Homework Assignment #5

Due date: Thursday, 12/10.

1. **GARCH Model (Model Property).**

   Consider the GARCH(1,1) model,

   $$a_t = \sigma_t \epsilon_t \qquad \text{and} \qquad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

   where $\epsilon_t \sim_{iid} N(0,1)$, ϵ_t and σ_t are independent for all $t = 1, \cdots, T$. Assume model stationary.

   (a) Compute unconditional mean $E(a_t)$.

   (b) Compute unconditional variance $\mathrm{Var}(a_t^2)$.

   (c) Show that the kurtosis of a_t is given by

   $$\frac{E a_t^4}{\left(E a_t^2\right)^2} = \frac{3[1-(\alpha_1+\beta_1)^2]}{1-(\alpha_1+\beta_1)^2 - 2\alpha_1^2}.$$

   (d) Assume forecast origin to be time n, let $\sigma_n^2(\ell)$ to be the ℓ-step ahead predicted value for $\sigma_{n+\ell}^2$. Show that

   $$\sigma_n^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_n^2(\ell - 1) \qquad \text{for } \ell > 1.$$

2. **GARCH Model (Empirical Application).**

   Consider the daily log returns $\{r_t\}_{t=1,\cdots,T}$ of Apple stock based on daily **adjusted price**. Sample period is from July 31, 2010 to July 31, 2020. The data can be downloaded from Yahoo via the *quantmod* package.

   (a) Is the expected value of r_t significantly different from 0?

   (b) Does any serial correlation appears to be significant for r_t series?

   (c) Build an appropriate Gaussian ARMA model to the r_t series. Test ARCH effect on squared residual series by applying Box.test statistics. ($H_0 : \rho_1 = \rho_2 = \cdots\cdots = \rho_8 = 0$ for squared residual series)

   (d) Build an appropriate Gaussian ARMA-GARCH(1,1) model to the r_t series. Write down the fitted model. Is the model adequate? Justify your answer.

   (e) Build an appropriate ARMA-GARCH(1,1) model with Student-t innovations to the r_t series. Write down the fitted model. Is the model adequate? Justify your answer.

   (f) Plot estimated volatilities based on fitted model in part (e).

   (g) Obtain 1-step to 5-step ahead mean and volatility forecasts based on fitted model in part (e).

1) a) $\mathbb{E}[a_t] = \mathbb{E}\left[\mathbb{E}[a_t \mid F_{t-1}]\right] = \mathbb{E}\left[\mathbb{E}[\sigma_t \varepsilon_t \mid F_{t-1}]\right] = \mathbb{E}\left[\sigma_t \mathbb{E}[\varepsilon_t \mid F_{t-1}]\right] = \mathbb{E}\left[\sigma_t \mathbb{E}[\varepsilon_t]\right] = 0$

b) $\mathrm{Var}(a_t) = \mathbb{E}[a_t^2] = \mathbb{E}\left[\mathbb{E}[a_t^2 \mid F_{t-1}]\right] = \mathbb{E}\left[\sigma_t^2 \mathbb{E}[\varepsilon_t^2]\right] = \mathbb{E}[\sigma_t^2]$

$$= \mathbb{E}\left[\alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2\right]$$

$$= \alpha_0 + \alpha_1 \mathbb{E}[a_{t-1}^2] + \beta_1 \mathbb{E}[\sigma_{t-1}^2]$$

$$= \alpha_0 + \alpha_1 \mathrm{Var}[a_{t-1}] + \beta_1 \mathrm{Var}[a_{t-1}]$$

$\mathrm{Var}(a_t) = \mathrm{Var}(a_{t-1})$ (stationary)

$$\mathrm{Var}(a_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

c)

$$\text{Kurtosis} = \frac{\mathbb{E}[a_t^4]}{\mathrm{Var}(a_t)^2} > 3$$ (fatter tails than normal dist) $$\mathrm{Var}(a_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

$$\mathbb{E}[a_t^4] = \mathbb{E}[\sigma_t^4 \varepsilon_t^4] = \mathbb{E}\left[\mathbb{E}[\sigma_t^4 \varepsilon_t^4 \mid F_{t-1}]\right] = \mathbb{E}\left[\sigma_t^4 \cdot \mathbb{E}[\varepsilon_t^4]\right] = 3 \cdot \mathbb{E}[\sigma_t^4]$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1} + \beta_1 \sigma_{t-1}^2$$

$$\sigma_t^4 = (\alpha_0 + \alpha_1 a_{t-1} + \beta_1 \sigma_{t-1}^2)^2$$

$$= \alpha_0^2 + \alpha_1^2 a_{t-1}^4 + \beta_1^2 \sigma_{t-1}^4 + 2\alpha_0 \alpha_1 a_{t-1}^2 + 2\alpha_0 \beta_1 \sigma_{t-1}^2 + 2\alpha_1 \beta_1 a_{t-1}^2 \sigma_{t-1}^2$$

$$\mathbb{E}[\sigma_t^4] = \alpha_0^2 + \alpha_1^2 \mathbb{E}[a_{t-1}^4] + \beta_1^2 \mathbb{E}[\sigma_{t-1}^4] + 2\alpha_0\alpha_1 \mathbb{E}[a_{t-1}^2] + 2\alpha_0 \beta_1 \mathbb{E}[\sigma_{t-1}^2] + 2\alpha_1 \beta_1 \mathbb{E}[a_{t-1}^2 \cdot \sigma_{t-1}^2]$$

$$= \alpha_0^2 + \alpha_1^2 \cdot 3\,\mathbb{E}[\sigma_{t-1}^4] + \beta_1 \mathbb{E}[\sigma_{t-1}^4] + 2\alpha_0\alpha_1 \mathrm{Var}(a_t) + 2\alpha_0\beta_1 \mathrm{Var}(a_t) + 2\alpha_1\beta_1 \underbrace{\mathbb{E}[\varepsilon_{t-1}^2 \sigma_{t-1}^4]}_{\mathbb{E}[\mathbb{E}[\sigma_{t-1}^4 \cdot \varepsilon_{t-1}^2 \mid F_{t-1}]] = \mathbb{E}[\sigma_{t-1}^4 \mathbb{E}[\varepsilon_{t-1}^2]] = \mathbb{E}[\sigma_{t-1}^4]}$$

(apply stationary property and solve for $\mathbb{E}[\sigma_t^4]$)

$$= \frac{\alpha_0^2 (1 + \alpha_1 + \beta_1)}{(1 - \alpha_1 - \beta_1)[1 - (\alpha_1 + \beta_1)^2 - 2\beta_1^2]}$$

$$\text{Kurtosis}(a_t) = 3 \cdot \frac{\alpha_0^2 (1 + \alpha_1 + \beta_1)}{(1 - \alpha_1 - \beta_1)[1 - (\alpha_1 + \beta_1)^2 - 2\beta_1^2]} \cdot \frac{(1 - \alpha_1 - \beta_1)^2}{\alpha_0^2} = 3 \cdot \frac{1 - (\alpha_1 + \beta_1)^2}{1 - (\alpha_1 + \beta_1)^2 - 2\beta^2} > 3$$

Stat 461 HW5

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```
#getSymbols("AAPL",from="2010-07-31",to="2020-07-31") # this won't work for me, so I am just goi
ng to download the csv from yahoo for the correct date range
AAPL = read.table("AAPL.csv", sep=",", header=TRUE)
adj_price = AAPL$Adj.Close
log_rt = diff(log(adj_price))
```

```
mean(log_rt)
```

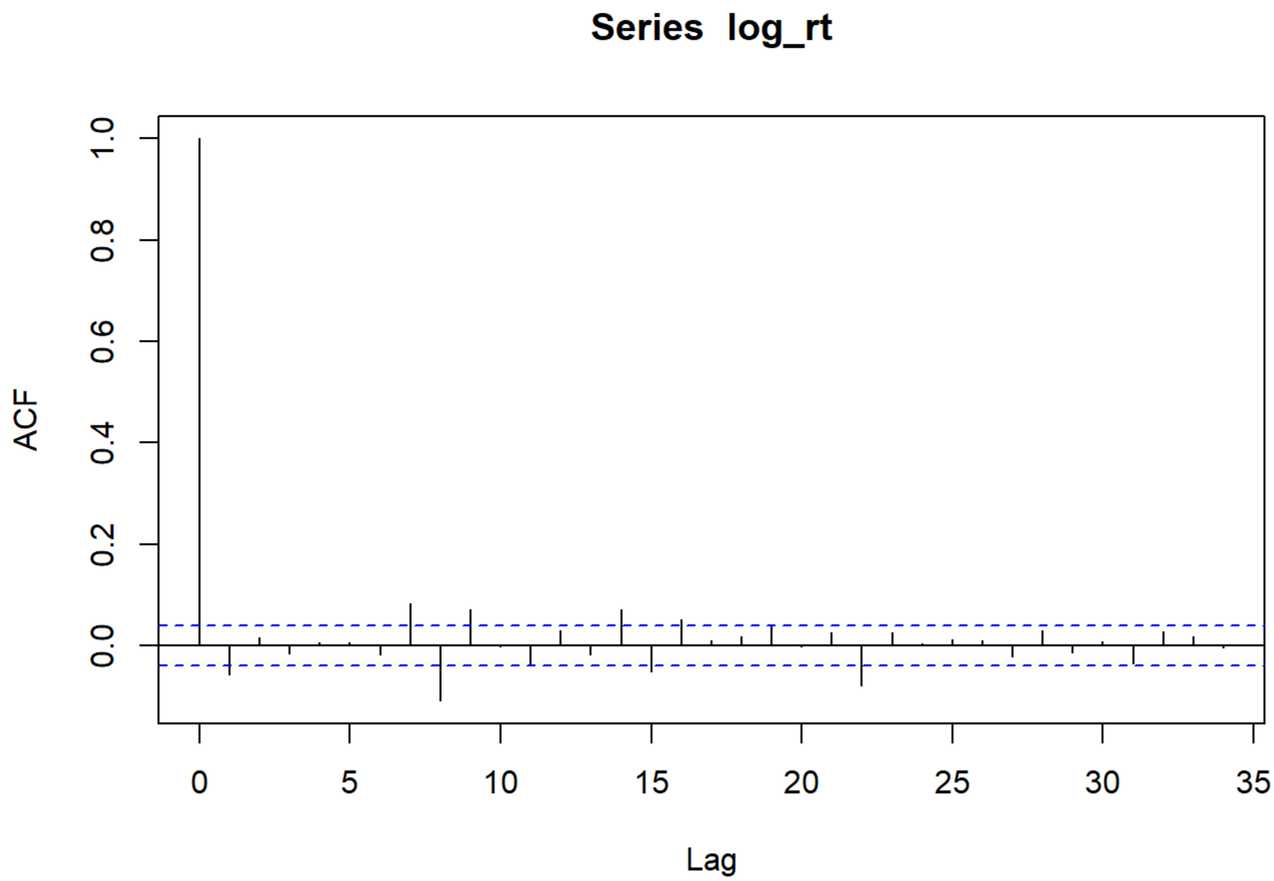
```
## [1] 0.0009869365
```

```
t.test(log_rt, mu=0, alternative="two.sided")
```

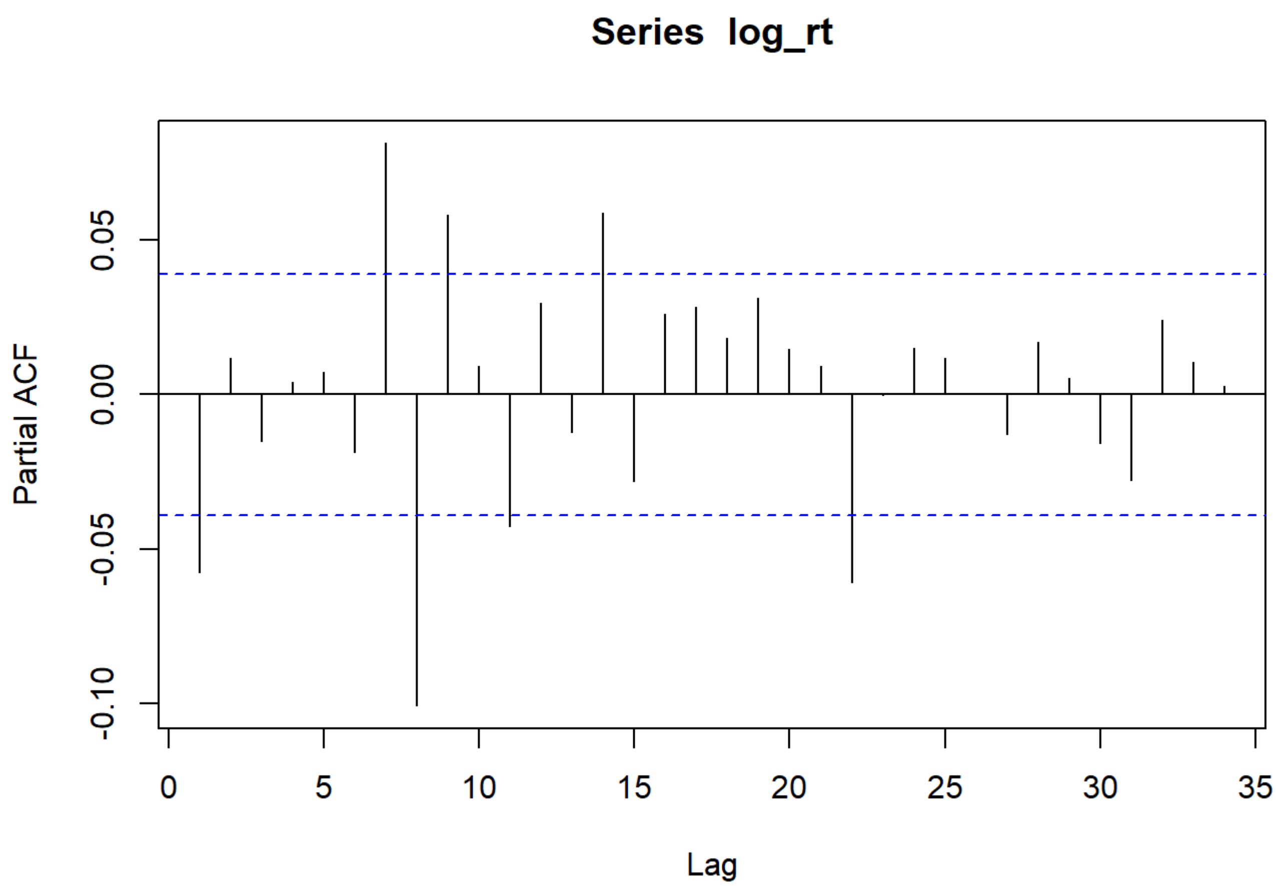
```
##
##  One Sample t-test
##
## data:  log_rt
## t = 2.8581, df = 2515, p-value = 0.004297
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  0.000309816 0.001664057
## sample estimates:
##    mean of x
## 0.0009869365
```

2a) It appears that the expected value of r_t is different from 0, so we should implement MA in our model.

```
acf(log_rt)
```




```
pacf(log_rt)
```




```
## This section takes a long time to run, so I have commented it out and written the important r
esults down
#lags = 10
#aic_m = matrix(0, nrow=lags, ncol=lags)
#bic_m = matrix(0, nrow=lags, ncol=lags)

#for (i in 1:lags) {
#  for (j in 1:lags) {
#    aic_m[i,j] = AIC(arima(log_rt, order=c((i-1),0,(j-1))))
#    bic_m[i,j] = BIC(arima(log_rt, order=c((i-1),0,(j-1))))
#  }
#}
#which(aic_m==min(aic_m), arr.ind=TRUE) # [10, 7]
#which(bic_m==min(bic_m), arr.ind=TRUE) # [6, 3]
```

2b) There appears to be some serial correlation (and the mean of returns is not 0). The AIC tells me that I should use ARMA(9,6), but the BIC suggests to use ARMA(5,2), which is simpler. So I will use ARMA(5,2).

```
ar = 5
ma = 2
m1 = arima(log_rt, c(ar,0,ma))
m1
```

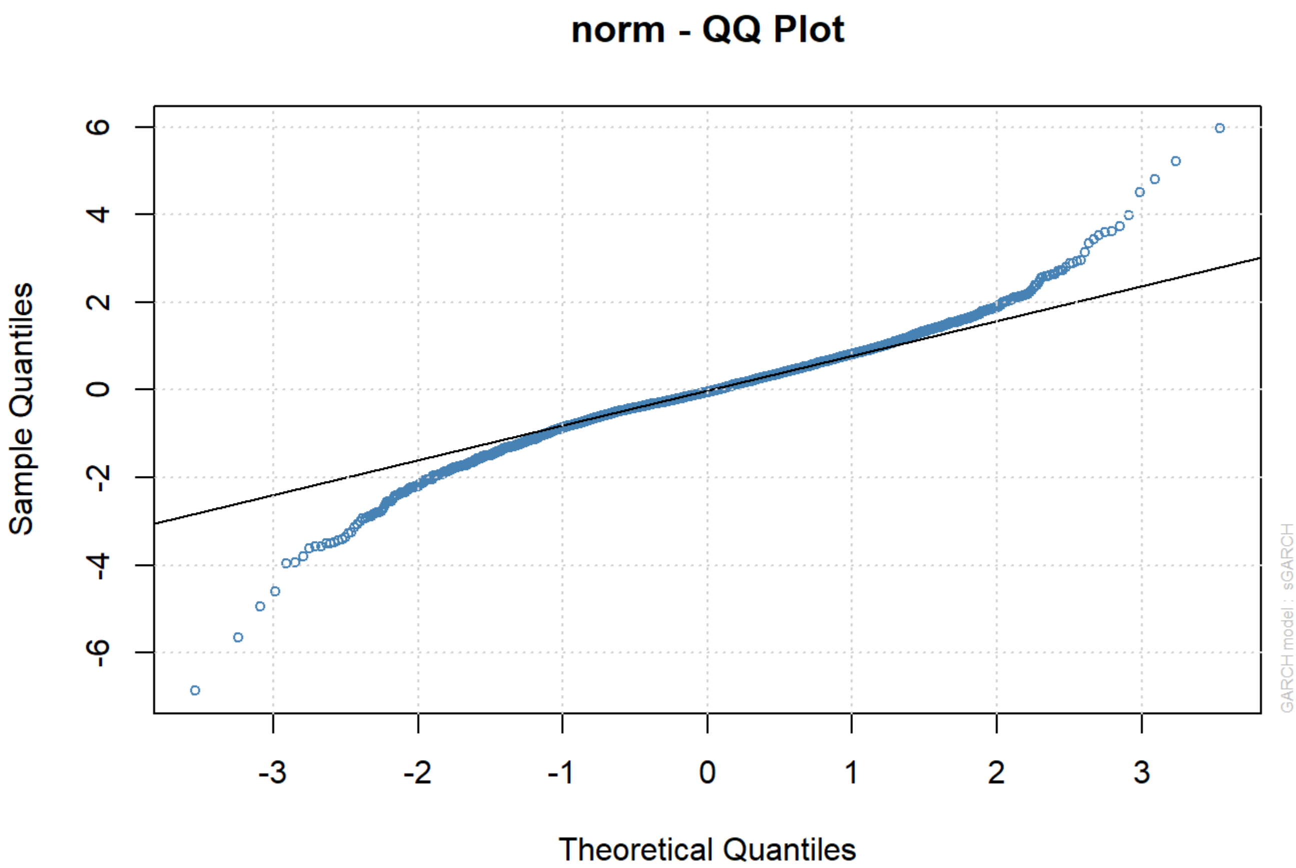
```
##
## Call:
## arima(x = log_rt, order = c(ar, 0, ma))
##
## Coefficients:
##           ar1      ar2      ar3     ar4     ar5     ma1     ma2  intercept
##       -1.6231  -0.8721  -0.0425  0.0116  0.0488  1.5848  0.8122      1e-03
## s.e.   0.0566   0.0592   0.0418  0.0399  0.0239  0.0535  0.0434      3e-04
##
## sigma^2 estimated as 0.0002924:  log likelihood = 6666.51,  aic = -13315.01
```

```
Box.test(m1$residuals^2, lag=8, fitdf=7, type="Ljung-Box")
```

```
##
##  Box-Ljung test
##
## data:  m1$residuals^2
## X-squared = 496.42, df = 1, p-value < 2.2e-16
```

2c) The result of the Ljung-Box test tells us that we reject $H_0 : \rho_1 = \rho_2 = \ldots\ldots = \rho_8 = 0$, so there appears to still be correlation with the squared term.

```
# using arma(0,0) per TA's recommendation
garch_spec_norm = ugarchspec(
  mean.model = list(armaOrder = c(0,0)),
  variance.model = list(model = "sGARCH", garchOrder = c(1,1)),
  distribution.model = "norm")
m2 = ugarchfit(log_rt, spec=garch_spec_norm)
plot(m2, which=9)
```

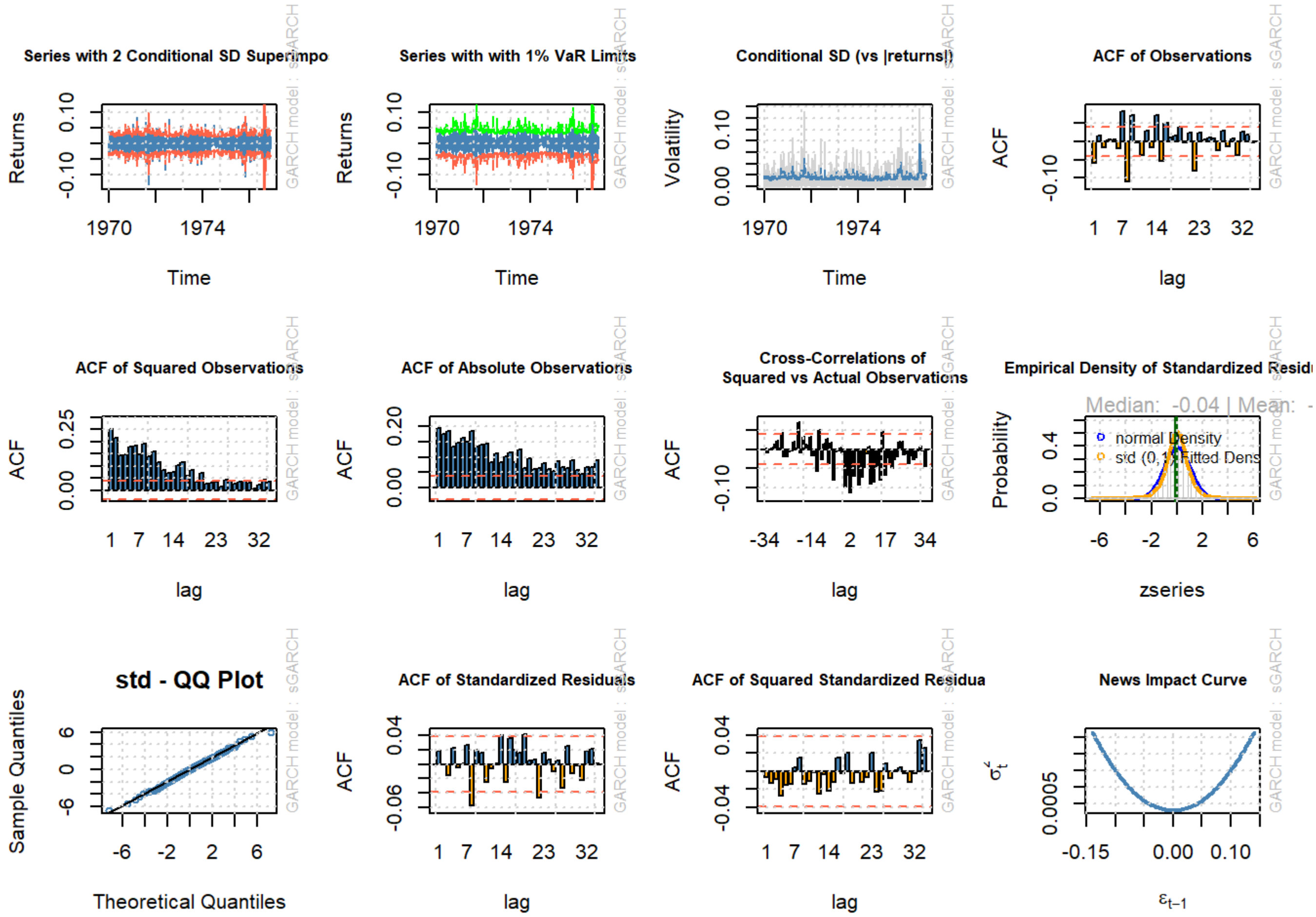


2d)

No, the model is not adequate. We can see from the norm-QQ Plot that the tails of the normal distribution are not heavy enough.

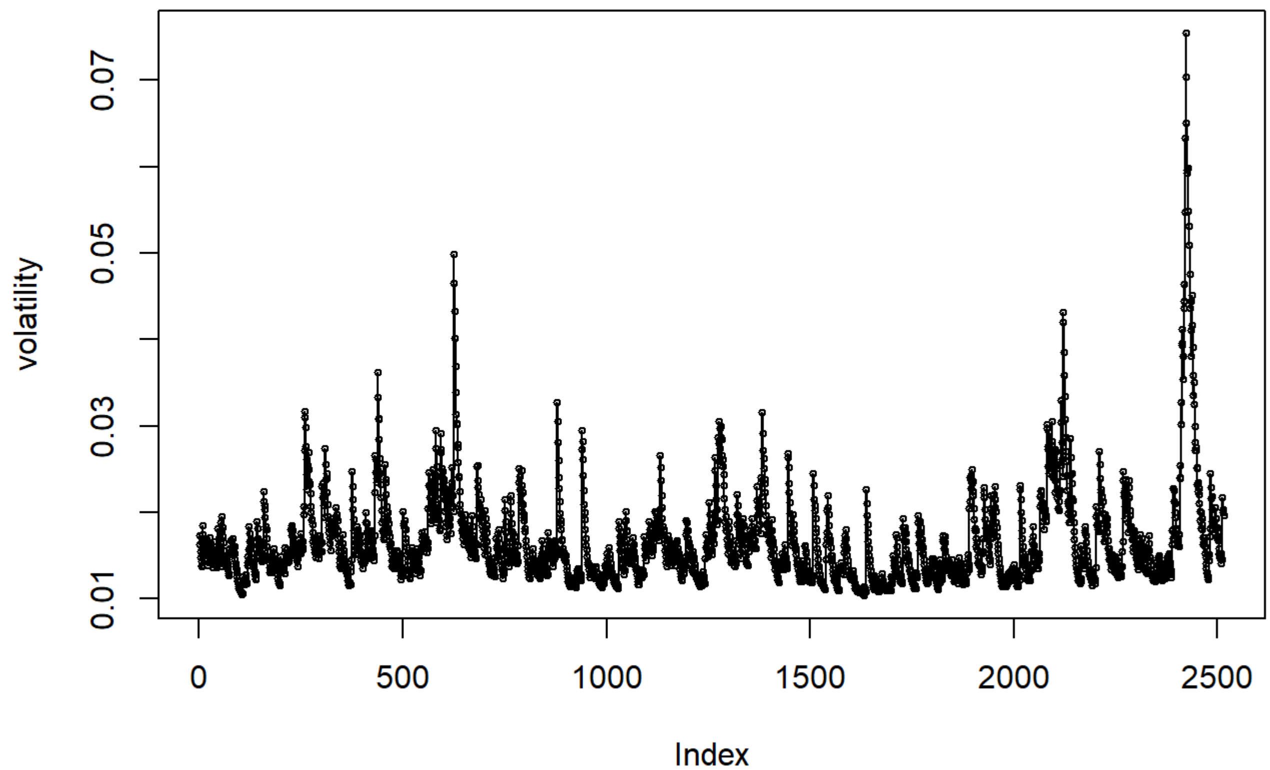
```
garch_spec_std = ugarchspec(
  mean.model = list(armaOrder = c(0,0)),
  variance.model = list(model = "sGARCH", garchOrder = c(1,1)),
  distribution.model = "std")
m3 = ugarchfit(log_rt, spec=garch_spec_std)
plot(m3, which="all")
```

```
##
## please wait...calculating quantiles...
```



2e) It looks like this model is adequate. The QQ Plot looks good, and the ACF plots of the standardized residuals and squared standardized residuals seem to indicate that the model has fit the data relatively well.

```
plot(as.numeric(sigma(m3)), main="", type="o", ylab="volatility", cex=0.5)
```




```
forecast = ugarchforecast(m3, n.ahead = 5)
forecast
```

```
##
## *------------------------------------*
## *       GARCH Model Forecast         *
## *------------------------------------*
## Model: sGARCH
## Horizon: 5
## Roll Steps: 0
## Out of Sample: 0
##
## 0-roll forecast [T0=1976-11-20 18:00:00]:
##       Series   Sigma
## T+1 0.001506 0.01877
## T+2 0.001506 0.01876
## T+3 0.001506 0.01876
## T+4 0.001506 0.01875
## T+5 0.001506 0.01874
```