Yablo's Paradox and Circularity

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Rohan Sikand[1]

Ranney School, Class of 2020

1 Introduction

The famous Liar paradox is thought to have created a gateway for about half of mathematical theory created after it. In fact, paradoxes have been derived from the great Liar paradox itself. A prime example of such is Yablo's Paradox. Crafted in Steve Yablo's *Paradox without self-reference*, the Yablo paradox attempts to do something that the Liar paradox, and all of it's derivatives, do not: an omega-sequence paradox that does not reference itself. Or, in other words, a paradox that is non-circular. However, it can be questioned as to whether the paradox is actually non-circular. This has lead to ubiquitous debate among philosophers of all kinds. Shortly after its debut in literature, the paradox was both argued for and against its circularity - or self-reference. These arguments have it roots in Graham Priest's work *Yablo's Paradox*, where he argues against Yablo and claims that his paradox is circular. A year later, Roy A. Sorenson published his work *Yablo's Paradox and Kindred Liars* in which he argues against Priest's claim and therefore upholding Yablo's non self-referential claim. However, various works of literature have been published since then with most supporting Priest's argument of circularity in Yablo's paradox. Mainly, J.C. Beall's work *Is Yablo's paradox non-circular?* provides an insightful analysis of both Priet's and Sorenson's work and shows that Sorenson's argument does not refute what Priest is

[1] I would like to personally thank Professor Rayo for providing insightful and intriguing instruction and Mr. Cosmo Grant for his endless help. Please email any questions regarding this paper to rosikand@gmail.com

saying. Thus, Beall comes in agreement with Priest in that Yablo's paradox is indeed circular. Although appearing non-circular explicitly, Yablo's paradox can be proved to be implicitly circular through fixed point predicates and indirect self-referential sentences.

2 The Paradox

Yablo's paradox is an reverse-omega sequence paradox, meaning that the set contains infinite many indices in reverse and contains no first member. The paradox has its roots from a plethora of other omega-sequence paradoxes. In fact, it is primarily analogous to the Bomber's paradox which itself is derived from Bernadete's paradox. However, Yablo's paradox differentiates itself from the Bomber's paradox through its widespread plausibility. Such, the sentences can actually be applied linguistically to the real world, whereas the bombs can't. The paradox world has had an ever lasting quest to craft paradoxes without self reference - in specific, omega-sequence paradoxes. Yablo attempted to create just that (but did he succeed?). The thought is based off the Liar's paradox in that the paradox only functions with references to itself. Thus, the paradox attempted to create a derived version of the Liar's paradox but without self-reference - a feat many omega-sequence paradoxes fail to accomplish. Yablo defines his paradox in an example designed to show that self-reference is not essential to paradox.

The paradox begins with an infinite sequence of sentences, one for each natural number. The sentences are labeled as S_0, S_1, S_2, ... S_k ... and so on. Each of these sentences claim that all sentences occurring later in the infinite sequence are not true. The figure below provides a mathematical view of the paradox:[2]

[2] The following figure was obtained from Professor Rayo's book *On the Brink of Paradox*

S_0	"For each $i > 0$, sentence S_i is false"
S_1	"For each $i > 1$, sentence S_i is false"
S_2	"For each $i > 2$, sentence S_i is false"
$\vdots$	$\vdots$
S_k	"For each $i > k$, sentence S_i is false"
$\vdots$	$\vdots$

The sequence creates a paradox similar to the Liar's paradox, but without circularity involved. No sentence in the sequence refers to itself or the sentences above it. The sequence gives rise to the question on which sentences are true and which are false. The paradox is that sentences can be true when they are deemed false and sentences can be false when they are deemed true. As Professor Rayo initially proves, $\mathrm{S_k}$ seems to be false. Here is his proof:

> "Suppose that sentence $\mathrm{S_k}$ is true. It follows from statement (k) above that $\mathrm{S_n}$ must be false for each $n > k$ (and therefore for each $n > k + 1$. So it follows from statement $(k + 1)$ that $\mathrm{S_{k+1}}$ must be true, contradicting our assumption that $\mathrm{S_k}$ is true." [3]

However, it can also be proved that sentence $\mathrm{S_k}$ must be true. Here is Professor Rayo's proof regarding this:

> "The previous argument shows that sentence $\mathrm{S_m}$ is false for arbitrary m. This means, in particular, that sentence $\mathrm{S_m}$ is false for every $m > k$. So follows from statement (k) above that $\mathrm{S_k}$ must be true." [4]

Thus, the paradox arises from these two contradicting proofs. Although it seems that these proofs, as well as the paradox itself, don't use self-reference, they actually implicitly do. In fact, mathematically, this can be proved.

[3] This proof was directly excerpted from Professor Rayo's book *On the Brink of Paradox*

[4] This proof was also directly excerpted from Professor Rayo's book *On the Brink of Paradox*

2.1 Circularity

The idea of self-reference gives rise to the definition of "circularity" or the concept that the paradox only functions when it references itself. This can best be explained through the Liar's paradox first. Suppose you have the following sentence:

(*) The starred sentence is false.

The paradox here questions whether the starred sentence is true. Professor Rayo views it as follows: If it is true, what it says is true, so it is false. if it is false, what it says is false, so it is true.[5] Though the paradox is deeply colored given how simple it appears, it's main "flaw" is that it references itself. Thus, the paradox is circular. As Yablo intended it to be, Professor Rayo claims that Yablo's paradox is indeed non-circular, or, without self-reference. But this if false.

The initial, and ostensible, reaction of Yablo's paradox seems to make it non-circular - each sentence only says, but does not reference, something about sentences below it in the list. However, If looked at in detail, proofs can be made regarding the circularity in the paradox. Shortly after Yablo published his paradox, Graham Priest challenged the non-circularity of the paradox in his work *Yablo's Paradox*. His main claim is that the paradox relates a predicate *Y(x)* of the form:

$$Y(x) = (\forall k > x)(\neg Sat(Y(z), k))$$

This shows that Yablo's paradox has a fixed point *Y(x)* here of the same kind as in the liar paradox - a self-referential one. Essentially, in Priests words, "*Y(x)* is the predicate no number greater than x satisfies this predicate". With this, the circularity now becomes mathematically clear - Yablo's paradox does indeed reference itself. Though Priest's argument is undoubtedly correct, it leaves some ambiguity as to what we consider a non-circular paradox, or a paradox

[5] This is directly obtained from Professor Rayo's book *On the Brink of Paradox*

at all. In fact, if every predicate in arithmetic is also a predicate fixed point, but with the majority these not creating paradoxes, then the world is left wondering what role circularity has on Yablo's paradox. However, circularity can be proved in different ways such as circular linguistic constructions. In his work *What is a Self-Referential Sentence?*, Hannes Leitgeb states that all sentences have some sort of self-reference in them, and thus furthers the claim that each sentence in Yablo's paradox is indeed self-referential. Though it can be proved in multiple ways, the fixed predicate proof seems to be the most definite. The construction of the sentences S_k requires a uniform fixed point result, and thus, the predicate *Y(x)* is equivalent (for variable x) to "for all y > x, *Y(y)* is not true".[6]

A counterargument to the circularity in Yablo's paradox is conducted by Roy A. Sorenson in his work *Yablo's Paradox and Kindred Infinite Liars*. Sorenson quotes his work as an extension and defense of Yablo's claim that self-reference is inessential to the Liar paradox. He argues that Yablo's technique of substituting infinity for self-reference applies to all self-referential paradoxes. In furthering his claim, Sorenson refutes objections (mainly Priest's) that Yablo's paradox is not an authentic liar by constructing a sequence of liars that blend into Yablo's paradox. But this argument is unviable because it bases it off a modified version of Yablo's paradox and doesn't accurately refute the claim Priest puts out.

Despite these arguments from Sorenson, J.C. Beall uses his work *Is Yablo's paradox a non-circular one?* to claim that Sorenson's work is rather a mute argument because it fails to take into account Priest's main claim. In his work, Beall uses careful analysis between the two works (Priest and Sorenson), and deems Priest's claim of circularity in Yablo's paradox the winner.

[6] This was obtained from Roy T. Cook's work *There are non-circular paradoxes (but Yablo's isn't one them!)*

3 A Way Out?

It seems to appear that Professor Rayo's score of 8 for this paradox is indeed fitting. Despite mathematical literature being known to have black and white answers, this paradox adds color and allows for viable, and debatable, arguments on either position. What this paradox also does is question whether there even exists a non-circular paradox. Further, it questions whether there is even a way out of this circumscribed paradox. The answer may be yes, but certainly not for a pure, untouched Yablo paradox. However, it seems to appear Yablo's paradox can be altered to make it circular. In fact, Beall argues that Yablo's paradox is circular because it references itself through demonstration or description. Mainly, he argues his point that Yablo's paradox is circular through description. But, Beall does not provide an adequate argument for demonstration, leaving the door open for non-circular Yabloesque paradoxes. Simply, if there is any way out of this paradox, it won't be with a pure Yablo paradox, but rather with a modified one.

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