PHY905 Project 4 - Ising Model

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Background: A fairly accurate model of the solar system can be achieved with the use of newtonian mechanics. However, this leads to many coupled differential equations which are difficult to solve analytically.
Purpose: The goal of this work is to solve numerically the aforementioned problem. We aim to study the numerical stability of various computation methods as well as the numerical precision.
Method: We approximate the derivatives in the coupled differential equations and solve them using the Euler method and the velocity Verlet method.
Results: We find the the Euler method is unstable even in the binary Earth-Sun system, and causes non-conservation of both energy and angular momentum. In contrast, the velocity Verlet method is found to be very stable and enforces conservation of energy and angular momentum.
Conclusions: Our results demonstrate both the importance of an effective differential equation solver, and that the sun (due to its large mass) essentially controls all of the dynamics of the other bodies in the system.

I. INTRODUCTION

While there are many types of magnetism in physics, the strongest and most common type one encounters in everyday life is ferromagnetism, in which a metal, such as iron or cobalt, becomes permanently magnetized when exposed to a magnetic field. This phenomenon is initiated by a quantum mechanical interaction which causes the spins of unpaired electrons to align, acting against the thermodynamic tendency to randomize the spins.

One of the most common ways to analyze magnetism is the Ising model, in which the magnetic material is treated as a lattice of atomic spins which interact with each of it's nearest neighbors. In this model, at low temperatures the system exhibits spontaneous magnetization where the average magnetization is nonzero. As the temperature increases the system undergoes a second order phase transition at a specific temperature, known as the critical temperature. In a second order phase transition the two phases on either side of the transition are identical and the transition manifests as a discontinuity in the derivative of the energy, which is different from a first order transition in which the two different phases coexist at the critical temperature and the transition manifests as a discontinuity of the energy.

In this work, we implement the metropolis algorithm with the ising model in two dimensions in order to study phase transitions of magnetic systems. Calculations of the average energy, heat capacity, average magnetization and the susceptibility are performed and compared to the expected values for a small system. In addition, the behavior of these quantities as a function of temperature is analyzed in order to extract the critical temperature of the phase transition. Finally, the performance of the metropolis algorithm is analyzed. In Sections II and III, the necessary theory and implementation of the algorithms are described. In Section IV the performance and accuracy of the code are analyzed. Finally, in Section V we give a summary and our conclusions.

II. THEORY

Number of spins up	Degeneracy	E_i	M_i
4	1	-8	4
3	4	0	2
2	4	0	0
2	2	8	0
1	4	0	-2
0	1	-8	-4

TABLE I: Energy and magnetization of all possible configurations for a 2x2 lattice.

For a simple 2x2 lattice the partition function is given by:

$$Z = \sum_i e^{-E_i/T} = 2e^{8/T} + 2e^{-8/T} + 12. \qquad (1)$$

Similarly the average energy and average squared energy are given by:

$$\langle E \rangle = \sum_i E_i e^{-E_i/T} = -16e^{8/T} + 16e^{-8/T} \qquad (2)$$

$$\langle E^2 \rangle = \sum_i E_i^2 e^{-E_i/T} = 128e^{8/T} + 128e^{-8/T}. \qquad (3)$$

From these quantities we can calculate the energy variance, which is related to the heat capacity by:

$$\begin{aligned} C_v &= \frac{1}{T^2}\left(\langle E^2 \rangle - \langle E \rangle^2\right) \\ &= \frac{1}{T^2}\left(128e^{-8/T} - 128e^{8/T} + 512\right) \end{aligned} \qquad (4)$$

Similarly, one can calculate the average magnetization, average square magnetization, and the magnetic susceptibility as:

$$\langle |M| \rangle = \sum_i |M_i| e^{-E_i/T} = 8e^{8/T} + 8 \tag{5}$$

$$\langle M^2 \rangle = \sum_i M_i^2 e^{-E_i/T} = 32e^{8/T} + 32 \tag{6}$$

$$\chi = \frac{1}{T}\left(\langle M^2 \rangle - \langle |M| \rangle^2\right) = \frac{1}{T}\left(-96e^{8/T} - 32\right). \tag{7}$$

These analytic values can then be compared to the calculated ones for a 2x2 lattice to ensure that the calculations are being performed correctly.

III. METHODS

IV. RESULTS

V. CONCLUSIONS