Discuss the relation between the syntactic and semantic rules of prepositional calculus. What is meant by compositionality of meaning in such a system? Why is such a system thought to be illuminating as a basis for natural language grammars?

Derived partly from the discovery that two syllogisms[1] are similar in abstract structure, propositional calculus is concerned with the soundness and unsoundness of arguments. Propositional calculus is a form of logic; it is comprised of a vocabulary; syntax that describes how to combine elements of the vocabulary to create a well-formed formula (wff) or argument[2]; the semantics of the language is the meaning that we give to these arguments and how they are interpreted – the symbols of an argument adopt a value of either true or false, and the meaning of the sentence is interpreted using the semantics accordingly.

For example using an informal proof method called reductio ad absurdum to discover whether a particular argument is valid, the components of the sentence are given a value of true and if the result is interpreted as false then the complete meaning of the sentence is false or 'zero', and the argument is described as being unsound. If the result is true, then the total meaning of the sentence is true or 'one', and the argument is described as being sound.

Propositional calculus is considered to be compositional in nature: A function of the truth-value of the atomic propositions together with the truth functional connectives by which they are combined together (Frege (1892)). The theory of compositionality is thought to extend to other context free grammars and into natural language grammars. In our own language we have a finite number of words, and we also have rules for combining these words to create sentences that are grammatical, it is thought that our ability to comprehend these symbols and understand these rules and how they act as functions on syntactically permitted sentences allows us to comprehend an infinite number of sentences, as such natural language grammars are also thought to be compositional.

Within this paper I shall discuss the relation between the syntax and semantics of propositional calculus explaining what compositionality means with regards to the system and how it is an illuminating basis for natural language grammars. I shall illustrate the discussion with a syntactic and semantic representation of a well-formed argument in propositional calculus. Before closing with the illustration of a number of factors for and against the theory of compositionality.

An argument in propositional calculus could look as follows:

$$((P \vee Q) \rightarrow (\neg (\neg P \wedge \neg Q)))$$

Fig 1.

Decomposing this sentence reveals it consists of the following elements:

$P, Q, \rightarrow, \wedge, \neg, \vee, (,)$

These are all part of the propositional calculus vocabulary, the vocabulary is formally described as follows:

Atomic Propositions (arbitrary symbols):	$p, q, r, s \ldots\ldots\ldots. x, y, z; \phi, \psi$
Propositional Connectives:	$\neg$ (not), $\leftrightarrow$ (biconditional), $\rightarrow$ (implies), $\wedge$ (and), $\vee$ (or)
Brackets:	(,); [,]

Fig 2.

Knowing the vocabulary of the grammar we can now look at how propositions are built and how these propositions are interpreted, the syntax and semantics of the grammar respectively.

[1] Deductive reasoning in which a conclusion is derived from two premises
[2] The terms argument, sentence and proposition shall be used interchangeably henceforth

Syntax

Syntax is considered to be a set of rules for the formation of sentences[3]. Syntax can also be defined as a set of rules for combining the elements of a language into permitted constructions[4]. Both of these definitions can comprehensively cover what syntax is in relation to propositional calculus.

Fig. 1 shows a well-formed formula. It follows a strict set of rules. Each atomic proposition or negated atomic proposition (both well-formed formulas alone) combined with another using a connective is surrounded in brackets, henceforth each well-formed formula or the negated well-formed formula combined using a connective symbol is also surrounded in brackets until the argument is complete. What is left is a well-formed formula – a proposition.

Thus we can say that the syntax of propositional calculus is a set of rules for combining the vocabulary of the grammar into permitted constructions. Sentences that follow these rules are called well-formed formula (wff).

The syntax is formally described as follows:

i) Any atomic proposition is itself a wff

ii) If ϕ is a wff, then $\neg\phi$ is a wff

iii) If ϕ and ψ are wff, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$ are wff

iv) Nothing else is a wff

Fig 3.

This syntax says that (i) an atomic proposition alone is a well-formed formula; (ii) if that is a well-formed formula then its negation is also a well-formed formula alone. (iii) If an atomic proposition and its negation are well-formed formulas then two well-formed formulas combined using a connective and surrounded by brackets are well-formed formula. (iv) Finally nothing else is a well-formed formula.

Using this syntax, propositions can grow to become very complicated, as the third rule shows that the formation of propositional sentences is clearly recursive. The third syntactical rule can be applied to each preceding formula to create a new well-formed formula, as long as it is maintained and ensured that the new formula is well formed. Atomic propositions are also shown to be recursive in nature as the negated atomic proposition can itself be negated according to the second rule, as with all well-formed formulas creating the negation of the negation, and so forth.

Knowing how to use the symbols of the language we need to add some form of meaning to the propositions that are created. Semantics are used to provide this meaning. Syntax is formulated independently of semantics but with the aim of creating some agreeable form, that when combined outline a functional language specification.

Semantics

Semantics is defined as pertaining to what something means[5]. Semantics can also be defined as the meaning attached to words or symbols.

The proposition in fig. 1 is described as being a tautology; it is true under all possible conditions. Therefore the entire meaning of the sentence is true or ‘one’, its atomic propositions can adopt any value without affecting this result. A truth table can be used to model this situation:

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[3] www.oed.com - accessed 28th December 2004
[4] Wordsworth Dictionary of Science and Technology
[5] www.wikipedia.org - accessed 16th January 2005

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg (\neg P \wedge \neg Q)$	$((P \vee Q) \rightarrow (\neg (\neg P \wedge \neg Q)))$
0	0	0	1	1	1	0	1
0	1	1	1	0	0	1	1
1	0	1	0	1	0	1	1
1	1	1	0	0	0	1	1

Table 1

The truth table represents every possible truth-value for a proposition where its constituent[6] parts have the value of being either true '1' or false '0'. From this table we cans see the formula is broken down to its atomic propositions. An atomic proposition is the only part of the formula that adopts a value of either true or false. The value of the entire proposition and its remaining constituent parts is interpreted from these values using the semantics of the language.

We can use the row highlighted in the truth table as an example to how semantics are used in propositional calculus. Within this row we define two values for P and Q, zero and one respectively. The negated values of P and Q and the compound propositions[7] that form the entire argument and containing the atomic propositions P and Q are interpreted accordingly.

For example in $(P \vee Q)$, P is false and Q is true. Propositional calculus semantics require only P or Q to be true for this proposition to be true; therefore the value of this compound proposition is true. Its semantic value is true, or '1'. The rest of the formula is computed in this way until we have a value for the entire proposition, which in this case and for every other value of P or Q will be true.

Thus we can say that the semantics of propositional calculus is the meaning given to atomic propositions, and the method of interpreting the value of compound propositions from those values.

Propositional Calculus semantics are described as follows[8]:

i) $V(\neg\phi) = 1$ iff $V(\phi)=0$

ii) $V(\phi \wedge \psi) = 1$ iff $V(\phi) = 1$ and $V(\psi) = 1$

iii) $V(\phi \vee \psi) = 1$ iff $V(\phi) = 1$ or $V(\psi) = 1$

iv) $V(\phi \rightarrow \psi) = 1$ iff $V(\phi) = 0$ or $V(\psi) = 1$

v) $V(\phi \leftrightarrow \psi) = 1$ iff $V(\phi) = V(\psi)$

Fig 4.

The following section models the construction and interpretation of a well-formed formula using the propositional calculus syntax and semantics:

Characterisation of the Well-Formed Formula: $(P \rightarrow (\neg(R \vee Q)))$

With the correct use of the semantics of propositional calculus it can help us to describe a possible real world state. If we give the formula $(P \rightarrow (\neg(R \vee Q)))$ some arbitrary real world values then we can model the outcome if these values are either true or false.

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[6] By constituent parts it is meant that the formula is broken down into the separate well-formed formulas that form the complete proposition from the level of atomic propositions

[7] Compound Propositions being two well-formed formulae combined using connective symbols

[8] V is a valuation function such that all atomic propositions have one and no more than one truth value, iff is used to denote if and only if

Using the card game snap as a basis for our model, the proposition could describe the following:

Snap is a game implies that it isn’t true that snap is serious or that snap is boring

These real world values can be given truth-values of either true or false to represent at random an assertion about a possible world, say:

Snap is a game	= False
Snap is serious	= True
Snap is boring	= False

Using these values we can model the interpretation of the entire statement. First lets look at how the symbolic representation of the model is constructed syntactically.

A construction tree is used to show the construction of a well-formed formula according to the syntax of the propositional calculus grammar. The syntactical rules used at each stage are denoted by (syn (x)), (x) representing the specific number as noted in fig.3.

Syntactic Characterisation:

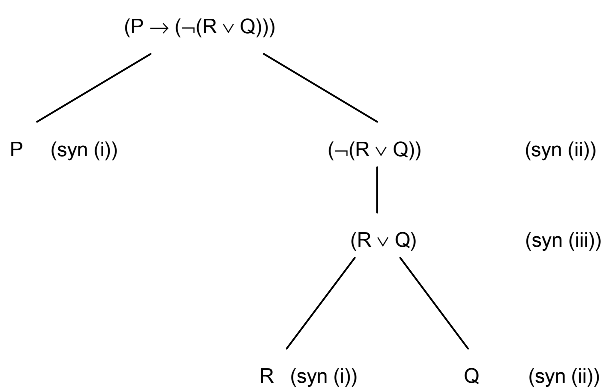

Diagram 1.

The syntactic characterisation of the proposition is built from the bottom up. It is constructed from the atomic propositions and each time a connective is used the statement becomes more and more complex. Even though this is a relatively simple proposition to construct, it can still describe valid and arguably complicated natural language statements.

Model M is used to show the interpretation of the proposition assuming P = ‘False’; Q = ‘True’ and R = ‘False’. The semantic rule used to interpret the formula at each stage is denoted by (se (x)), (x) representing the specific number as noted in fig.4.

Semantic Characterisation:

$[[\ (P \rightarrow (\neg(R \vee Q)))\ ]]^M = ?$

$[[\ P\ ]]^M = F$

$[[\ Q\ ]]^M = T$

$[[\ R\ ]]^M = F$

$[[\ (R \vee Q)\ ]]^M = T$ se(iii)

$[[\ (\neg(R \vee Q)\ ]]^M = F$ se(i)

$[[\ (P \rightarrow (\neg(R \vee Q)))\ ]]^M = \mathbf{T}$ se(iv)

This notation represents the interpretation of proposition (ϕ | ψ) with regards to model M

Fig 5.

Therefore by referring to the semantic characterisation of the proposition $(P \rightarrow (\neg(R \vee Q)))$ or “snap is a game implies that it isn’t true that snap is serious or that snap is boring”. We can say that the meaning, its semantic value of the statement is true or ‘1’ in one possible world.

Compositionality of Meaning and Propositional Calculus

The Principle of Compositionality is the principle that the meaning of a complex expression is determined by the meanings of its constituent expressions and the rules used to combine them[9].

Frege (1892) described compositionality with regards to propositional calculus as:

> “A function of the truth-value of the atomic propositions together with the truth functional connectives by which they are combined together.”

As noted above, in the interpretation of a propositional statement, only the atomic propositions adopt a truth-value. The compound propositions that form the argument and contain the atomic propositions are interpreted accordingly to the language semantics – as a function of the truth-value of the atomic propositions together with the truth functional connectives by which they are combined together.

Fig 5. Shows how a proposition is interpreted at each stage of syntactic construction with regards to a specific semantic rule. At each stage from the atomic proposition level the meaning of the proposition is interpreted as a function of a semantic rule relating to that permitted construct.

Before looking further into how propositional calculus is considered to be compositional in nature, I shall explain how formal grammars and the principle of compositionality are illuminating with regards to natural language grammars. Within the discussion I shall draw examples from propositional calculus to help illustrate evidence for compositionality in natural language, which will simultaneously show further evidence of how propositional calculus is compositional. I shall also attempt to cover some arguments against the compositionality of natural language grammars such as English.

Propositional Calculus and Compositionality of meaning, illuminating as a basis for natural language grammars?

The study of meaning in natural language sentences is a continuous one. Philosophers in the field of study have fixed upon no dominant theory. But the theory of compositionality is a recurring one, and there are factors within a system such as propositional logic that are similar to natural language and can help to provide support that understanding of a complex expression is a function of the meanings of its constituent parts together with their mode of combination[10].

A natural language grammar consists of a vocabulary. The typical vocabulary for an American graduate student is said to be around 60,000 words. English grammar also consists of syntax, however deciding on a finite set of rules for formation of sentences is a difficult task, as we shall see. Semantics also exist within natural language, and this is the meaning we give to sentences and words, however the intangible nature of natural language semantics is what leads us to this discussion.

An analysis of the English language can help to reveal its constituent parts further. Hodge 1977, used an analysis of English to illustrate the effect of syntax on the semantics of a sentence, however it will help to reveal the elements that constitute English grammar:

> Hodge identified natural groups and grammatical phrases of the language. A grammatical phrase occurring as a natural group is called a constituent of the sentence. If the grammatical phrase occurs twice in a sentence they are counted as different constituents.
>
> A single word is considered to be a grammatical phrase, which in turn is considered to be a constituent of a sentence.

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[9] www.wikipedia.org - accessed 16th January 2005
[10] http://plato.stanford.edu/archives/fall2004/entries/compositionality/ - accessed 16th January 2005

Traditional English grammars group grammatical phrases of English into phrase-classes, four of these classes are:

Noun: john, room, answer, play
Adjective: happy, steady, new, large, round
Adverb: steadily, completely, really, very, then
Verb: search, grow, play, be, have, do

The analysis of a sentence can be represented using a phrase marker. Using an example from Hodge 1977:

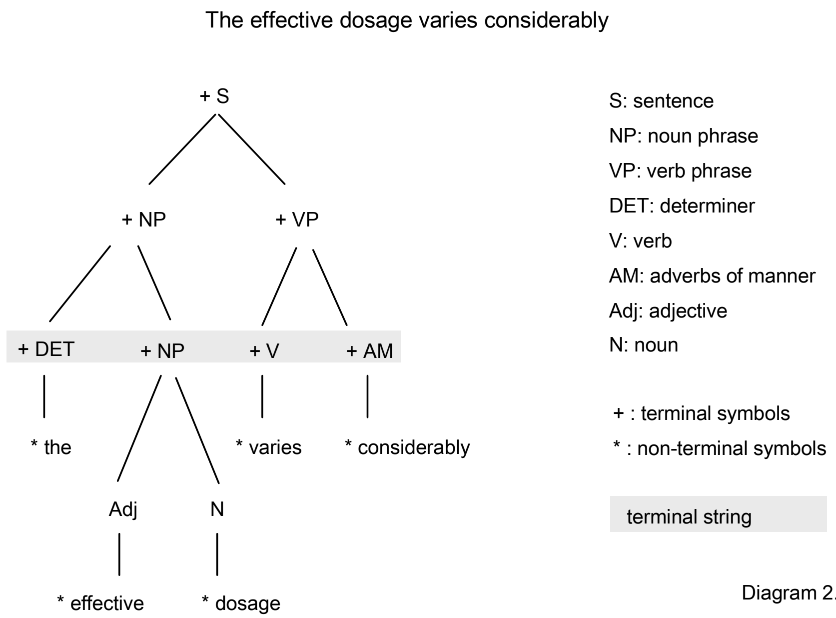

Diagram 2.

Whenever a non-terminal symbol appears in a phrase marker, it tells us that a certain part of the terminal string belongs in stated phrase class. Any part of string assigned to a phrase class in this way is a constituent of the terminal string.

Terminal String: Nine occurrences of non-terminal symbols each represent a constituent.

By deconstructing sentences this way we can begin to construct syntax for natural language grammar. Diagram 2. Reveals a number of patterns, for example a noun phrase consists of a determiner and a noun phrase, or it consists of an adjective or a noun. Similarly a verb phrase consists of a verb and an adverb of manner. We could reinforce these examples with syntax and experiment to see whether it holds for larger sentences. A very basic example follows[11]:

Using the following vocabulary[12]:

Determiners: the, a, my, your, our, this
Nouns: cat, mouse, house, dog
Pronouns: i, you, he, she, it, we, they
Transitive Verbs: ate, saw, likes

[11] See appendix i
[12] Note the use of phrase classes

The syntax is defined as follows:

Sentence:

A sentence is a sentence iff it is constructed as:

(S (Noun Phrase Transitive Verb Noun Phrase))

Noun Phrase:

A noun phrase is a noun phrase iff it is constructed as:

(NP (Determiner Noun)) Or (NP (Pronoun))

Combining the vocabulary and syntax allow the following example permutations:

i) The cat saw the mouse (S ((NP (The cat)) (TV (saw)) (NP (the mouse)))

ii) You saw the house (S ((NP (You)) (TV (saw)) (NP (the house)))

iii) They saw the house (S ((NP (They)) (TV (saw)) (NP (the house)))

iv) She likes my dog (S ((NP (She)) (TV (likes)) (NP (my dog)))

Although far from perfect this example begins to illustrate how a natural language can be specified formally. The syntax and grammar could be built upon for example; to include tenses, adjectives, and form (singular/plural), thus making it more true to life, while maintaining a strict well formed structure, and preventing unwanted permutations of sentences.

The meaning of the sentences created depends on the semantics adopted. As noted earlier the syntax and semantics of a language are derived separately from each other but with the aim of agreeing. Agreeing on a formal rule for natural language semantics is difficult. This is where the principle of compositionality comes in.

In terms of natural language the theory of compositionality is as follows:

> "The meaning (or semantic value) of a sentence is a function of the meanings of its constituents together with their mode of combination."

According to Larson and Segal (1995) the rules needed to understand the sentence "Boris kissed Natasha" are compositional. Giving the semantic content of the sentence by specifying the semantic contributions of its parts and the semantic significance of putting those parts together according to a definite mode of syntactic combination.

It apparently explains three principle phenomena:

1: Sentence comprehension is systematic

There is a definite and predictable form to sentences we understand, i.e. we understand Boris kissed Natasha, therefore we can understand Natasha kissed Boris, simply by reforming the constituent parts of the sentence we already understood and reinterpreting its meaning accordingly.

2: We understand new sentences that we have never come across before

It is possible that there is a body of rules that allow us to infer the meanings of new sentences from prior knowledge, the meaning of its constituent parts, and the significance of its semantic combination.

3: We have the capacity to understand each of an indefinitely large number of sentences

We can see examples of these three phenomena in propositional calculus:

1: Sentence comprehension is systematic

In propositional calculus when computing the value of a proposition we carefully separate each wff from the rest of the propositions constituent parts. We separate the proposition into atomic propositions and begin to work out the meaning of the formula from there; Understanding how the semantics works in accordance with the well formed formulas defined by the semantics allows us to calculate a truth value for the entire sentence.

2: We understand new sentences that we have never come across before

The vocabulary of propositional calculus is finite; it has a limited number of possible symbols and connectives. The rules that govern what is considered to be a well-formed formula are also finite, as with the semantics, the rules for interpretation of the formula. Therefore whenever we come across a new proposition we look at the rules for combining the vocabulary and we can gain an understanding of the sentence, we can also interpret it giving a precise binary value for the meaning of the proposition.

Within natural language there is also a finite vocabulary, and as we have seen we can also begin to derive a finite (although indefinitely large) number of syntactic rules for combination of the symbols contained within this vocabulary. If semantics is taken purely as the meaning attributed to a word or symbol then meaning is also within a finite domain. We can look at a sentence and understand its composition and its constituent parts, and we can derive its meaning through prior knowledge, i.e. words in our vocabulary, or learn of the meaning through other methods.

3: We have the capacity to understand each of an indefinitely large number of sentences

Propositional logic is recursive in nature. That is a proposition is not limited to a single set of atomic propositions and a single functional connective. A proposition takes the form (ϕ, ψ), where ϕ is a well formed formula, and ψ is a well-formed formula, combined through a functional connective of some sort. Thus if you had a moderately large formula similar to that in fig 1. It would be possible to recombine it with itself using a single connective to create a new proposition. Also the negation of a formula in propositional calculus is recursive, allowing the negation of a negation. Both of these factors mean that a well-formed formula can be infinitely large, but also there are an infinite number of well-formed formulae.

An analogy in natural language is the use of an adjective and a quantified noun. As an adjective is a descriptive term it can be applied to a noun or pronoun over and over again. It is a recursive function. This we could create sentences such as:

“The big, big, big, big, big house.”

Or

“The big, fat, round cat, sat on the big, round, brown mat.”

This could continue indefinitely. Therefore adjectives alone in natural language ensure we can create an infinitely large sentence, and an infinitely large number of sentences.

Propositional calculus is compositional in nature. It is a context free grammar that simply tells us how to construct grammatical sentences of the language and nothing else. The interpretation of proposition involves truth-conditions (binary values, true or false); therefore we do not rely on having to make assumptions about intangible or abstract features.

Supporters of compositionality of natural language point to the three phenomena outlined above. Challenges to the theory point to examples of context and where the meanings of complex expressions rely on the intentions of the speaker.

i) Pedro jumped from the top of the bank
ii) Is it snowing?

Larson and Segal point out in (i) that a bank could mean a fluvial embankment, or a financial institution. They point out that (ii) could possibly be a serious question or a sarcastic question, depending on the speakers intentions and the context.

While (ii) is a difficult proposition to explain (i) could be explained as a failure of natural language grammars rather than compositionality. While the English vocabulary is finite, it is also extremely large therefore to have separate words for every abstract concept could cause difficulties. If X represents a financial institution and Y a fluvial embankment we could write:

iii) Pedro jumped from the top of the X
iv) Pedro Jumped from the top of the Y

If the financial institution alone was used, as the grammatical phrase in the sentence then the sentence would no longer be ambiguous to the extent of knowing which 'bank' it was. Therefore the question is posed as to whether meaning is purely an arbitrary value, a symbol representing a single abstract concept at any one time.

Either way, the issue raised in points (i) and (ii) do not detract from the theory of compositionality as in (iii) and (iv) the meanings are explicit. If we have a tangible grasp on the components of a sentence then there are no alternative methods of interpreting the components, and therefore the sentence.

The study of natural language grammars in the same way as one would study formal grammars such as propositional calculus brings about a number of advantages. Outlined above the ability to give real values (numerical or otherwise) to the constituents of sentences allows are more objective study of the concept of sentence meaning. This is indicative of the method being more scientific in nature. Being more scientific in nature allows the testing of theories, and the re-testing against theories. This allows for comparisons, and possible the combining of empirical evidence that could help give insight into the concept being studied. Another important feature of a scientific study is its objectivity. Before a conclusion is finalised within a scientific study it is tested thoroughly through objective methods, and only if there is concrete evidence for the existence of a particular phenomena is it accepted as a valid theory and re-tested in an attempt to disprove its problem solving capability.

Ultimately while the theory of compositionality may or may not apply to natural language grammars, the realisation of formal semantics for a comprehensive grammar will nonetheless give much more insight into this area than subjective introspection and debate. And will hopefully help us to understand the methods we apply ourselves in the comprehension of complex sentences and propositions.

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Appendix i