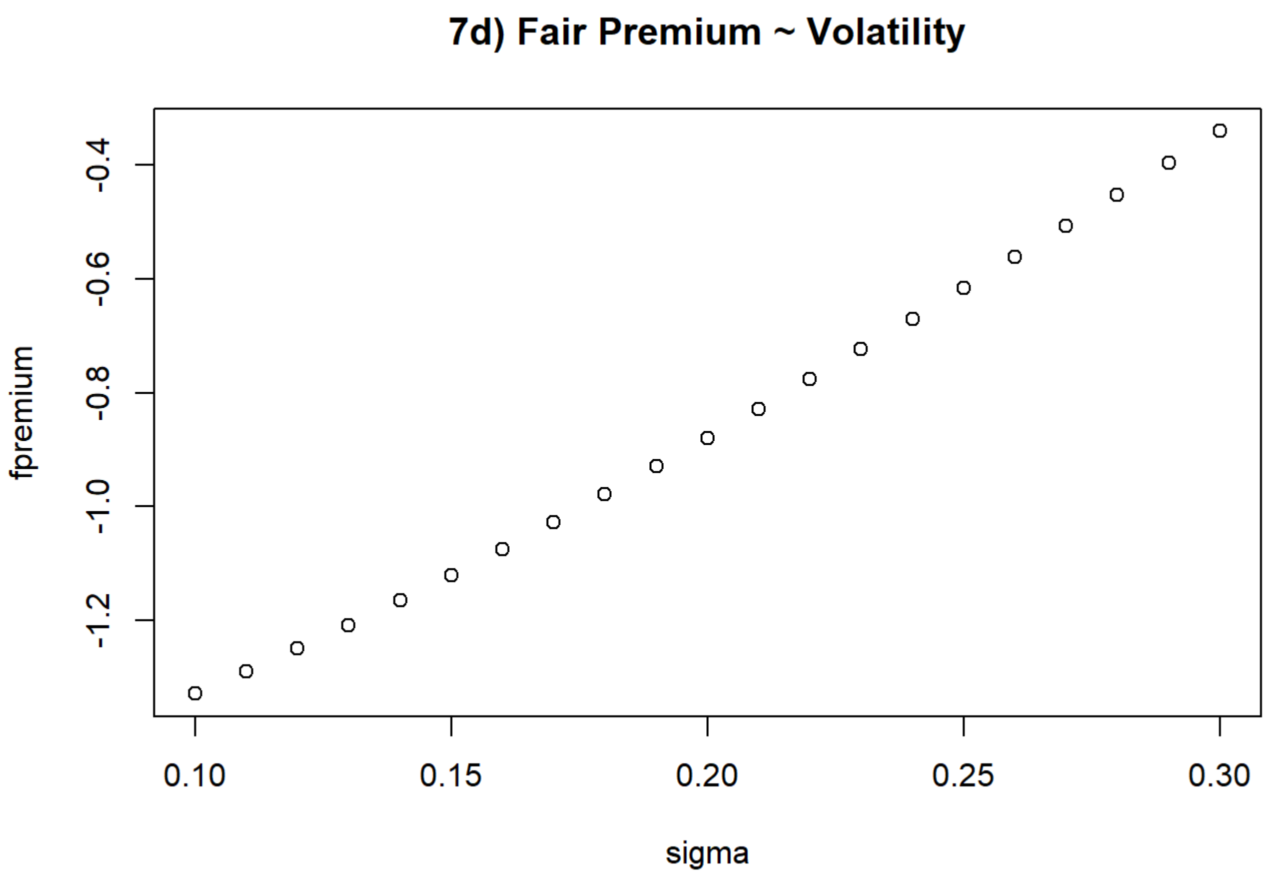
high). If the stock isn't going to get much more expensive to buy, there is less incentive to bu
y a call option that sets a price for you to buy at in the future (likely the call option will n
ot be exercised)."
```

```
S0 = 60
K = 68
r = .06
sigma = seq(.1, .3, .01)
T = 1
fpremium = C0(S0, K, r, sigma, T)
plot(sigma, fpremium, main="7d) Fair Premium ~ Volatility")
```




```
paste("7d) As volitility increases, the fair premium increases. If the price of a stock is more
 volatile, then it makes sense to want to set a price you can buy the stock at later by purchasi
ng a call (more demand = higher price). If the price of a stock is not very volatile, then there
is not much need for a call option because it is easier to get a more reliable estimate for the
 stock price in the futur.")
```

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