Topological Simplification

Guy Rosman

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- In many signal and image processing algorithms, the main emphasis is on the geometric structure of the signal, or another aspect of it. Changes occuring in this structure are often modeled, or ignored.
- Sometimes, all you can say about a signal is in terms of connectivity and topology (for example – dimensionality reduction – in class).
- In this talk I present a glimpse of two algorithms aimed at taking topological insights into signal processing.
  - *Optimal topological simplification of discrete functions on surfaces*, Bauer, Lange and Wardetzky, arXiv.org, 2010.
  - *Total variation meets topological persistence: A first encounter*, Bauer, Schonlieb and Wardetsky, ICNAAM 2010.

The paper [1] handles discrete Morse functions defined on CW complexes.
A CW (closure-finite, weak-topology) complex is a much more general definition than an abstract simplicial complex.
The paper describes a solution for the following problem

Definition (Topological simplification on surfaces)

Given a function f on a surface, and $\delta > 0$, find a function f_δ subject to $\|f_\delta - f\|_\infty < \delta$ has a minimum number of critical points.

Main contribution:

- Treating multiple cancellations in the simplification process by suggesting that on a surface, there exists a nesting order for cancellation pairs.
- Treating degenerate functions via pseudo-morse functions
- As a post-processing - combining the discrete optimization and a continuous optimization to choose among feasible solutions.

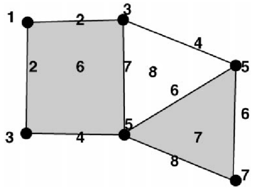
For discrete morse functions, we define:

A function $f : K \to R$ is a *discrete Morse function* on K if for every $\sigma^{(p)} \in K_p$, we have

$$\#\{\tau^{(p+1)} > \sigma \,|f(\tau) \leq f(\sigma)\} \leq 1 \qquad (1)$$

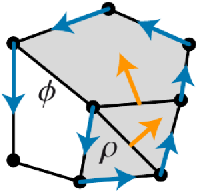
$$\#\{\nu^{(p-1)} < \sigma \,|f(\nu) \leq f(\sigma)\} \leq 1 \qquad (2)$$

Intuitively, at most one counterbalancing face $\tau^{(p+1)}$ of codimension 1 and one counterbalancing bounding cell $\nu^{(p-1)}$ for every cell $\sigma^{(p)}$.



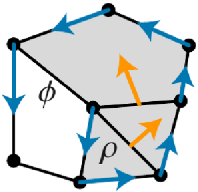
The paper goes through defining and simplifying the discrete gradient vector field associated with f.
A discrete vector field V on a regular CW complex K is a set of pairs of cells $(\sigma, \tau) \in K \times K$, where

- σ is a facet of τ.
- Each cell appears at most once in a pair in V.

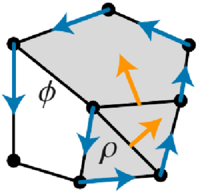


Similar to morse theory, we define in discrete morse theory:

- A cell $\sigma \in K$ is *critical* w.r.t. V if σ is not contained in any pair of V. For example: ϕ, ρ
- A V-path Γ from a cell σ_0 to a cell σ_r is a sequence $(\sigma_0, \sigma_1, \cdots, \sigma_r)$ of cells such that
  - σ_i is a facet of τ_i with $(\sigma_i, \tau_i) \in V$
  - σ_{i+1} is a facet of τ_i with $(\sigma_{i+1}, \tau_i) \notin V$



- A discrete gradient vector field V is a discrete vector field that contains no nontrivial closed V-paths.
- A discrete gradient vector field defines a partial order $\prec_V$ on the cells in K whenever σ is a facet of τ,
  - $(\sigma, \tau) \notin V$ implies $\sigma \leftarrow_V \tau$
  - $(\sigma, \tau) \in V$ implies $\sigma \rightarrow_V \tau$



Relating discrete Morse functions and their gradient

A function $f : K \rightarrow R$ is a discrete Morse function on K if there is a gradient vector field V_f such that whenever σ is a facet of τ,

- $(\sigma, \tau) \notin V_f$ implies $f(\sigma) < f(\tau)$ and
- $(\sigma, \tau) \in V_f$ implies $f(\sigma) \geq f(\tau)$.

V_f is known as the gradient vector field of f.

Informally, a discrete Morse function can be assigned a gradient vector field so that it mimics a Morse function in terms of its minima, maxima and the gradient paths connecting them.

Actually, we'd like to deal with pseudo-Morse functions

A function $f : K \to R$ is a discrete pseudo-Morse function on K if there is a gradient vector field V_f such that whenever σ is a facet of τ,

- $(\sigma, \tau) \notin V_f$ implies $f(\sigma) \leq f(\tau)$ and
- $(\sigma, \tau) \in V_f$ implies $f(\sigma) \geq f(\tau)$.

we call f and V consistent.

$\prec_V$ can be viewed as: for any pseudo-Morse function f consistent with V, $\sigma \prec_V \tau \to f(\sigma) \leq f(\tau)$.

Another interpretation for pseudo-Morse functions is that of approximating Morse functions:

- $f : K \to R$ is a discrete pseudo-Morse function on K iff for every $\epsilon > 0$ there is a Morse function $f_\epsilon : K \to R$ close enough (L_∞) to f.

This will allows treating degenerate functions on K as well as Morse functions, but in our case, where a total order of the critical cells is required, it's not enough.
Another step is needed, of constructing a function whose g.v.f. is the same as V, but that defines a total order.

Theorem ([1], Theorem 5)

Let V be a g.v.f. on $\mathcal{K}$, and $\prec$ a linear extension of $\prec_V$. If $\rho \prec \psi$ are two cells where there is no critical cell ϕ w.r.t V such that $\rho \prec \phi \preceq \psi$. Then ρ, ψ define homotopic order subcomplexes.

Given $f : \mathcal{K} \to \mathbb{R}$, a pseudo-Morse function, let the carrier of a subset $L \subset K$ be the smallest subcomplex of $\mathcal{K}$ containing L. We define the level subcomplex $\mathcal{K}(t)$ as the carrier of the cells whos value is below t:

$$\mathcal{K}(t) = carrier\left(\cup_{\rho \in K : f(\rho) \leq t} \rho\right)$$

Similarly, if $\prec$ is a strict total order on cells K of $\mathcal{K}$, then for a cell $\sigma \in K$, the order subcomplex is

$$\mathcal{K}(\sigma) = carrier\left(\cup_{\rho \in K : \rho \prec \sigma} \rho\right)$$

These both, approximately, represent sublevel sets in the discrete setting.

Removing persistence pairs in $\mathcal{K}$

The following theorem is quite handy:

Theorem (D. Attali et al., Theorem 4)

The persistence pairs of dimension $(p, p+1)$ in $\mathcal{F}^*$ correspond to pairs of dimension $(d-1-p, d-p)$ in $\mathcal{F}$,

or in our case, $(\mathcal{F}^* \rightarrow \mathcal{K}, p- > 1, d- > 2)$, we have
The persistence pairs of dimension $(1, 2)$ on $\mathcal{K}$ correspond to the $(0, 1)$ persistence pairs for the dual complex $\mathcal{K}^*$
Homology groups $H_0(\mathcal{K}(\rho_i))$ and the $(0, 1)$ persistence pairs are determined by the 1-skeleton (primal graph) of $\mathcal{K}$.

The proof of the simplification suggested in [1] hinges on reversing V-paths. Specifically:

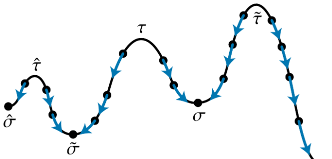
Theorem ([1],Theorem 1)

Let ϕ, ρ be two critical cells of a g.v.f. V with exactly one V-path Γ from $\partial\rho$ to ϕ. Then, there exists a g.v.f. $\tilde{V}$ obtained by reversing V along Γ. Critical cells of $\tilde{V}$ are sames as with V, except for ϕ, ρ. Furthermore, $V = \tilde{V}$ except on Γ.

Cancelling one pair is okay, but how do we use [[1], Theorem 1] to remove all small pairs?

Definition (descendant cells)

Consider a pair (σ, τ) on a combinatorial surface $\mathcal{K}$, with $dim\sigma = 0$, and $[\sigma] = H_0(\mathcal{K}(\sigma))$ be the equivalence class of σ.
Let $\tilde{\sigma}$ be the unique cell creating the class $[\tilde{\sigma}] = H_0(\mathcal{K}(\tau))$ into which σ is merged by τ.
Then $\tilde{\sigma}$ is called the parent of σ and σ is the child of $\tilde{\sigma}$. The transitive closure of the child relation is called a descendant.



Definition (nested pairs)

Let (σ, τ) and $(\tilde{\sigma}, \tilde{\tau})$ be two persistence pairs such that $\tilde{\sigma} \prec \sigma \prec \tau \prec \tilde{\tau}$. We say that (σ, τ) is nested in $(\tilde{\sigma}, \tilde{\tau})$

Lemma

Let (σ, τ) be a descendant of $(\tilde{\sigma}, \tilde{\tau})$. Then (σ, τ) is nested in $(\tilde{\sigma}, \tilde{\tau})$.

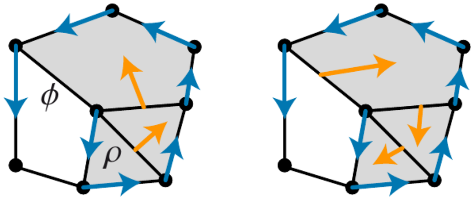
And, the main claim:

Theorem ([1],Theorem 13)

Let f be a pseudo-Morse function on a combinatorial surface $\mathcal{K}$ and $\delta > 0$, then there exists a nested delta-persistence cancellation sequence.

(this is used only for the proof, not in the actual algorithm).

Hence, we can use [[1], Theorem 1] and find several V-paths that can be reversed in a right order, so as to cancel critical cells and obtain a simpler g.v.f.



Finally, the algorithm itself..

The algorithm of [Bauer/Lange/Wardetzky] consists of the following steps

1. Compute persistence pairs on $\mathcal{K}$ using its graph, dual graph, and limited persistence graphs.
2. Construct a simplified gradient vector field on $\mathcal{K}$.
3. Construct a simplified function on $\mathcal{K}$ by traversing the Hasse diagram of the partial order induced by the simplified g.v.f.
4. Finally, since the functions conforming to the simplified g.v.f form a convex polytop, minimization of the Dirichlet energy is performed on this set of solutions.

And some results..

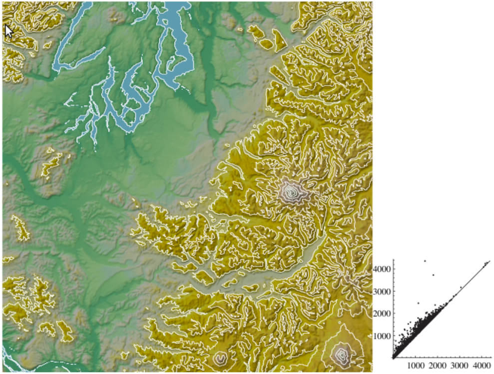


Figure: Map and persistence diagram, before simplification

And some results..

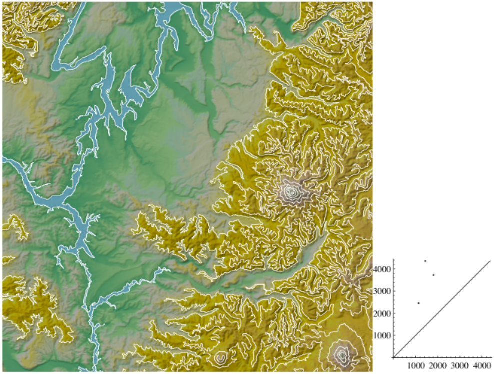


Figure: Map and persistence diagram, after simplification

Computing persistence pairs on surfaces

According to [D. Attali et al., Theorem 4], The persistence pairs of the complex can be detected by going over the 1-skeleton of $\mathcal{K}$ and the 1-skeleton of the dual complex $\mathcal{K}^*$. These give us the $(0,1)$ and $(1,2)$ persistence pairs. Obtaining the pairs for a graph is done by a Kruskal-like algorithm, adding edges that are negative cells while keeping track of the lowest birth time at each component. In addition to the MST, we compute the MST of the graph without the smaller pairs, $M_\delta(G)$, and also for the dual graph, $M_\delta(G^*)$.

Computing V_δ

We now go over the MST $M_\delta(G)$, when we encounter an edge (1-cell) ψ connecting a previously visited vertex ϕ to an unvisited vertex ρ, we add (ϕ, ψ) to the gradient vector field.
Then we go over the dual graph MST $M_\delta(G^*)$ and do the same, adding links between 1-cells and 2-cells, to obtain V_δ.

Computing the simplified function

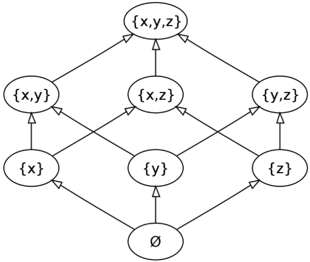
We now have a partial order of the cells, given by V_δ, and we need to find our simplified function f_δ (not the same as the plateau functions).

We now go over all the points based on the order given by V_δ, and update their weight according to adjacent points, and the limitation (L_∞) on f_δ:

$f_\delta(\sigma) = max(f(\sigma) - \delta, max_{\rho \leftarrow_{V_\delta} \sigma} f_\delta(\rho))$

Hasse diagram

Given a partial order, a Hasse diagram is a directed graph with the set elements as vertices, and partial order dependencies as directed edges. A DFS on the graph starting only at nodes with inward degree 0 can create a conforming traversal order - or in our case, build a simplified function (in terms of critical points) conforming to the simplified V.

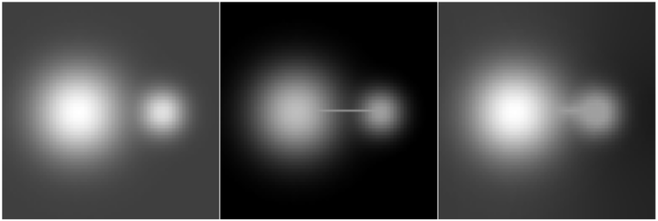


In our case, the diagram connects all the elements in the CW complex, with edges based on the order relation found. In the construction of the function f, after returning from each node σ in the graph we set $f_{delta}(\sigma) = max\left(f(\sigma) - \delta, max_{\rho \leftarrow v_\delta} f_\delta(\rho)\right)$

Since the constraints given by f and V_δ give us a polytop of solutions, we can choose among them. The authors do so by minimizing the Dirichlet energgy,

$$\int \|\nabla f\|^2 d\Omega$$

among feasible solutions. This is done in order to minimize carving artifacts such as the one below:



Looking at persistence diagrams, Cohen-Steiner et. al. give the following theorem:

Theorem

Let $f, g : K \to \mathbb{R}$ be two discrete pseudo-Morse functions. Then the respective persistence diagrams satisfy $d_B(D(f), D(g)) \leq \|f - g\|_\infty$

- Well-defined, unlike persistence pairs.
- Independent of the total order chosen, or a specific choice of gradient vector field consistent with f.
- Leads to the following corollary, a lower bound on the complexity of simplifications, and linking the two optimization problems

Corollary ([1], corollary 15)

For any pseudo-Morse function f_δ with $\|f_\delta - f\|_\infty \leq \delta$, the number of persistence pairs of f_δ is bounded from below by the number of persistence pairs of f with persistence $> 2\delta$.

..and naturally, we'd like to achieve this bound.

Definition (perfect δ-simplification)

Let f be a pseudo-Morse function on a combinatorial surface $\mathcal{K}$. A perfect δ-simplification of f is a pseudo-Morse function f_δ such that $\|f_\delta - f\|_\infty \leq \delta$, and the number of persistence pairs of f_δ equals the number of persistence pairs of f that have persistence $> 2\delta$.

Theorem ([1], theorem 16)

Let f be a pseudo-Morse function on a combinatorial surface. Then there exists a perfect δ-simplification of f.

The proof, turns out, is constructive.

Furthermore:

Corollary ([1], corollary 17)

Given a discrete pseudo-Morse function f on a surface and $\delta > 0$, there exists a discrete pseudo-Morse function f_δ consistent with a GVF V_δ such that $\|f_\delta - f\|_\infty \leq \delta$, and the number of critical points of V_f equals the number of critical points of f that have persistence $> 2\delta$.

And also for discrete Morse functions.

During the simplification sequence we need to verify we are consistent with V_i, and δ-close to the original f.
This can be shown when looking at a set of plateau functions f_i resulting after the ith simplification step.
Let (σ, τ) be the pair cancelled at step i.
Define $m_i = \frac{f(\sigma)+f(\tau)}{2}$ to be the value replacing cell values in the (attracting/repelling set):

$$f_i(\rho) = \begin{cases} m_i & \text{if } \sigma \prec_{i-1} \rho \text{ and } f_{i-1}(\rho) < m_i \\ & \text{or } \rho \prec_{i-1} \tau \text{ and } f_{i-1}(\rho) > m_i \\ f_{i-1}(\rho) & \textit{otherwise} \end{cases}$$

Lemma ([1], lemma 19)

The plateau function f_i is consistent with both V_{i-1} and V_i.

And

Corollary ([1], lemma 20)

Each plateau function satisfies
$\|f_i - f\|_\infty < \delta$

Following [1], Bauer, Schonlieb, and Wardetzky [2] look at total variation denoising:

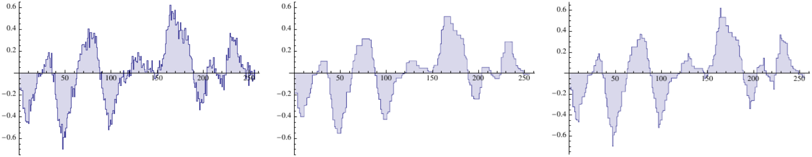
Definition (Total variation denoising)

Given a function $f : \Omega \to \mathbb{R}$ with bounded variation, total variation denoising seeks
$f_\alpha = \operatorname{argmin}_u \int_\Omega (u - f)^2 + 2\alpha \|\nabla u\| d\Omega.$

This simplification prior is used in many applications, including image denoising, optical flow, target location / segmentation and so forth.
Specifically, it is convex, and has a global solution.

In [2], we seek a connection between TV denoising and topological approaches..
First, [2] go on to further suggest looking at denoising by filling-up critical points with larger and larger fill volumes, rather than levels. This algorithm still allows us to control $\|f_\alpha - f\|_\infty$.

They then link the solution of the volume filling algorithm and the solution for the TV denoising problem.



Specifically, they claim that the function obtained by the filling process identifies with f whenever $\tilde{u}^{(\alpha)}$ does.

Specifically, the TV solution $\tilde{u}^{(\alpha)}(x)$ is of the form:

$$\tilde{u}^{(\alpha)}(x) = \begin{cases} f_{c_k(i)} \pm \delta_i^k(\alpha) & \text{if } x \in \Omega_i^{(k)} \\ & \text{for some critical point } i \text{ of } u^k \\ \mu^{(m)} & \text{if } x \notin \Omega_i^{(k)} \\ & \text{and } x \in \Omega_i^{(m)} \text{ for } m = max_{l<k} l : x \in \Omega_i^{(l)} \\ f(x) & \textit{otherwise} \end{cases}$$

In the $2D$ case, they present the following results of filling-based persistence removal: