Augmented Lagrangian Regularization for Non-Standard Data

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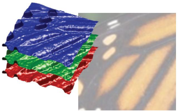
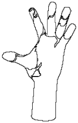




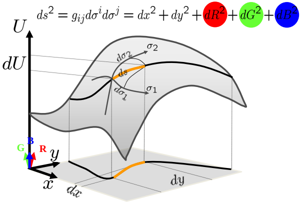
Motivation

- Augmented Lagrangian regularization have proven useful for total-variation (TV) regularization of scalar and color image.
- We demonstrate augmented-Lagrangian regularization in a variety of other applications, such as
  - Beltrami color image regularization,
  - Motion segmentation, and
  - DTI reconstruction.

The Beltrami Framework

- Treats color images as a 2-manifold embedded in a spatial-color space $\mathbb{R}^5$
- Irregularity defined by area of the embedded manifold.
- The Beltrami color flow and related anisotropic processes in image processing use finite, or semi-implicit difference schemes.

$$U : \Sigma \to \mathbb{R}^5, U\left(x,y\right) = \left(x, y, \beta U^1\left(x,y\right), \beta U^2\left(x,y\right), \beta U^3\left(x,y\right)\right)$$



The Beltrami Framework

- The area term can be written as

$$S(U) = \int \sqrt{g}\,dxdy = \int \sqrt{\det(G)}\,dxdy =$$

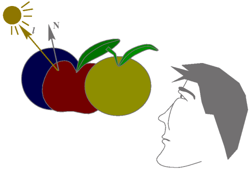
$$\int \sqrt{1 + \beta^2 \sum_{a=1}^{3} \left\|\nabla U^a\right\|^2 + \frac{1}{2}\beta^4 \underbrace{\sum_{a,b=1}^{3} \left\|\nabla U^a \times \nabla U^b\right\|^2}_{\text{color alignment}}}\,dxdy$$

(known as the Polyakov action functional)

- β - An intensity / space aspect ratio
- For $\beta \to 0$, the flow degenerates into a decoupled channel by channel linear diffusion
- For $\beta \to \infty$, we get a non-linear flow

The Beltrami Framework

- Piecewise constant model - *Mondrian world* [Land and McCann, 1971]



- Assume a Lambertian reflection model for each object

$$U^a(x,y) = \rho^a(x,y)\langle N(x,y), l(x,y)\rangle$$

- $N(x,y)$ - the unit normal to the image surface
- $l(x,y)$ - light source direction, approximately constant $l(x,y) \approx l$
- Assume piecewise constant albedo: $\rho^a(x,y) = \rho^a$

The Beltrami Framework

- Under the Lambertian model assumptions, we have

$$\left\|\nabla U^a \times \nabla U^b\right\|^2 = 0 \quad \forall a \neq b$$

  which means the gradients directions should agree.

- Minimizing the term $\left\|\nabla U^a \times \nabla U^b\right\|^2$ in the functional enforces the Lambertian model for every smooth surface patch.

  The alignment term is an outcome of the color image formation model.

Augmented-Lagrangian Total-Variation

- In the case of total-variation, several optimization schemes are available for the functional

  ([Chan/Golub/Mulet '99],[Chambolle '04],[Wang et. al '08],[Bresson and Chan '09],[Wu/Zhang/Tai '09] among others)

- Many of them use an auxiliary field for the gradient of the image, resulting in fast minimization procedures for the cost function.
- For example, for TV regularization, [Wu and Tai '09] minimize

$$\int \rho_{TV}(q) + \frac{\alpha}{2}(k * f - u)^2 + \mu^T(q - \nabla u) + \frac{r}{2}\|q - \nabla u\|^2 =$$

$$\int |q| + \frac{\alpha}{2}(k * f - u)^2 + \mu^T q - \langle div\mu, u\rangle + \frac{r}{2}\left(q^2 - 2\langle divq, u\rangle + \|\nabla u\|^2\right)$$

  - $\rho_{TV}(q)$ - TV regularization prior
  - μ - Lagrange multipliers, q - the auxiliary field
  - k - the blur kernel, f - the original image

The Beltrami Framework

- We take a similar path, minimizing

$$\int \rho_{BEL}\left(q\right) + \frac{\alpha}{2}(k * f - u)^2 + \mu^T\left(q - \nabla u\right) + \frac{r}{2}\|q - \nabla u\|^2$$

- $\rho_{BEL}\left(q\right)$ - Beltrami regularization prior
- Update by alternating over u, q, μ
- Update u either by SOR by in Fourier domain

The Beltrami Framework

- Updating w.r.t u, the Euler-Lagrange equation become

$$\alpha k^* * (k * f - u) + (div\mu - rdivq) + rdiv\left(\nabla u\right) = 0$$

- Update by alternating over u, q, μ
- Update u either by SOR by in Fourier domain

Updating the auxiliary field

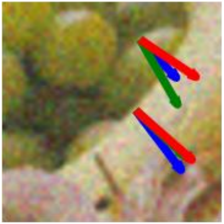
- We seek to optimize the (partial) expression

$$\sqrt{1+\beta^2\sum_i\left(\left(p_i^l\right)^2+\left(q_i^l\right)^2\right)+\frac{\beta^4}{2}\sum_i\sum_{j\neq i}\left(\left(q_j^l\right)\left(p_i^l\right)-\left(q_i^l\right)\left(p_j^l\right)\right)^2}+$$
$$\mu^T\left(\begin{pmatrix}q_i\\p_i\end{pmatrix}-\nabla u\right)+\frac{r}{2}\left\|\begin{pmatrix}q_i\\p_i\end{pmatrix}-\nabla u\right\|_2^2$$

- Solve for the aux. field q by IRLS over each component of the gradient.
- Typically, only a few (1-2) inner iterations suffice.

Channel Consistency

Original Image

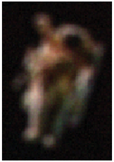


Gaussian Noise $\sigma = 20$

Denoised Image

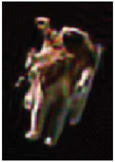
Image Deblurring

Original Image



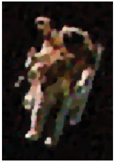
Blurred+ Noisy

Lucy - Richardson



BM3D Restored

FTVD Restored



Beltrami Restored

FTVD Restored

Beltrami Restored

FTVD Restored

Beltrami Restored

Image Deblurring

Blurred + Noisy

Beltrami Restored

Image Deblurring

Blurred +
Noisy

Beltrami
Restored

Fast Matrix-valued Regularization for Motion Segmentation

- Another important aspect of regularization is that of matrix- / group- valued regularization.
- We describe a fast algorithm for matrix-valued regularization.
- We demonstrate several applications of this algorithm in motion segmentation, orientation denoising and diffusion tensor images (DTI) reconstruction.

Group-valued Regularization for Motion Segmentation

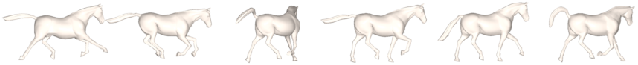
- One example for group-valued regularization - articulated motion segmentation and estimation on depth video.
- We have 2 or more partial scans of the same $3D$ object, $\mathcal{S}_1, \mathcal{S}_2, \cdots$.
- We would like to obtain the rigid segments and their transformations with minimal additional assumptions.

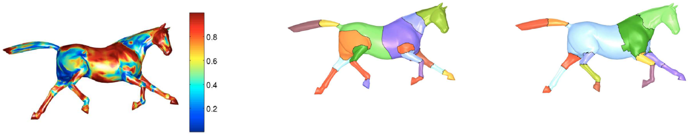
Articulated Motion Segmentation

- Many algorithms are available for articulated motion segmentation:
  - [Anguelov et al., UAI, '04] define EM steps over labeling and transformations,
  - [James et al., SIGGRAPH '05] perform mean-shift clustering,
  - [Huang et al., SIGGRAPH'08] perform clustering and deformation of the cluster points,
  - [Thierny et al., ICPR '08] measure changes in edge-lengths,
  - [Wuhrer and Brunton, VC '10] perform a segmentation of the dual graph based on the dihedral angles on neighboring triangles.
- Many algorithms incorporating cues other than motion in segmentation.

Articulated Motion Segmentation

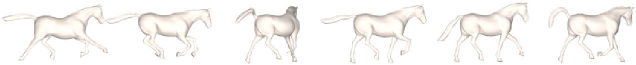
- We seek an implicit, bottom-up approach to segmentation,
- Unknown number of rigid parts, rigid parts are not always well-defined.
- Inside a rigid part - all points undergo the same rigid transformation.
- This leads to edge-preserving regularization of maps from the object' surface to the group of rigid motions ($SE(3)$)

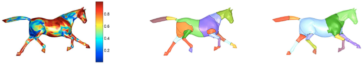




Articulated Motion Segmentation

In previous work ([R., Bronstein, Bronstein, Wolf, Kimmel, SSVM'11],[R., Bronstein, Bronstein, Kimmel, 3DOR'12]) - we formulated as Mumford-Shah segmentation of $SE(3)$-valued maps on surfaces.





Fast Regularization of Matrix-Valued Images

- Slow to compute - requires explicit iterations using the exponent map. We want an efficient solution.
- We have depth images as data in many cases - Cartesian grids are available.
- We re-formulate the problem as edge-preserving regularization for matrix-valued images.

Fast Regularization of Matrix-Valued Images

Matrix/group valued data is everywhere!

- Robotics and motion ([Park et al., '95], [Zefran and Kumar, '98], [Tuzel et al., '08], [Kobilarov et al., '09])
- Computer vision ([Perona, '98], [Tang and Sapiro '00], [Sochen and Kimmel '01], [Brox and Weickert '02])
- Medical images ([Stejskal and Tanner '65], [Basser et al., '94], [Tschumperle and Deriche '05], [Burgeth et al., '07], [Franken et al. '07], [Gur and Sochen '09], many more)

Functional Description

1. Instead of total-variation regularity term defined according to the Lie-algebra:

$$\min_{u \in \mathcal{G}} \int \|u^{-1}\nabla u\| + \lambda \|u - u_0\|^2 dx,$$

2. We simplify (Justified for isometry Lie-groups):

$$\min_{u \in \mathcal{G}} \int \|\nabla u\| + \lambda \|u - u_0\|^2 dx,$$

(Also possible for $SPD(n)$ images, and other matrix manifolds)

Another option – second-order regularization

$$\min_{u \in \mathcal{G}} \int \|Hu\| + \lambda \|u - u_0\|^2 dx,$$

Algorithm Description

3. We now add an auxiliary variable $v \in \mathcal{G}$, and constrain $u = v$ (but remove the restriction on $u \in \mathcal{G}$).

4. We constrain v to equal u using an augmented Lagrangian term,

$$\min_{v \in \mathcal{G},\ \ u} \int \|\nabla u\| + \lambda \|u - u_0\|^2 + \langle \mu, u - v \rangle + \frac{r}{2} (u - v)^2 dx,$$

5. Vectorial total-variation denoising w.r.t. u - (we used Wen/Goldfarb/Yin '09, Tai '09, other possibilities as well)
   (We can use other quadratic fitting/reconstruction terms)

Algorithm Description

6. Optimizing w.r.t. v under constraint is a projection step per pixel,

$$\underset{v \in \mathcal{G}}{\operatorname{argmin}} \langle \mu, u - v \rangle + \frac{r}{2}(u-v)^2 = \underset{\mathcal{G}}{\operatorname{Proj}}\left(u + \frac{\mu}{r}\right) \qquad (1)$$

Algorithm Description

The overall algorithm is as follows
Repeat for $k = 0, 1, \ldots$ until convergence

1. update $F(u, v, \mu)$ w.r.t u - fast TV regularization
2. update $F(u, v, \mu)$ w.r.t v using eq. 1
3. update The lagrange multipliers using

$$\mu^k = \mu^{k-1} + r\left(v^k - u^k\right) \tag{2}$$

Convergence Properties

We look at the case where the Lagrange multipliers are fixed,

- The projection operator is discontinuous, making a straightforward convergence proof difficult
- Instead, we slightly modify the algorithm (as suggested by Attouch et al. , '10)

$$(u^{k+1}, v^{k+1}) = \underset{v \in \mathcal{G},\, u}{\operatorname{argmin}} \int \begin{array}{c} \|\nabla u\| + \lambda \|u - u_0\|^2 + \langle \mu, u - v \rangle + \frac{r}{2}(u - v)^2 \\ + \frac{1}{\theta}\|u - u^k\|^2 + \frac{1}{\theta}\|v - v^k\|^2 \end{array} dx,$$

We can show convergence given μ.

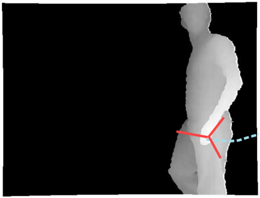
Results

Turns out, this formulation is important in several applications:

1. $SE(3)$ - Segmentation and motion scale-space from depth sensors
2. $SO(2)$ - Direction diffusion ($SO(2)$ is isomorphic to S^1)
3. $SPD(3)$ - DTI regularization and regularized reconstruction

Motion Scale-space

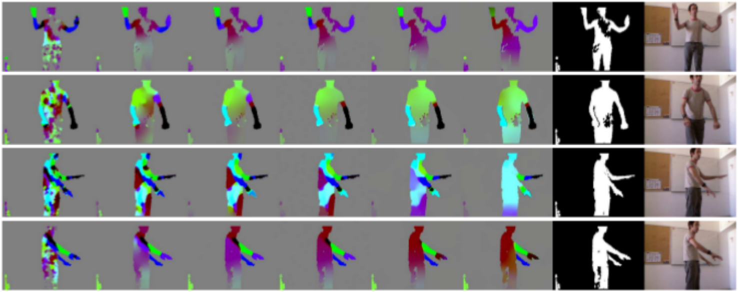
1. 3D frames are taken with a depth sensors.



2. Local rigid motion is computed at each point
3. $SE(3)$ diffusion scale-space is used to discern major moving parts.
4. Fast regularization: for a 320×240 image, convergence takes $49ms$ (GTX 580).

Motion Scale-space

Simple (local) motion estimation:

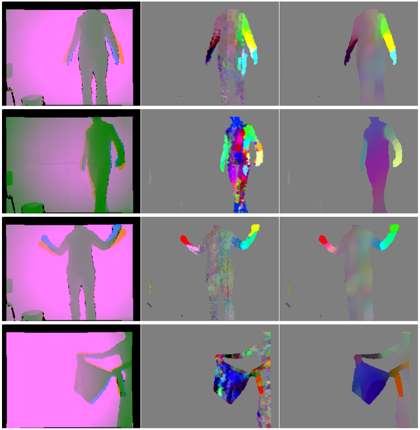


(Different colors visualize different rigid motions)

Fast regularization: for a 320×240 image, convergence takes $49ms$ (GTX 580).

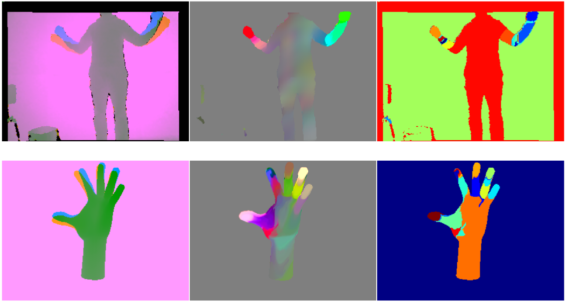
Motion Scale-space

Using linear nonrigid deformation for initial motion:



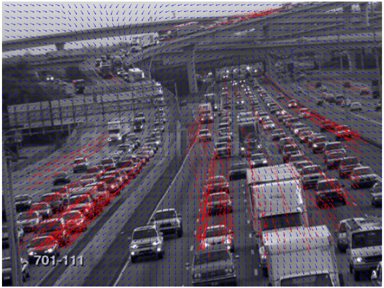
Motion Scale-space

But, can we really get segmentation out of it, in an implicit manner?
Yes we can – allows to obtain articulated parts (for example, using mean-shift clustering)



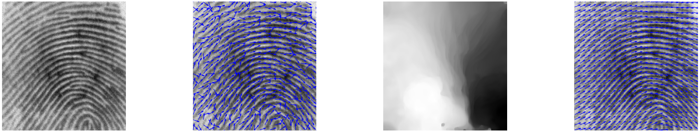
Orientation Diffusion

1. Another example - diffusion of orientations, since $SO(2)$ is homeomorphic to S^1.
2. Tracklets are gathered from a stationary surveillance camera, using a simple correlation tracker.
3. We obtain a map of principle Motion directions.



Orientation Diffusion

1. Another example - fingerprint curve smoothing



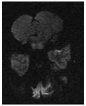
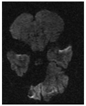
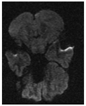
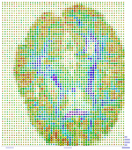
MSE for noised fingerprint image was
0.0270 for the second-order regularization,
0.0317 for the 1st-order / TV functional,
0.0324 by [SK'01],
0.0449 in the original image.

Diffusion Tensor Imaging

1. Another application - diffusion-tensor images (DTI) reconstruction (and denoising)
2. The convex domain allows us to show global convergence
3. Basically our data term comes from the Stejskal-Tanner equation

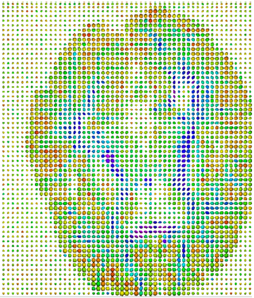
$$\underset{v\in SPD(n)}{\operatorname{argmin}} \int \begin{array}{c} \sum_i \left\| b_i g_i^T u g_i - \log\left(\frac{S_i}{S_0}\right)\right\|^2 + \\ \lambda\|\nabla u\| + \frac{r}{2}\|v-u\|^2 + \langle \mu, v-u\rangle dx. \end{array}$$

(g_i - measurement vectors, S_i - DWI measurement from MR scan i)

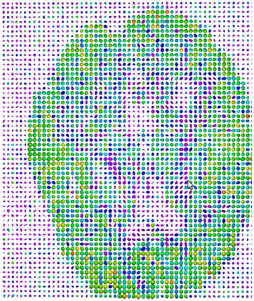
 ,  ,  .. $\xrightarrow{\textit{reconstruction}}$ 

Diffusion Tensor Imaging

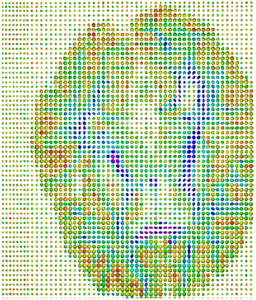
Here we show regularized reconstruction from a noisy (AGWN) set of diffusion-weighted images (linear reconstruction data term)



Original



Noisy Reconstruction



Regularized Recon.

Conclusion

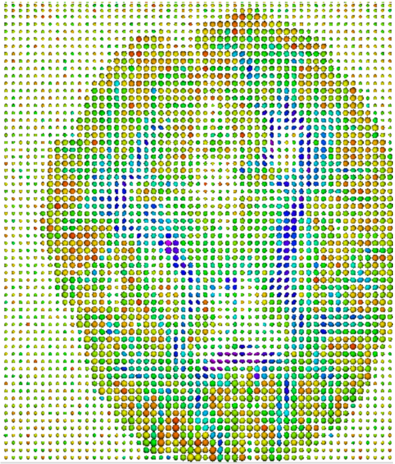
1. We formulate bottom-up articulated motion segmentation as matrix-valued TV regularization.
2. We obtain a simple and efficient algorithm for regularization of matrix-valued images in general.
3. Achieves real-time performance for bottom-up articulated motion segmentation with minimal assumptions.
4. Examples show its usefulness for several problems in computer vision / image processing.

Thank you for your attention!

Backup Slides

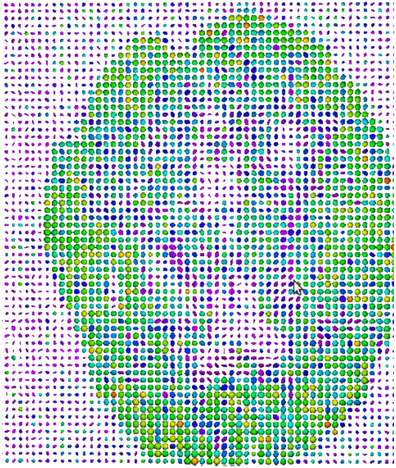
Diffusion Tensor Imaging

close-up: Original image



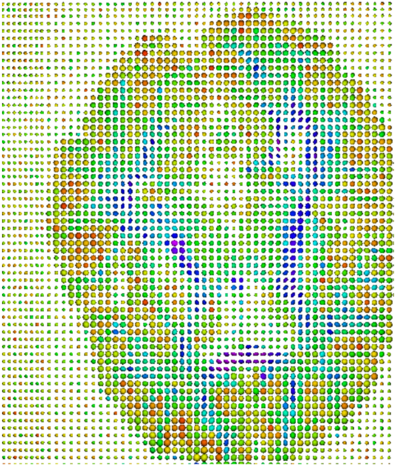
Diffusion Tensor Imaging

Reconstruction from noisy DWI



Diffusion Tensor Imaging

Regularized reconstruction



The Stejskal-Tanner equation

The Stejskal-Tanner equation

$$\frac{S}{S_0} = \exp\left[-\gamma^2 G^2 \delta^2 \left(\Delta - \frac{\delta}{3}\right) D\right]$$

where

- γ - the Gyromagnetic constant, G - gradient strength
- δ - pulse length
- Δ - inter-pulse interval
- D - the diffusion coefficient, according to $g^T ug$