Asset Returns

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Compounding

Money borrowing and lending involve interest. Interest rates ordinary people work with are compounded annually, semiannually, quarterly, monthly, weekly or daily.

Example: suppose that \$100 is invested for a year with interest rate 10% annum. If we compound once per annum, at the end of one year the investment value is equal to

$$\$100 \times 1.1 = \$110.$$

If we compound twice a year, the investment value at the end of the year becomes

$$\$100 \times 1.05 \times 1.05 = \$110.25.$$

10.25%

If we compound quarterly (i.e. four times a year), the terminal investment value is

10.38%

$$\$100 \times 1.025^4 = \$110.38.$$

Compounding

We easily compute the dollar amount of the terminal investment for monthly, weekly and daily compounding as follows:

$$\$100 \times \left(1 + \frac{0.1}{12}\right)^{12} = \$110.47,$$

$$\$100 \times \left(1 + \frac{0.1}{52}\right)^{52} = \$110.51,$$

$$\$100 \times \left(1 + \frac{0.1}{365}\right)^{365} = \$110.52.$$

Compounding

In general, consider an amount of D dollars invested for T years at an interest rate r per annum. If the interest rate is compounded m times per annum, after T years the investment will have dollar value

$$D \times \left(1 + \frac{r}{m}\right)^{mT}.$$

As m goes to infinity, the investment value has a limit

$$D \times \exp(rT).$$

The limit of discrete compounding as compounding frequency tends to infinity is called **continuous compounding**.

- Continuous compounding: $\exp(rT)$;
- Continuous discounting: $\exp(-rT)$.

Compounding

The terminal value for a \$100 investment for a year at interest rate 10% per annum based on continuous compounding is

$$\$100 \times \exp(0.1 \times 1) = \$100.52,$$

which agrees up to two decimal places the value using daily compounding.

Suppose r_m is an interest rate that is compounded m times per annum and r_c is the equivalent continuous compounding interest rate. Then

$$\exp(r_c) = \left(1 + \frac{r_m}{m}\right)^m$$

which implies that

$$r_c = m\log\left(1 + \frac{r_m}{m}\right),$$

where $\log(a)$ denotes the natural logarithm of a.

Asset Returns

Let P_t be the price of an asset at time index t and assume that no dividends are paid.

One-period simple return:

- **Simple gross return**

$$1 + R_t = \frac{P_t}{P_{t-1}}$$

- **Simple net return**

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}$$

Multi-period simple return:

- **Simple gross return**

$$
\begin{aligned}
1 + R_t[k] =& \frac{P_t}{P_{t-k}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}} \\
=& (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) \\
=& \prod_{j=1}^{k} (1 + R_{t-j+1}).
\end{aligned}
$$

  Thus, the k-period simple gross return is just the product of the k one-period simple gross returns involved.

- **Simple net return**

$$
R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1.
$$

Example: Table 1 provides five daily adjusted prices of Apple stock from June 12 to June 16, 2017.

Date	06/12	06/13	06/14	06/15	06/16
Price($)	145.42	146.59	145.16	144.29	142.27

Table: Daily adjusted prices for Apple stock from June 12 to June 16, 2017

The 1-day gross return of holding the stock from June 15 to June 16 is $1 + R_t = 142.27/144.29 \approx 0.9860$ so that the corresponding daily simple net return is -0.0140, which is $(142.27 - 144.29)/144.29$. On the other hand, the simple gross return for the 5 days period from June 12 to June 16 is $1 + R_t[4] = 142.27/145.42 \approx 0.9783$ so that the corresponding simple net return for the 5 days period is -0.0217.

Continuously compounded return: natural logarithm of the gross return, often referred to as **log return**.

$$r_t = \log(1 + R_t) = \log \frac{P_t}{P_{t-1}} = p_t - p_{t-1},$$

where $p_t = \log(P_t)$.

Advantages:

$$\begin{aligned} r_t(k) &= \ln\left[1 + R_t(k)\right] \\ &= \ln\left[(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})\right] \\ &= \ln(1 + R_t) + \ln(1 + R_{t-1}) + \cdots + \ln(1 + R_{t-k+1}) \\ &= r_t + r_{t-1} + \cdots + r_{t-k+1}. \end{aligned}$$

Thus, the continuously compounded multi-period return is simply the sum of continuously compounded one-period returns involved.

Stochastic process: a **stochastic process** $\{X_t\}$ is a *collection* $\{X_t : t \in \mathscr{I}\}$ of random variables where the index t belongs to the index set $\mathscr{I}$.

- $\{X_t\}$ is a continuous time stochastic process if $\mathscr{I}$ is an interval in $\mathcal{R}$;
- $\{X_t\}$ is a discrete time stochastic process if $\mathscr{I}$ is a subset of $\{1, 2, \cdots, n \cdots\}$.

In general, the introduction of a stochastic process is intended to model some aspects of stochastic evolution and, of course, return time series present such kind of evolutions.

Well-known stochastic models include random walk, GARCH models, and stochastic volatility models in discrete time, Brownian motion, diffusion, and jump-diffusion models in continuous time.

Random Walk

Suppose that r_t are the log returns from an asset, recall that

$$p_t - p_0 = \log(P_t) - \log(P_0) = r_t + r_{t-1} + \cdots + r_1.$$

r_1 = return from t ∅ to 1

In other words, (recover the stock price)

$$P_t = P_0 \exp\left(r_t + r_{t-1} + \cdots + r_1\right).$$

- Assume that $r_t \sim_{iid}$ Dist, then log price process p_t is a random walk, and price process P_t a geometric random walk.
- Assume that $r_t \sim_{iid}$ Normal, then p_t follow normal distributions, and the price process P_t is a lognormal geometric random walk.
- As the time length of each step shrinks to zero, appropriately normalized random walk p_t converges to Brownian motion and geometric random walk P_t converges to geometric Brownian motion, under which the Black-Scholes formula is derived.

Random Walk: Simulation

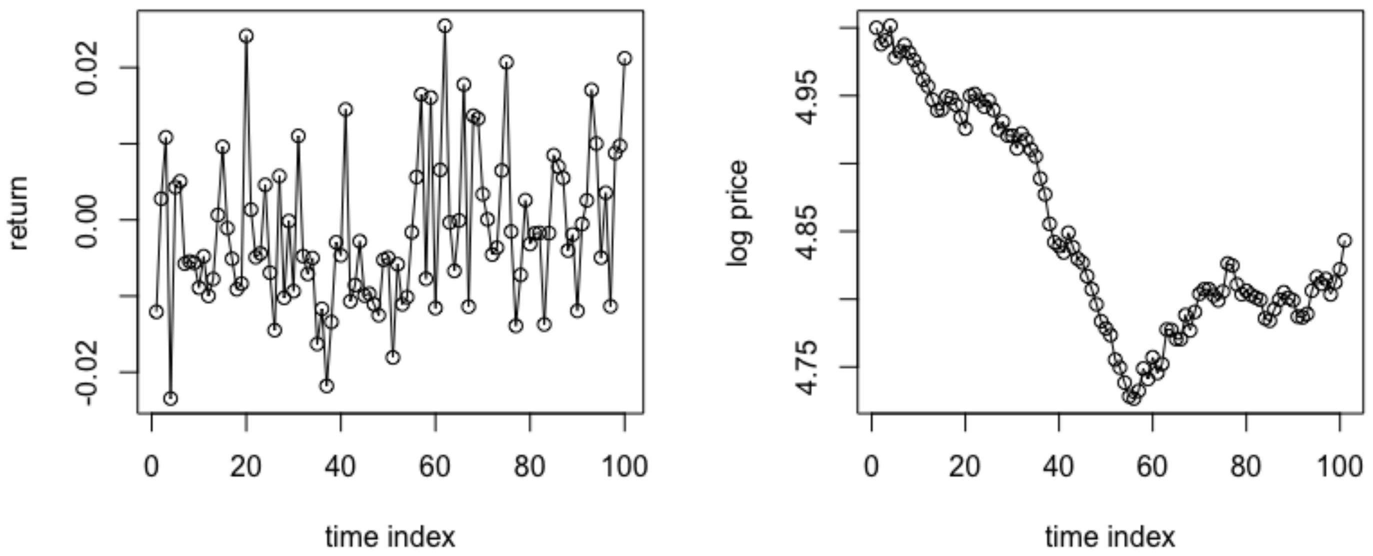

Figure: One realization of random walk process where $p_0 = 5$ and $r_1, \cdots, r_{100} \sim_{iid} N(0, 0.01)$.

Brownian Motion

Brownian Motion: a stochastic process with continuous trajectories that have independent and stationary increments.

For a continuous stochastic process to be **standard Brownian motion (Wiener process)**, it must satisfy the following properties:

- $W_0 = 0$;
- for any $0 < t_1 < \cdots < t_n$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}})$ are independent;
- for any $0 \leq s < t$, the increments $W_t - W_s$ is a centered (mean-zero) normal random variable with variance $E(W_t - W_s)^2] = t - s$. In particular, $W_t = W_t - W_0 \sim N(0, t)$.

Efficient Market Hypothesis

A market is said to be **efficient** with respect to an information set if prices fully reflect the available information set. (all information for the company's performance is reflected by the stock price)

In other words, in efficient markets there would be no change in prices when the information is revealed to all participants, and prices change only when there is new and unanticipated information. As information flow is unanticipated, prices evolve randomly and are not predictable. Under the efficient market assumption, it is not possible to predict prices and hence make economic profits from any price prediction. Price processes following geometric random walks are efficient markets, as their returns are independent under the models.

A weak assumption is that the returns are dependent but uncorrelated. While market efficiency is retained for the random walk models with this weaker assumption, the models allow for heteroskedasticity and dependence in higher moments such as volatility.

Stylized Facts

Empirical evidence show that returns

- are not normally distributed: their marginal distributions have high peaks around their mean and fatter tails;
- have little correlation among themselves but strong correlation among their absolute values or squares;
- their volatilities (or conditional variance) change and appear in clusters

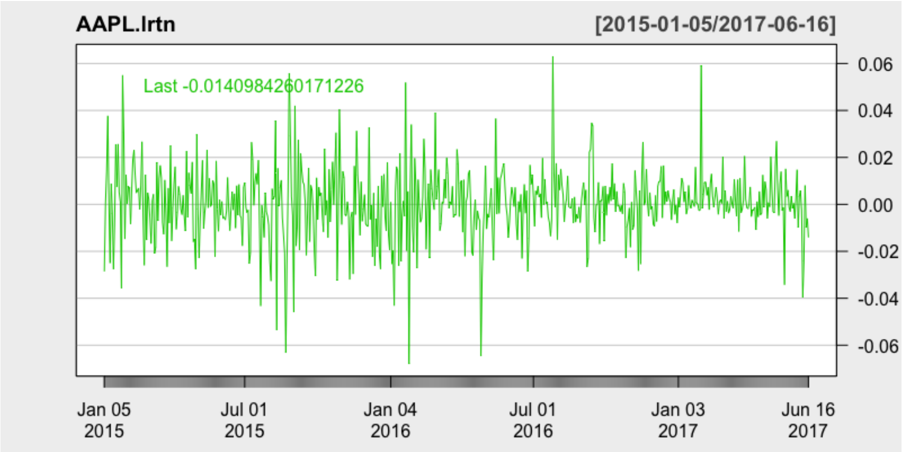

Figure: Time series plot of daily log returns for Apple stock from January 2, 2015 to June 16, 2017.

Stylized Facts

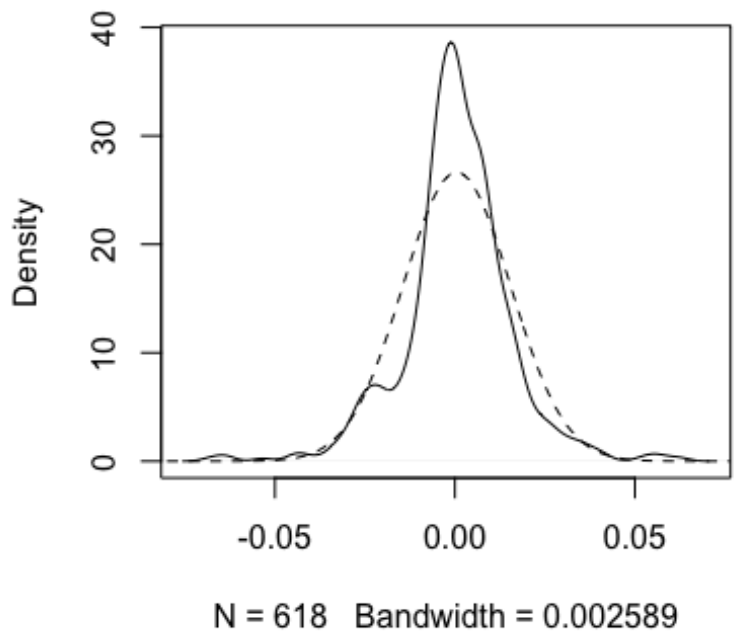

Figure: Empirical density of daily log returns for Apple stock from January 2, 2015 to June 16, 2017. The dashed line denotes the density function of a normal distribution with the same mean and standard deviation.