Augmented Lagrangian for Polyakov Action Minimization in Color Images

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INTRODUCTION

Variational, nonlinear diffusion filters have been extensively used for different image processing tasks. While many of them are tuned towards greyscale images, a small portion has a specific meaning for vector-valued signals such as color images.

The Beltrami framework [1] describes such a regularizing functional. Images in this framework are described as 2-manifolds embedded in a hybrid spatial-feature space. They are regularized by minimizing the manifold area measure. The resulting Beltrami filter is related to the bilateral filter (see [2], [3], [4]), and to the nonlocal means filter [5]. Minimization of the associated functional is usually done by evolving the image according to its Euler-Lagrange equation [1] with an explicit scheme. This scheme is limited in its time-step, resulting in high computational complexity. Another possibility [6] is to perform fixed-point iterations based on the Euler-Lagrange equation. Recently, several approaches were suggested for improving the speed of the minimization. These include a finite time approximation of the Beltrami filter kernel [7], vector extrapolation techniques [8], and operator splitting methods [9].

In [10], the augmented Lagrangian method [11, 12] is used to perform TV regularization of images. We propose to use a similar constrained optimization for regularization of color images. Instead of discretizing the continuous optimality condition or the resulting Beltrami flow, we minimize the discretized functional itself. The resulting method is shown to be more efficient and accurate for image denoising and deblurring, compared to existing methods for Beltrami regularization in image processing.

THE BELTRAMI FRAMEWORK

We now briefly review the Beltrami framework for non-linear diffusion in computer vision [13, 1, 14]. The basic notions used in this introduction are taken from Riemannian geometry, and we refer the reader to [15] for further reading.

In the Beltrami framework, images are expressed as maps between two Riemannian manifolds. Denote such a map by $X : \Sigma \to M$, where Σ is a two-dimensional manifold, parameterized by global coordinates (σ^1, σ^2), and M is the spatial-feature manifold, embedded in $\mathbb{R}^{d+2}$, where d is the number of image channels. For example, for color images, X is given by $X(\sigma^1,\sigma^2) = (\sigma^1,\sigma^2,I^1(\sigma^1,\sigma^2),I^2(\sigma^1,\sigma^2),I^3(\sigma^1,\sigma^2))$, where I^1,I^2,I^3 are the color components (for example, R,G,B, for color images). The canonical choice of coordinates in image processing uses Cartesian coordinates $\sigma^1,\sigma^2 = x,y$.

We now define a measure on this space of embedding maps. Our measure is the area of the resulting embedded manifold, expressed in terms of coordinates on the manifold Σ. This functional, for $\dim(\Sigma) = 2$, is known in string theory as the Polyakov action [16], and extends the action functional from classical mechanics to the relativistic case.

In the case of color images, where both the spatial and the color spaces are assumed to be Cartesian, the functional becomes

$$S(X) = \int \sqrt{g}\, d\sigma^1\, d\sigma^2, \quad g = \det(G) = 1 + \beta^2 \sum_a \|\nabla I^a\|^2 + \frac{\beta^4}{2} \sum_{a,b} \|\nabla I^a \times \nabla I^b\|^2, \tag{1}$$

where ∇I^a denotes the gradient of one of the color channels, and the parameter $\beta > 0$ determines the ratio between the spatial and color coordinates. In the case of color images, $a, b \in \{R, G, B\}$.

The functional S is usually minimized by time evolution of the image according to the Euler-Lagrange equations (see [1] for a detailed derivation). Evolution according to these equations results in the Beltrami scale-space.

In the variational framework, the reconstructed image minimizes a functional of the form

$$\Psi = \frac{\alpha}{2} \sum_a \|KI^a - I_0^a\|^2 + S(X),$$

where K is a bounded linear operator. In the denoising case, K is the identity operator $Ku = u$, and in the deblurring case, $Ku = k * u$, where $k(x,y)$ is the blurring kernel. We denote by I^a and I_0^a the color channels of the solution image and noisy input image, respectively. The parameter α controls the smoothness of the solution. This functional has been used for various image processing applications [1, 17, 18, 6]. We introduce an approach for optimizing the functional Ψ using the augmented Lagrangian method.

AN AUGMENTED LAGRANGIAN APPROACH FOR BELTRAMI REGULARIZATION

In recent years, several attempts have been made of optimizing total variation (TV) functionals [19] using dual variables (we refer the reader to [20, 21, 22, 23, 24, 10, 25] and references therein, as well to the technical report [26]). These algorithms are considered to be among state-of-the-art methods for TV restoration.

Specifically, in [10], TV regularization is obtained by decoupling the optimization problem into a constrained optimization problem

$$\min_{u, \mathbf{q}} \int |\mathbf{q}| + \frac{\alpha}{2} \|Ku - f\|^2 \quad s.t. \quad \mathbf{q} = \nabla u, \tag{2}$$

where $\mathbf{q}$ is a auxiliary field approximating the gradient of u. The constraint $\mathbf{q} = \nabla u$ is incorporated using an augmented Lagrangian penalty function of the form $\rho_{\mu,r}(u, \mathbf{q}) = \mu^T (\nabla u - \mathbf{q}) + \frac{r}{2} \left(\|\nabla u - \mathbf{q}\|^2\right), r > 0$, where μ approximates the Lagrange multiplier. The penalty term enforces the constraint, without making the problem severely ill-conditioned.

We now describe a similar construction for the Polyakov action. Again, it is important to stress we are minimizing the functional itself, rather than discretizing the resulting minimizing PDE as in [1, 17, 7, 6, 8, 9].

Specifically, we replace the gradient norm penalty used in TV regularization by the action functional of Equation (1). This is done by replacing the first term in Equation (2) by a term expressing an area element of the image manifold in term of $\mathbf{q}$. We then trivially extend the rest of the functional to the vectorial (per-pixel) case, obtaining the following functional

$$\mathscr{L}_{BEL}(u, \mathbf{q}, \mu) = \int \left\{ \begin{array}{c} \sqrt{1 + \beta^2 \sum_a \|\mathbf{q}_a\|^2 + \frac{\beta^4}{2} \sum_a \sum_{b \neq a} \|\mathbf{q}_a \times \mathbf{q}_b\|^2} \; + \\ \sum_a \mu_a^T (\mathbf{q}_a - \nabla u_a) + \frac{\alpha}{2} \|Ku - f\|^2 + \frac{r}{2} \sum_a \|\mathbf{q}_a - \nabla u_a\|^2 \end{array} \right\} dxdy,$$

where β is the spatial-intensity aspect ratio, and $\{\mathbf{q}_a\}$ denote components of the field $\mathbf{q}$, parallel to the gradient of each of the image channels. The expressions optimizing u and μ are replaced by their per-channel equivalents, $\{u_a\}$ and $\{\mu_a\}$. The augmented Lagrangian algorithm for regularizing an image using the Polyakov action is given as Algorithm 1.

Algorithm 1 Augmented Lagrangian optimization of the Beltrami framework

1: $\mu^0 \longleftarrow 0$
2: **for** k=0,1,... **do**
3: Update $\{u_a\}^k, \{\mathbf{q}_a\}^k$:

$$\left(\{u_a\}^k, \{\mathbf{q}_a\}^k\right) = \operatorname{argmin}_{\{u_a\},\{\mathbf{q}_a\}} \mathscr{L}_{BEL}(\{u_a\}, \{\mathbf{q}_a\}, \{\mu_a^k\}) \tag{3}$$

by updating u_a in the Fourier domain, and updating $\mathbf{q}_a$ according to Equation (5).
4: Update the Lagrange multipliers according to Equation (4)
5: **end for**

At each inner iteration k, $\{u_a\}$ is updated in the Fourier domain, as in [10], according to the EL equation. The Lagrange multipliers μ_a are updated, according to the augmented Lagrangian method, by

$$(\mu_a)^k = (\mu_a)^{k-1} + r\left((\mathbf{q}_a)^k - (\nabla u_a)^k\right). \tag{4}$$

Finally, the coefficient r is updated between each outer iteration by multiplying r with a scalar $\gamma > 1$. We note r needs not become very large, thus avoiding ill-conditioning of the functional $\mathscr{L}_{BEL}(u, \mathbf{q}, \mu)$.

Updating the auxiliary field q

For optimizing $\mathbf{q}$, a short inner-loop of a fixed-point solver with *iterative reweighted least squares* (IRLS) allows us to efficiently obtain a solution. In numerical experiments, optimization over $\mathbf{q}$ takes less than half the CPU time of the algorithm. Furthermore, since this problem is solved per pixel, it can be easily parallelized, for example on a GPU. The update of $\mathbf{q}_a = (p_a, q_a)$, the components of $\mathbf{q}$ at each pixel, is done by optimizing the function

$$\sqrt{1+\beta^2\sum_a (p_a^2+q_a^2) + \frac{\beta^4}{2}\sum_a\sum_{b\neq a}(p_a q_b - q_a p_b)^2} + \frac{r}{2}\sum_a \|\mathbf{q}_a - (\nabla u_a)\|^2 + \sum_a (\mu_a^k)^T(\mathbf{q}_a - \nabla u_a),$$

where $(\nabla u)_a = ((u_a)_x, (u_a)_y)^T$ denote the components of the various image channel gradients. Each fixed-point sub-iteration updates the elements of $\mathbf{q}_a$ according to the IRLS approach, by replacing the square root with a weighted version of the quadratic expression inside. Thus, the equation used to update p_a is of the form

$$\frac{2\left(\frac{1}{\beta^2}+\frac{1}{2}\sum\limits_{b\neq a}\left(q_b^l\right)^2\right)p_a + \frac{1}{2}\left(-\sum\limits_{b\neq a}\left(q_b^l\right)\left(p_b^l\right)\left(q_a^l\right)\right)}{\sqrt{\frac{1}{\beta^4}+\frac{1}{\beta^2}\sum\left(\left(p_a^{l-1}\right)^2+\left(q_a^{l-1}\right)^2\right)+\sum\limits_a\sum\limits_{b\neq a}\left(\left(p_a^{l-1}\right)\left(q_b^{l-1}\right)-\left(q_a^{l-1}\right)\left(p_b^{l-1}\right)\right)^2}} + r\left(\left(p_a^l\right)-(u_a)_x\right)+(\mu_a^k)_x = 0, \tag{5}$$

and similarly for q_a, where l denotes the IRLS iteration number. These iterations resemble the lagged diffusivity approach for TV regularization [27].

RESULTS

We now demonstrate the minimization of the Polyakov functional using the augmented Lagrangian method, for image denoising. More examples, including deblurring examples are shown in the Technical report [26]. A comparison of the results of the augmented Lagrangian method and splitting schemes [9] shows that the augmented Lagrangian method converges faster, as can be seen in Figure 2. In this experiment, α was set for optimal results for both the augmented Lagrangian and the splitting methods. The *peak signal-to-noise ratio* (PSNR) plot also demonstrates the more accurate discretization obtained in practice by the proposed method. Tested on a variety of images, the speedups obtained are by at least of a factor of two compared to additive operator splitting (AOS) [28], one of the fastest methods for Beltrami regularization [9], even when time-steps in the AOS method were large enough to cause visible artifacts in the solution.

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CONCLUSIONS

We present an extension of the augmented Lagrangian method for color image processing with Beltrami regularization. Unlike existing Beltrami regularization techniques, we discretize the functional itself. Our experiments demonstrate the algorithm's efficiency and accuracy compared to existing methods for Beltrami regularization.

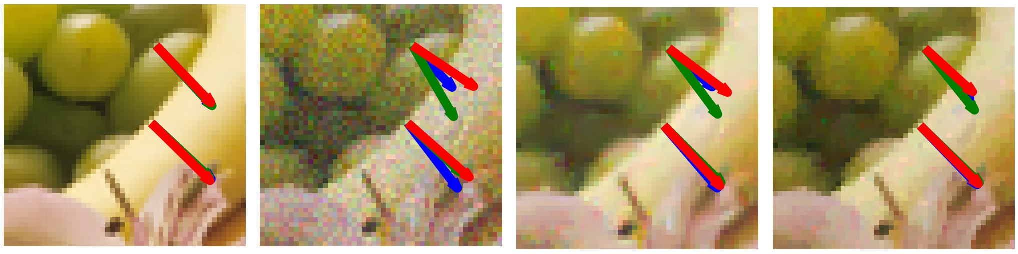

FIGURE 1. A comparison of the results for the AOS scheme, and the augmented Lagrangian method.

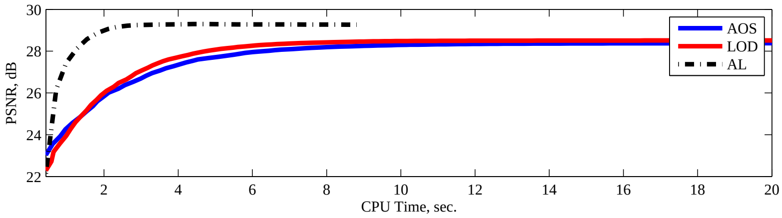

FIGURE 2. Comparison in terms of PSNR vs. CPU-time of the AOS scheme, and the augmented Lagrangian method

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