# On Semi-implicit Splitting Schemes for the Beltrami Color Flow

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## Contribution

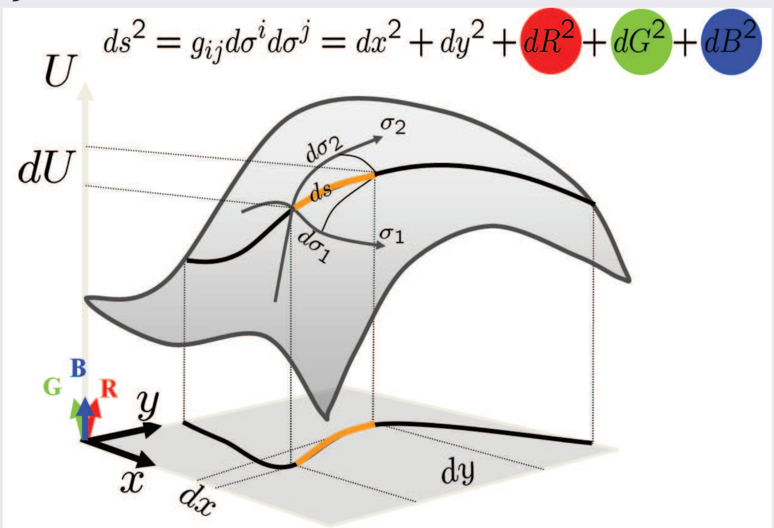
- The Beltrami color flow [1] and related anisotropic processes use explicit finite difference schemes.
- Explicit schemes are stable only for small time steps and require many iterations.
- Due to its non-separability and anisotropic nature, there are no multigrid or implicit schemes for the Beltrami color flow.
- We propose a modified semi-implicit splitting scheme in order to accelerate the convergence of the flow.

## The Beltrami Flow

- The Beltrami flow regards color images as 2-manifolds embedded in $\mathbb{R}^5$, $U(x,y) = (x, y, R(x,y), G(x,y), B(x,y))$
- This manifold is equipped with the induced metric

$$(g_{ij}) = \begin{pmatrix} 1+\beta^2\sum_a (U^a_{x_1})^2 & \beta^2\sum_a U^a_{x_1}U^a_{x_2} \\ \beta^2\sum_a U^a_{x_1}U^a_{x_2} & 1+\beta^2\sum_a (U^a_{x_2})^2 \end{pmatrix},$$

with $\beta$ defining the chromal-spatial aspect ratio, and $a \in \{R, G, B\}$.

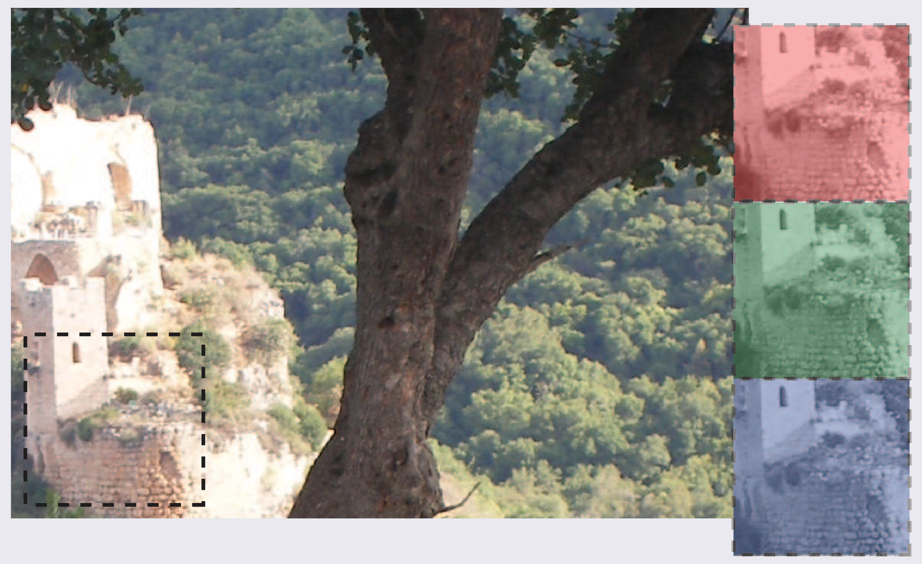



The functional we seek to minimize is

$$S(U) = \int \sqrt{g}\, dx^1\, dx^2$$

$$= \int \sqrt{1 + \beta^2 \sum_a |\nabla U^a|^2 + \frac{\beta^4}{2}\sum_{a,b} |\nabla U^a \times \nabla U^b|^2}\, dx^1\, dx^2$$

- This functional enforces the assumptions of the Mondrian world image formation model with Lambertian lighting.
- The color intensity of each surface in the image is determined according to $U^a = \rho^a\langle N, l\rangle$. $N$ is the surface normal, $l$ is the light direction, and $\rho^a$ is the albedo coefficient.
- According to the Lambertian model, color gradients should agree, and $\|\nabla U^a \times \nabla U^b\|$ should be small.



## Splitting Scheme Used

The modified Euler-Lagrange equations are
i) for denoising:

$$U^a_t = \Delta_g U^a - \frac{\alpha}{\sqrt{g}}(U^a - F^a),$$

$$\Delta_g U^a = \frac{1}{\sqrt{g}} div\left(\sqrt{g}G^{-1}\nabla U^a\right)$$

ii) for deblurring:

$$U^a_t = \Delta_g U^a - \alpha\bar{k} * (k * U^a - F^a).$$

In order to stabilize the flow in practice and use larger time steps, we employ a Crank-Nicolson scheme. The semi-implicit part is inverted by either AOS or LOD.

$$(U^a)^{n+1} = \left((1+\Delta t\frac{\lambda}{\sqrt{g^n}})I - \frac{\Delta t}{2}\bar{A}^n_{22}\right)^{-1} \left((1+\Delta t\frac{\lambda}{\sqrt{g^n}})I - \frac{\Delta t}{2}\bar{A}^n_{11}\right)^{-1} \left[\left((I+\frac{\Delta t}{2}\bar{A}^n_{11})(I+\frac{\Delta t}{2}\bar{A}^n_{22}) + \Delta t\sum_{q=1}^{2}\sum_{r\neq q}\bar{A}^n_{qr}\right)(U^a)^n + 2\Delta t F^a\frac{\lambda}{\sqrt{g^n}}\right].$$

Where $\bar{A}^n_{ij}$ denote the partial derivative components of the flow, $\bar{A}^n_{ij} = \frac{\partial}{\partial x_i}\left(g^{ij}\frac{\partial}{\partial x_j}(\cdot)\right)$.

- A slightly different formulation was required in order to stabilize the flow when a large fidelity term was used.
- As the explicit part becomes more dominant, simply moving the fidelity term to the semi-implicit part did not give satisfactory results.
- A straightforward trick was used to re-normalize the matrices and to better handle cases involving a fidelity term.

Denote $S^n = \left(1 + 2\Delta t\frac{\lambda}{\sqrt{g^n}}\right)I$. The resulting LOD scheme becomes

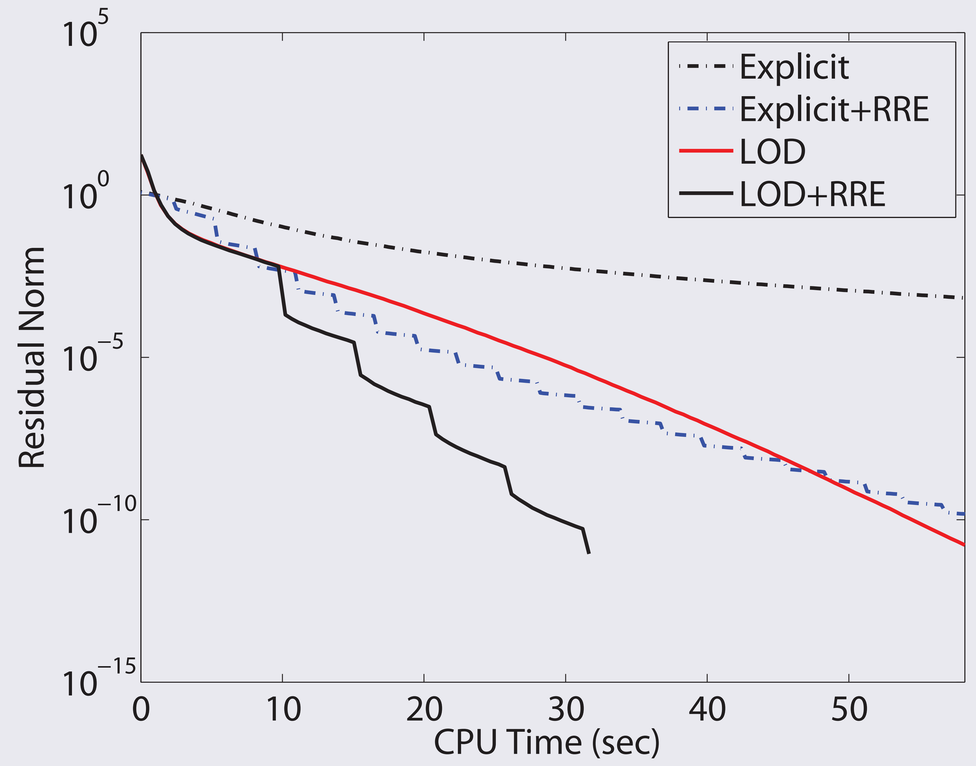
$$(U^a)^{n+1} = \left(I - \frac{\Delta t}{2}(S^n)^{-1}\bar{A}^n_{22}\right)^{-1}\left(I - \frac{\Delta t}{2}(S^n)^{-1}\bar{A}^n_{11}\right)^{-1} \left[(S^n)^{-1}\left((I+\frac{\Delta t}{2}\bar{A}^n_{11})(I+\frac{\Delta t}{2}\bar{A}^n_{22}) + \Delta t\sum_{q=1}^{2}\sum_{r\neq q}\bar{A}^n_{qr}\right)(U^a)^n + 2(S^n)^{-1}\Delta t F^a\frac{\lambda}{\sqrt{g^n}}\right].$$

## References

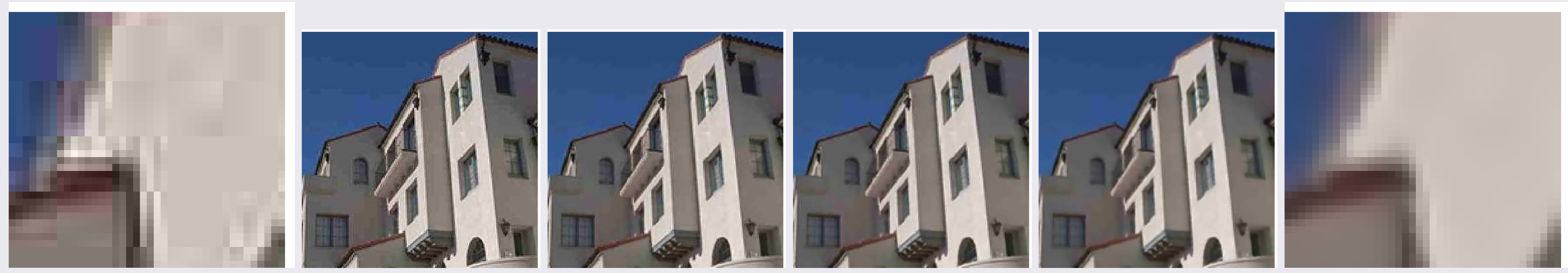
[1] R. Kimmel, R. Malladi, and N. Sochen. Images as embedding maps and minimal surfaces: Movies, color, texture, and volumetric medical images. *Intl. J. of Comp. Vision*, 39 (2):111–129, 2000.

[2] G. Rosman, L. Dascal, R. Kimmel, and A. Sidi. Efficient beltrami image filtering via vector extrapolation methods. *SIAM J. Imag. Sci.*, 2008. submitted.

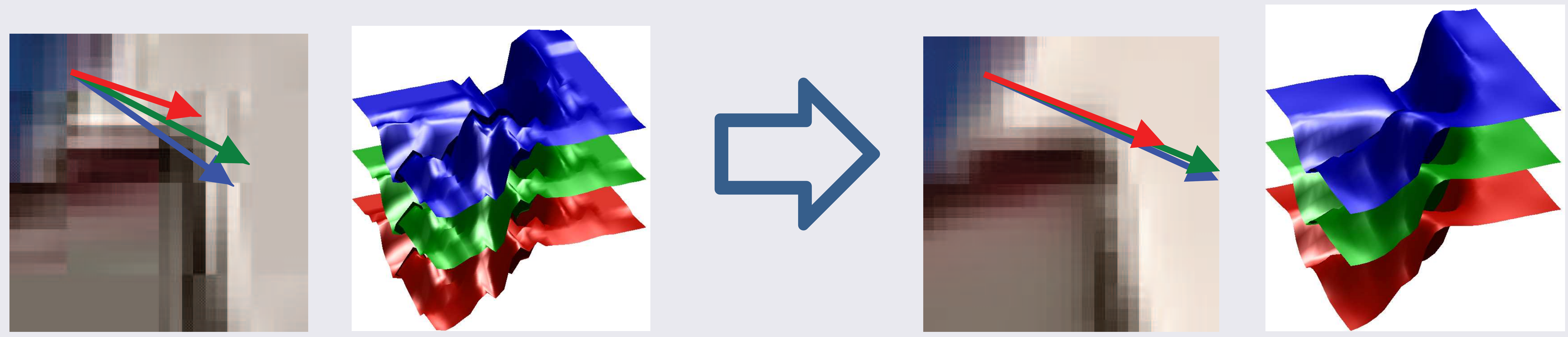
## Results



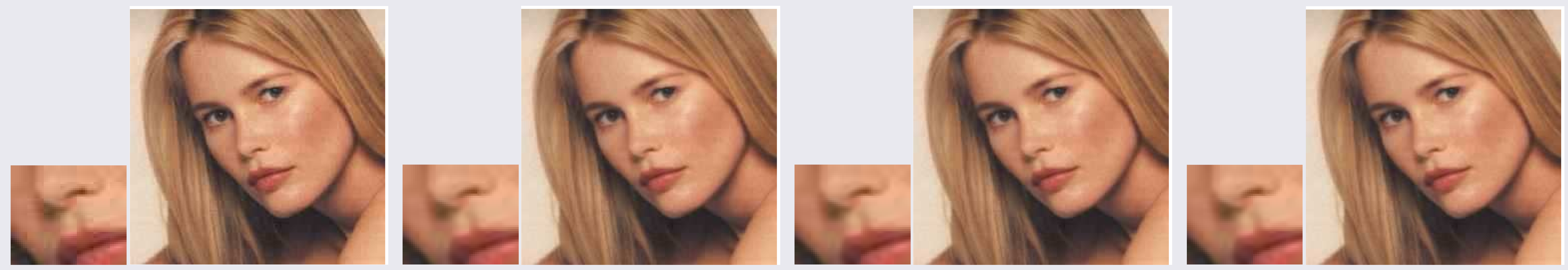

- The proposed scheme allowed us to significantly increase the time steps used, resulting in a more efficient implementation, despite mixed derivative components.
- The modification to the scheme helps stabilize convergence in practice even when a large fidelity term is involved.
- The resulting scheme can be combined with extrapolation techniques [2] for further speedup, or a better speed/accuracy trade-off.
- This can be seen in the figure to the right, in the graph of the residuals for the various schemes (explicit, LOD, explicit+RRE and LOD+RRE) versus CPU times.
- Parameters: $\Delta t = 0.05$ for the explicit scheme, $\Delta t = 2.5$ for LOD, $\lambda = 0.5$, $\beta = \sqrt{500} \approx 22.36$.



The 4 central images, left to right The original image which contains JPEG artifacts, and results of the LOD splitting scheme with $\Delta t = 1$, after 1, 2 and 4 iterations, $\beta = \sqrt{10^3}$, $\lambda = 0$. The outermost images give a close-up of the original image and the resulting image after 4 iterations.



Left to right: A small image patch and its surface representation before and after denoising.



Left to right: An image with lossy compression artifacts, the reference image – Beltrami-based denoising by explicit scheme, run with 4000 explicit iterations, $\Delta t = 0.0005$, denoising by LOD, $\Delta t = 0.02$, denoising by AOS, $\Delta t = 0.02$. $\lambda = 1$, $\beta = \sqrt{2000}$.

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