1.

Show that

$$f(\mathbf{x}) = (x_2 - x_1^2)^2 + x_1^5$$

has only one stationary point which is neither a minimizer or a maximizer.

First differentiate w/ respect to $x_1$

$$\frac{df}{dx_1} = \frac{d([x_2 - x_1^2]^2 + x_1^5)}{dx_1} =$$

$$2(x_2 - x_1^2)^2(-2x_1) + 5x_1^4$$

Now differentiate w/ respect to $x_2$

$$\frac{df}{dx_2} = \frac{d([x_2 - x_1^2]^2 + x_1^5)}{dx_2}$$

$$= 2(x_2 - x_1^2)$$

For stationary point:

$$\frac{df}{dx_1} = 0 \qquad \frac{df}{dx_2} = 0$$

$$2(x_2 - x_1^2)(-2x) + 5x_1^4 = 0$$

$\hookrightarrow 2(x_2 - x_1^2) = 0$

$$x_2 = x_1^2$$

Stationary point is $(0,0)$

$$f_{x_1 x_1} = \frac{df}{dx_1}\left[2(x_2 - x_1^2)(-2x_1) + 5x_1^4\right]$$

$$= -4(x_2 - x_1^2) + 8x_1^2 + 20x_1^3$$

$f_{x_2 x_2} = 2 \qquad f_{x_1 x_2} = -4x_1$

Calculate $f_{x_1 x_1} f_{x_2 x_2} - f_{x_1 x_2}^2$ @ $(0,0)$

$$= \left(-4(x_2 - x_1^2) + 8x_1^2 + 20x_1^3\right)(2) - (-4x_1)$$

at $(0,0)$

$f_{x_1 x_1} f_{x_2 x_2} - f_{x_1 x_2}^2 = 0$ and therefore has no minimizer or maximizer at the stationary point

2. Check if the following functions are convex. Show your computations.

(a) $f(\mathbf{x}) = e^{x_1} + x_2^2 + 5$
(b) $f(\mathbf{x}) = 3x_1^2 - 5x_1x_2 + x_2^2$
(c) $f(\mathbf{x}) = \frac{1}{4}x_1^4 - x_1^2 + x_2^2$
(d) $f(\mathbf{x}) = 50 + 10x_1 + x_2 - 6x_1^2 - 3x_2^2$

- use Hessian matrix to determine convexility

a) $f(x) = e^{x_1} + x_2^2 + 5$

$\frac{\partial f}{\partial x_1} = e^{x_1}$ $\quad \frac{\partial f}{\partial x_2} = 2x$ $\quad \frac{\partial^2 f}{(\partial x_1)^2} = e^{x_1}$

$\frac{\partial^2 f}{(\partial x_2)^2} = 2$ $\quad \frac{\partial^2 f}{(\partial x_2)(\partial x_1)} = 0$

$\begin{bmatrix} e^{x_1} & 0 \\ 0 & 2 \end{bmatrix} = 2e^{x_1}$ The function will be convex

b) $f(x) = 3x_1^2 - 5x_1 x_2 + x_2^2$

$f_1'(x_1, x_2) = 6x_1 - 5x_2$
$f_2'(x_1, x_2) = -5x_1 + 2x_2$

$H = \begin{bmatrix} 6 & -5 \\ -5 & 2 \end{bmatrix} = 12 - 25 = -13 < 0$ so the function is not convex

c) $f(x_1, x_2) = \frac{1}{4} x_1^4 - x_1^2 + x_2$

$f_{x_1} = x_1^3 - 2x_1$

$f_{x_2} = 2x_2$

$f_{x_1 x_1} = 3x_1^2 - 2$

$f_{x_2 x_2} = 2$

$f_{x_1 x_2} = 0$

$f_{x_2 x_1} = 0$

$$H = \begin{bmatrix} 3x_1^2 - 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$= (3x_1^2 - 2)(2) = 6x_1^2 - 4$

$6x_1^2 - 4 > 0 \qquad 6x_1^2 > 4$

$x_1^2 = \frac{2}{3}$

function is convex if $6x_1^2 - 4 > 0$

d) $f(x) = 50 + 10(x_1) + x_2 - 6x_1^2 - 3x_2^2$

$f_{x_1} = 10 - 12x$

$f_{x_2} = 1 - 6x_2$

$f_{x_1 x_1} = -12$

$f_{x_1 x_2} = 0$

$f_{x_2 x_1} = 0$

$$H = \begin{bmatrix} -12 & 0 \\ 0 & 6 \end{bmatrix}$$

$= (-12)(-6) - 0^2 = 72$

Therefore the function is convex

3. Dr. C wants to invest in two projects, A and B. The total investment budget is $100. He does not want to invest more than $40 in project A. The investment goal is the maximization of satisfaction measured as the product of the amount invested in projects A and B. (a) Does this problem have a maximum? Give an argument to support your answer and (b) compute the optimal solution, if it exists.

a) Does this problem have a maximum

$x \rightarrow$ amount invested in Project A

$y \rightarrow$ amount invested in Project B

Given:

1) $x \leq 40$ (does not want to invest more than $40)

2) $x+y = 100$ (total budget)

Express y in terms of x:

$y = 100 - x$

Satisfaction, S, is the product of the investment in A and B:

$S = x(100 - x)$

$S = 100x - x^2$ → downward facing parabola indicates a max exists

b) Compute the optimal solution:

Differentiate w/ respect to $x$ to find value of $S$:

$$\frac{dS}{dx} = 100 - 2x \quad \Rightarrow \quad 0 = 100 - 2x$$

$$x = 50$$

however we have a constraint $x \leq 40$,

so $x = 40$

$$y = 100 - 40 = 60$$

The maximum satisfaction achieved w/ an investment of \$40 in A & \$60 in B, w/ a product of $(40)(60) = 2400$