A probabilistic approach to the accuracy and stability of numerical algorithms

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***Abstract*—**

I. Introduction

- Traditional approach to accuracy and stability: worst-case analysis
- We are interested analysis of *typical* behaviour. This can be made precise by using probabilities: e.g. accuracy and stability guarantees in 99% of executions
- Recent probabilistic approaches to numerical accuracy (Nick Higham, Ilse Ipsen) focus on large/high-dimensional problems and rely on concentration of measure inequalities which provide useful information in this context.
- Our approach is exact (no concentration of measure inequalities) but tailored to smaller programs.
- Test-cases: 'small' scalar products for accuracy and ray tracing algorithm (Slabs method) for stability.

II. A probabilistic model of rounding errors

A. Rounding error distribution

- Compute the distribution of the random variable $\frac{X-\mathrm{Round}(X)}{X}$ given the random variable X
- Show that under mild assumptions on the distribution of X, the distribution of rounding errors is given by the roughly trapezoidal distribution of Fig 1.

B. Probabilistic version of the IEEE 754 standard

- Replace the usual non-deterministic

$$x +_{\mathrm{fp}} y = (x + y)(1 + \varepsilon), |\varepsilon| \leq u$$

with

$$x +_{\mathrm{fp}} y = (x + y)(1 + \varepsilon)$$

where ε is a random variable of known distribution.
- The 'typical' distribution of ε is described in § II-A.

III. Rounding error distribution of simple programs

A. Probabilistic programs

- What they are: (1) programs which can sample from known probability distributions, (2) Programs whose inputs can be probabilistic.
- The probabilistic model of IEEE 754 of § II-B turns any deterministic program into a probabilistic one.

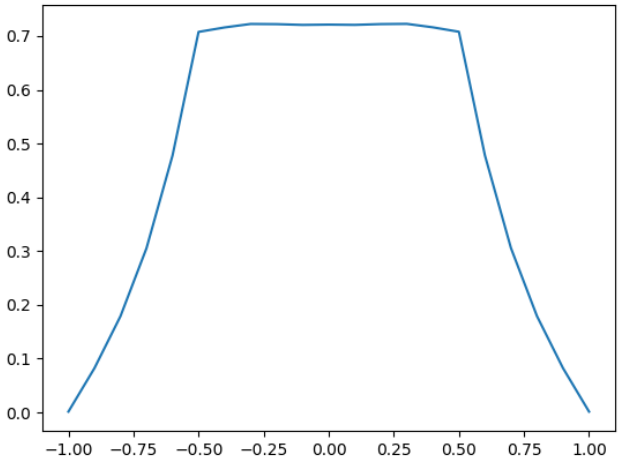

Fig. 1. Typical distribution of rounding errors (in unit roundoffs)

B. How probabilistic programs process probabilistic inputs

- Pushing a distribution through a deterministic function
- Pushing a distribution through a probabilistic function
- Pushing a distribution through an `if then else` statement
- Pushing a distribution through a simple program
- Application to programs with probabilistic rounding errors (§ II-B)

IV. Probabilistic accuracy and stability

A. Accuracy: scalar products

B. Stability: ray tracing via the slabs method