# Probabilistic (Graphical) Models

## and inference

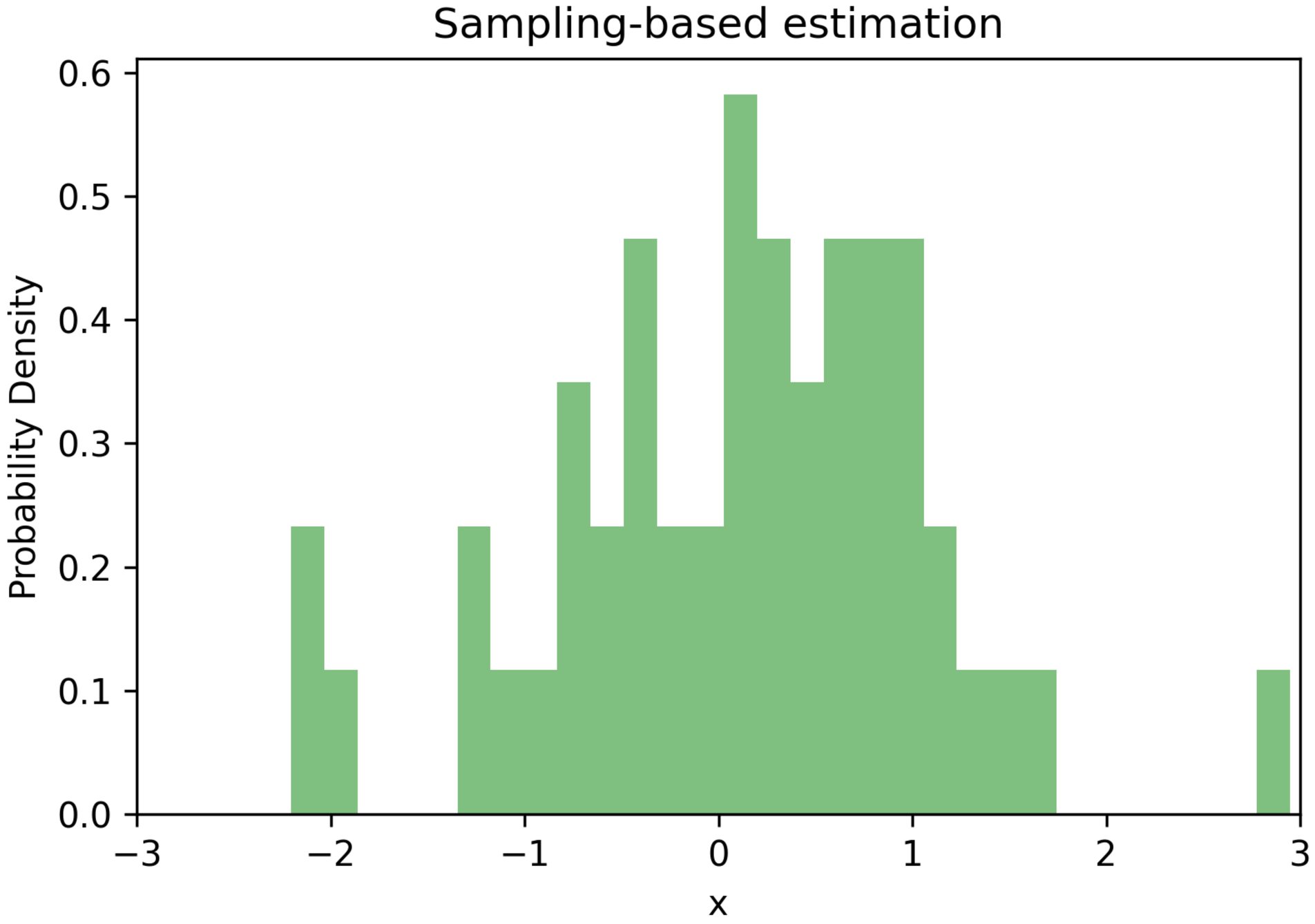
Oliver Obst · Autumn 2024

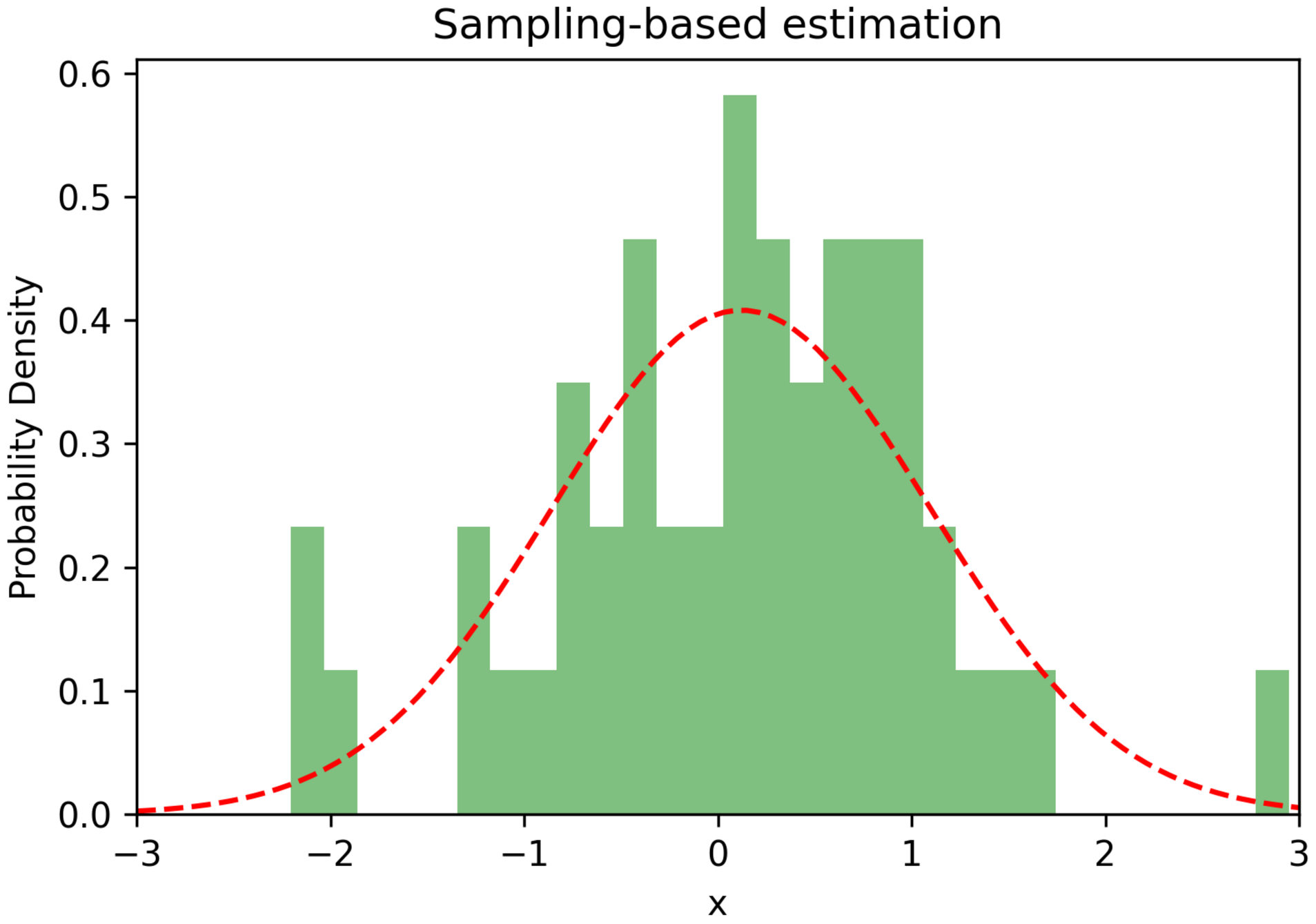
WESTERN SYDNEY
UNIVERSITY

# Probabilistic (Graphical) Models and Inference

(PGM: Probabilistic Graphical Models: Principles and Techniques by Daphne Koller and Nir Friedman. MIT Press)
(PMLI: Probabilistic Machine Learning: An introduction by Kevin Murphy. MIT Press)

| Week | Lecture | Required reading | Assessment |
|---|---|---|---|
| 1<br>Monday, 4 March 2024 | Introduction, Probability Theory | PGM Chapter 2, PMLI Chapter 6.1 | |
| 2<br>Monday, 11 March 2024 | Directed and undirected networks introduction | PGM Chapter 3 & 4 | Quiz 1 |
| 3<br>Monday, 18 March 2024 | Variable elimination | PGM Chapter 9 | |
| 4<br>Monday, 25 March 2024 | Belief propagation | PGM Chapter 10/11 | Quiz 2 |
| 5<br>Monday, 1 April 2024 | public holiday | | 5 April 2024: census date |
| 6<br>Monday, 8 April 2024 | Message passing / Graph neural networks | https://distill.pub/2021/gnn-intro/ | |
| 7<br>Monday, 15 April 2024 | Sampling | PGM Chapter 12 | Quiz 3 |
| 8<br>Monday, 22 April 2024 | Mid-term break | | |
| 9<br>Monday, 29 April 2024 | Variational inference | https://leimao.github.io/article/Introduction-to-Variational-Inference/ | Intra-session exam |
| 10<br>Monday, 6 May 2024 | Autoregressive models | | Quiz 4 |
| 11<br>Monday, 13 May 2024 | Variational Auto-Encoders | | |
| 12<br>Monday, 20 May 2024 | GANs | | Quiz 5 |
| 13<br>Monday, 27 May 2024 | Energy-based models | | |
| 14<br>Monday, 3 June 2024 | Evaluating generative models | | Quiz 6 |
| Monday, 17 June 2024 | | | Project due |


Sampling-based estimation
Probability Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−3
−2
−1
0
1
2
3
x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0.1
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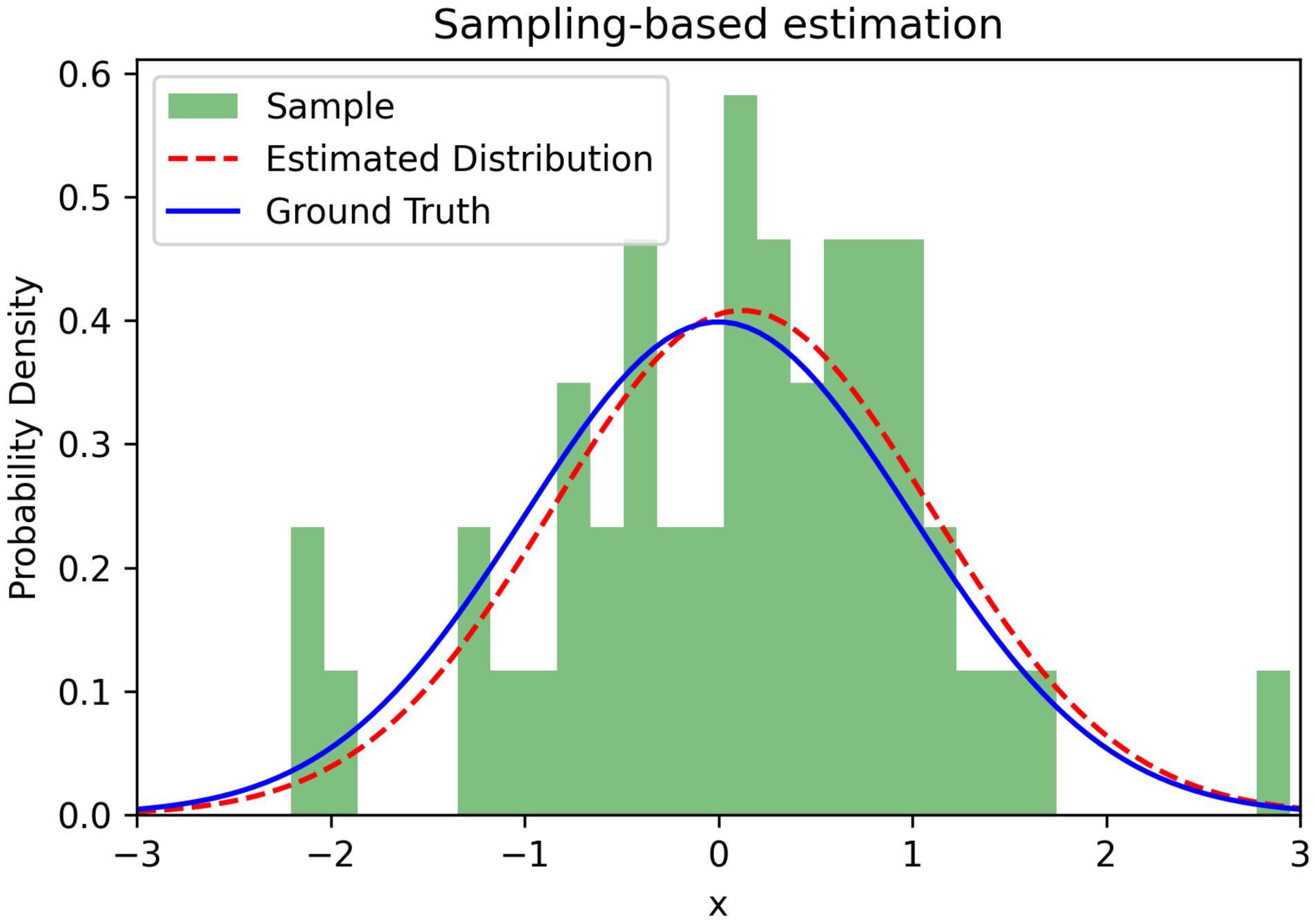
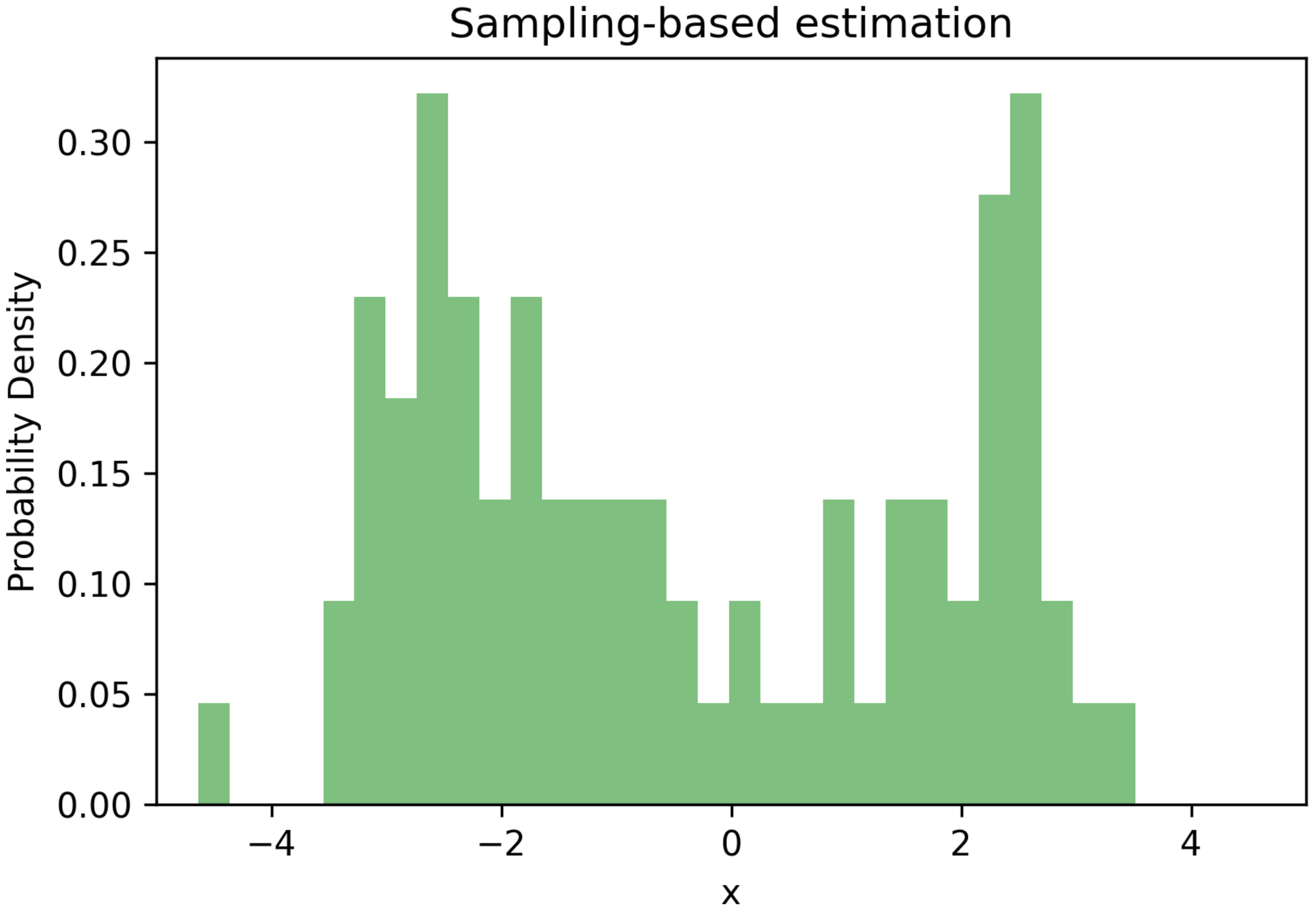
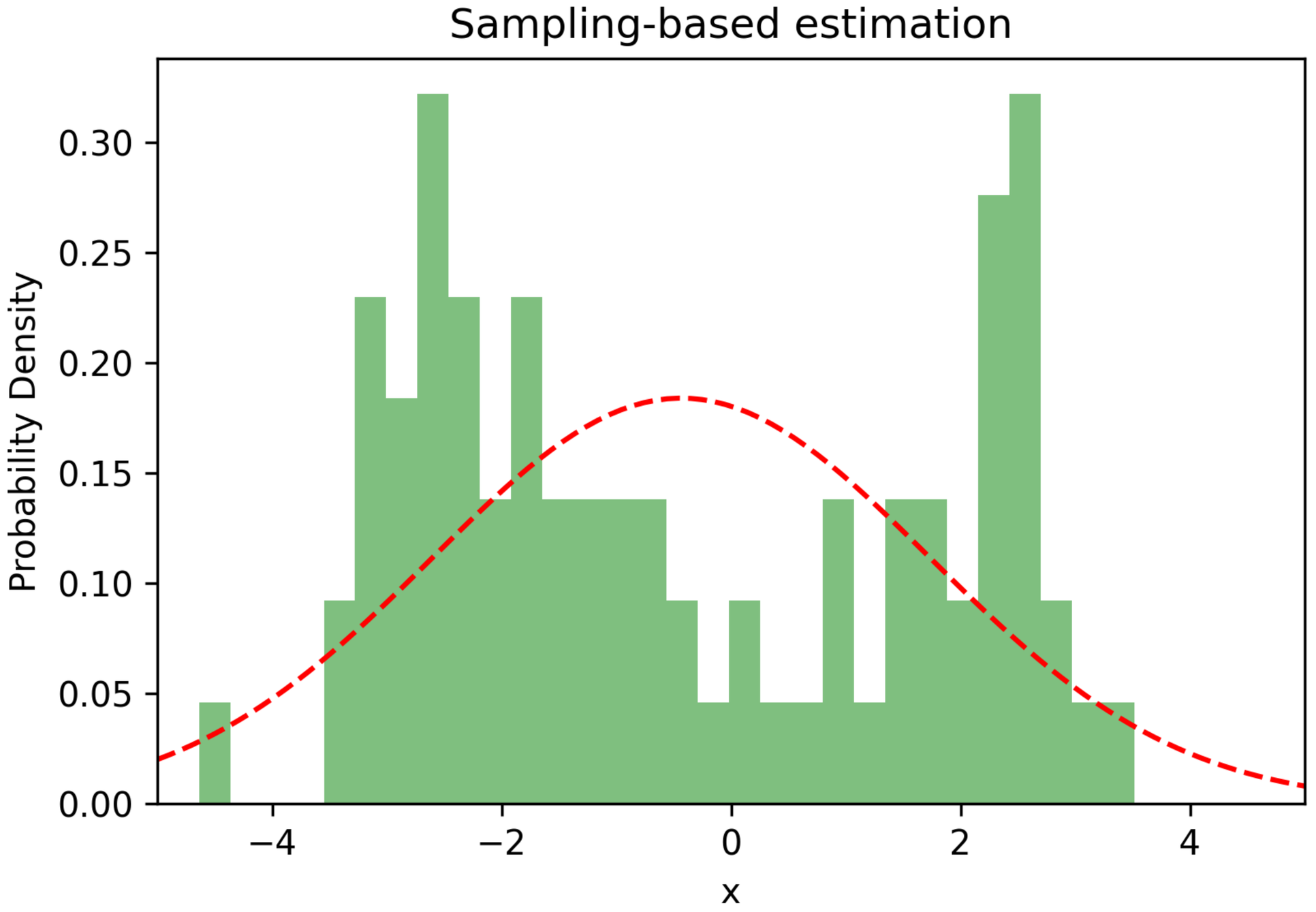

Sampling-based estimation
Sample
Estimated Distribution
Ground Truth
Probability Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−3
−2
−1
0
1
2
3
x


Sampling-based estimation
Probability Density
0.00
0.05
0.10
0.15
0.20
0.25
0.30
−4
−2
0
2
4
x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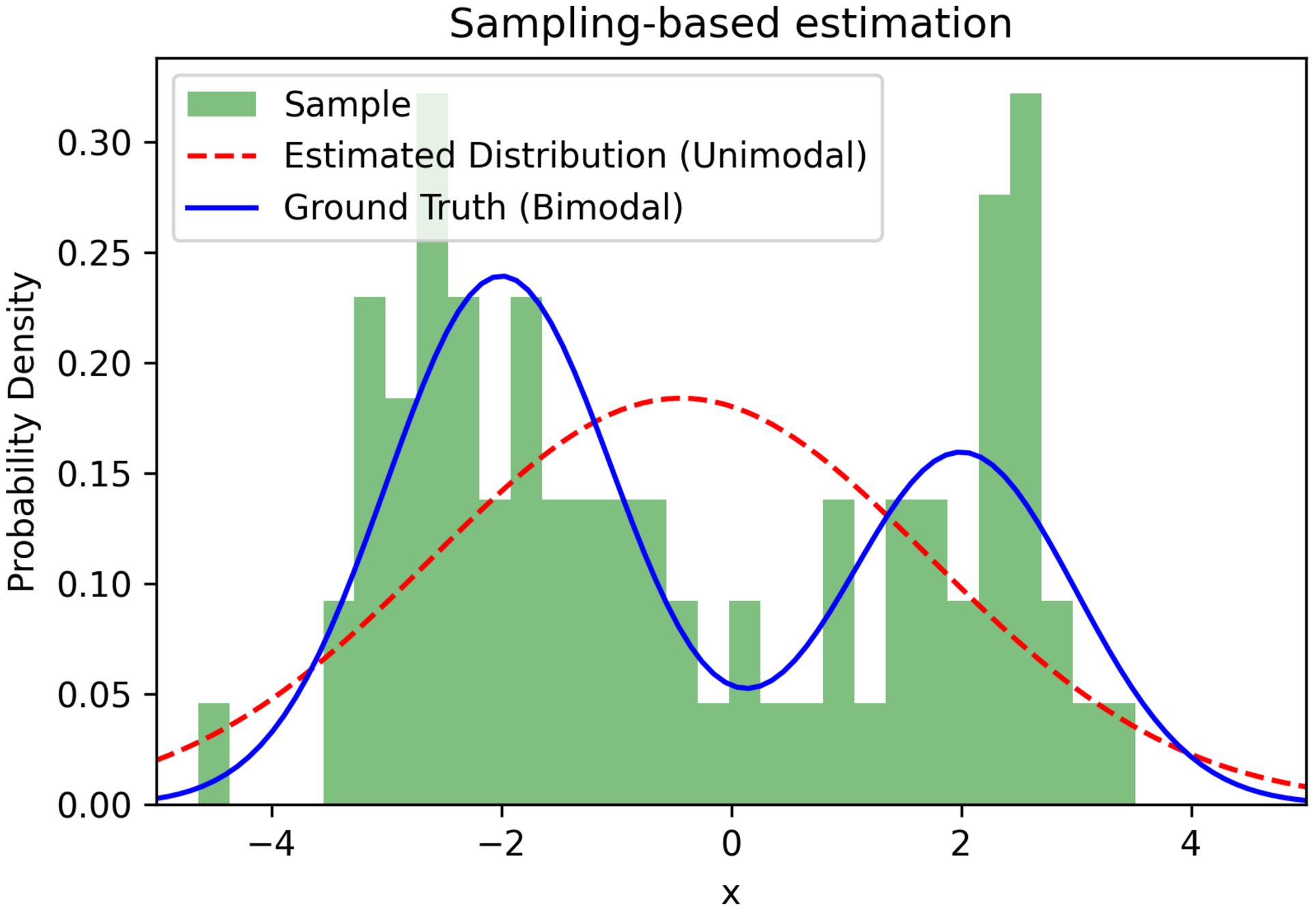
Sampling-based estimation
Sample
Estimated Distribution (Unimodal)
Ground Truth (Bimodal)
Probability Density
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# Simple sampling

# Sampling-based estimation

$\mathscr{D} = \{x[1], \ldots, x[M]\}$ sampled IID from $P$

# Sampling-based estimation

$\mathscr{D} = \{x[1], \ldots, x[M]\}$ sampled IID from $P$

- Independent: The independence aspect means that the random variables in the collection do not influence or depend on each other.
- Identically distributed: This aspect means that all random variables share the same probability distribution.

# Sampling-based estimation

$\mathscr{D} = \{x[1], \ldots, x[M]\}$ sampled IID from $P$

If $P(X = 1) = p$

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Empirical expectation

# Sampling from discrete distribution

$\text{Val}(X) = \{x^1, \ldots, x^k\}$ $\quad$ $P(x^i) = \theta^i$

# Sampling from discrete distribution

$$\theta^1, \ldots, \theta^k$$

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**Uniformly in [0...1]**

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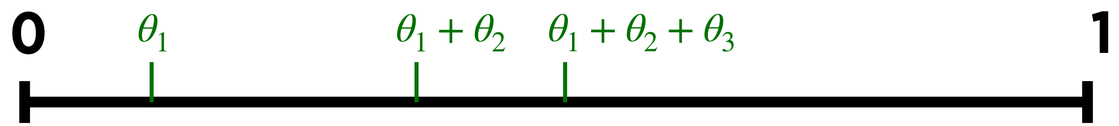


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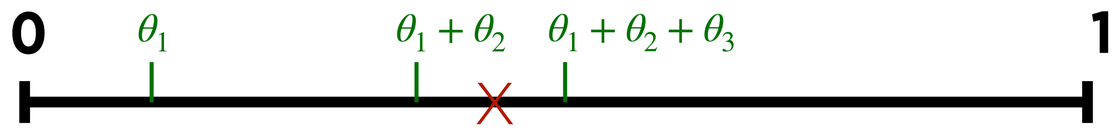


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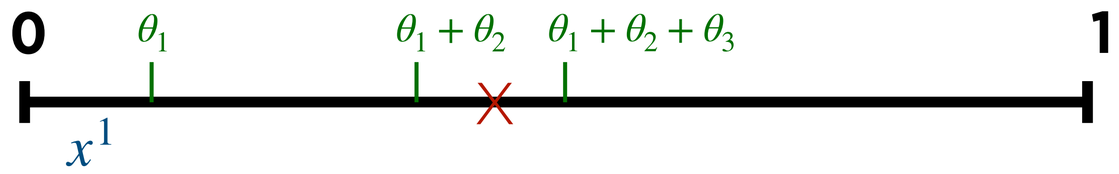


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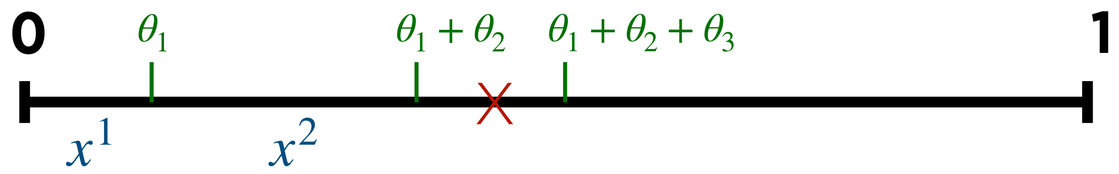


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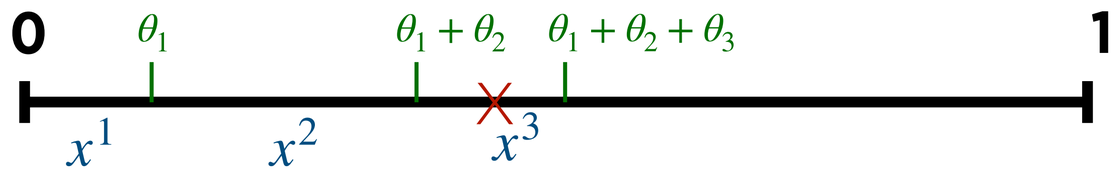


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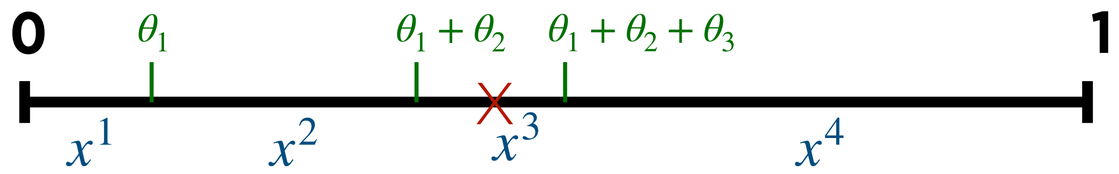


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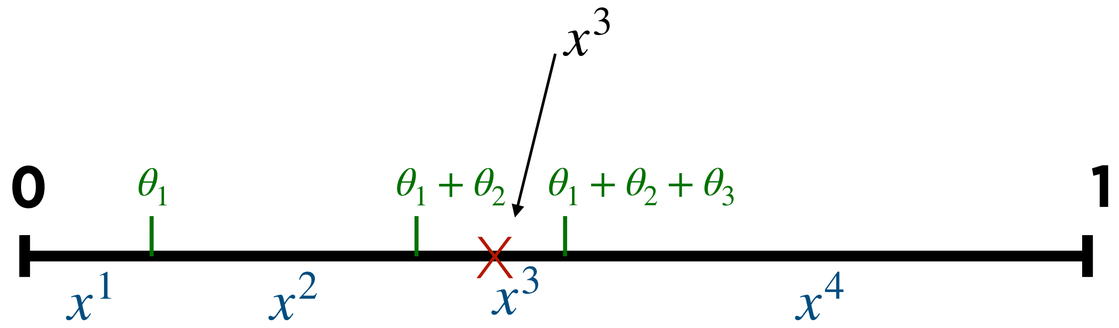


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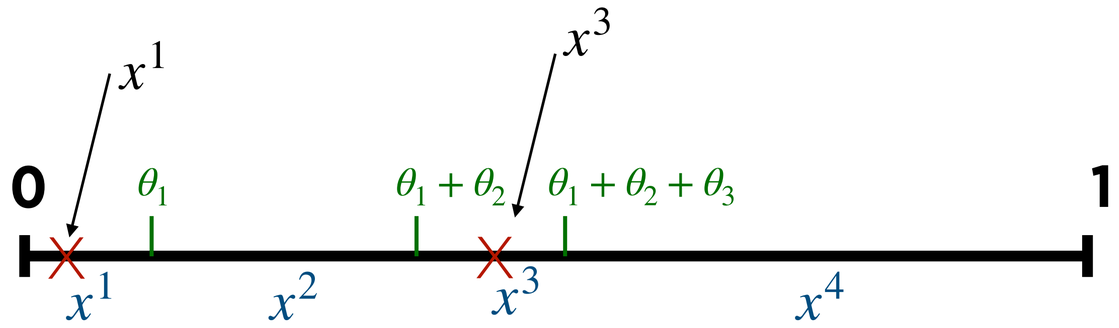


**Uniformly in [0...1]**

# How good is our estimator?

# Sampling-based estimation

**Properties — how far off is our estimate from the true value?**

$$T_{\mathscr{D}} = \frac{1}{M} \sum_{m=1}^{M} x[m]$$

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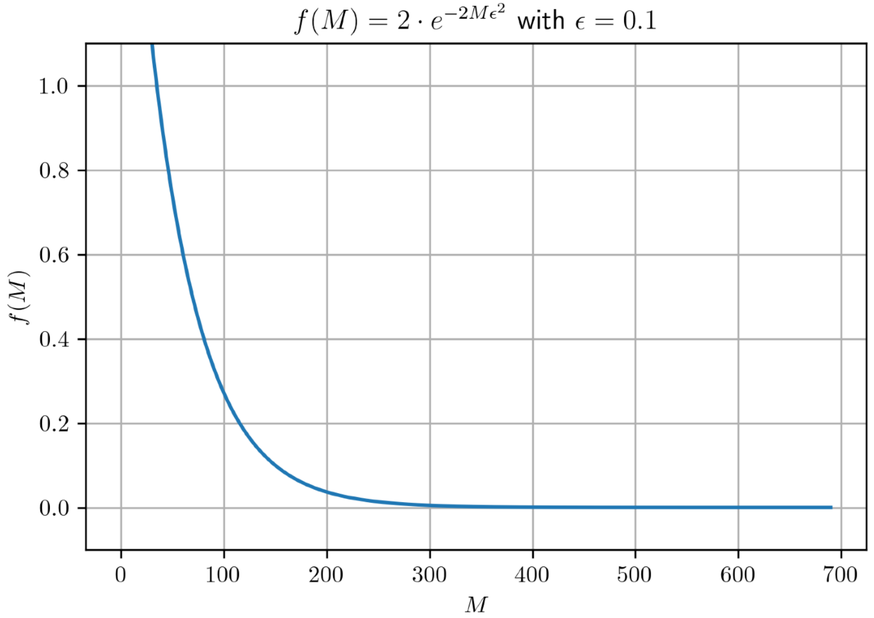

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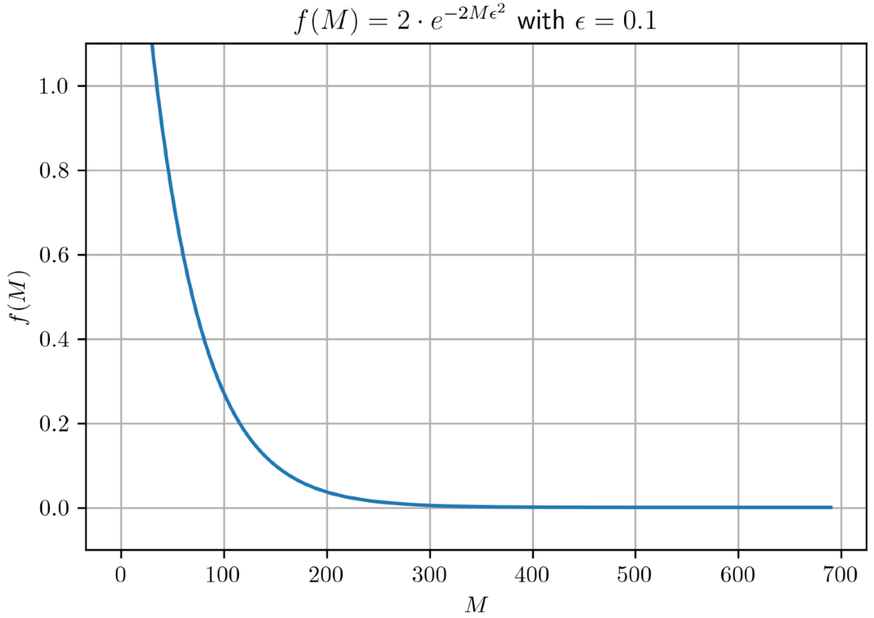


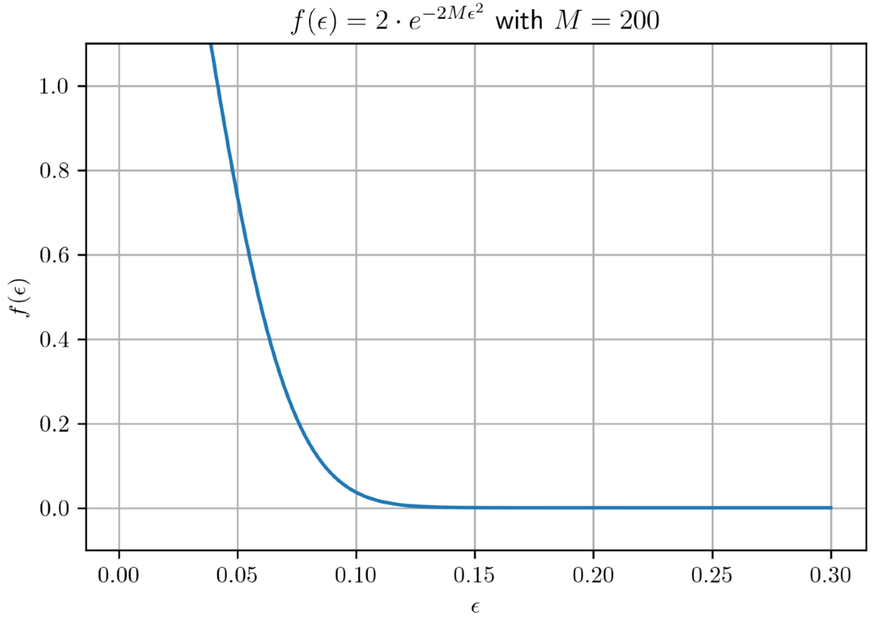

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Chernoff Bound:

$$P_{\mathscr{D}}(T_{\mathscr{D}} \notin [p(1-\epsilon), p(1+\epsilon)]) \leq 2e^{-2Mp\epsilon^2/3}$$

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For additive bound $\epsilon$ on error with probability $> 1-\delta$:

$$M \geq \frac{\ln(2/\delta)}{2\epsilon^2}.$$

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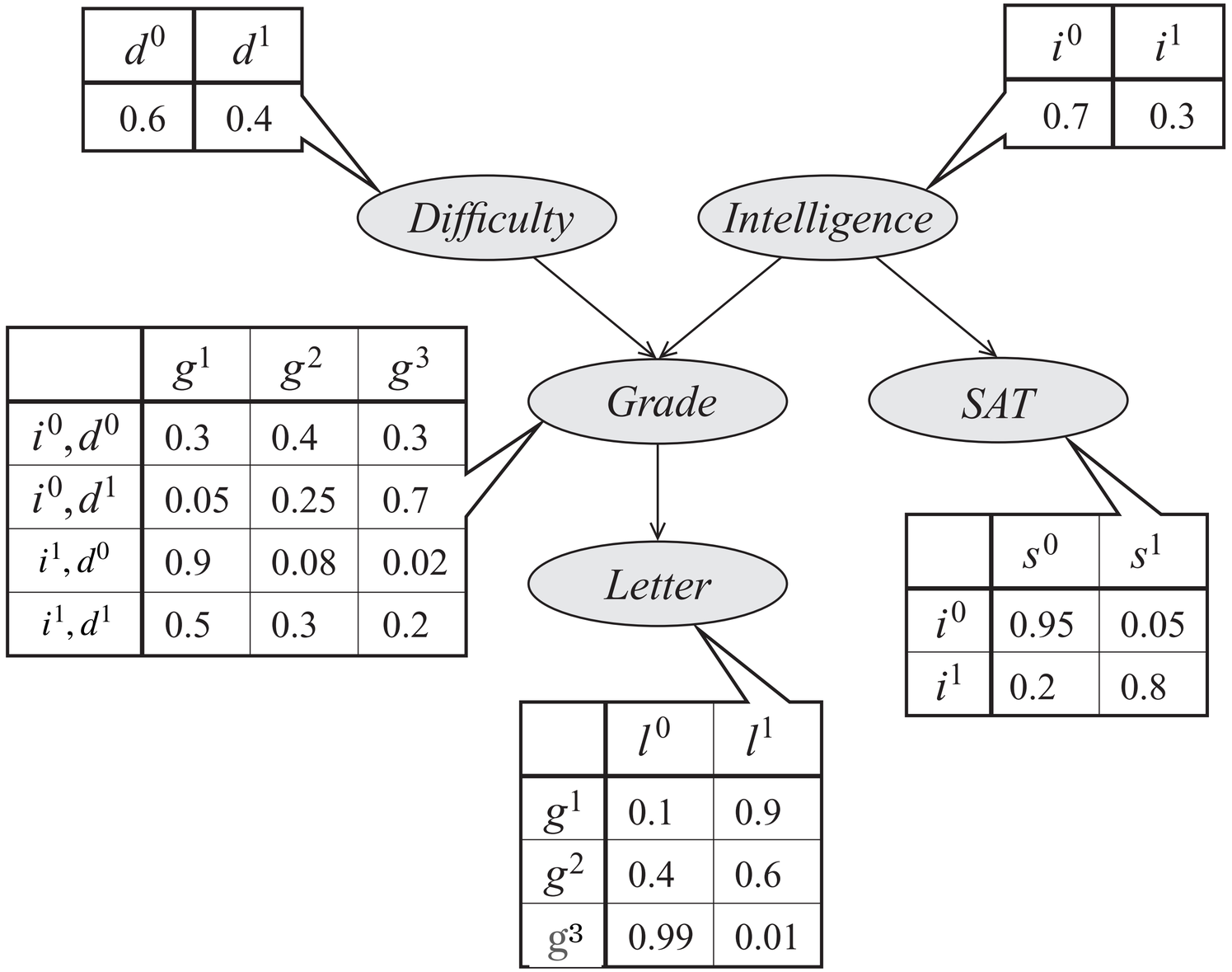
For multiplicative bound $\epsilon$ on error with probability $> 1 - \delta$:

$$M \geq 3\frac{\ln(2/\delta)}{2p\epsilon^2}.$$

# Forward sampling from a BN




| $d^0$ | $d^1$ |
|---|---|
| 0.6 | 0.4 |

| $i^0$ | $i^1$ |
|---|---|
| 0.7 | 0.3 |

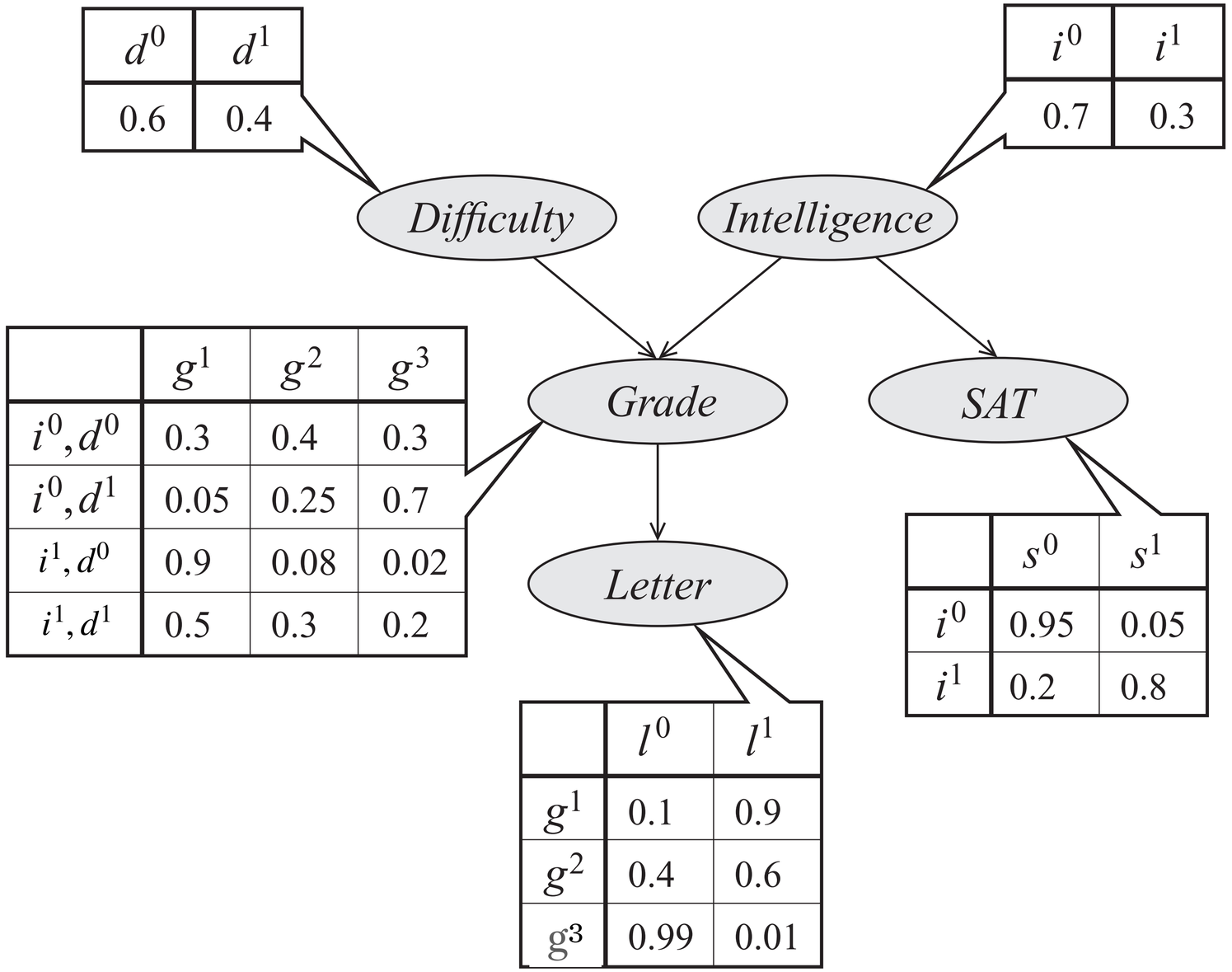
| | $g^1$ | $g^2$ | $g^3$ |
|---|---|---|---|
| $i^0, d^0$ | 0.3 | 0.4 | 0.3 |
| $i^0, d^1$ | 0.05 | 0.25 | 0.7 |
| $i^1, d^0$ | 0.9 | 0.08 | 0.02 |
| $i^1, d^1$ | 0.5 | 0.3 | 0.2 |

| | $s^0$ | $s^1$ |
|---|---|---|
| $i^0$ | 0.95 | 0.05 |
| $i^1$ | 0.2 | 0.8 |

| | $l^0$ | $l^1$ |
|---|---|---|
| $g^1$ | 0.1 | 0.9 |
| $g^2$ | 0.4 | 0.6 |
| $g^3$ | 0.99 | 0.01 |

# Forward sampling from a BN

$P(D, I, G, S, L)$

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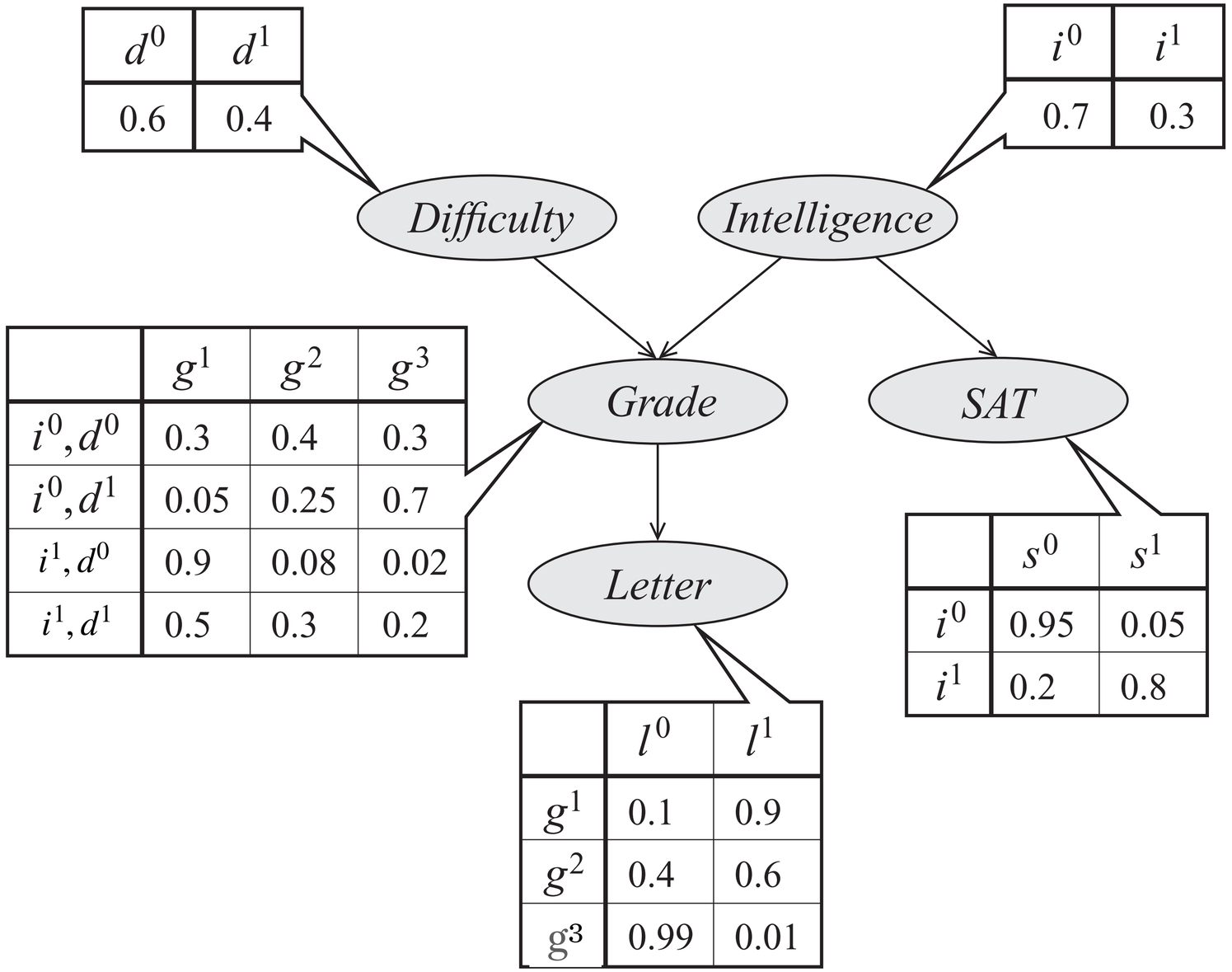


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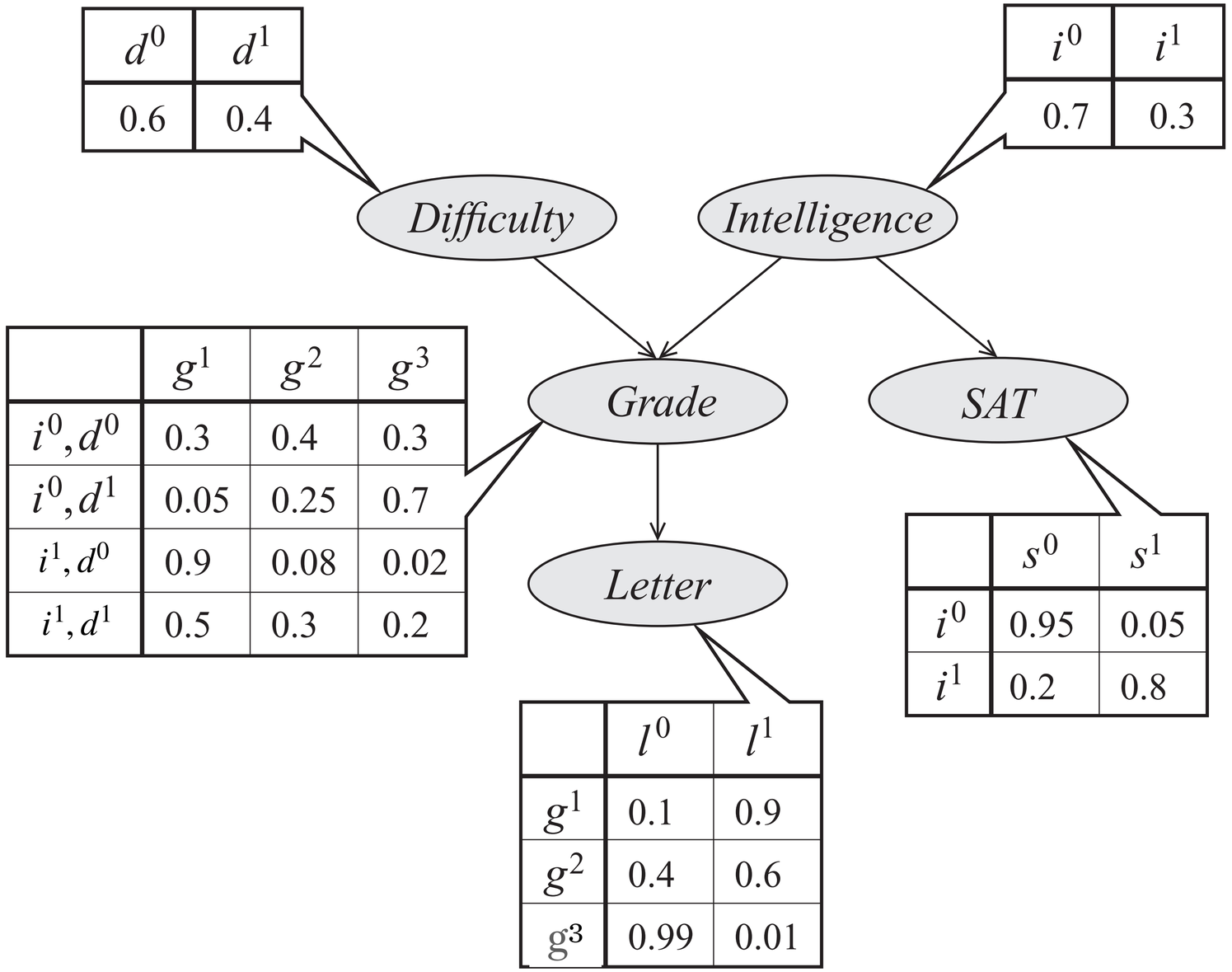
|---|---|---|
| $i^0$ | 0.95 | 0.05 |
| $i^1$ | 0.2 | 0.8 |

| | $l^0$ | $l^1$ |
|---|---|---|
| $g^1$ | 0.1 | 0.9 |
| $g^2$ | 0.4 | 0.6 |
| $g^3$ | 0.99 | 0.01 |

$d^0$

# Forward sampling from a BN

$P(D, I, G, S, L)$




$d^0 i^1$

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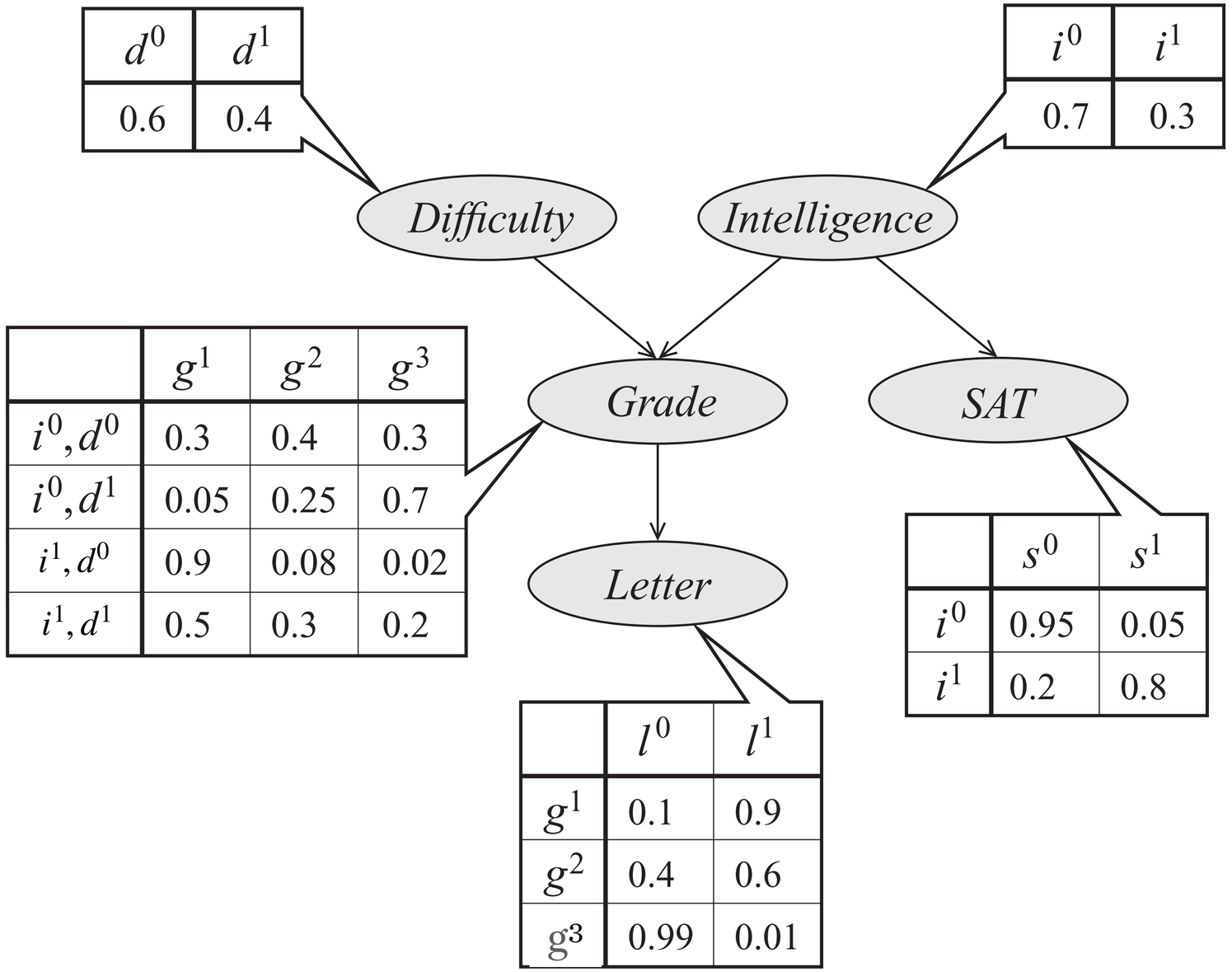
$P(D, I, G, S, L)$




| $d^0$ | $d^1$ |
|---|---|
| 0.6 | 0.4 |

| $i^0$ | $i^1$ |
|---|---|
| 0.7 | 0.3 |

| | $g^1$ | $g^2$ | $g^3$ |
|---|---|---|---|
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| $i^0, d^1$ | 0.05 | 0.25 | 0.7 |
| $i^1, d^0$ | 0.9 | 0.08 | 0.02 |
| $i^1, d^1$ | 0.5 | 0.3 | 0.2 |

| | $s^0$ | $s^1$ |
|---|---|---|
| $i^0$ | 0.95 | 0.05 |
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$d^0 i^1 g^1$

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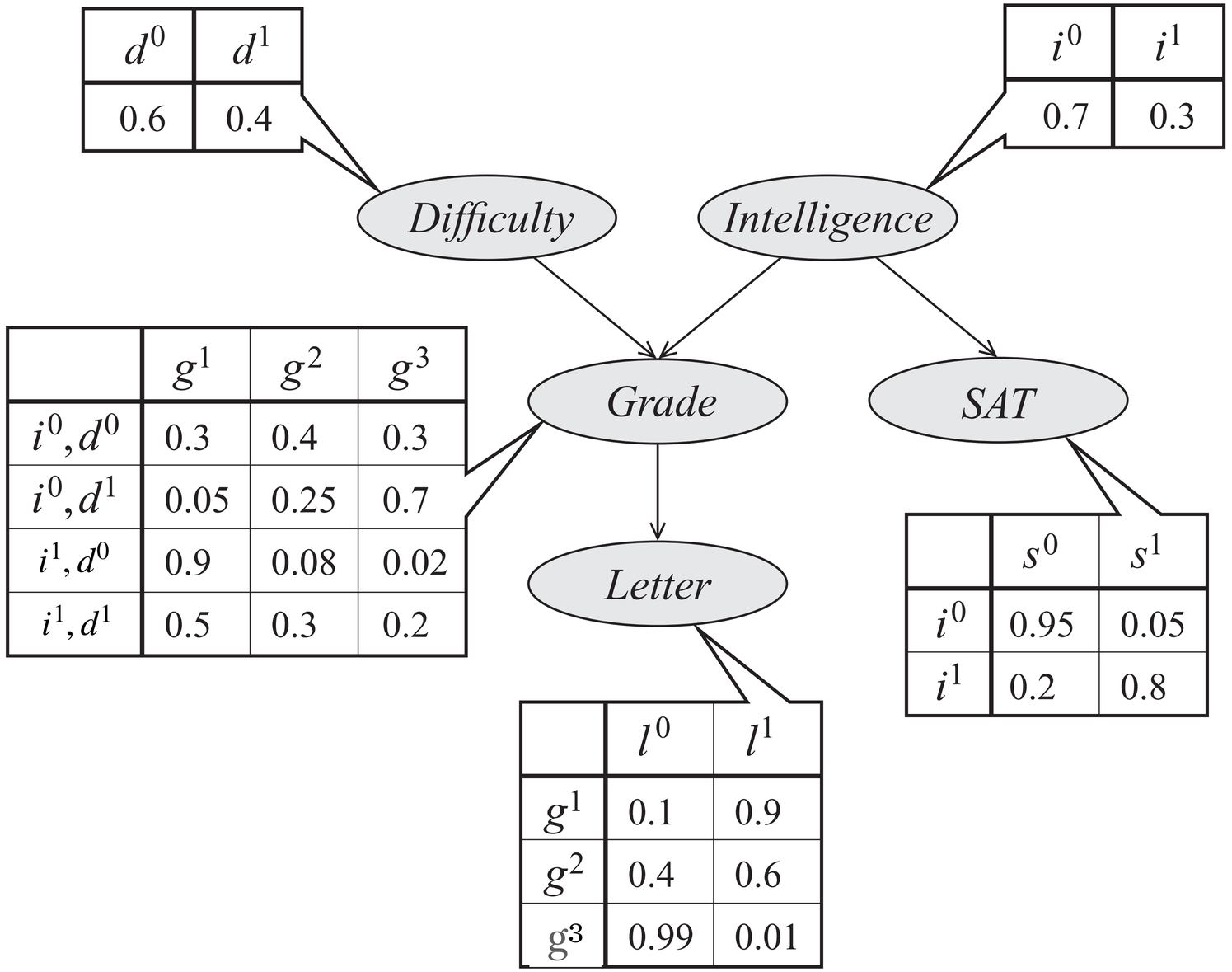
$P(D, I, G, S, L)$

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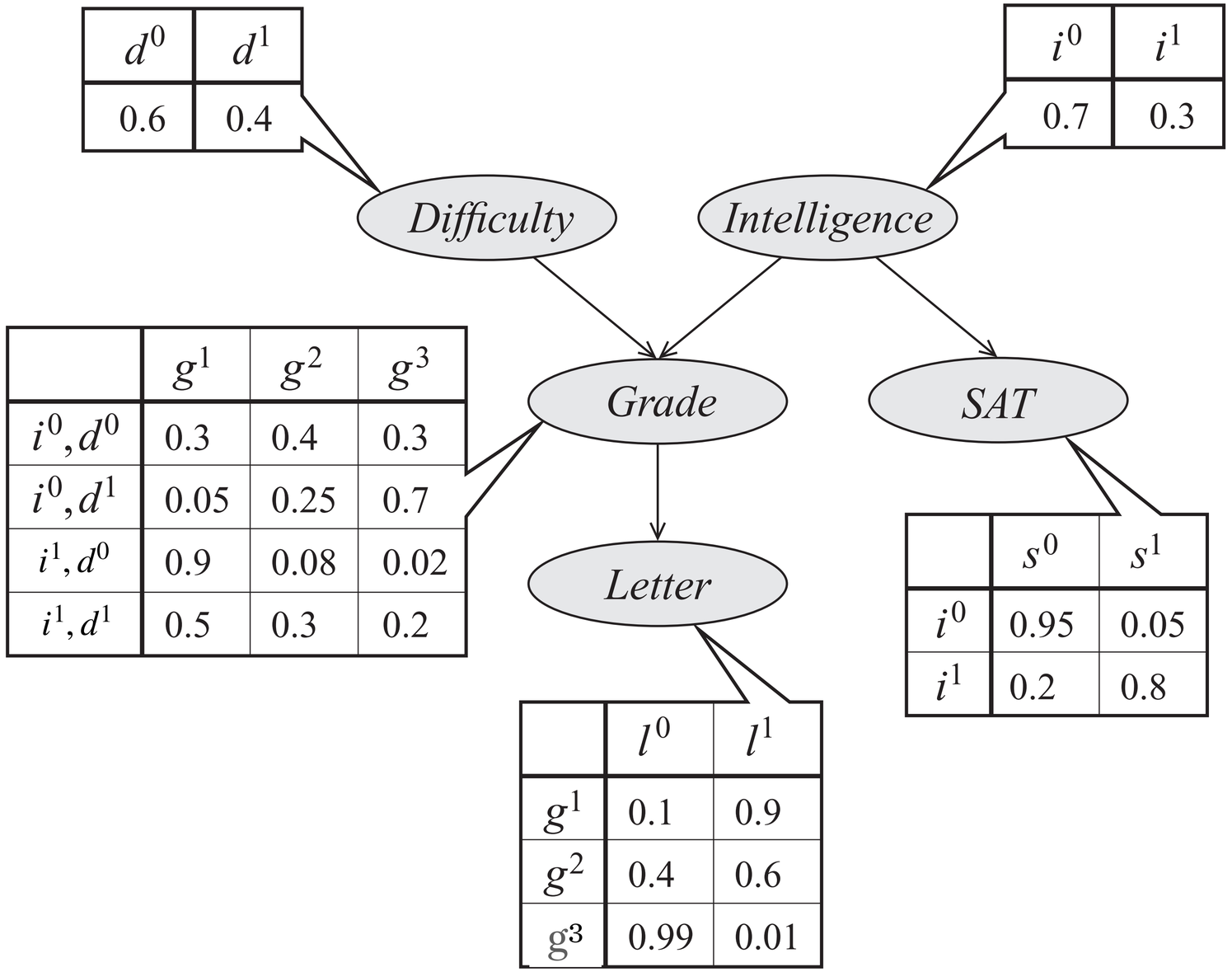


$d^0 i^1 g^1 s^0 l^1$

# Forward sampling from a BN

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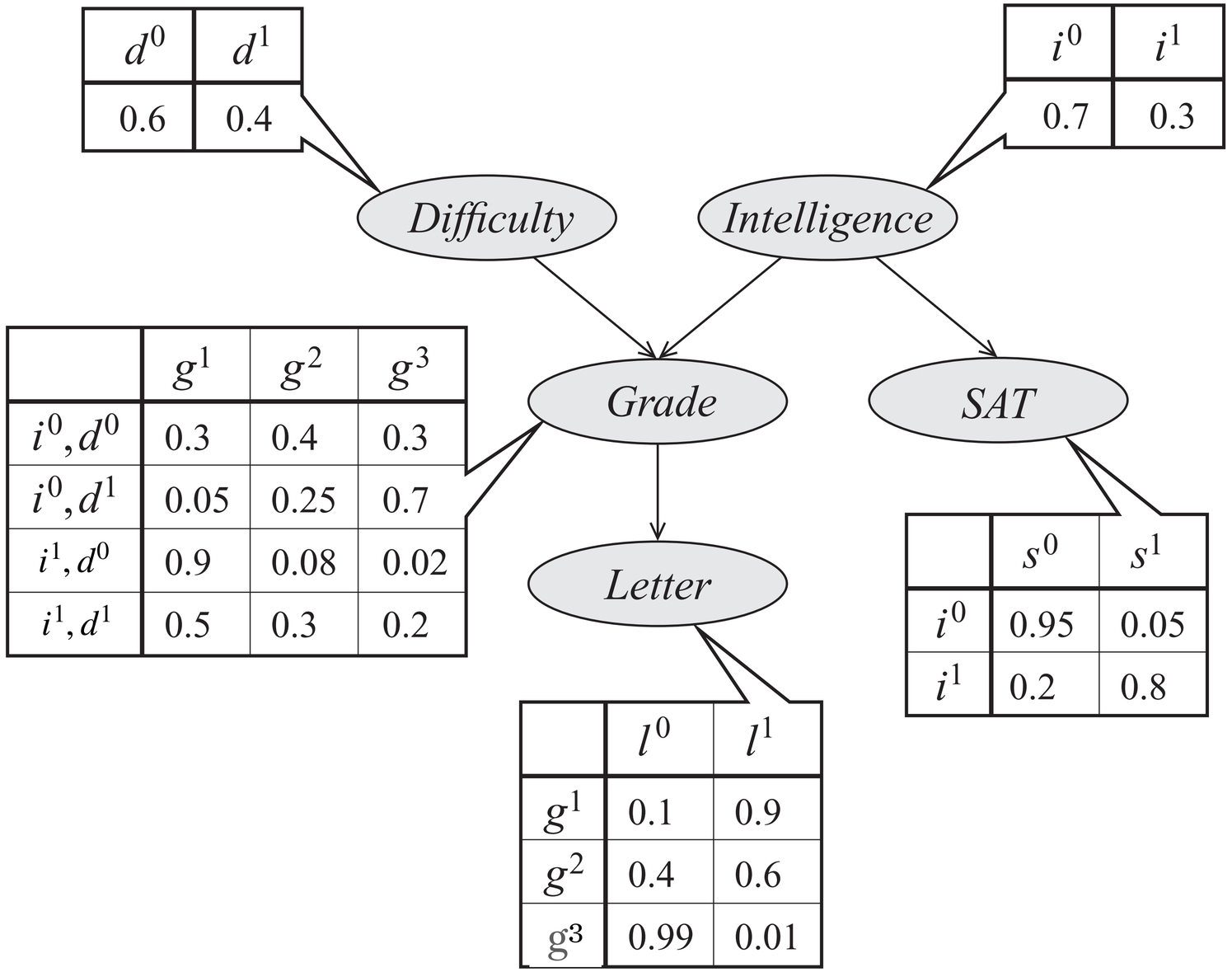


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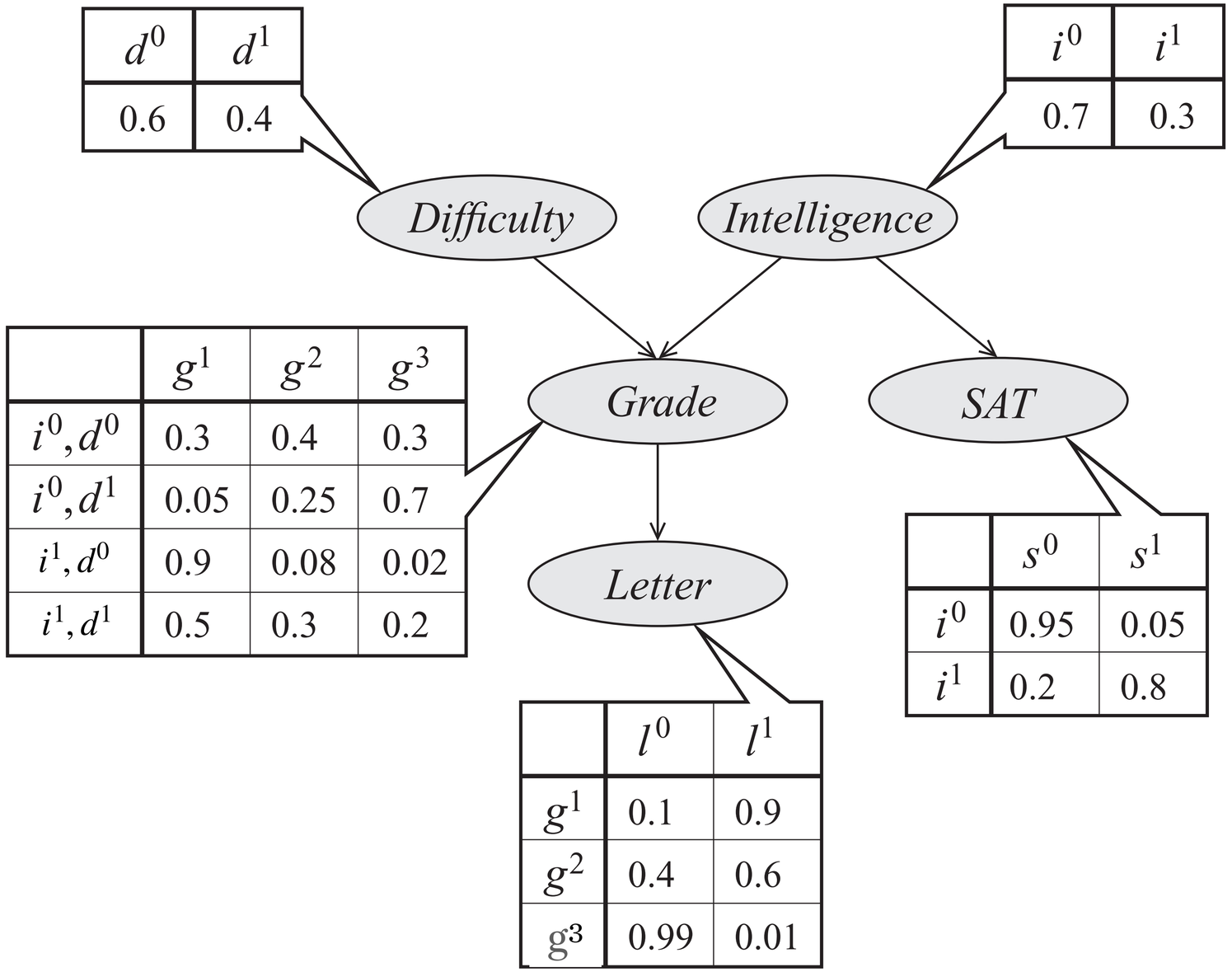
$d^0 i^1 g^1 s^0 l^1$

$d^1$

# Forward sampling from a BN

$P(D, I, G, S, L)$

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$d^0 i^1 g^1 s^0 l^1$

$d^1 i^1$

# Forward sampling from a BN

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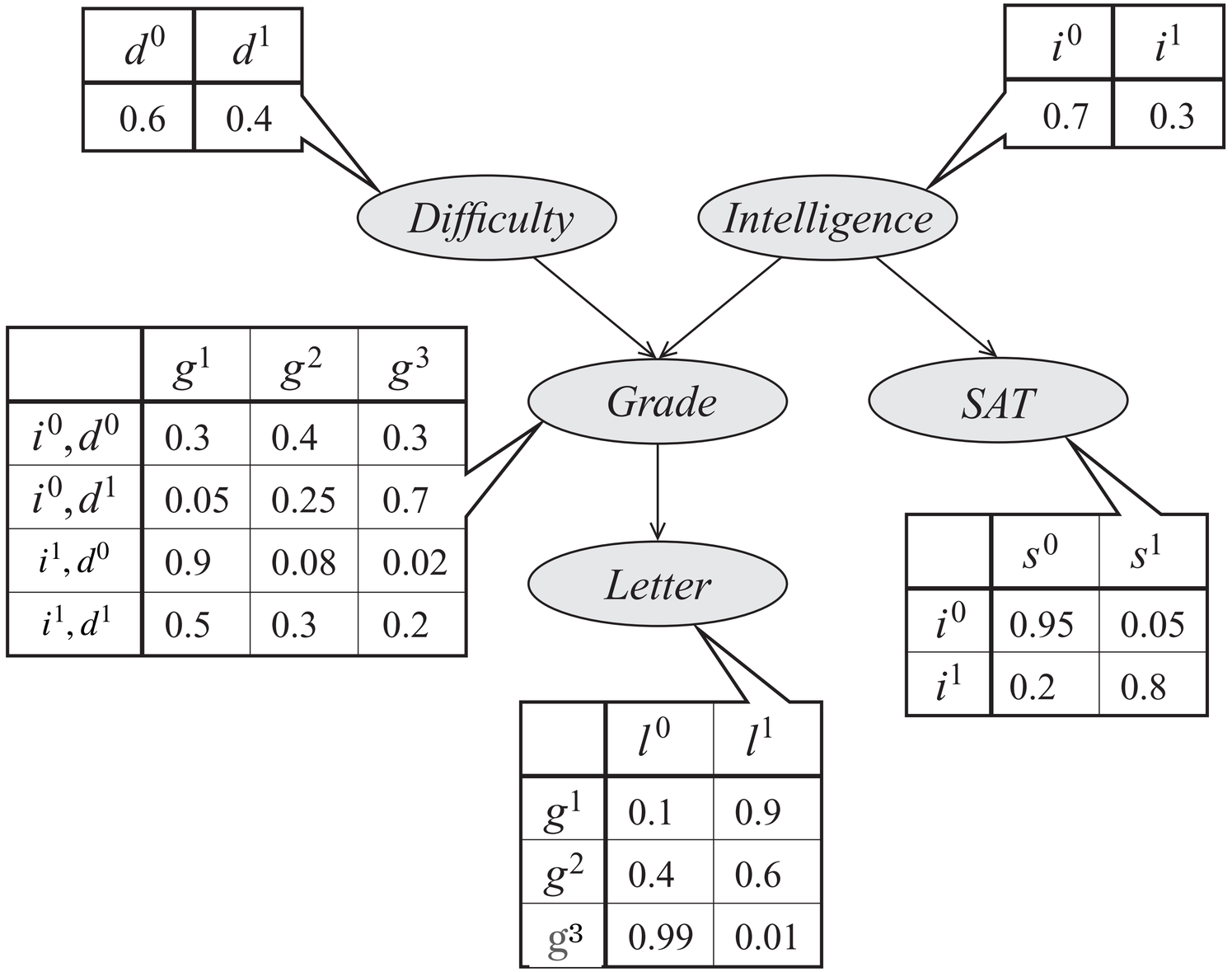


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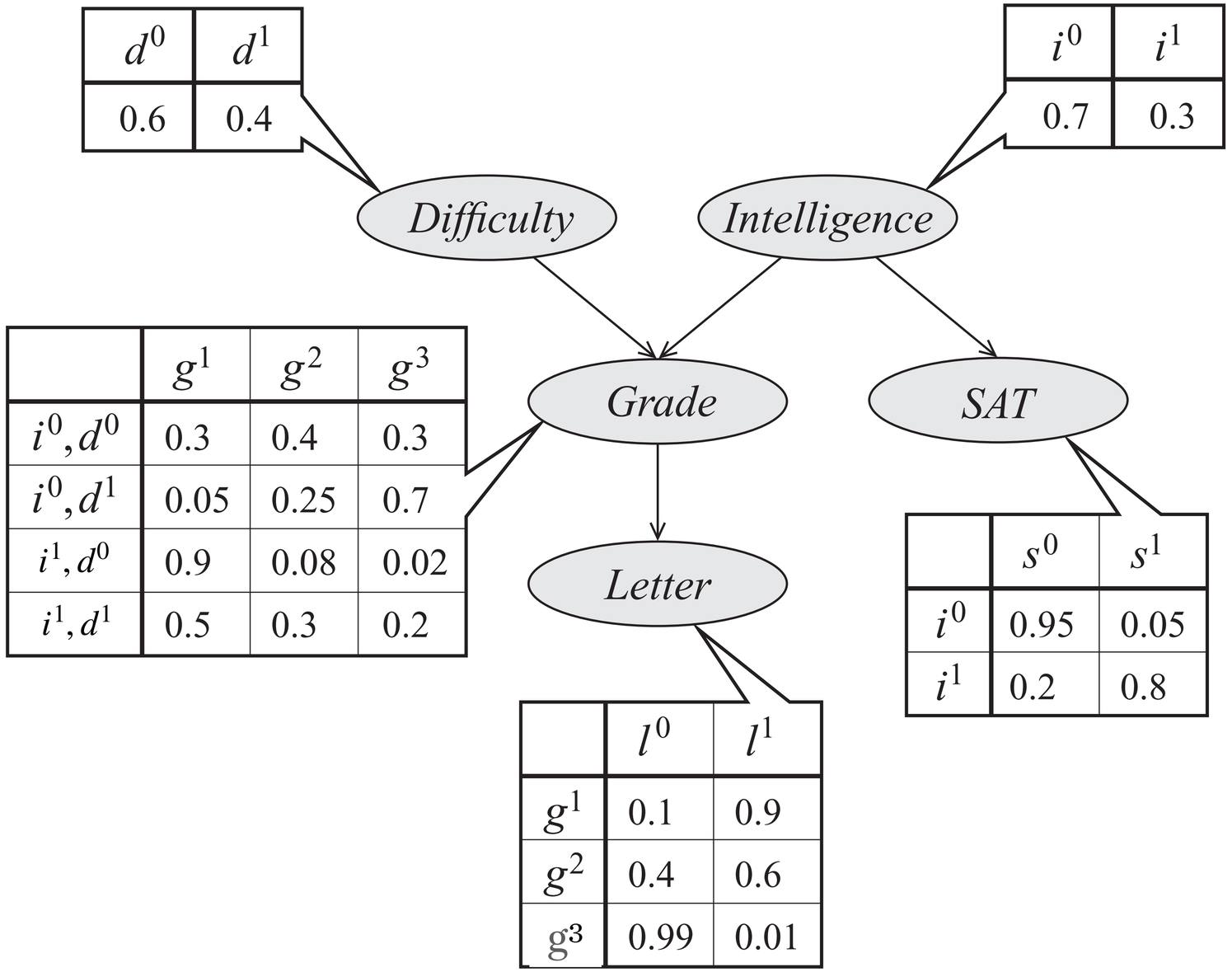
$d^0 i^1 g^1 s^0 l^1$

$d^1 i^1 g^2$

# Forward sampling from a BN

$P(D, I, G, S, L)$

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$d^0 i^1 g^1 s^0 l^1$

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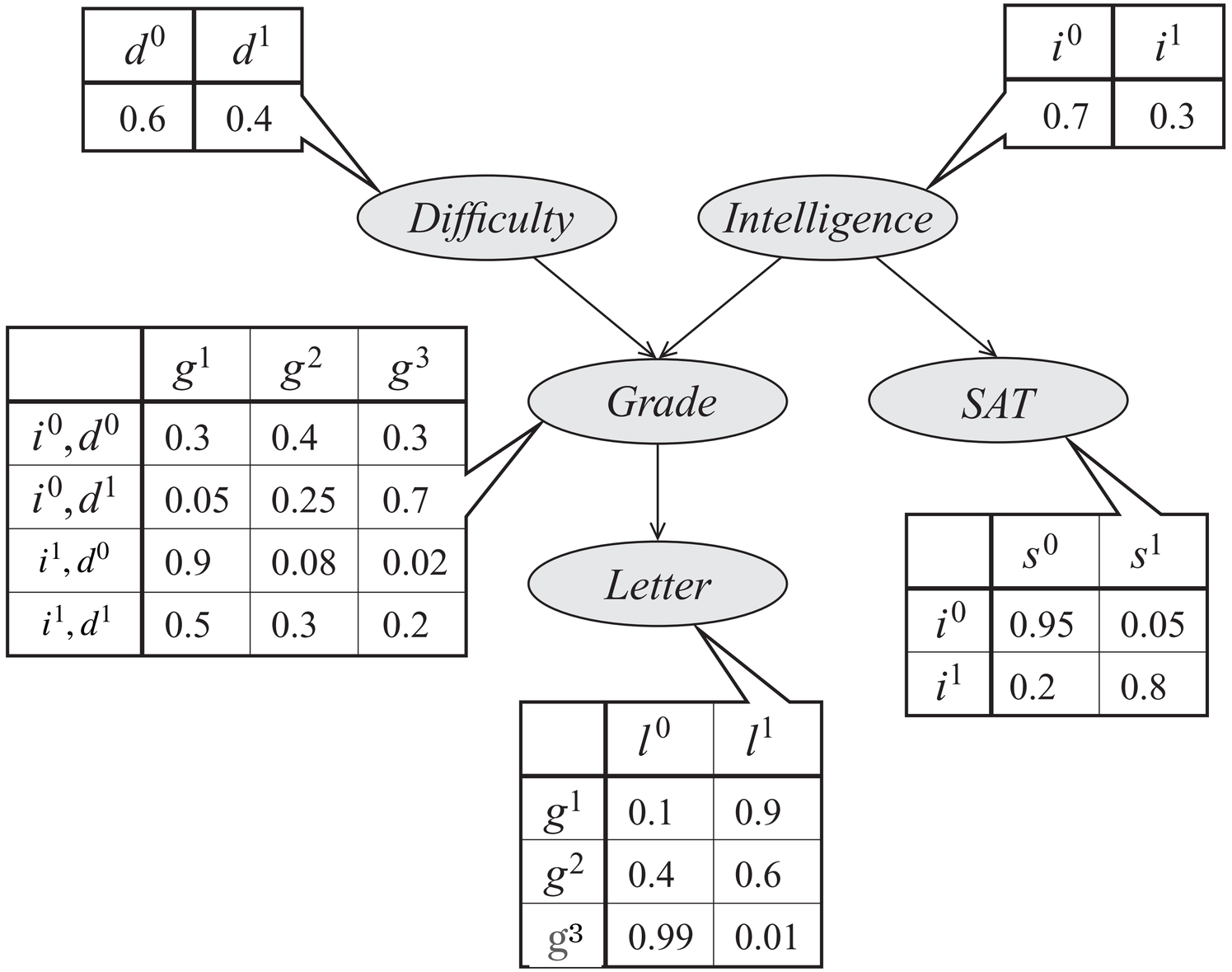
# Forward sampling from a BN

$P(D, I, G, S, L)$

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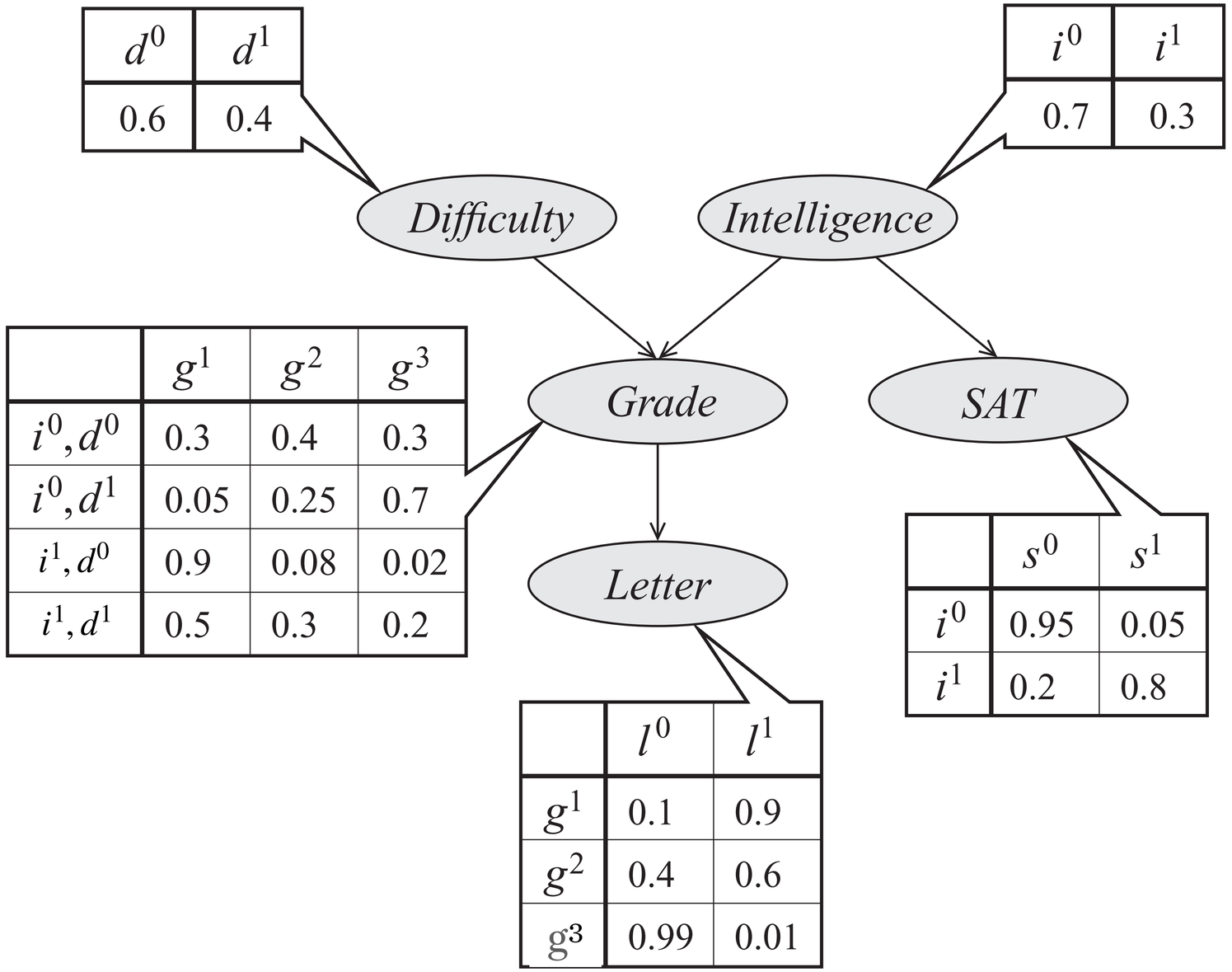


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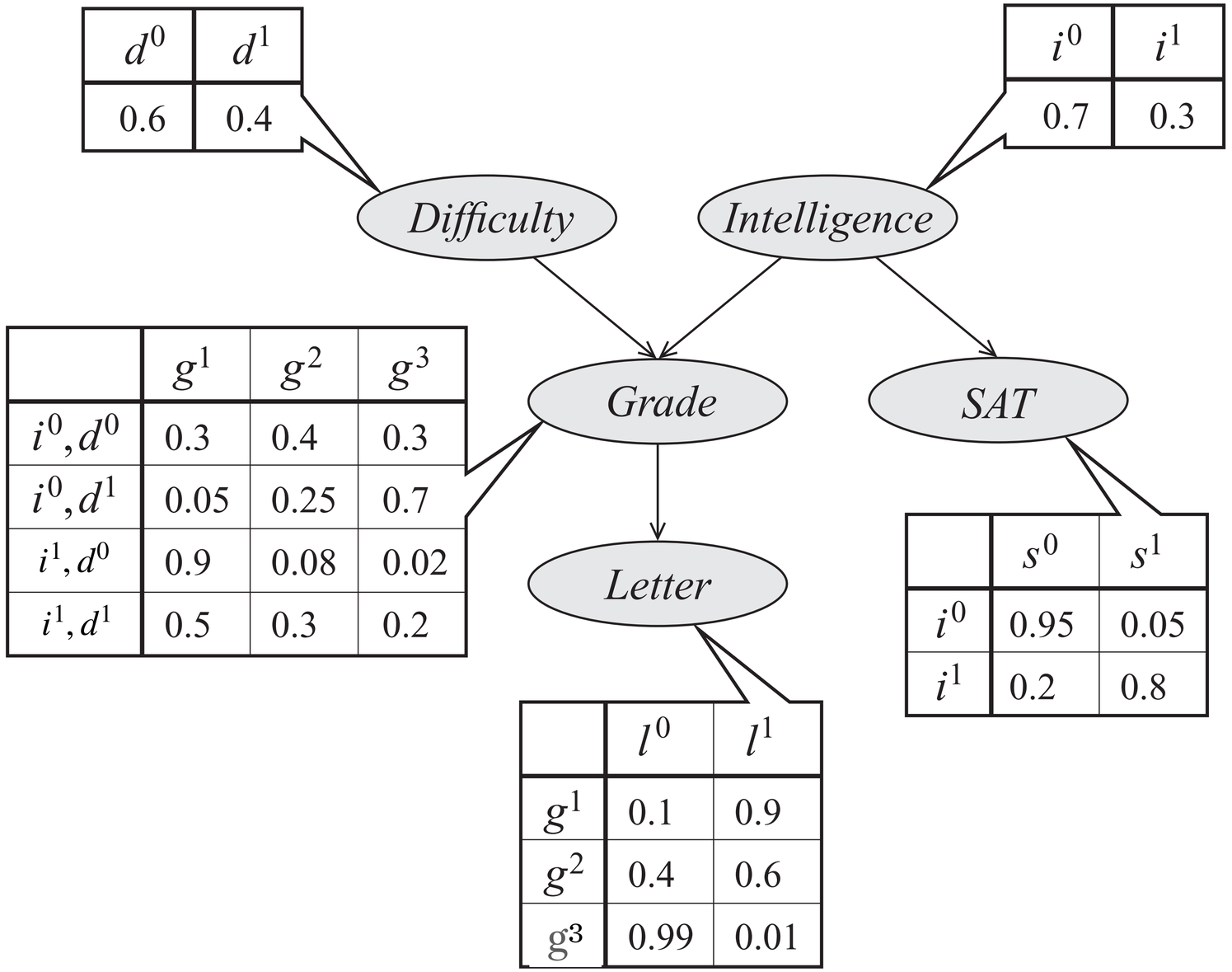
$d^1 i^1 g^2 s^1 l^0$

Sampling in topological order

# Forward sampling from a BN

$P(D, I, G, S, L)$

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$d^0 i^1 g^1 s^0 l^1$

$d^1 i^1 g^2 s^1 l^0$

Sampling in topological order

Parents appear before children

# Forward sampling for Querying

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  - Generate samples from BN
  - Compute fraction where $Y = y$

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For multiplicative bound $\epsilon$ on error with probability $> 1 - \delta$: $M \geq 3\frac{\ln(2/\delta)}{P(y)\epsilon^2}$

# Queries with evidence

- Goal: estimate $P(Y = y \mid \boldsymbol{E} = \boldsymbol{e})$
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  - Throw away all those where $\boldsymbol{E} \neq \boldsymbol{e}$
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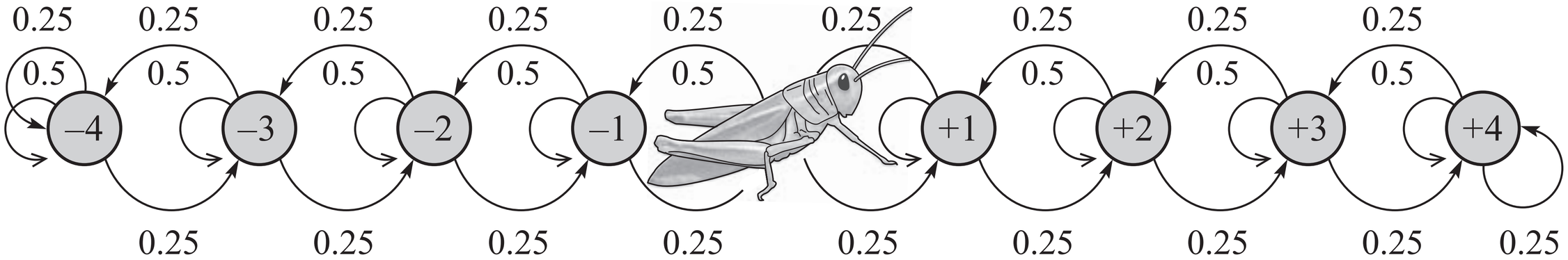
- Forward sampling generally infeasible for Markov Networks


# Markov Chain Monte Carlo

Photo by Matthew Hartley from Helmshore, Lancashire, United Kingdom - Casino de Monte-Carlo, CC BY-SA 2.0, wikimedia

# Markov Chain

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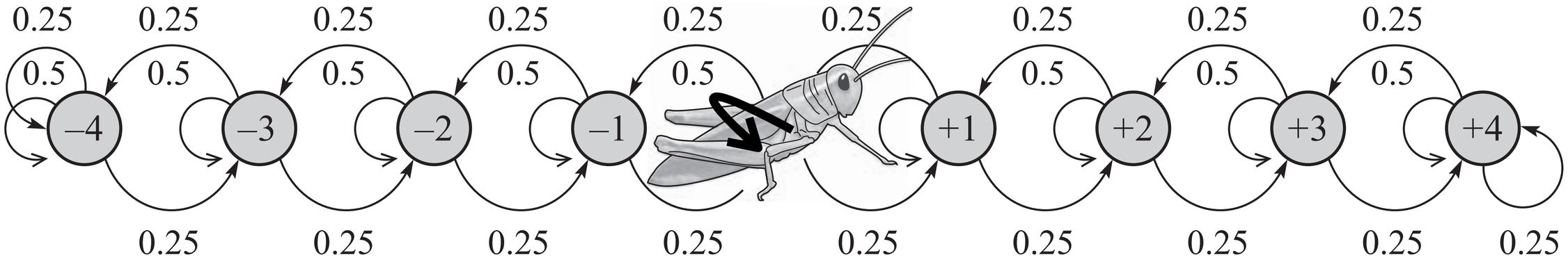
  - For all $x$: $\sum_{x'} T(x \rightarrow x') = 1$

# Markov Chain

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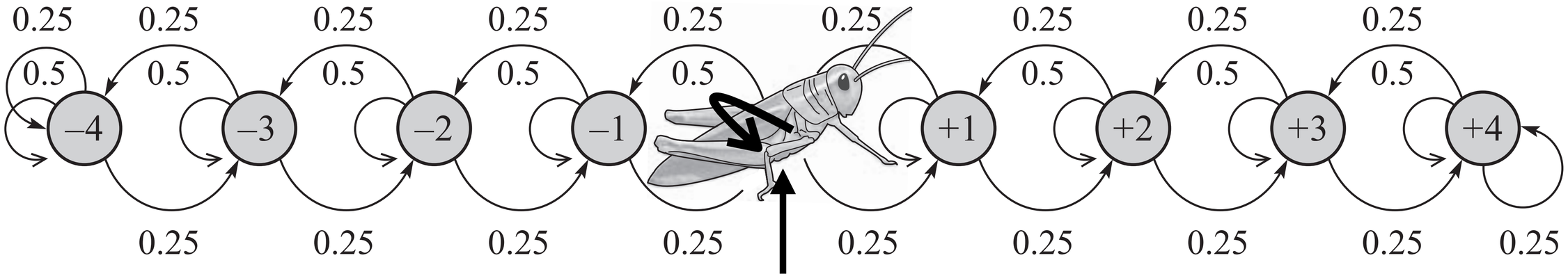


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# Temporal dynamics

$$p^{(t+1)}(X^{(t+1)} = x') = \sum_x P^{(t)}(X^{(t)} = x)\, T(x \to x')$$

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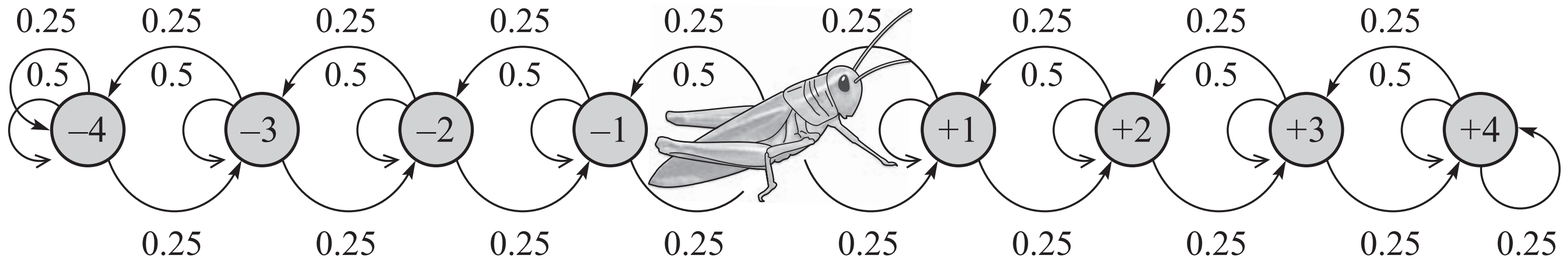


| | -2 | -1 | 0 | +1 | +2 |
|---|---|---|---|---|---|
| $\mathbf{p^{(0)}}$ | 0 | 0 | 1 | 0 | 0 |
| $\mathbf{p^{(1)}}$ | 0 | 0.25 | 0.5 | 0.25 | 0 |
| $\mathbf{p^{(2)}}$ | $0.25^2 = 0.0625$ | $2 \times (0.5 \times 0.25) = 0.25$ | $0.5^2 + 2 \times 0.25^2 = 0.375$ | $2 \times (0.5 \times 0.25) = 0.25$ | $0.25^2 = 0.0625$ |

# Stationary distribution

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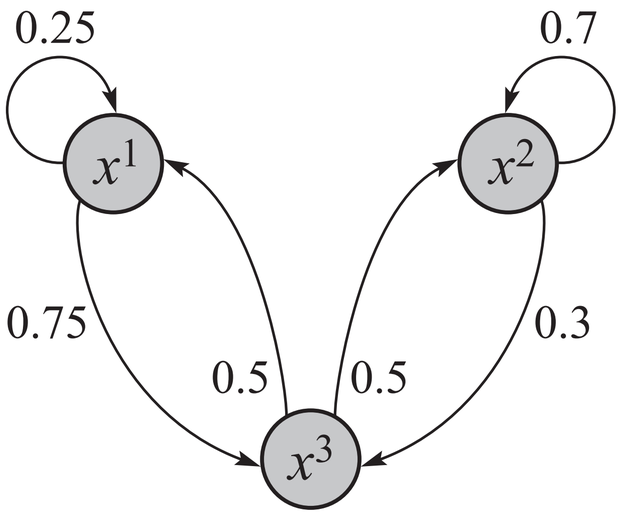

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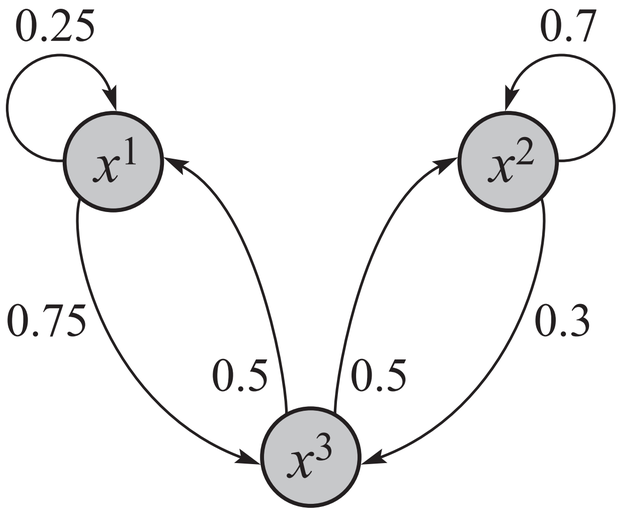


$$\pi(x^1) = 0.25\pi(x^1) + 0.5\pi(x^3)$$
$$\pi(x^2) = 0.7\pi(x^2) + 0.5\pi(x^3)$$
$$\pi(x^3) = 0.75\pi(x^1) + 0.3\pi(x^2)$$

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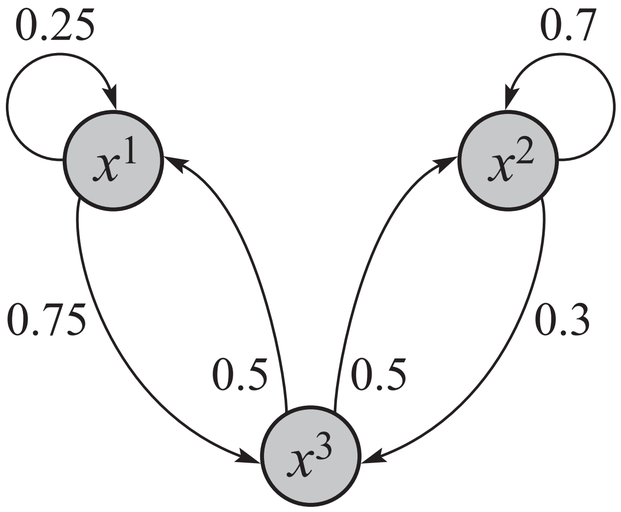


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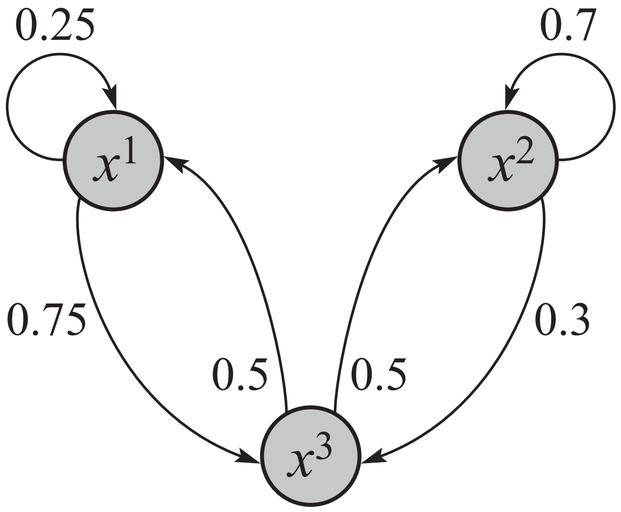


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# Regular Markov chains

- A Markov chain is *regular* if there exists a $k$ such that, for every $x, x'$, the probability of getting from $x$ to $x'$ in exactly $k$ steps is $> 0$

- Theorem: A regular Markov chain converges to a unique stationary distribution regardless of start state

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# Summary Markov chains

- A Markov chain defines a dynamical system from which we can sample trajectories
- Under certain conditions (e.g., regularity), this process is guaranteed to converge to a stationary distribution at the limit
- This allows us to sample from a distribution indirectly, and thereby provides a mechanism for sampling from an intractable distribution

# Using a Markov chain

# Using a Markov chain

- Goal: compute $P(x \in S)$
  - but $P$ is too hard to sample from directly
- Construct a (regular) Markov chain $T$ whose unique stationary distribution is $P$
- Sample $x^{(0)}$ from some initial distribution $P^{(0)}$
- For $t = 0,1,2,\ldots$
  - Generate $x^{(t+1)}$ from $T(x^{(t)} \to x')$

# Using a Markov chain

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**$P^{(t)}$ close enough to $\pi$**

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  - In general, you can never “prove” a chain has mixed
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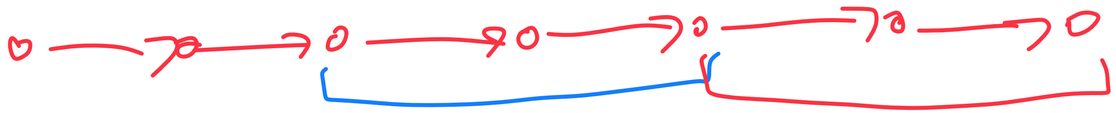

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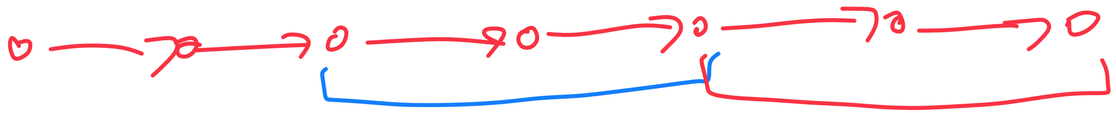

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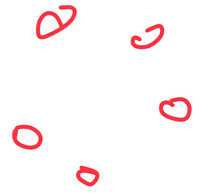

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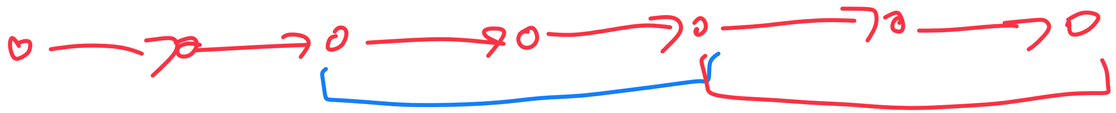

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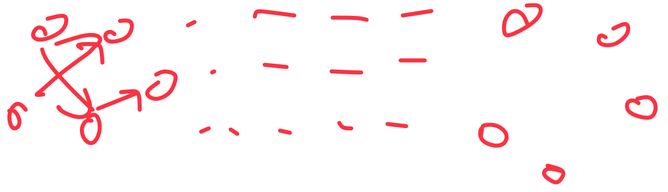
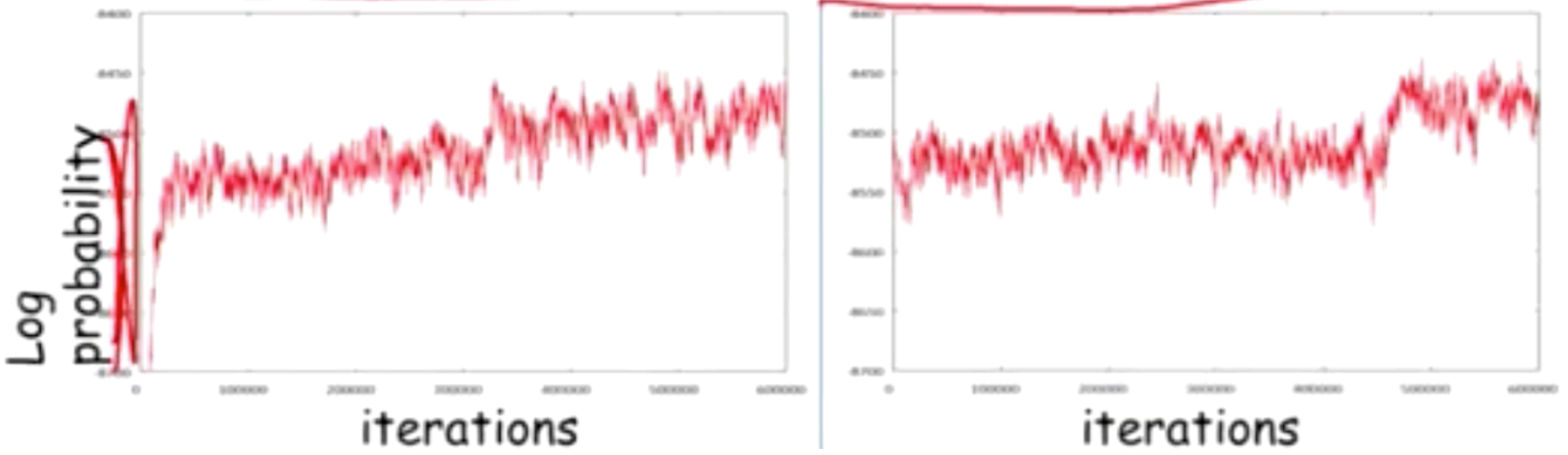
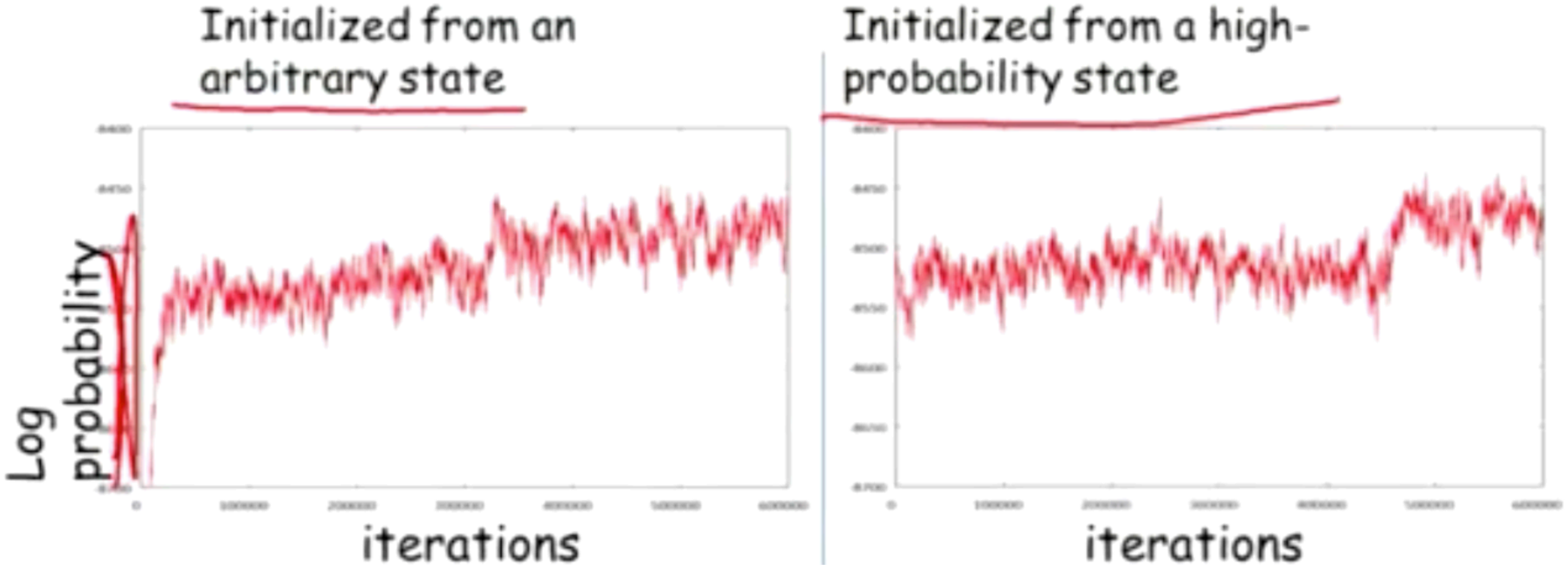


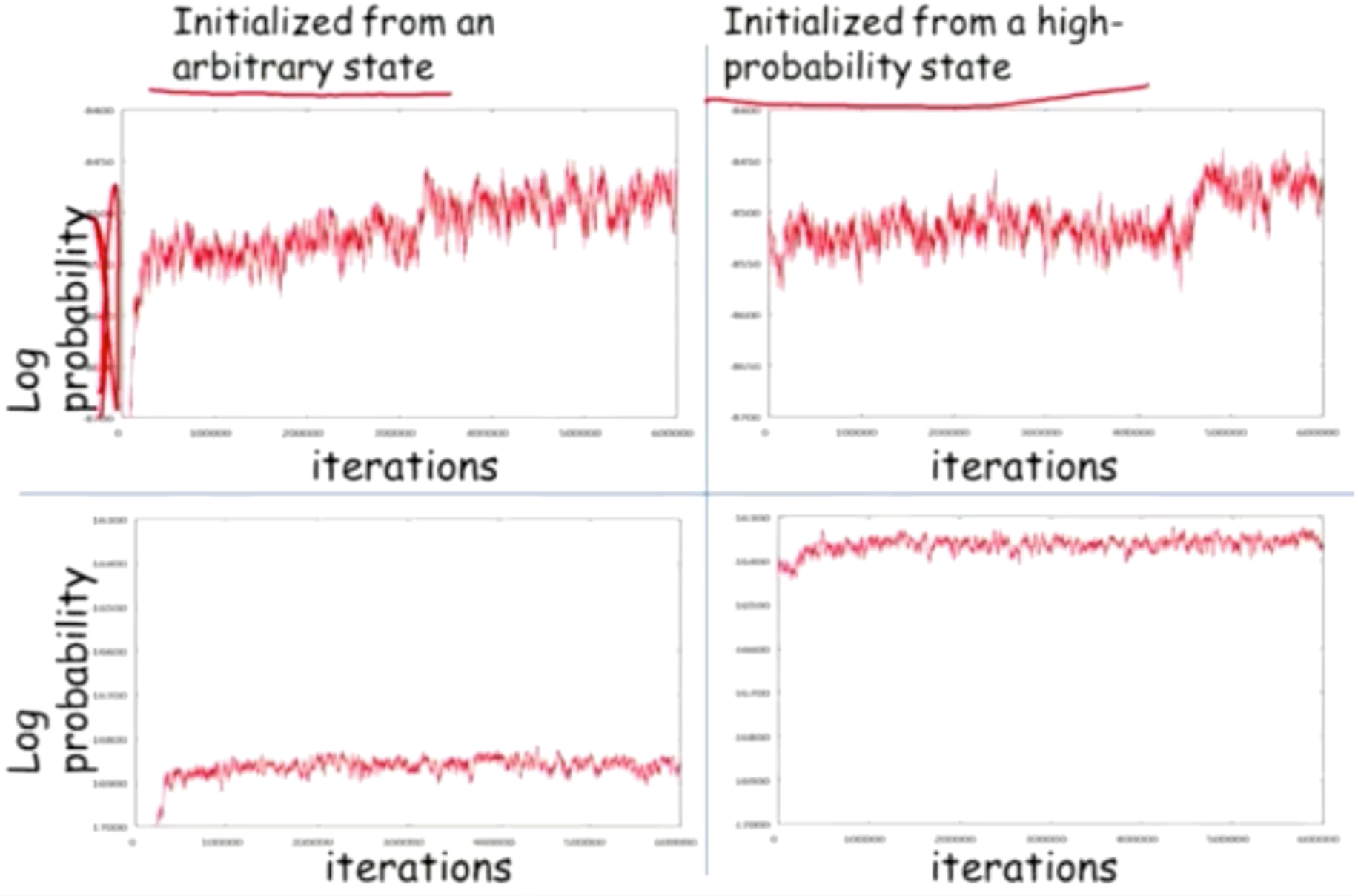
Initialized from an arbitrary state

Initialized from a high-probability state






**Possibly mixed**

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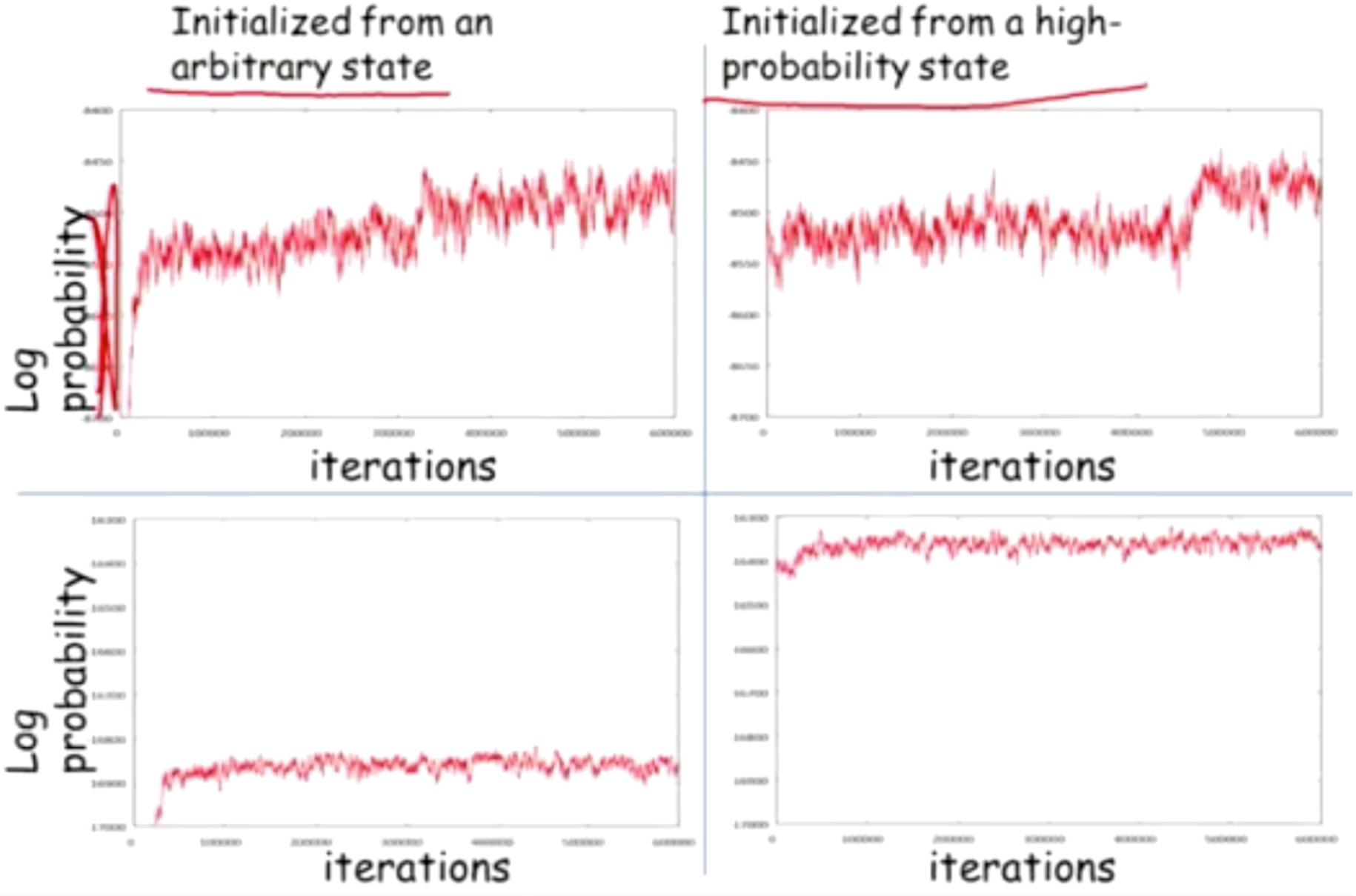


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- The faster the chain mixes, the less correlated (more useful) the samples

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$$\boldsymbol{E}_P[f] \approx \frac{1}{M} \sum_{m=1}^{M} f(x[m])$$

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  - Good theoretical guarantees as $t \to \infty$
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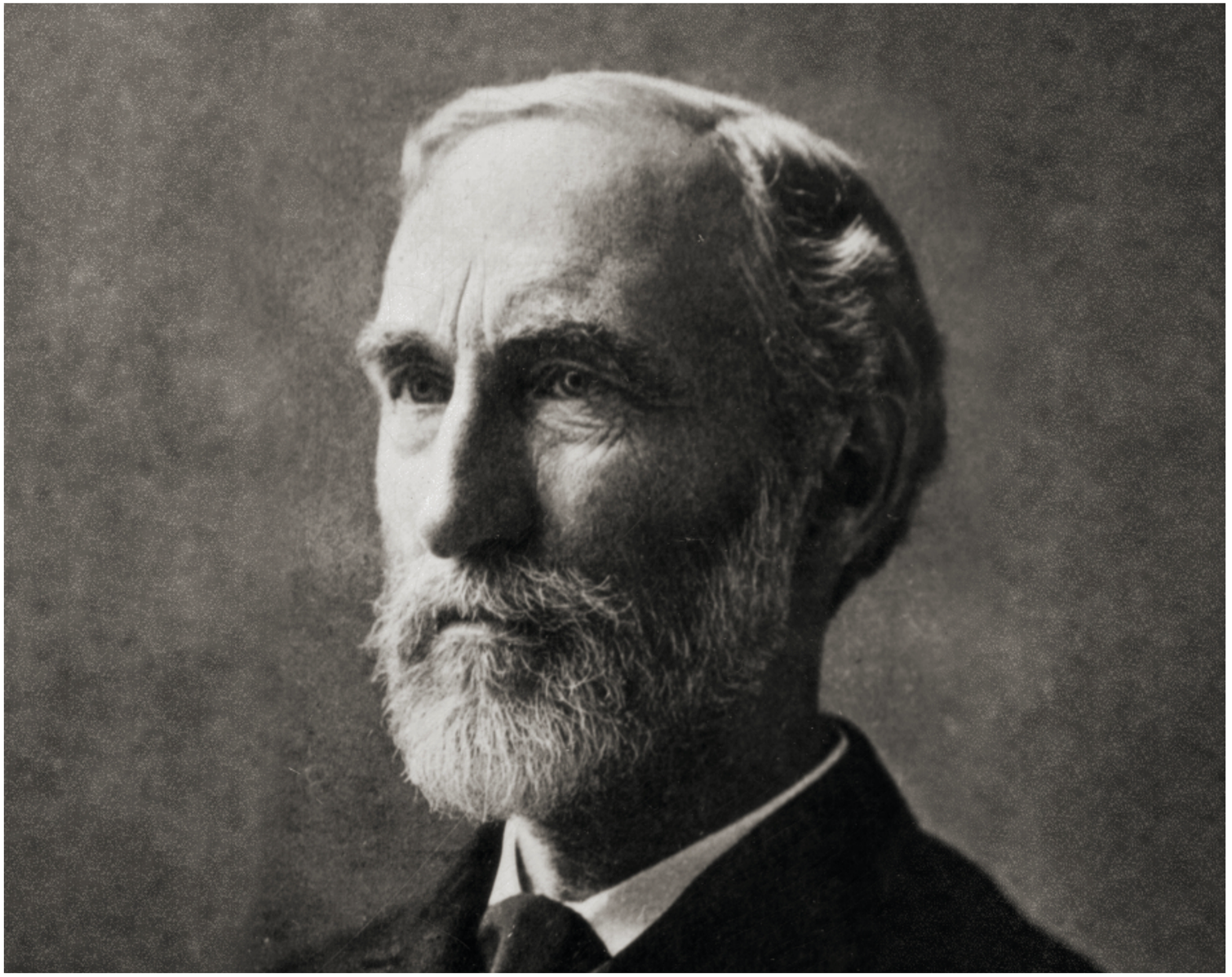
  - Lots of tunable parameters / design choices
  - Can be quite slow to converge
  - Difficult to tell whether it's working

# Gibbs sampling



# MCMC for PGM: The Gibbs Chain

J. W. Gibbs
(1839 — 1903)

# Gibbs chain

- Target distribution $P_\Phi(X_1, \ldots, X_n)$
- Markov chain state space: complete set of assignments $\boldsymbol{x}$ to $\boldsymbol{X} = \{X_1, \ldots, X_n\}$
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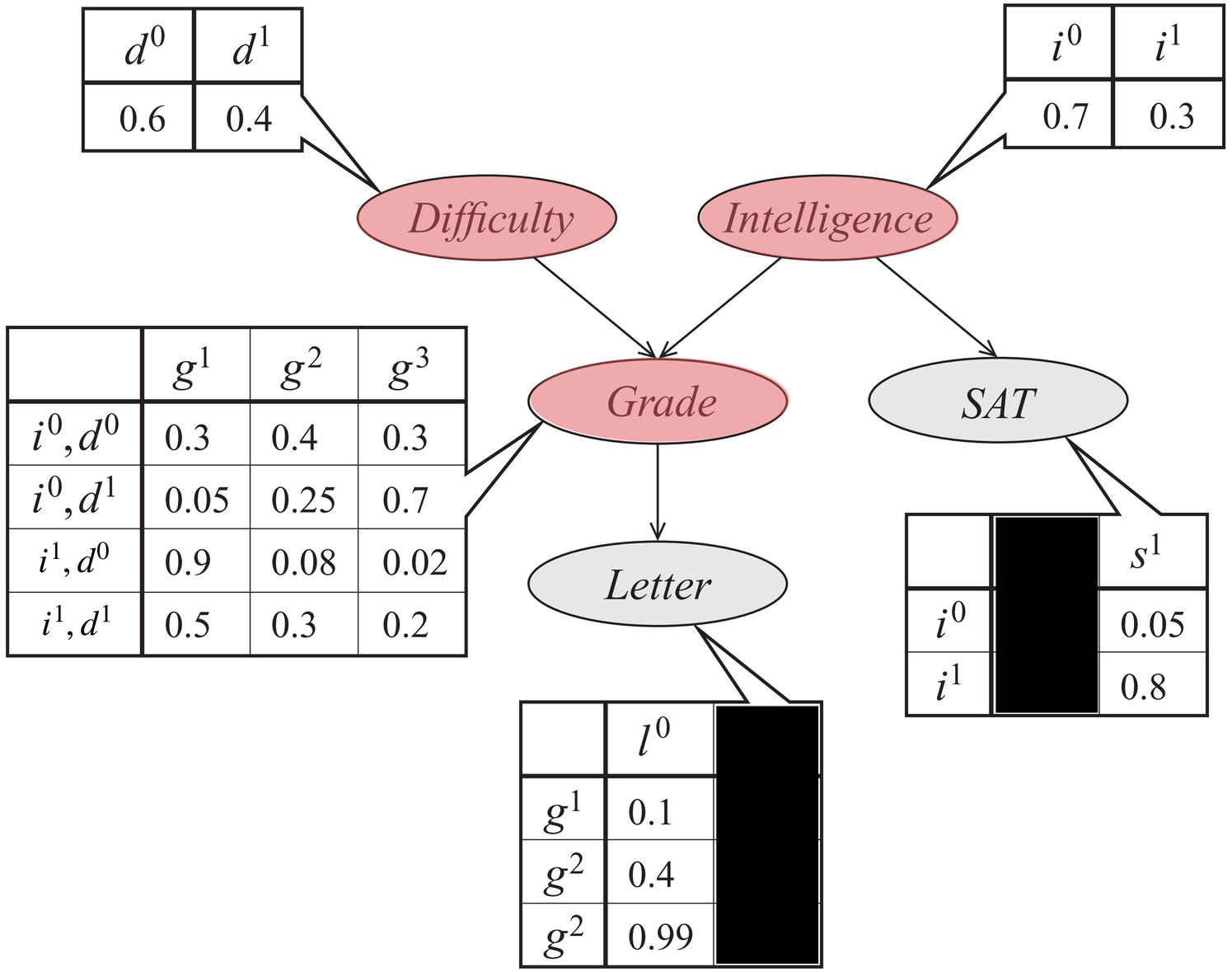
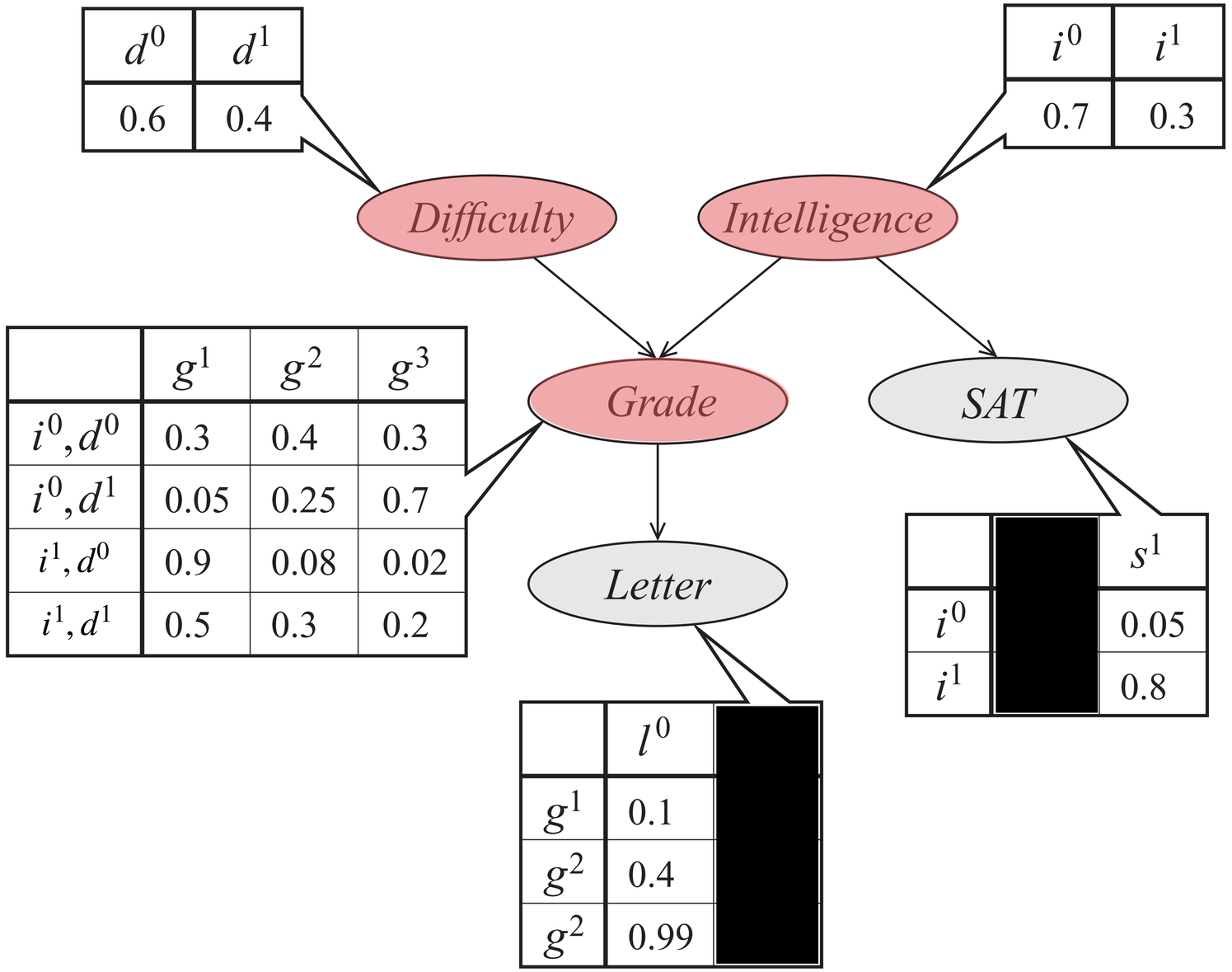
# Example


$d^0$ $d^1$
0.6 0.4
$i^0$ $i^1$
0.7 0.3
Difficulty
Intelligence
Grade
SAT
Letter
$g^1$ $g^2$ $g^3$
$i^0, d^0$ 0.3 0.4 0.3
$i^0, d^1$ 0.05 0.25 0.7
$i^1, d^0$ 0.9 0.08 0.02
$i^1, d^1$ 0.5 0.3 0.2
$s^1$
$i^0$ 0.05
$i^1$ 0.8
$l^0$
$g^1$ 0.1
$g^2$ 0.4
$g^2$ 0.99

# Example

$\tilde{P}_{\Phi}(D, I, G \mid s^1, l^0)$



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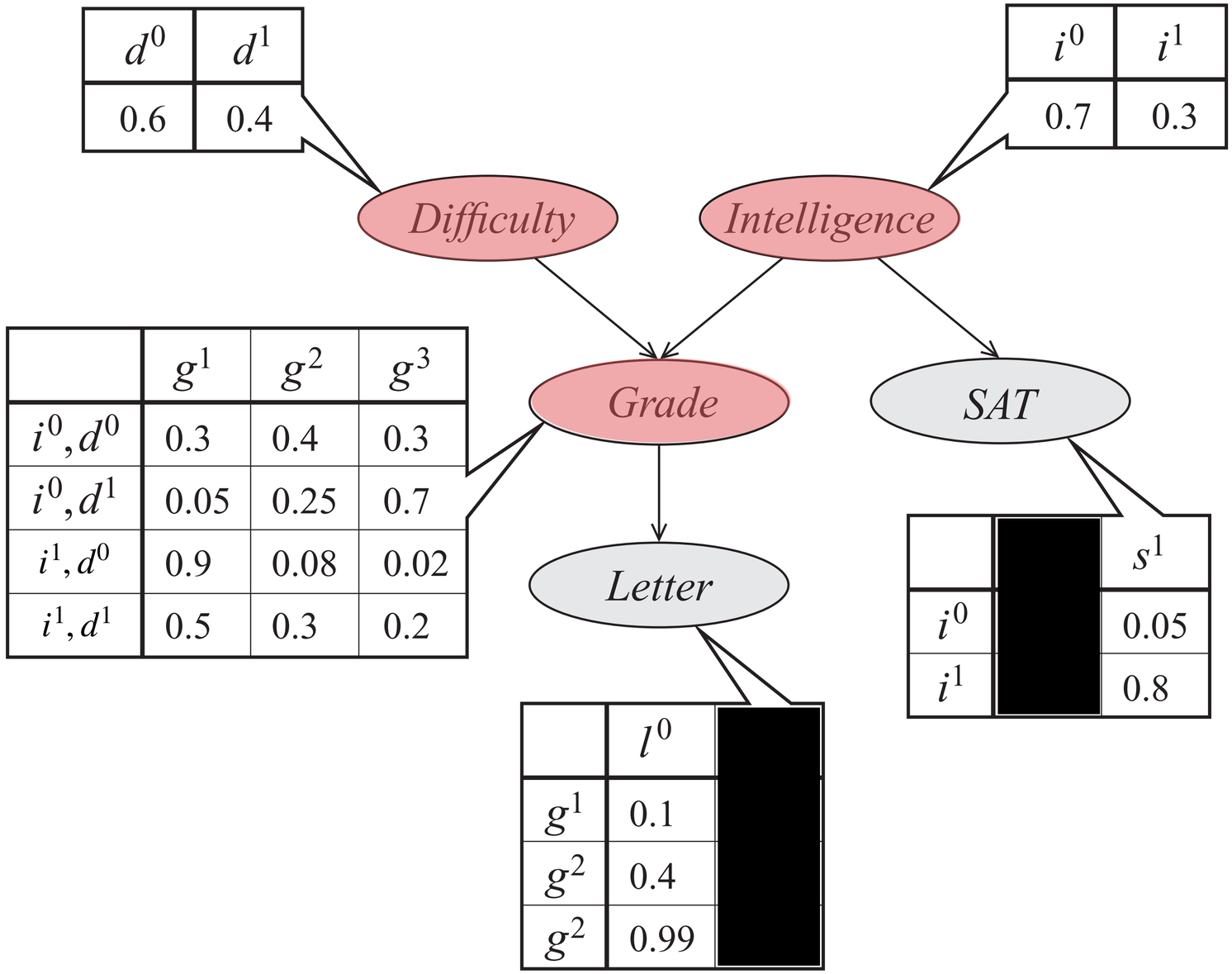

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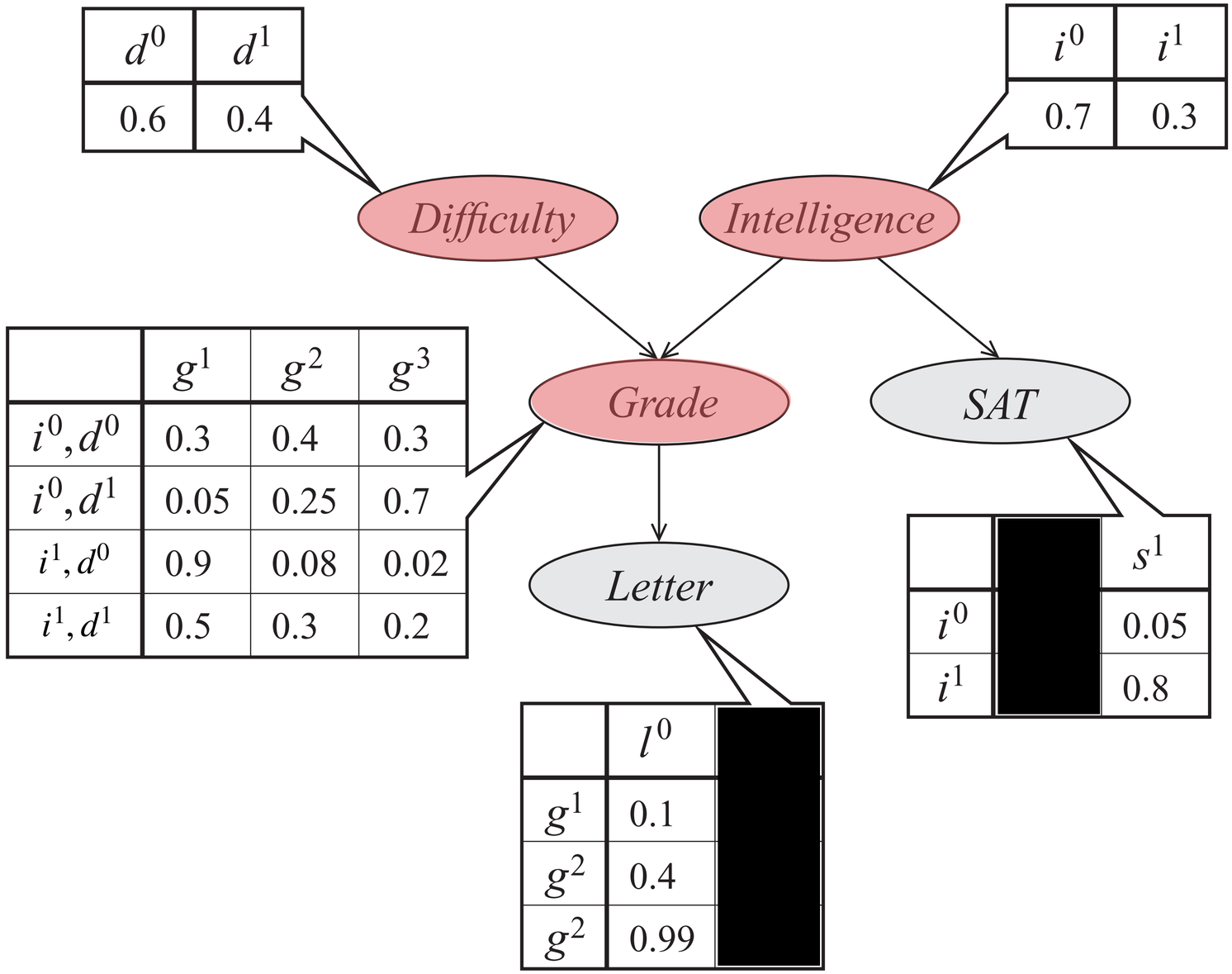


| $d^0$ | $d^1$ |
|---|---|
| 0.6 | 0.4 |

| $i^0$ | $i^1$ |
|---|---|
| 0.7 | 0.3 |

| | $g^1$ | $g^2$ | $g^3$ |
|---|---|---|---|
| $i^0, d^0$ | 0.3 | 0.4 | 0.3 |
| $i^0, d^1$ | 0.05 | 0.25 | 0.7 |
| $i^1, d^0$ | 0.9 | 0.08 | 0.02 |
| $i^1, d^1$ | 0.5 | 0.3 | 0.2 |

| | | $s^1$ |
|---|---|---|
| $i^0$ | | 0.05 |
| $i^1$ | | 0.8 |

| | $l^0$ |
|---|---|
| $g^1$ | 0.1 |
| $g^2$ | 0.4 |
| $g^2$ | 0.99 |

$P(D \mid i^0, g^1, l^0, s^1)$

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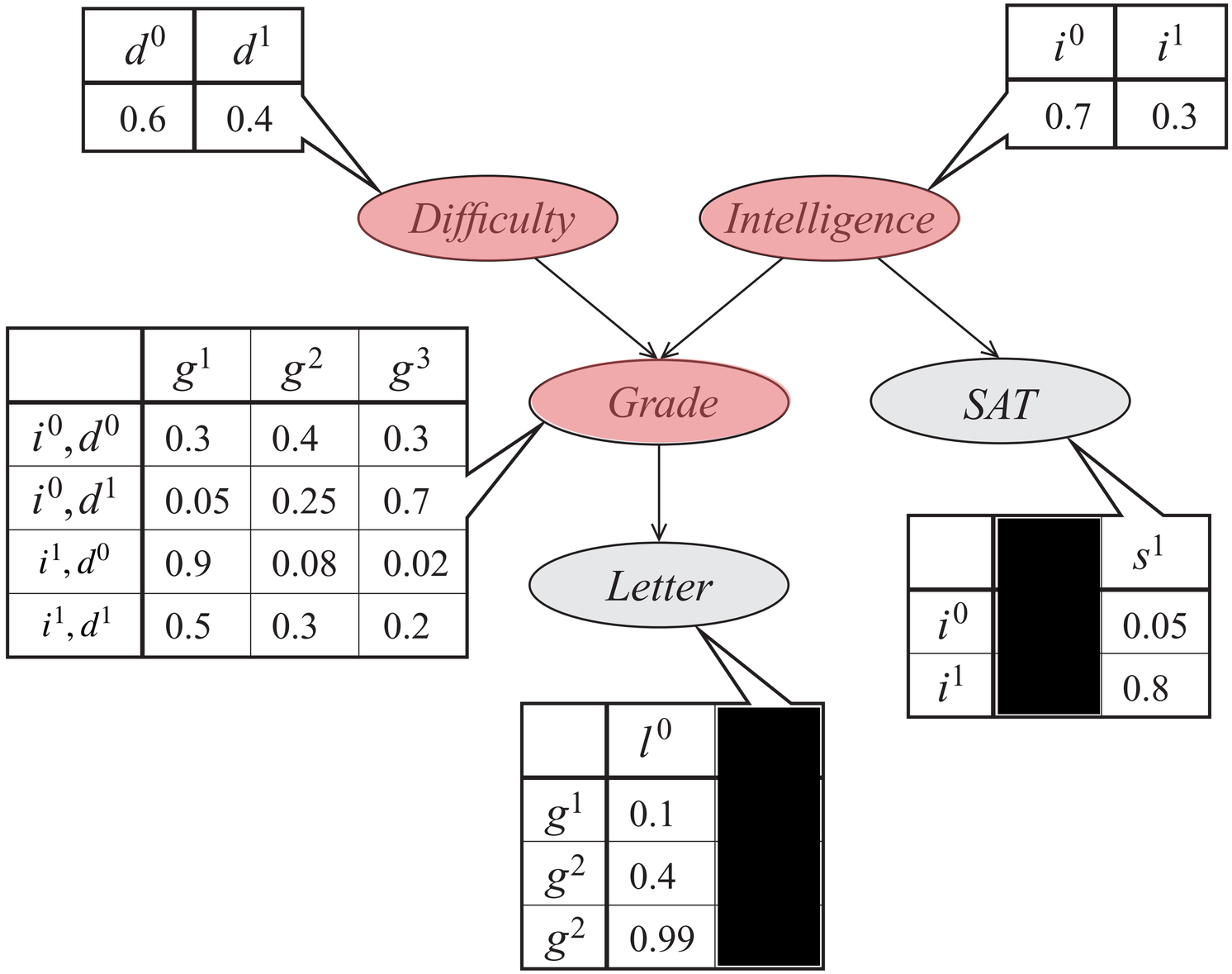
$d^0\ i^0\ g^1$

$d^1\ i^0\ g^1$



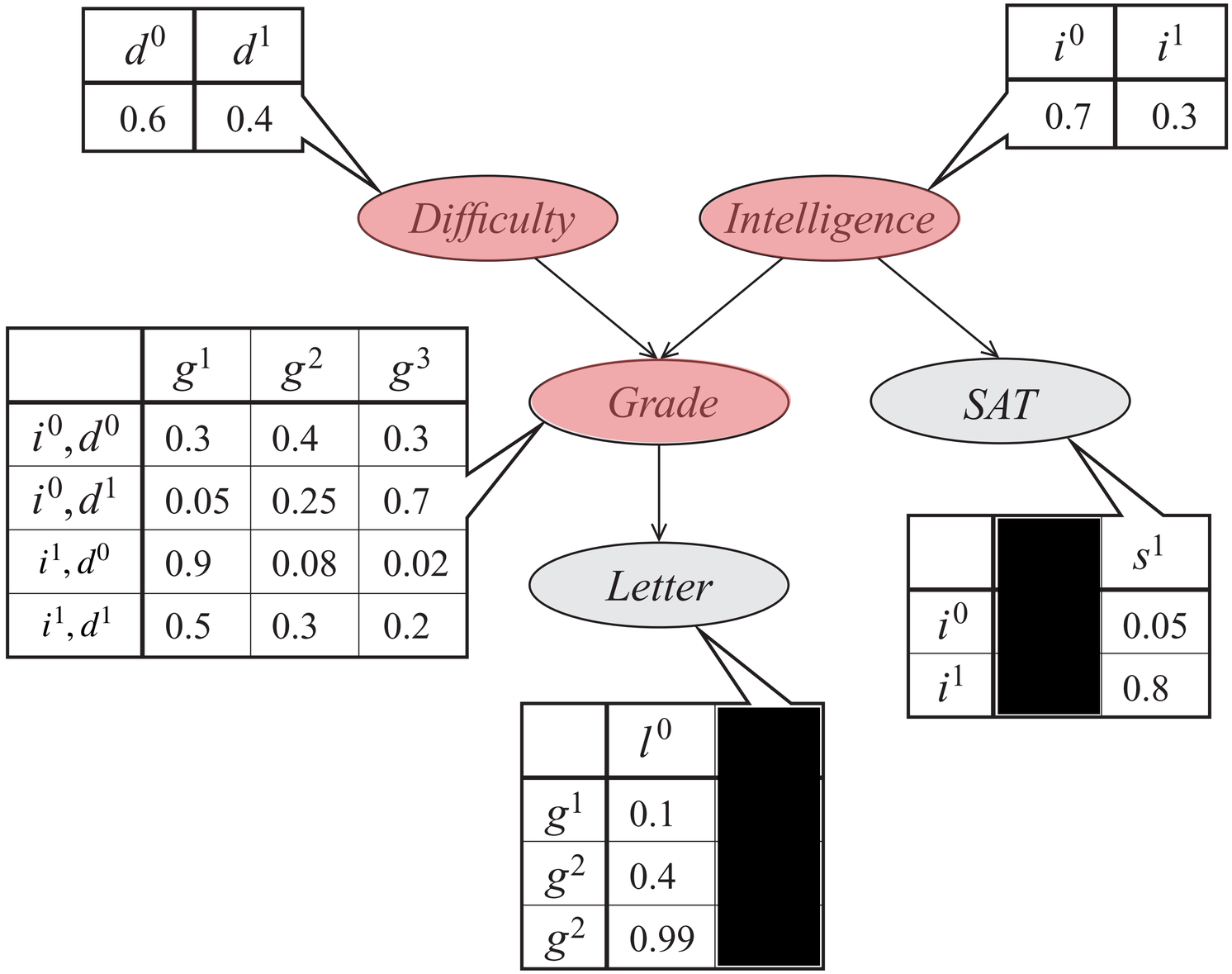

$P(D \mid i^0, g^1, l^0, s^1)$

# Example

$\tilde{P}_\Phi(D, I, G \mid s^1, l^0)$

$d^0\ i^0\ g^1$

$d^1\ i^0\ g^1$




$P(D \mid i^0, g^1, l^0, s^1)$

$P(I \mid d^1, g^1, l^0, s^1)$

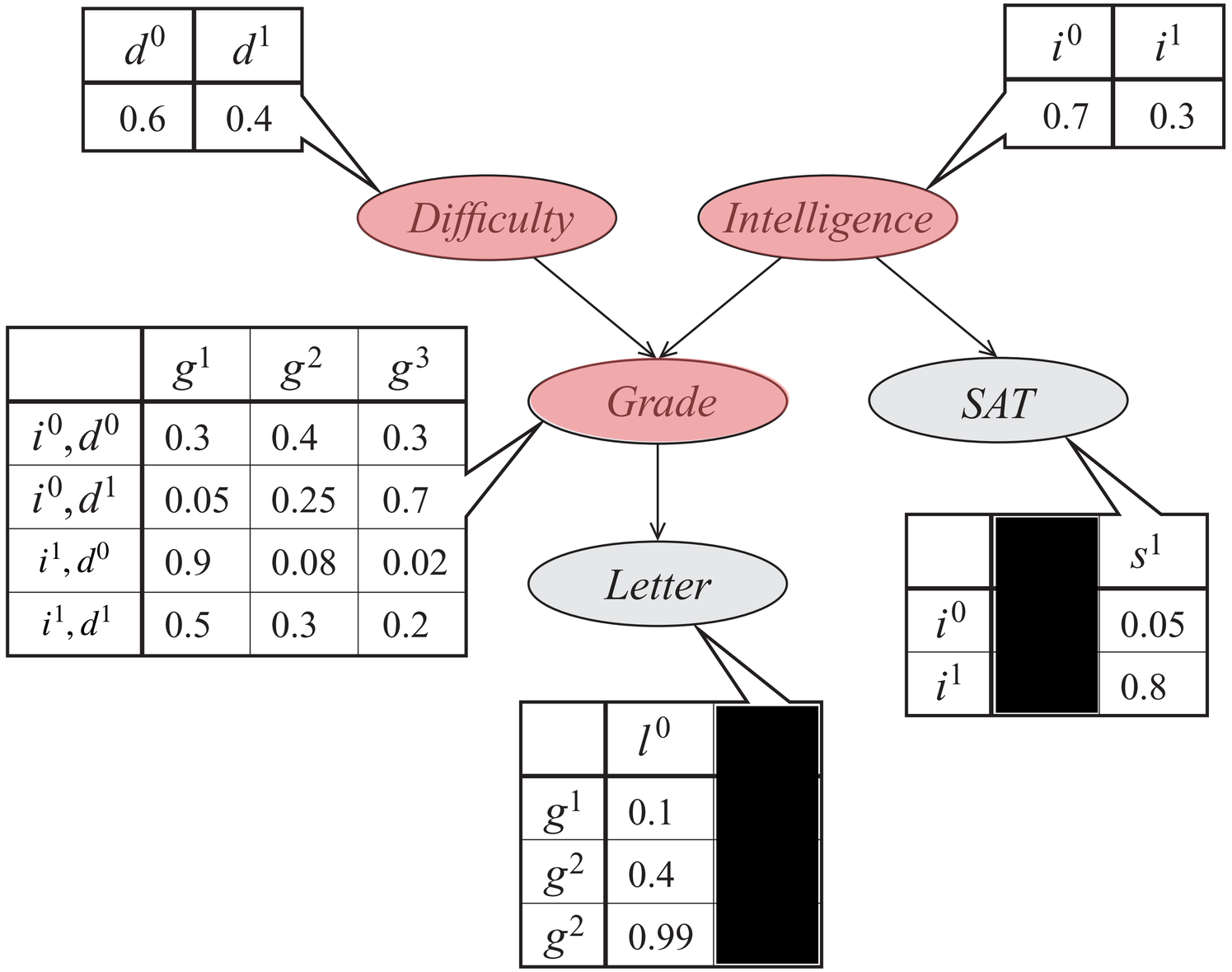
# Example

$\tilde{P}_\Phi(D, I, G \mid s^1, l^0)$

$d^0\ i^0\ g^1$

$d^1\ i^0\ g^1$

$d^1\ i^1\ g^1$




| $d^0$ | $d^1$ |
|---|---|
| 0.6 | 0.4 |

| $i^0$ | $i^1$ |
|---|---|
| 0.7 | 0.3 |

| | $g^1$ | $g^2$ | $g^3$ |
|---|---|---|---|
| $i^0, d^0$ | 0.3 | 0.4 | 0.3 |
| $i^0, d^1$ | 0.05 | 0.25 | 0.7 |
| $i^1, d^0$ | 0.9 | 0.08 | 0.02 |
| $i^1, d^1$ | 0.5 | 0.3 | 0.2 |

| | | $s^1$ |
|---|---|---|
| $i^0$ | | 0.05 |
| $i^1$ | | 0.8 |

| | $l^0$ |
|---|---|
| $g^1$ | 0.1 |
| $g^2$ | 0.4 |
| $g^2$ | 0.99 |

$P(D \mid i^0, g^1, l^0, s^1)$

$P(I \mid d^1, g^1, l^0, s^1)$

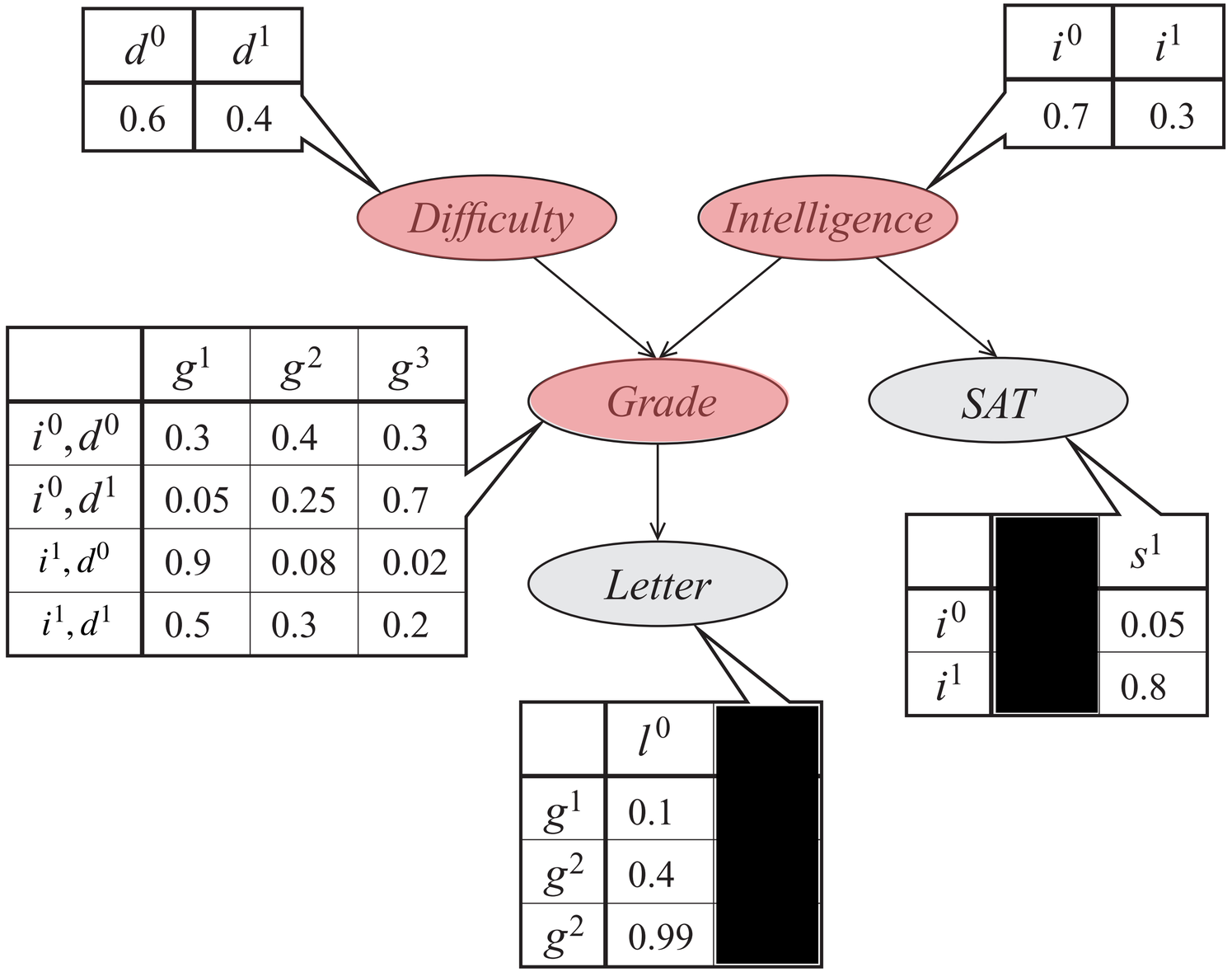
# Example

$\tilde{P}_\Phi(D, I, G \mid s^1, l^0)$

$d^0\ i^0\ g^1$

$d^1\ i^0\ g^1$

$d^1\ i^1\ g^1$

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| $g^2$ | 0.99 |

$P(D \mid i^0, g^1, l^0, s^1)$

$P(I \mid d^1, g^1, l^0, s^1)$

$P(G \mid d^1, i^1, l^0, s^1)$

# Example

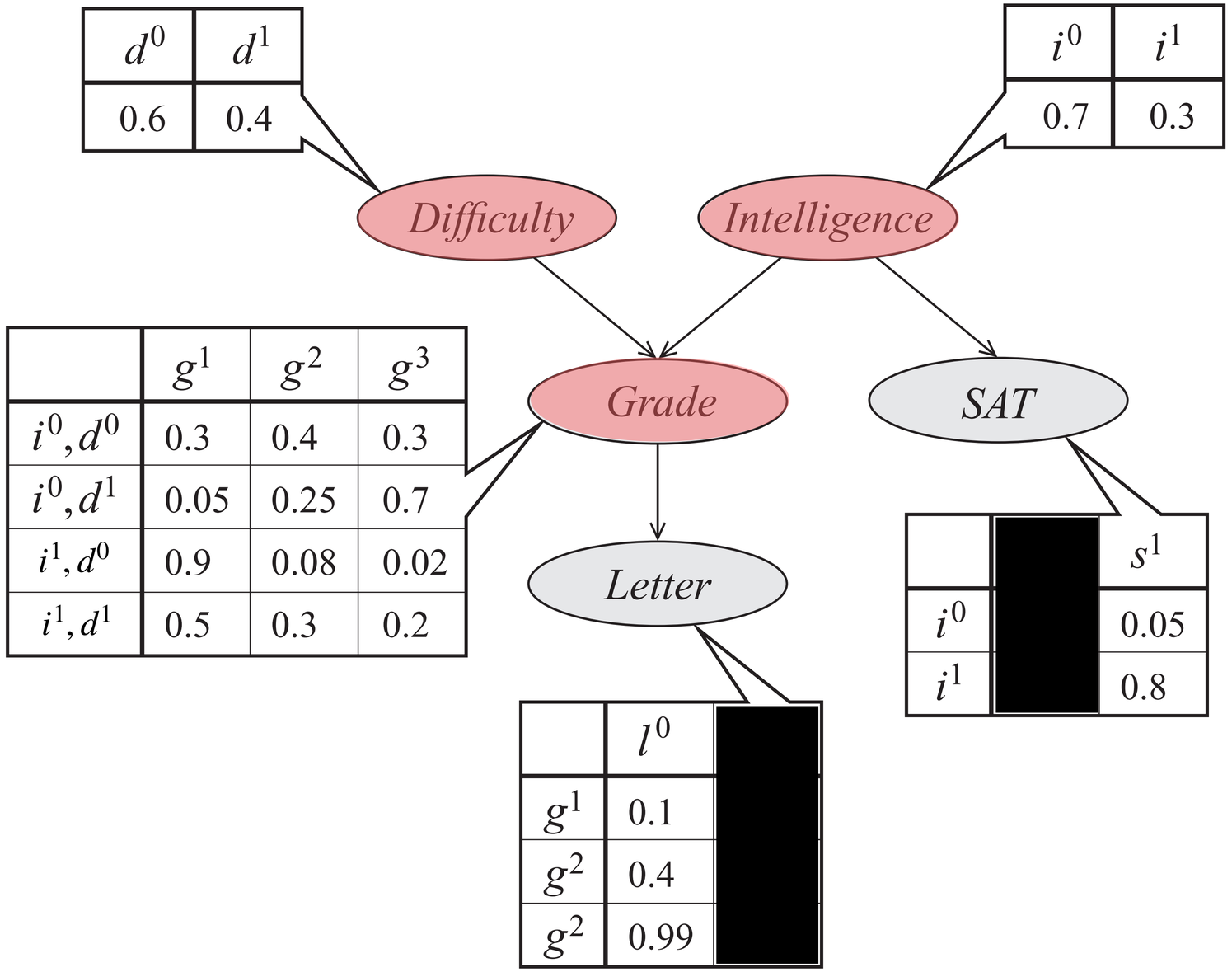
$\tilde{P}_\Phi(D, I, G \mid s^1, l^0)$

$d^0\ i^0\ g^1$

$d^1\ i^0\ g^1$

$d^1\ i^1\ g^1$

$d^1\ i^1\ g^3$




| $d^0$ | $d^1$ |
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| 0.6 | 0.4 |

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|---|---|
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| $i^0, d^0$ | 0.3 | 0.4 | 0.3 |
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| $i^1, d^1$ | 0.5 | 0.3 | 0.2 |

| | [redacted] | $s^1$ |
|---|---|---|
| $i^0$ | | 0.05 |
| $i^1$ | | 0.8 |

| | $l^0$ | [redacted] |
|---|---|---|
| $g^1$ | 0.1 | |
| $g^2$ | 0.4 | |
| $g^2$ | 0.99 | |

$P(D \mid i^0, g^1, l^0, s^1)$

$P(I \mid d^1, g^1, l^0, s^1)$

$P(G \mid d^1, i^1, l^0, s^1)$

# Example

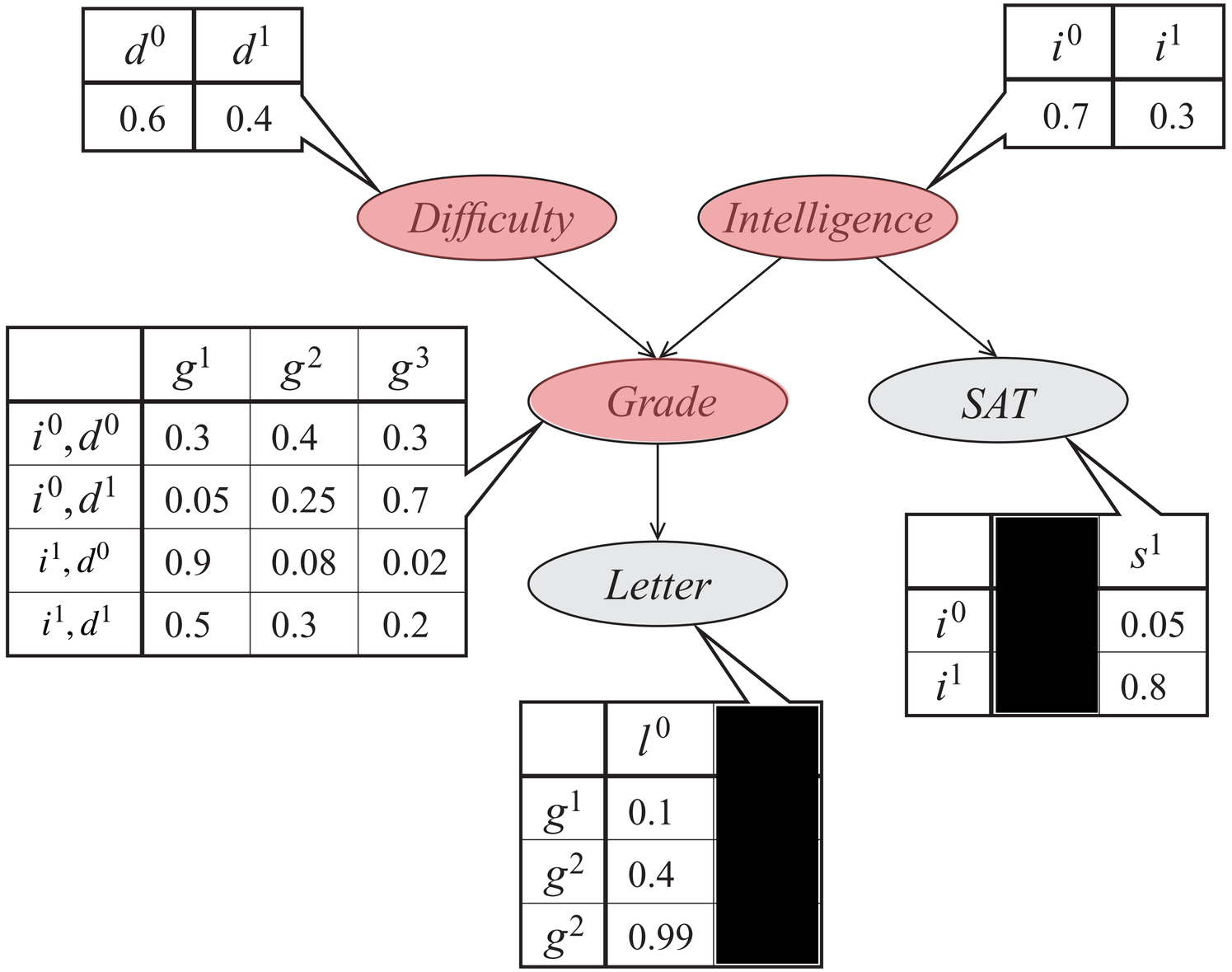
$\tilde{P}_\Phi(D, I, G \mid s^1, l^0)$

$d^0\ i^0\ g^1$

$d^1\ i^0\ g^1$

$d^1\ i^1\ g^1$

$d^1\ i^1\ g^3$




| $d^0$ | $d^1$ |
|---|---|
| 0.6 | 0.4 |

| $i^0$ | $i^1$ |
|---|---|
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| | [redacted] | $s^1$ |
|---|---|---|
| $i^0$ | | 0.05 |
| $i^1$ | | 0.8 |

| | $l^0$ | [redacted] |
|---|---|---|
| $g^1$ | 0.1 | |
| $g^2$ | 0.4 | |
| $g^2$ | 0.99 | |

$P(D \mid i^0, g^1, l^0, s^1)$

$P(I \mid d^1, g^1, l^0, s^1)$

$P(G \mid d^1, i^1, l^0, s^1)$

# Computational cost

- For $i = 1,\ldots,n$
  - Sample $x_i \sim P_\Phi(X_i \mid \boldsymbol{x}_{-i})$

$$P_\Phi(X_i \mid \boldsymbol{x}_{-i}) = \frac{P_\Phi(X_i, \boldsymbol{x}_{-i})}{P_\Phi(\boldsymbol{x}_{-i})} = \frac{\tilde{P}_\Phi(X_i, \boldsymbol{x}_{-i})}{\tilde{P}_\Phi(\boldsymbol{x}_{-i})}$$

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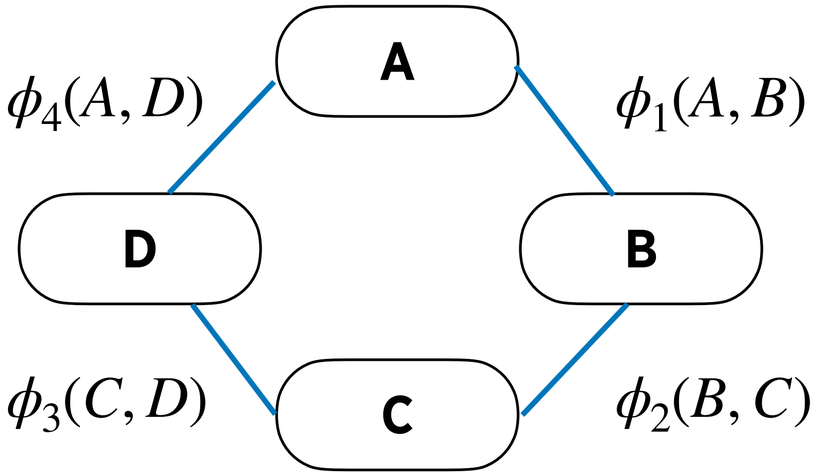
**Complete assignment**
**Product of factors**

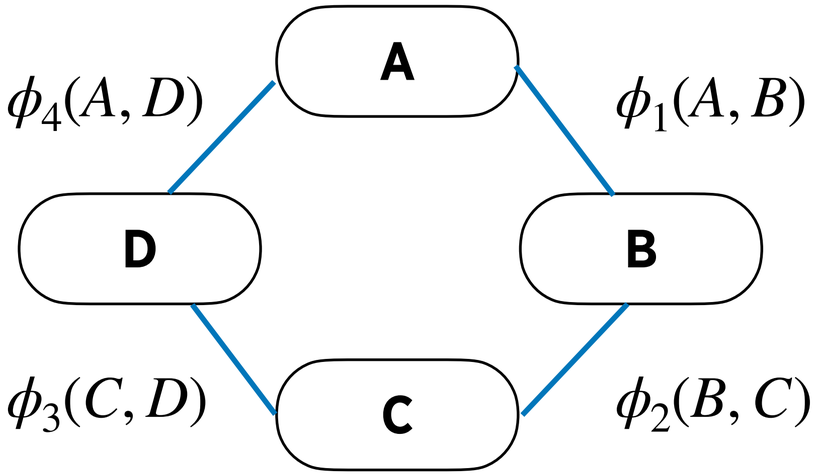
# Another example




$$P_\Phi(A \mid b, c, d) = \frac{\tilde{P}_\Phi(a, b, c, d)}{\sum_{A'} \tilde{P}_\Phi(A', b, c, d)}$$

$$\frac{\phi_1(A, b)\, \phi_2(b, c)\, \phi_3(c, d)\, \phi_4(A, d)}{\sum_{A'} \phi_1(A', b)\, \phi_2(b, c)\, \phi_3(c, d)\, \phi_4(A', d)}$$

# Another example

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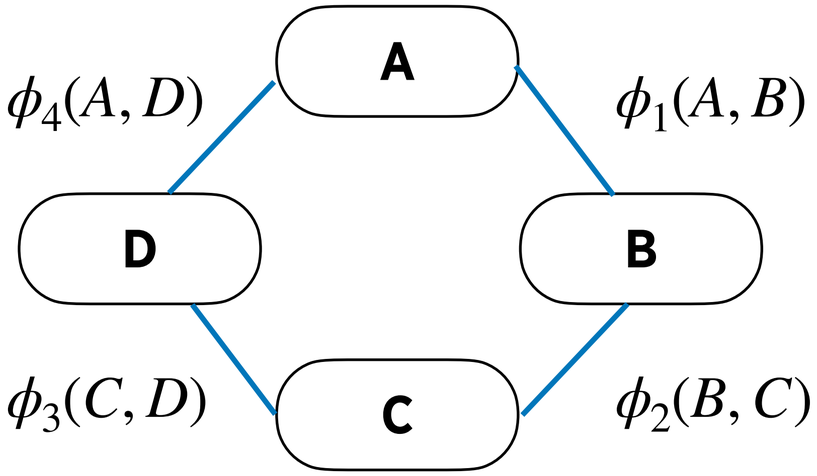
# Another example

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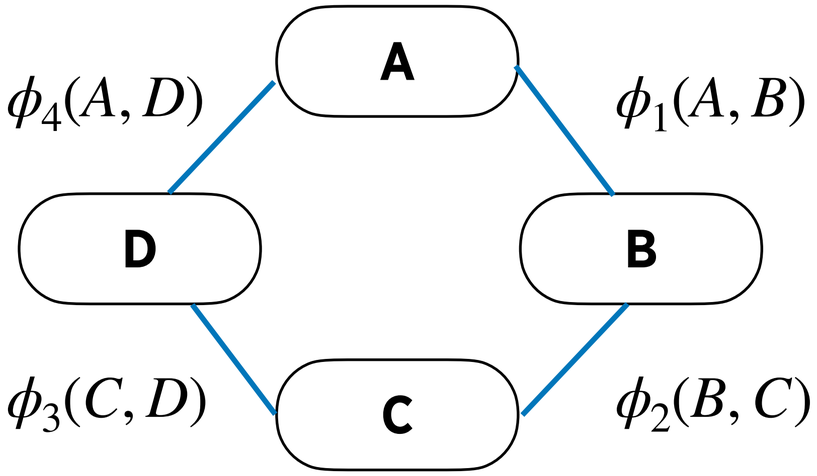
# Another example

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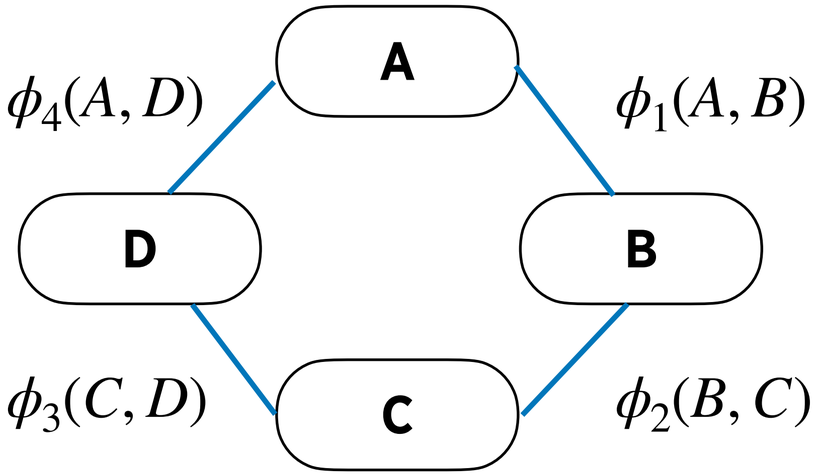


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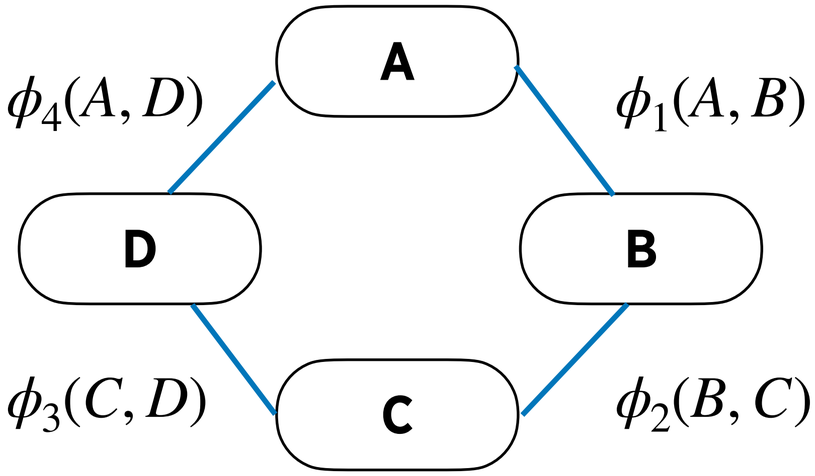
# Another example

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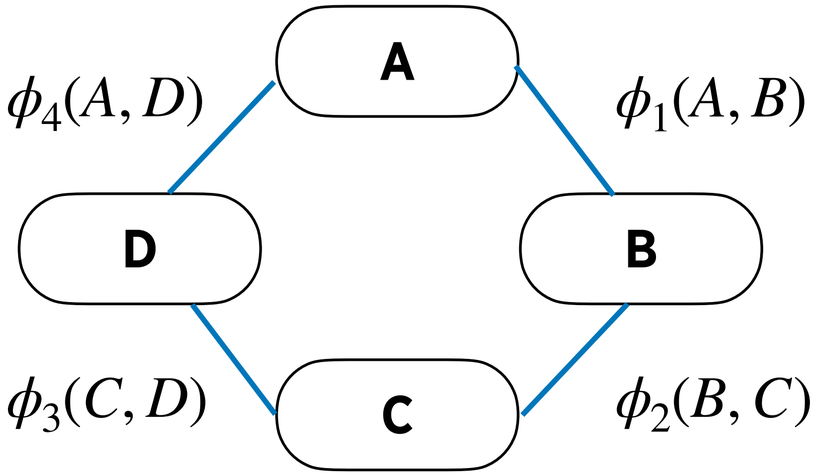
**Factors that involve $A$**

# Another example

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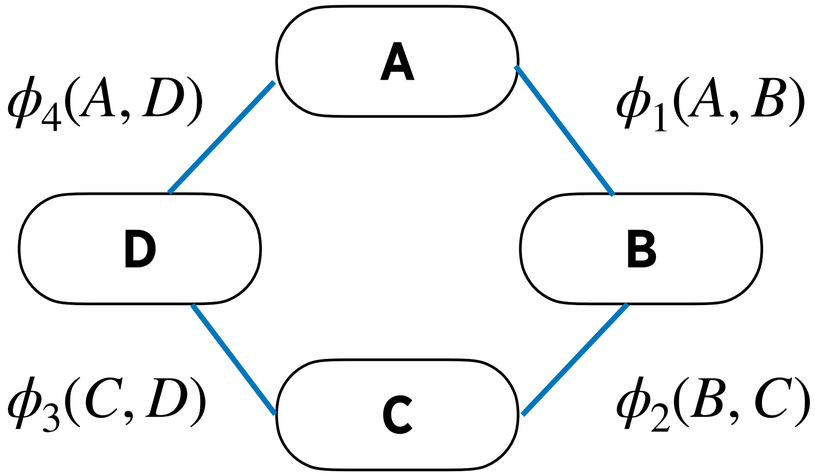
**Factors that involve $A$**

**normalising constant**

# Another example

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**Factors that involve $A$** **normalising constant**

$$\propto \phi_1(A, b)\, \phi_4(A, d)$$

# Computational cost revisited

- For $i = 1,\ldots,n$
  - Sample $x_i \sim P_\Phi(X_i \mid \boldsymbol{x}_{-i})$

$$P_\Phi(X_i \mid \boldsymbol{x}_{-i}) = \frac{P_\Phi(X_i, \boldsymbol{x}_{-i})}{P_\Phi(\boldsymbol{x}_{-i})} = \frac{\tilde{P}_\Phi(X_i, \boldsymbol{x}_{-i})}{\tilde{P}_\Phi(\boldsymbol{x}_{-i})}$$

$$\propto \prod_{j : X_i \in \text{scope}[C_j]} \phi_j(X_i, \boldsymbol{x}_{j,-i})$$

# Computational cost revisited

- For $i = 1,\ldots,n$
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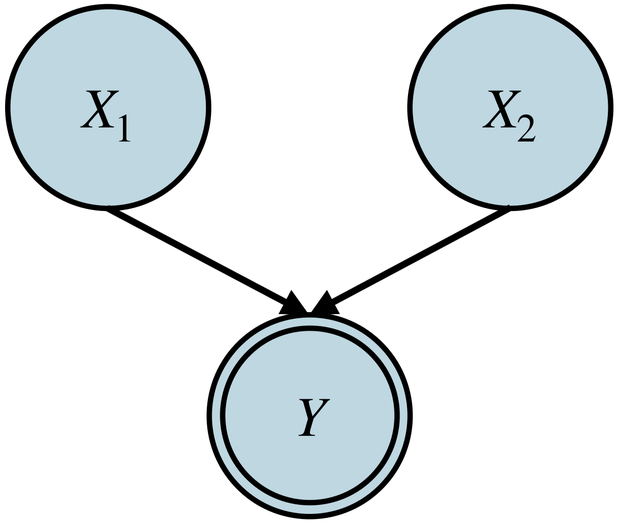
only $X_i$ and its neighbours

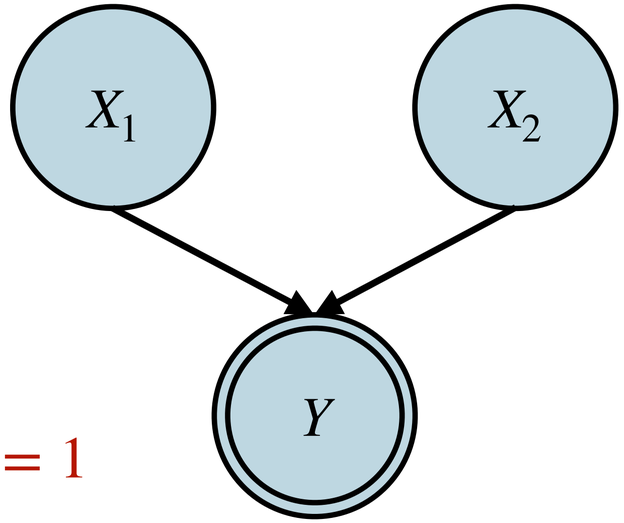
$$\propto \prod_{j:X_i \in \text{scope}[C_j]} \phi_j(X_i, \boldsymbol{x}_{j,-i})$$

# Gibbs chain and regularity




| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| 0 | 0 | 0 | 0.25 |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| 1 | 1 | 0 | 0.25 |

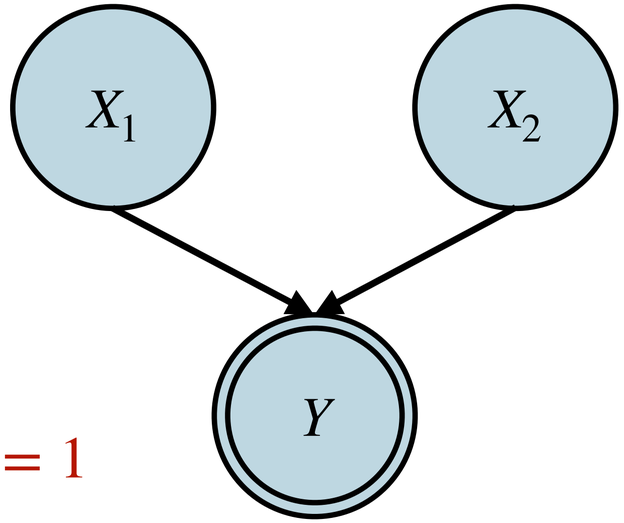
# Gibbs chain and regularity




$Y = 1$

| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| 0 | 0 | 0 | 0.25 |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| 1 | 1 | 0 | 0.25 |

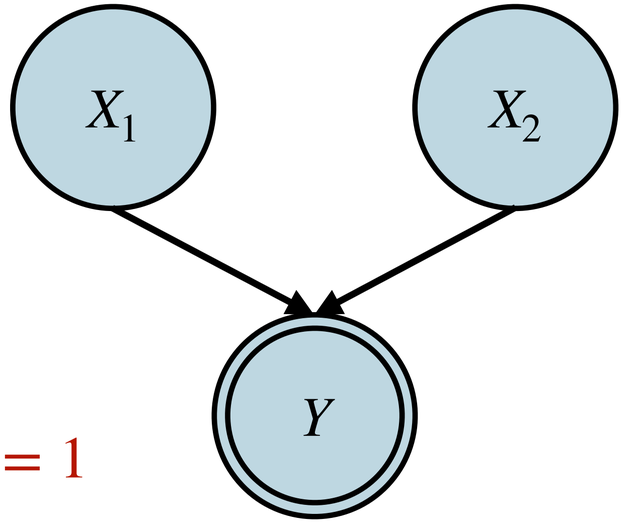
# Gibbs chain and regularity




$Y = 1$

| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

# Gibbs chain and regularity

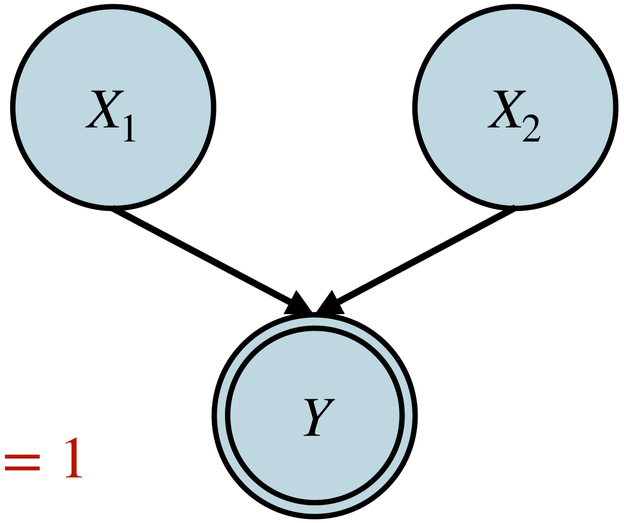



$Y = 1$

| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

# Gibbs chain and regularity

$X_1$ $X_2$ $Y = 1$



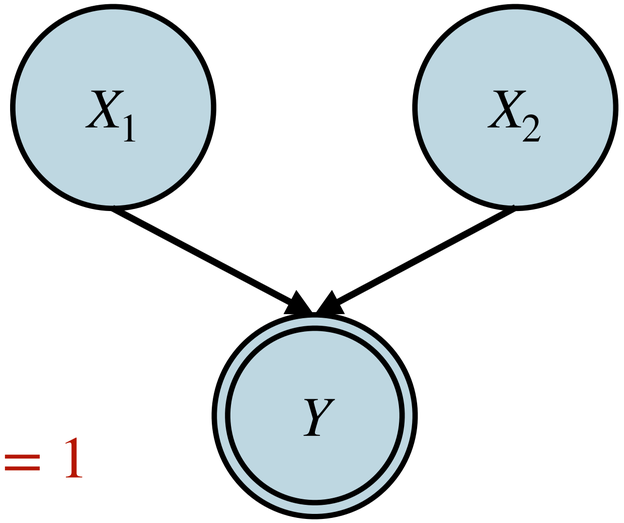

$Y = 1$

| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

# Gibbs chain and regularity

$X_1$ $X_2$ $Y = 1$

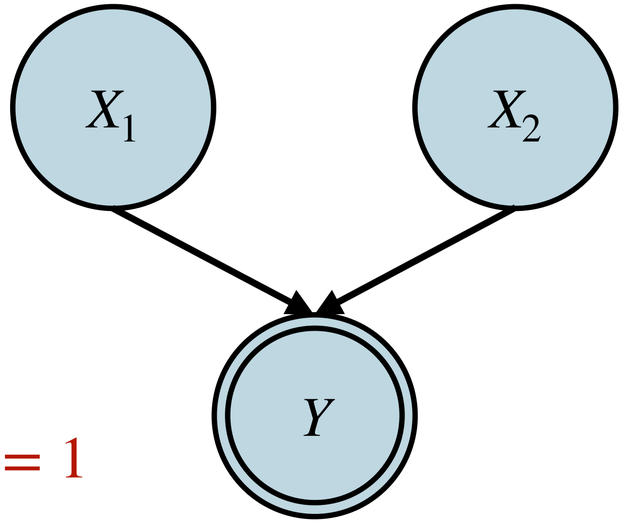
0 1




$Y = 1$

| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

# Gibbs chain and regularity




$Y = 1$

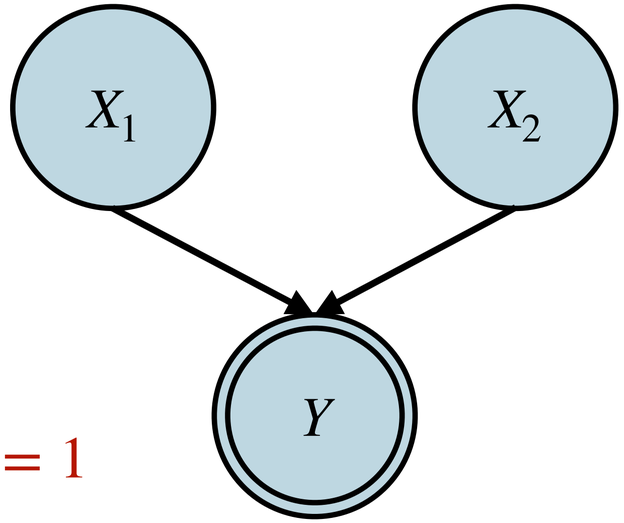
| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

$X_1$ $X_2$ $\quad Y = 1$

0 1

0 1

# Gibbs chain and regularity




$Y = 1$

| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

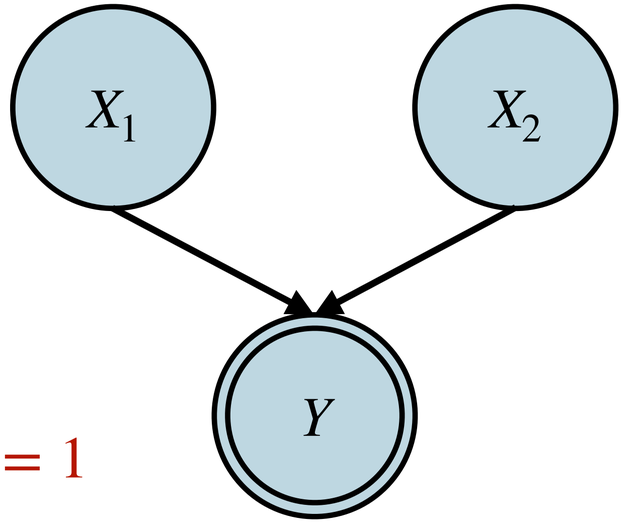
$X_1$ $X_2$ $\quad Y = 1$

0 1

0 1

0 1

# Gibbs chain and regularity




$Y = 1$

| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

$X_1$ $X_2$ $\quad Y = 1$

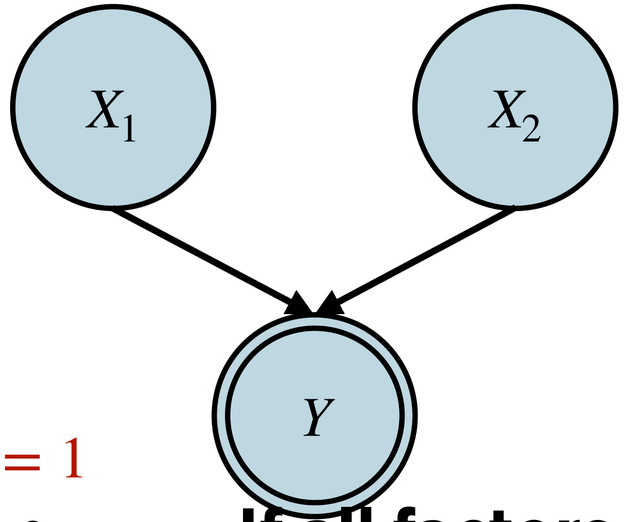
0 1

0 1

0 1

0 1

# Gibbs chain and regularity

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0 1

0 1

0 1

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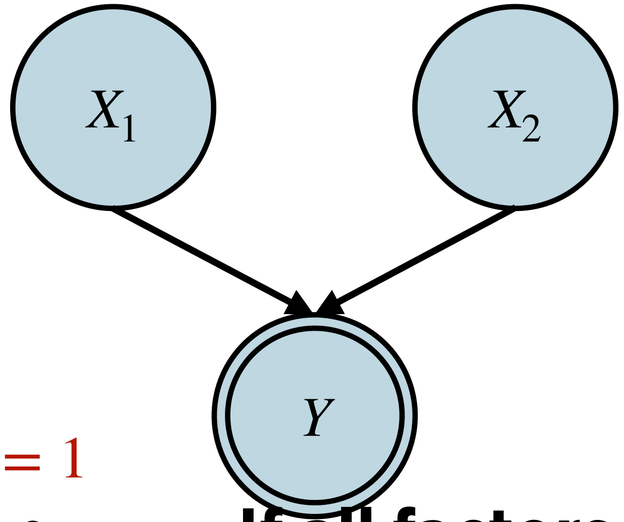


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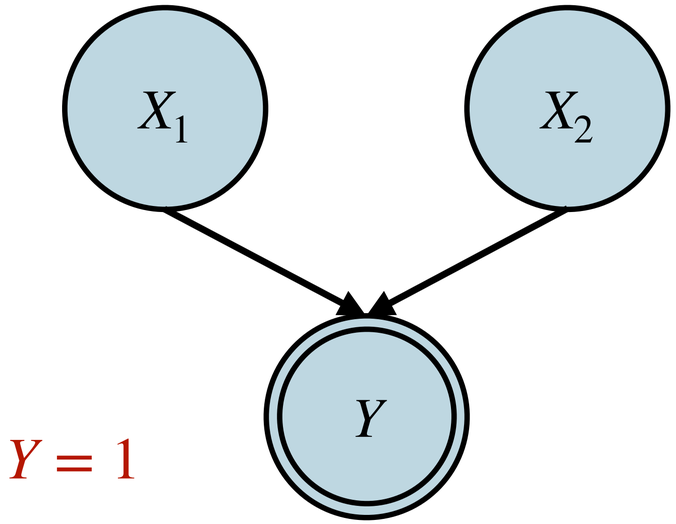
- However, mixing can still be very slow

# Gibbs chain and regularity




| $X_1$ | $X_2$ | Y | Prob |
|---|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ | ~~0.25~~ |
| 0 | 1 | 1 | 0.25 |
| 1 | 0 | 1 | 0.25 |
| ~~1~~ | ~~1~~ | ~~0~~ | ~~0.25~~ |

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Image: https://lime.data-imaginist.com

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  - Computationally efficient to sample
- Cons:
  - Often slow to mix, esp. when probabilities are peaked
  - Only applies if we can sample from product of factors