Homework 3 Solution

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Problem 1: we first show that the put-call parity $C + K\exp(-rT) = P + S_0$ does not hold. Here

$$LHS = C + K\exp(-rT) = 3 + 22\exp(-0.1 \times 0.5) = 23.93,$$

and

$$RHS = P + S_0 = 4 + 20 = 24.$$

To arbitrage, we need to purchase the cheaper side and sell the more expensive side. That is, we need to take the following actions today:

- borrow 1 share of stock and sell (+$20)
- sell 1 put option (+$4)
- buy 1 call option (-$3)
- deposit the rest money into the bank (-$21)

Similar to the example discussed in lecture, no matter what happens with the stock price S_T at expiration, we end up buying one share of stock at the strike price, on the other hand, we receive money from the bank, so the total net profit is

$$21 \exp(rT) - K = 21 \exp(0.1 \times 0.5) - 22 = 0.0767.$$

Problem 2: $S_0 = 10$, $S_u = 11$, $S_d = 9$, $K = 10.5$, $T = 1/4 = 0.25$, $r = 0.12$.

(a) The risk neutral probability is

$$q = \frac{\exp(0.12 \times 0.25) - 0.9}{1.1 - 0.9} = 0.6523.$$

The fair price of the call option is

$$C_0 = \exp(-0.12 \times 0.25) [(11 - 10.5) \times 0.6523 + 0 \times (1 - 0.6523)] = 0.3165.$$

(b) Consider a portfolio that consists of 1) buy $\triangle$ shares of stock and 2) sell 1 call option. Portfolio value when the stock price moves up to 11 is $11\triangle - 0.5$ and when the stock price moves down to 9 is $9\triangle$. To make the portfolio risk free, we need

$$11\triangle - 0.5 = 9\triangle$$

which provides $\triangle = 0.25$. Value of the risk-free portfolio at expiration is $11 \times 0.25 - 0.5 = 2.25$ and value of the risk-free portfolio at the time of purchase is $2.25 \exp(-0.12 \times 0.25) = 2.1835$.

If the call option can be sold/bought at 0.8, price needs to pay today to set-up the portfolio is $10 \times 0.25 - 0.8 = 1.7$. We note that the portfolio is priced cheaper than its value, therefore, we need to purchase the whole portfolio to arbitrage. That is, we need to take the following actions today

- buy 0.25 shares of stock (-$2.5)
- sell 1 call option (+$0.8)
- borrow rest money from the bank (+$1.7)

The net profit at expiration is

$$2.25 - 1.7\exp(0.12 \times 0.25) = 0.4982,$$

where 2.25 is the payoff of the risk-free portfolio (no matter what the stock price S_T is at the expiration date, the payoff of the whole portfolio is always 2.25).

Problem 3:

(a)

$$1 - q = 1 - \frac{\exp(r\tau) - d}{u - d} = \frac{u - \exp(r\tau)}{u - d}.$$

(b)

$$\begin{aligned} & C_0 \\ =& \exp(-r \cdot 3\tau) \sum_{j=0}^{3} \left(S_0 u^j d^{3-j} - K\right)_+ \cdot \binom{3}{j} q^j (1-q)^{3-j} \\ =& \exp(-3r\tau)\left[(S_0u^3 - K)_+ q^3 + (S_0u^2d - K)_+ \cdot 3q^2(1-q) + (S_0ud^2 - K)_+ \cdot 3q(1-q)^2 + (S_0d^3 - K)_+(1-q)^3\right] \end{aligned}$$

(c) $n = 10$, $\tau = 1$, $r = 0.06$, $S_0 = 100$, $u = 1.1$, $d = 0.9$, $K = 110$.

$$q = \frac{\exp(r\tau) - d}{u - d} = \frac{\exp(0.06 \times 1) - 0.9}{1.1 - 0.9} = 0.8092.$$

Probability stock price moves up 5 times and down 4 times is

$$\binom{9}{5}(0.8092)^5(1 - 0.8092)^{9-5} = 0.05795$$

and the corresponding price value is $100 \times 1.1^5 \times 0.9^4 = 105.6656$.

(d) R script attached below. Fair price of the call option is 39.90502.

```r
S0=100; K=110; r=0.06; u=1.1; d=0.9
n=10; tau=1; T=n*tau
q=(exp(r*tau)-d)/(u-d)
C=0
  for (j in 0:n){
    C=C+max(c(S0*u^j*d^(n-j)-K,0))*dbinom(j, n, q)
  }
C*exp(-r*T)
```

```
## [1] 39.90502
```

Problem 4:

(a) Let X be the number of steps where stock price moves up during te first total k steps, then

$$\begin{aligned}S_k =& S_0 u^X d^{k-X}\\ =& S_0 \exp\left(\frac{\sigma}{\sqrt{n}}X\right)\left(-\frac{\sigma}{\sqrt{n}}(k-X)\right)\\ =& S_0 \exp\left(\frac{\sigma}{\sqrt{n}}(2X-k)\right)\end{aligned}$$

Let W_j, $j = 1, \cdots, k$ be a bernoulli trial where $W_j = 1$ when stock price moves up and $W_j = 0$ when stock price moves down. Let Y_j, $j = 1, \cdots, k$ be a bernoulli trial where $Y_j = 1$ when stock price moves up and $Y_j = -1$ when stock price moves down. Then we have

$$Y_j = 2W_j - 1$$

and

$$2X - k = 2\sum_{j=1}^{k} W_j - \sum_{j=1}^{k} 1 = \sum_{j=1}^{k}(2W_j - 1) = \sum_{j=1}^{k} Y_j$$

Therefore,

$$S_k = S_0 \exp\left(\frac{\sigma}{\sqrt{n}}(2X-k)\right) = S_0 \exp\left(\frac{\sigma}{\sqrt{n}}\sum_{j=1}^{k} Y_j\right) = S_0 \exp\left(\frac{\sigma}{\sqrt{n}}Y\right)$$

where $Y = \sum_{j=1}^{k} Y_j$ and takes values $-k, -k+2, \cdots, k+2, k$.

(b) Since $2X - k = Y$, then we have $X = \frac{Y+k}{2}$ which is a sum of k iid bernoulli trials W_j, therefore, $X \sim \text{Binomial}(k, q)$. For the risk-neutral probability, we have

$$q = \frac{\exp(r\tau) - d}{u - d} = \frac{1 - \exp\left(-\frac{\sigma}{\sqrt{n}}\right)}{\exp\left(\frac{\sigma}{\sqrt{n}}\right) - \exp\left(-\frac{\sigma}{\sqrt{n}}\right)} \approx \frac{1 - \left(1 - \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n}\right)}{\left(1 + \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n}\right) - \left(1 - \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n}\right)} = \frac{1}{2}\left(1 - \frac{\sigma}{2\sqrt{n}}\right).$$

where the approximation is by Taylor expansion $\exp(x) \approx 1 + x + \frac{x^2}{2}$.

Problem 5:

(a) Let $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ be the pdf function for a standard normal distribution.

$$\begin{aligned}& e^x \phi\left(\frac{x - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right)\\ =& \exp(x)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{[x - (r - \sigma^2/2)T]^2}{2\sigma^2 T}\right)\\ =& \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{-2\sigma^2 Tx + x^2 - 2x(r - \sigma^2/2)T + (r - \sigma^2/2)^2 T^2}{2\sigma^2 T}\right)\\ =& \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2 - 2(r + \sigma^2/2)Tx + (r - \sigma^2/2)^2 T^2}{2\sigma^2 T}\right)\\ =& \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2 - 2(r + \sigma^2/2)Tx + (r + \sigma^2/2)^2 T^2 - 2r\sigma^2 T^2}{2\sigma^2 T}\right)\\ =& \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{[x - (r + \sigma^2/2)T]^2}{2\sigma^2 T} + rT\right)\\ =& \exp(rT)\frac{1}{\sqrt{2\pi}}\left(-\frac{[x - (r + \sigma^2/2)T]^2}{2\sigma^2 T}\right)\\ =& \exp(rT)\phi\left(\frac{x - (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right)\end{aligned}$$

(b) Put-call parity suggests that

$$\begin{aligned}P_0 =& C_0 + K\exp(-rT) - S_0 \\ =& S_0\Phi(d_1) - K\exp(-rT)\Phi(d_2) + K\exp(-rT) - S_0 \\ =& K\exp(-rT)\left[1 - \Phi(d_2)\right] - S_0\left[1 - \Phi(d_1)\right] \\ =& K\exp(-rT)\Phi(-d_2) - S_0\Phi(-d_1)\end{aligned}$$

where the last equality is due to the fact that

$$1 - \Phi(d_2) = 1 - P(Z \le d_2) = P(Z > d_2) = P(Z < -d_2) = P(Z \le -d_2) = \Phi(-d_2);$$

$$1 - \Phi(d_1) = 1 - P(Z \le d_1) = P(Z > d_1) = P(Z < -d_1) = P(Z \le -d_1) = \Phi(-d_1).$$

Problem 6:

(a) $S_0 = 60$, $T = 1$, $K = 68$, $r = 0.06$ and $\sigma = 0.1$. Recall that in the risk-neutral world,

$$S_T = S_0 \ \exp\left[(r - \frac{\sigma^2}{2})t + \sigma W_T\right].$$

That is,

$$S_1 = 60 \ \exp\left[(0.06 - \frac{0.1^2}{2})1 + 0.1W_1\right] = 60 \ \exp(0.055 + 0.1W_1).$$

where $W_1 \sim N(0, 1)$. Let $X = 0.055 + 0.1W_1$, then $X \sim N(0.055, 0.1^2 \times 1) = N(0.0055, 0.01)$. Now

$$P(S_1 > 68) = P\left(\frac{S_1}{60} > \frac{68}{60}\right) = P\left(log(\frac{S_1}{60}) > log(\frac{68}{60})\right) = P\left(X > log(\frac{68}{60})\right) = 0.2415.$$

(b) 95% confidence interval for X defined in part (a) is

$$(0.055 - 1.96 \times \sqrt{0.01}, 0.055 + 1.96 \times \sqrt{0.01}) = (-0.141, 0.251).$$

Since $S_1 = 60\exp(X)$, 95% confidence interval for S_1 is

$$(60\exp(-0.141), 60\exp(0.251)) = (52.109, 77.119).$$

(c)

$$d_1 = \frac{log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{log(60/68) + (0.06 + 0.1^2/2)T}{0.1\sqrt{1}} = -0.6016$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -0.6016 - 0.1\sqrt{1} = -0.7016.$$

$$C_0 = S_0\Phi(d_1) - K\exp(-rT)\Phi(d_2) = 60\Phi(-0.6016) - 68\exp(-0.06 \cdot 1)\Phi(-0.7016) = 0.9598.$$

(d)

$$P_0 = C_0 + K\exp(-rT) - S_0 = 0.9598 + 68\exp(-0.06 \cdot 1) - 60 = 4.9998.$$

Problem 7:

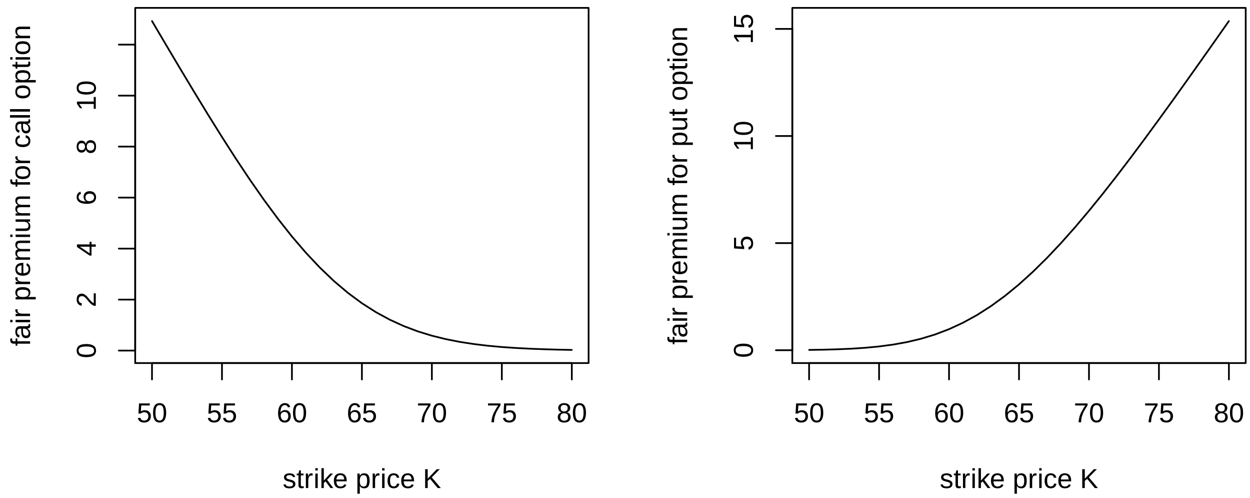
```r
########(a)########
###############################
# Call Option Price Function #
###############################
call.price <- function(x = 1, t = 0, T = 1, r = 1, sigma = 1, K=1){
  d2<-(log(x/K)+(r-0.5*sigma^2)*(T-t))/(sigma*sqrt(T-t))
  d1<-d2+sigma*sqrt(T-t)
  x*pnorm(d1)-K*exp(-r*(T-t))*pnorm(d2)
}

###############################
# Put Option Price Function #
###############################
put.price <- function(x = 1 ,t = 0, T =1 , r = 1, sigma=1, K=1){
  d2<-(log(x/K)+(r-0.5*sigma^2)*(T-t))/(sigma* sqrt(T-t))
  d1<-d2+sigma*sqrt(T-t)
  K*exp(-r*(T-t))*pnorm(-d2)-x*pnorm(-d1)
}

# specify parameters
S0 <- 60; K <- 68; r <- 0.06; T <- 1; sigma <- 0.1
# Call option price
C <- call.price(x=S0, t=0, T=T, r=r, K=K, sigma=sigma)
C
```

```
## [1] 0.9598403
```

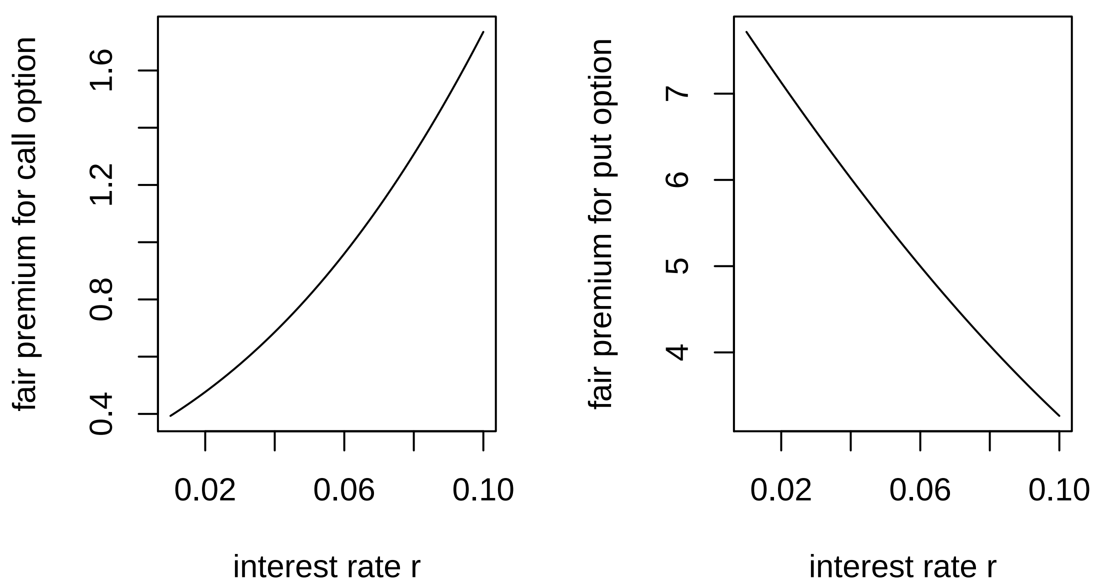
```r
# Put option price
P <- put.price(x=S0, t=0, T=T , r=r, K=K, sigma=sigma)
P
```

```
## [1] 4.999829
```

```r
########(b)########
# specify parameters
S0 <- 60; K <- 50:80; r <- 0.06; T <- 1; sigma <- 0.1
# Call option price
C <- call.price(x=S0, t=0, T=T, r=r, K=K, sigma=sigma)
P <- put.price(x=S0, t=0, T=T , r=r, K=K, sigma=sigma)
par(mfrow=c(1,2))
plot(K,C,type="l", xlab="strike price K", ylab="fair premium for call option")
plot(K,P,type="l", xlab="strike price K", ylab="fair premium for put option")
```




```r
########(c)########
# specify parameters
S0 <- 60; K <- 68; r <- seq(0.01,0.1,length.out = 100); T <- 1; sigma <- 0.1
# Call option price
C <- call.price(x=S0, t=0, T=T, r=r, K=K, sigma=sigma)
P <- put.price(x=S0, t=0, T=T , r=r, K=K, sigma=sigma)
par(mfrow=c(1,2))
plot(r,C,type="l", xlab="interest rate r", ylab="fair premium for call option")
plot(r,P,type="l", xlab="interest rate r", ylab="fair premium for put option")
```




```r
########(d)########
# specify parameters
S0 <- 60; K <- 68; r <- 0.06; T <- 1; sigma <- seq(0.1,0.3,length.out = 100)
# Call option price
C <- call.price(x=S0, t=0, T=T, r=r, K=K, sigma=sigma)
P <- put.price(x=S0, t=0, T=T , r=r, K=K, sigma=sigma)
par(mfrow=c(1,2))
plot(sigma,C,type="l", xlab="volatility sigma", ylab="fair premium for call option")
```

```
plot(sigma,P,type="l", xlab="volatility sigma", ylab="fair premium for put option")
```