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Born and Inverse Born Series for a Special Case of the Second Harmonic Generation Problem

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Abstract

We study the Born and Inverse Born series for a special case of the second harmonic generation system of PDEs. We give a recursive formula for the forward operators and prove boundedness conditions that guarantee the convergence of the Born and inverse Born series. We also use fixed point theory to give explicit conditions for convergence of the Born series.

1 Introduction: The Forward Problem

Let $\Omega \subseteq \mathbb{R}^d$ with smooth boundary. The second harmonic generation problem is to find a unique solution to the coupled system

$$\begin{aligned} \Delta u^{(1)} + k^2 \left(1 + 4\pi\eta\right) u^{(1)} &= -4\pi k^2 \left(\eta_1 u^{(1)} + 2\eta_2 u^{(2)} \left(u^{(1)}\right)^*\right) \\ \Delta u^{(2)} + \left(2k\right)^2 \left(1 + 4\pi\eta\right) u^{(2)} &= -4\pi \left(2k\right)^2 \left(\eta_1 u^{(2)} + \eta_2 \left(u^{(1)}\right)^2\right) \end{aligned} \qquad \text{in } \Omega$$

$$\frac{\partial u^{(1)}}{\partial \nu} = g, \qquad \frac{\partial u^{(2)}}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

where η is a known constant and ν is the unit outward normal vector to $\partial\Omega$. In this paper, we analyze the Born and inverse Born series and prove the existence and uniqueness of a special case of the second harmonic generation problem given by

$$\begin{aligned} \Delta u^{(1)} + k^2 u^{(1)} &= -4\pi k^2 \left(2\eta_2 u^{(2)} u^{(1)}\right) \\ \Delta u^{(2)} + \left(2k\right)^2 u^{(2)} &= -4\pi \left(2k\right)^2 \left(\eta_2 \left(u^{(1)}\right)^2\right) \\ \frac{\partial u^{(1)}}{\partial \nu} = g, &\qquad \frac{\partial u^{(2)}}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

Since $\eta = 0$ and $\eta_1 = 0$, we will abuse notation and denote $\eta \coloneqq \eta_2$ for the rest of the paper.

In order to prove the existence and uniqueness of solutions to eq. (1), we convert the system of PDEs into a transformation between vector spaces. Let $G\left(k; x, y\right)$ be the Green's function for the operator $\Delta u^{(1)} + k^2$ and $G\left(2k; x, y\right)$ be the Green's function for the operator $\Delta u^{(1)} + \left(2k\right)^2$. Then we can transform the system of PDEs in eq. (1) into an operator

$$T \colon C\left(\overline{\Omega}\right)^2 \to C\left(\overline{\Omega}\right)^2$$

given by

$$T\begin{pmatrix}u^{(1)}\\u^{(2)}\end{pmatrix}=\begin{pmatrix}u_0^{(1)}\\u_0^{(2)}\end{pmatrix}+\begin{pmatrix}-8\pi k^2\int_\Omega G(k;x,y)\,\eta(y)\,u^{(1)}(y)\,u^{(2)}(y)\,dy\\-4\pi(2k)^2\int_\Omega G(2k;x,y)\,\eta(y)\left(u^{(1)}\right)^2(y)\,dy\end{pmatrix}. \tag{2}$$

Note that a fixed point $u\in C\left(\overline{\Omega}\right)^2$ satisfies the PDE eq. (1). The following rest gives conditions for the existence of such a u.

Theorem 1. *Let $T:C\left(\overline{\Omega}\right)^2\to C\left(\overline{\Omega}\right)^2$ be defined by eq. (16). Define*

$$\mu_1:=8\pi k^2\sup_{x\in\Omega}\int_\Omega|G(k;x,y)|\,dy$$
$$\mu_2:=16\pi k^2\sup_{x\in\Omega}\int_\Omega|G(2k;x,y)|\,dy$$

and $\mu=\max\{\mu_1,\mu_2\}$. Then if

$$\|\eta\|\le\frac{1}{4\mu\|u_0\|},$$

T has a unique fixed point if the ball of radius $\|u_0\|$ about $u_0\in\left(C\overline{\Omega}\right)^2$.

The proof is given in Section 4

2 The Forward Born Series

We wish to compute the function u on $\partial\Omega$ given a source g on $\partial\Omega$. Our presentation follows that in [1]. The solution to the forward problem is derived by iteration of the integral equation eq. (16). We seek a series representation of u of the form

$$u=u_0+K_1(\eta)+K_2(\eta,\eta)+K_3(\eta,\eta,\eta)+\cdots. \tag{3}$$

The forward operators are of the form

$$K_n(\eta,\dots,\eta):=\begin{pmatrix}K_n^{(1)}(\eta,\dots,\eta)\\K_n^{(2)}(\eta,\dots,\eta)\end{pmatrix}:\ [L^\infty(\Omega)]^n\to C(\partial\Omega\times\partial\Omega)^2.$$

Now, K_n is n-linear on $[L^\infty(\Omega)]^n$. The fixed point iteration gives:

$$u^{(1)}(x):=T(u_0)(x)=\begin{pmatrix}u_0^{(1)}\\u_0^{(2)}\end{pmatrix}\begin{pmatrix}-8\pi k^2\int_\Omega G(k;x,y)\,\eta(y)\,u_0^{(1)}(y)\,u_0^{(2)}(y)\,dy\\-4\pi(2k)^2\int_\Omega G(2k;x,y)\,\eta(y)\,u_0^{(1)}u_0^{(1)}(y)\,dy\end{pmatrix}$$

which implies that

$$K_1(\eta)(x)=\begin{pmatrix}-8\pi k^2\int_\Omega G(k;x,y)\,\eta(y)\,u_0^{(1)}(y)\,u_0^{(2)}(y)\,dy\\-4\pi(2k)^2\int_\Omega G(2k;x,y)\,\eta(y)\,u_0^{(1)}u_0^{(1)}(y)\,dy\end{pmatrix}.$$

To make our lives easier, define

$$h_1\left(\begin{pmatrix}v^{(1)}\\v^{(2)}\end{pmatrix},\eta\right)=-8\pi k^2\int_\Omega G(k;x,y)\,\eta(y)\,v^{(1)}(y)\,v^{(2)}(y)\,dy,$$
$$h_2\left(\begin{pmatrix}v^{(1)}\\v^{(2)}\end{pmatrix},\eta\right)=-4\pi(2k)^2\int_\Omega G(2k;x,y)\,\eta(y)\,v^{(1)}(y)\,v^{(1)}(y)\,dy.$$

Definition 2. Let T_i and T_j be multilinear operators of order i and j respectively. Define the tensor product $T_i\otimes T_j$ to be

$$T_i\otimes T_j(\alpha_1,\dots,\alpha_i,\alpha_{i+1},\dots,\alpha_{i+j})=T_i(\alpha_1,\dots,\alpha_i)\,T_i(\alpha_1,\dots,\alpha_j),$$

which is of order $i+j$. ⌟

Definition 3. Let $T_k(\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a multilinear operator of order k. Define the $k+1$ order multilinear operators $H_1 T_k$ and $H_2 T_k$ by

$$H_1 T_k(\alpha_1, \ldots, \alpha_{k+1}) = h_1 \left(\begin{pmatrix} T_k^{(1)}(\alpha_1^{(1)}, \ldots, \alpha_k^{(1)}) \\ T_k^{(2)}(\alpha_1^{(2)}, \ldots, \alpha_k^{(2)}) \end{pmatrix}, \alpha_{k+1} \right)$$

and

$$H_2 T_k(\alpha_1, \ldots, \alpha_{k+1})\, h_2 \left(\begin{pmatrix} T_k^{(1)}(\alpha_1^{(1)}, \ldots, \alpha_k^{(1)}) \\ T_k^{(2)}(\alpha_1^{(2)}, \ldots, \alpha_k^{(2)}) \end{pmatrix}, \alpha_{k+1} \right).$$

⌟

Thus,

$$T \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 v^{(1)} \otimes v^{(2)} \\ H_2 v^{(1)} \otimes v^{(1)} \end{pmatrix}$$

becomes a sum of multilinear operators.

Lemma 4. *considering the nth iterate as a sum of multilinear operators, we have*

$$\begin{pmatrix} u_{n+1}^{(1)} \\ u_{n+1}^{(2)} \end{pmatrix} = \begin{pmatrix} u_n^{(1)} \\ u_n^{(2)} \end{pmatrix} + \begin{pmatrix} \textit{multilinear operators of order} \geq n+1 \\ \textit{multilinear operators of order} \geq n+1 \end{pmatrix}. \tag{4}$$

Proof. Proof by induction on n. For the base case, we have

$$\begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 u_0^{(1)} \otimes u_0^{(2)} \\ H_2 u_0^{(1)} \otimes u_0^{(1)} \end{pmatrix}.$$

which obviously satisfies eq. (4). For the induction step, we assume that

$$\begin{pmatrix} u_n^{(1)} \\ u_n^{(2)} \end{pmatrix} = \begin{pmatrix} u_{n-1}^{(1)} \\ u_{n-1}^{(2)} \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where w_1, w_2 are sums of multilinear operators of order at least n. Then,

$$\begin{aligned} u_n^{(1)} \otimes u_n^{(2)} &= \left(u_{n-1}^{(1)} + w_1\right) \otimes \left(u_{n-1}^{(2)} + w_2\right) \\ &= u_{n-1}^{(1)} \otimes u_{n-1}^{(2)} + u_{n-1}^{(1)} \otimes w_2 + w_1 \otimes u_{n-1}^{(2)} + w_1 \otimes w_2 \\ &= u_{n-1}^{(1)} \otimes u_{n-1}^{(2)} + \text{multilinear operators of order at least } n. \end{aligned}$$

A similar calculation gives

$$u_n^{(1)} \otimes u_n^{(2)} = u_{n-1}^{(1)} \otimes u_{n-1}^{(1)} + \text{multilinear operators of order at least } n.$$

Now, applying H_1 (resp. H_2) to an operator increases the order by 1 so any product containing a w

Thus, we combine our results to obtain

$$\begin{aligned} \begin{pmatrix} u_{n+1}^{(1)} \\ u_{n+1}^{(2)} \end{pmatrix} &= \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 u_n^{(1)} \otimes u_n^{(2)} \\ H_2 u_n^{(1)} \otimes u_n^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 u_{n-1}^{(1)} \otimes u_{n-1}^{(2)} + \text{multilinear operators of order at least } n \\ H_2 u_{n-1}^{(1)} \otimes u_{n-1}^{(1)} + \text{multilinear operators of order at least } n \end{pmatrix} \\ &= \begin{pmatrix} u_n^{(1)} \\ u_n^{(2)} \end{pmatrix} + \begin{pmatrix} \text{multilinear operators of order} \geq n+1 \\ \text{multilinear operators of order} \geq n+1 \end{pmatrix} \end{aligned}$$

Where the last equality holds because applying H_1 and H_2 to the operators $u_n^{(1)} \otimes u_n^{(2)}$ and $u_n^{(1)} \otimes u_n^{(1)}$ respectively, we increase the order of the operators by one.

giving us eq. (4). □

Lemma 4 shows that the fixed point iteration that generates the sequence $\{u_n\}$ also generates the series of the form eq. (3). For each K_n, the operator $K_n^{(1)}$ and $K_n^{(2)}$ are sums of multilinear operators of order n. Lemma 4 also tells us that the nth iterate u_n contains all terms of degree n. From this, we obtain that

$$\begin{pmatrix} u_n^{(1)} \\ u_n^{(2)} \end{pmatrix} = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \sum_{i=1}^{n} \begin{pmatrix} K_i^{(1)}(\eta,\dots,\eta) \\ K_i^{(2)}(\eta,\dots,\eta) \end{pmatrix} + \begin{pmatrix} \text{terms of order} \geq n+1 \\ \text{terms of order} \geq n+1 \end{pmatrix}.$$

2.1 A General Formula for the Forward Operators

Our fixed point iteration is given by the scheme

$$\begin{aligned} \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} &= \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 u_0^{(1)} \otimes u_0^{(2)} \\ H_2 u_0^{(1)} \otimes u_0^{(1)} \end{pmatrix} \\ \begin{pmatrix} u_2^{(1)} \\ u_2^{(2)} \end{pmatrix} &= \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 u_1^{(1)} \otimes u_1^{(2)} \\ H_2 u_1^{(1)} \otimes u_1^{(1)} \end{pmatrix} \\ \begin{pmatrix} u_{n+1}^{(1)} \\ u_{n+1}^{(2)} \end{pmatrix} &= \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 u_n^{(1)} \otimes u_n^{(2)} \\ H_2 u_n^{(1)} \otimes u_n^{(1)} \end{pmatrix}. \end{aligned}$$

To compute an explicit form for the forward operators K_n, we define

$$\begin{pmatrix} U_n^{(1)} \\ U_n^{(2)} \end{pmatrix} \coloneqq \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \sum_{i=1}^{n} \begin{pmatrix} K_i^{(1)}(\eta,\dots,\eta) \\ K_i^{(2)}(\eta,\dots,\eta) \end{pmatrix}.$$

That is, U_n is the sum of the first n forward operators. Now Lemma 4 gives

$$\begin{pmatrix} u_n^{(1)} \\ u_n^{(2)} \end{pmatrix} = \begin{pmatrix} U_n^{(1)} \\ U_n^{(2)} \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

where w_1, w_2 are sums of multilinear operators of order $\geq n+1$. Now, we compute U_{n+1}:

$$\begin{pmatrix} u_{n+1}^{(1)} \\ u_{n+1}^{(2)} \end{pmatrix} = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} H_1 \left(U_n^{(1)} + w_1 \right) \otimes \left(U_n^{(2)} + w_2 \right) \\ H_2 \left(U_n^{(1)} + w_1 \right) \otimes \left(U_n^{(1)} + w_1 \right) \end{pmatrix}.$$

Similarly to Lemma 4, after expanding out the tensor product, any terms containing w_1, w_2 may be dropped as they will be of higher order than $n+1$. Thus, all of the terms of $\begin{pmatrix} K_{n+1}^{(1)} \\ K_{n+1}^{(2)} \end{pmatrix}$ will be contained in

$$\begin{pmatrix} H_1 U_n^{(1)} \otimes U_n^{(2)} \\ H_2 U_n^{(1)} \otimes U_n^{(1)} \end{pmatrix}.$$

Now, H_1 and H_2 add one to the order, $K_{n+1}^{(1)}$ will be the sum of all terms of the form

$$H_1 K_i^{(1)} \otimes K_j^{(2)} \quad \text{where} \quad i+j=n$$

and $K_{n+1}^{(2)}$ will be the sum of terms of the form

$$H_2 K_i^{(i)} \otimes K_j^{(2)}, \quad \text{where} \quad i+j=n.$$

Thus, we obtain

$$K_{n+1}^{(1)} = H_1 \sum_{\substack{0 \leq i,j \leq n; \\ i+j=n}} K_i^{(1)} \otimes K_j^{(2)} \qquad \text{and} \qquad K_{n+1}^{(2)} = H_2 \sum_{\substack{0 \leq i,j \leq n; \\ i+j=n}} K_i^{(1)} \otimes K_j^{(1)}$$

and we have derived the following recurrence for the forward operators:

$$K_1 = \begin{pmatrix} K_0^{(1)} \\ K_0^{(2)} \end{pmatrix} = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix},$$
$$K_2 = \begin{pmatrix} K_1^{(1)} \\ K_1^{(2)} \end{pmatrix} = \begin{pmatrix} H_1 u_0^{(1)} \otimes u_0^{(2)} \\ H_2 u_0^{(1)} \otimes u_0^{(1)} \end{pmatrix},$$
$$K_{n+1} = \begin{pmatrix} K_{n+1}^{(1)} \\ K_{n+1}^{(2)} \end{pmatrix} = \begin{pmatrix} H_1 \sum\limits_{\substack{0\le i,j\le n \\ i+j=n}} K_i^{(1)} \otimes K_j^{(2)} \\ H_2 \sum\limits_{\substack{0\le i,j\le n \\ i+j=n}} K_i^{(1)} \otimes K_j^{(1)} \end{pmatrix}. \tag{5}$$

2.2 Bounds on the Forward Operators

Definition 5. Let K be a multilinear operator of order n on $[L^\infty(\Omega)]^n$. Define

$$|K|_\infty := \sup_{\substack{\alpha_i \neq 0 \\ 1\le i\le n}} \frac{\|K(\alpha_1,\ldots,\alpha_n)\|_{C(\partial\Omega\times\partial\Omega)}}{\|\alpha_1\|_\infty \cdots \|\alpha_n\|_\infty}.$$

⌟

Note that if T and K is are multilinear operators of order n, then the triangle inequality

$$|T+K|_\infty \le |T|_\infty + |K|_\infty$$

holds.

Define μ as it was defined in Section 4, that is $\mu := \max\{\mu_1,\mu_2\}$ where

$$\mu_1 := 8\pi k^2 \sup_{x\in\Omega} \int_\Omega |G(k;x,y)|\, dy,$$
$$\mu_2 := 8\pi k^2 \sup_{x\in\Omega} \int_\Omega |G(2k;x,y)|\, dy.$$

Lemma 6. *The components $K_n^{(1)}$ and $K_n^{(2)}$ of K_n defined by eq. (5) are bounded multilinear operators and*

$$\left|K_n^{(1)}\right|_\infty \le \nu_n^{(1)}\mu^n \quad \text{and} \quad \left|K_n^{(2)}\right|_\infty \le \nu_n^{(2)}\mu^n,$$

where

$$\nu_0^{(1)} = \|u_0^{(1)}\|_{C(\overline{\Omega}\times\delta\Omega)} \quad \text{and} \quad \nu_0^{(2)} = \|u_0^{(2)}\|_{C(\overline{\Omega}\times\delta\Omega)},$$

and for each $n\ge 0$,

$$\nu_{n+1}^{(1)} = \sum_{\substack{0\le i,j\le n \\ i+j=n}} \nu_i^{(1)}\nu_j^{(2)}, \tag{6}$$
$$\nu_{n+1}^{(2)} = \sum_{\substack{0\le i,j\le n \\ i+j=n}} \nu_i^{(1)}\nu_j^{(1)}. \tag{7}$$

First, note that for the product given in Definition 3, we have

$$|H_1 T_k|_\infty \le \mu\,|T_k|_\infty \quad \text{and} \quad |H_2 T_k|_\infty \le \mu\,|T_k|_\infty.$$

Also, we have

$$|T_k \otimes T_j|_\infty \le |T_k|_\infty\,|T_j|_\infty.$$

Proof. Proof by induction on n. For the base case, it's obvious that $\left|K_0^{(1)}\right|_\infty = \nu_0^{(1)}$ and $|K_0|_\infty = \nu_0^{(2)}$.

For the induction step, suppose that for each $i \le n$, we have

$$\left|K_i^{(1)}\right|_\infty \le \nu_i^{(1)} \mu^i \quad \text{and} \quad \left|K_i^{(2)}\right|_\infty \le \nu_i^{(2)} \mu^i.$$

Then, by eq. (5),

$$\begin{aligned}
\left|K_{n+1}^{(1)}\right|_\infty &= \left| H_1 \sum_{\substack{0\le i,j\le n \\ i+j=n}} K_k^{(1)} \otimes K_j^{(2)} \right|_\infty \\
&\le \mu \sum_{\substack{0\le i,j\le n \\ i+j=n}} \left|K_i^{(1)}\right|_\infty \left|K_j^{(2)}\right|_\infty \\
&\le \mu \sum_{\substack{0\le i,j\le n \\ i+j=n}} \nu_i^{(1)} \mu^{(i)} \nu_j^{(2)} \mu^{(j)} \\
&= \mu^{n+1} \sum_{\substack{0\le i,j\le n \\ i+j=n}} \nu_i^{(1)} \nu_j^{(2)} \\
&= \mu^{n+1} \nu_{n+1}^{(1)}.
\end{aligned}$$

A similar computation reveals that $\left|K_{n+1}^{(2)}\right|_\infty \le \mu^{n+1} \nu_{n+1}^{(2)}$. □

Lemma 7. *For the sequences $\{\nu_n^{(1)}\}$ and $\{\nu_n^{(2)}\}$ defined in Lemma 6, there exist constants $\nu^{(1)}, \nu^{(2)}$ and $\kappa^{(1)}, \kappa^{(2)}$ such that*

$$\nu_n^{(1)} \le \nu^{(1)} \left(\kappa^{(1)}\right)^n \tag{8}$$

$$\nu_n^{(2)} \le \nu^{(2)} \left(\kappa^{(2)}\right)^n. \tag{9}$$

$$\nu^{(1)} = \nu^{(2)} = \|u_0\|_{C\left(\overline{\Omega}\times\partial\Omega\right)^2}$$

and

$$\kappa^{(1)} = \kappa^{(2)} = 4\|u_0\|_{C\left(\overline{\Omega}\times\partial\Omega\right)^2}.$$

Proof. Let

$$\begin{aligned}
P(x) &= \sum_{n=0}^{\infty} \nu_n^{(1)} x^n \\
Q(x) &= \sum_{n=0}^{\infty} \nu_n^{(2)} x^n.
\end{aligned}$$

Then we have that

$$P(x)\,Q(x) = \sum_{n=0}^{\infty} \sum_{i+j=n} \nu_i^{(1)} \nu_j^{(2)} x^n$$

and

$$(P(x))^2 = \sum_{n=0}^{\infty} \sum_{i+j=n} \nu_i^{(1)} \nu_j^{(1)} x^n.$$

Thus, we can multiply eqs. (6) and (7) by x^n and sum to obtain

$$\frac{P(x) - \nu_0^{(1)}}{x} = P(x)\,Q(x) \tag{10}$$

$$\frac{Q(x) - \nu_0^{(2)}}{x} = (P(x))^2\,. \tag{11}$$

We may transform *eq.* (11) into the form $Q = xP^2 + \nu_0^{(2)}$ and plug back into eq. (10) to obtain

$$x^2P^3 + x\nu_0^{(2)}P - P + \nu_0^{(1)} = 0.$$

Differentiate with respect x and obtain

$$3x^2P^2P' + 2xP^3 + \nu_0^{(2)}P + x\nu_0^{(2)}P' - P' = 0.$$

Thus,

$$P'\left(3x^2P^2 + x\nu_0^{(2)} - 1\right) = -2xP^3 - \nu_0^{(2)}P.$$

and so

$$P' = -\frac{2xP^3 + \nu_0^{(2)}P}{3x^2P^2 + x\nu_0^{(2)} - 1}. \tag{12}$$

Now, as long as $3x^2P^2 + x\nu_0^{(2)} - 1 \neq 0$, P' is an analytic function of x and P which means that there is a unique analytic solution for eq. (12) which is exactly $P(x)$. Thus, the power series $P(x)$ has a positive radius of convergence.

Since $Q = xP^2 + \nu_0^{(2)}$, whenever $P(x)$ is an analytic function, so is $Q(x)$. Thus, both $P(x)$ and $Q(x)$ have positive radii of convergence and the proof is complete. □

Proposition 8. *The forward operator K_n given by eq. (5) is a bounded multilinear operator from $[L^\infty(\Omega)]^n$ to $C(\partial\Omega)^2$, and*

$$|K_n| \leq \nu\,(\kappa\mu)^n \tag{13}$$

where $\kappa = \max\left\{\kappa^{(1)}, \kappa^{(2)}\right\}$, $\nu = 2\max\left\{\nu^{(1)}, \nu^{(2)}\right\}$, and $\mu = \max\{\mu_1, \mu_2\}$ for

$$\mu_1 = 8\pi k^2 \sup_{x\in\Omega}\int_\Omega |G(k;x,y)|\,dy \qquad \text{and} \qquad \mu_2 = 16\pi k^2 \sup_{x\in\Omega}\int_\Omega |G(2k;x,y)|\,dy$$

as in section 4.

Proof.

$$\begin{aligned}
|K_n|^2 &= \left|\begin{pmatrix} K_n^{(1)} \\ K_n^{(2)} \end{pmatrix}\right|^2 \\
&= \left|K_n^{(1)}\right|_\infty^2 + \left|K_n^{(1)}\right|_\infty^2 \\
&\leq \left(\nu^{(1)}\left(\kappa^{(1)}\mu\right)^n\right)^2 + \left(\nu^{(2)}\left(\kappa^{(2)}\mu\right)^n\right)^2 \\
&\leq \left(\nu^{(1)}\left(\kappa^{(1)}\mu\right)^n + \nu^{(2)}\left(\kappa^{(2)}\mu\right)^n\right)^2 \\
&\leq \nu^2\,(\kappa\mu)^{2n}\,.
\end{aligned}$$

□

Corollary 9. *The Born series*

$$u = u_0 + \sum_{n=1}^{\infty} K_n(\eta, \ldots, \eta)$$

where K_n are given by eq. (5) converges in $C\left(\overline{\Omega}\right)^2$ for

$$\|\eta\|_\infty \leq \frac{1}{\kappa\mu}.$$

3 Inverse Born Series

The inverse problem is to reconstruct the coefficient η from measurements on the boundary $\phi = u - u_0$ on $\partial\Omega$. We define the Inverse Born Series (IBS) as

$$\tilde{\eta} = \mathcal{K}_1(\phi) + \mathcal{K}_2(\phi) + \mathcal{K}_3(\phi) + \cdots \tag{14}$$

where the data $\phi \in C(\partial\Omega)^2$. We have already seen the IBS analyzed in [2, 3]. The inverse operators $\mathcal{K}_m$ are given by

$$\begin{aligned}
\mathcal{K}_1(\phi) &= K_1^+(\phi), \\
\mathcal{K}_2(\phi) &= -\mathcal{K}_1(K_2(\mathcal{K}_1(\phi), \mathcal{K}_1(\phi))), \\
\mathcal{K}_m(\phi) &= -\sum_{n=2}^{m} \sum_{i_1+\cdots+i_n=m} \mathcal{K}_1(K_n(\mathcal{K}_{i_1}(\phi), \ldots, \mathcal{K}_{i_n}(\phi))).
\end{aligned}$$

Note that the operator K_1 does not, in general, have a bounded inverse. Thus, we take $\mathcal{K}_1$ to be the regularized pseudoinverse K_1^+ of K_1 as described in [4]. Now, our bounds on the forward operators given in proposition 8 in combination with theorems 2.2 and 2.4 of [2] gives the next two theorems on convergence and approximation error of the IBS.

Note. We denote by $\|\mathcal{K}_1\|$ the operator norm of $\mathcal{K}_1$ as a map from $C(\partial\Omega)^2$ to $L^\infty(\Omega)$.

Theorem 10 (convergence of the IBS). *If $\|\mathcal{K}_\infty(\phi)\| < r$ where*

$$r = \frac{1}{2\kappa\mu}\left(\sqrt{16C^2+1} - 4C\right),$$

$C = \max\{2, \|\mathcal{K}_1\|\nu\kappa\mu\}$, and ν, κ are the same as in Proposition 8, then the IBS eq. (14) converges.

Theorem 11 (approximation error). *Suppose that the hypotheses of Theorem 10 hold and that the Born and IBS converge. Let $\overline{\eta}$ denote the sum of the IBS and $\eta_1 = \mathcal{K}_1(\phi)$. Let $\mathcal{M} = \max\{\|\eta\|_\infty, \|\tilde{\eta}\|_\infty\}$. Also assume that*

$$\mathcal{M} < \frac{1}{2\kappa\mu}\left(1 - \sqrt{\frac{\nu\kappa\mu\|\mathcal{K}_1\|}{1+\nu\kappa\mu\|\mathcal{K}_1\|}}\right). \tag{15}$$

Then the approximation error of reconstruction can be estimated as follows:

$$\begin{aligned}
\|\eta - \sum_{m=1}^{N} \mathcal{K}_m(\phi)\|_\infty \le{}& M\left(\frac{\|\eta_1\|_\infty}{r}\right)^{N+1} \frac{1}{1-\frac{\|\eta_1\|_\infty}{r}} \\
&+ \left(1 - \frac{\nu\kappa\mu\|\mathcal{K}_1\|}{(1-\mu\mathcal{M})^2} + \nu\kappa\mu\|\mathcal{K}_1\|\right)^{-1} \|(I - \mathcal{K}_1 K_1)\eta\|_\infty,
\end{aligned}$$

where

$$M = \frac{2\mu\kappa}{\sqrt{16C+1}}.$$

4 Appendix: Existence and Uniqueness

Our main tool in proving the existence and uniqueness of solutions is the Banach fixed point theorem. Before stating the theorem, we give an important definition.

Definition 12. Let X be a normed vector space and let B be a closed ball in X. A map $T\colon X \to X$ is called a *contraction* on the closed ball B if for any $f, g \in B$, we have

$$\|T(f) - T(g)\| \le p\|f - g\|$$

where $0 < p < 1$. ⌟

Theorem 13 (Banach). *Let X be a Banach space and let $T\colon X \to X$ be a contraction on a closed ball $B \subseteq X$. If $T(B) \subseteq B$, then T has a unique fixed point in B, that is, some $x \in B$ satisfies $T(x) = x$.*

Let $G(k; x, y)$ be the Green's function for the operator $\Delta u^{(1)} + k^2$ and $G(2k; x, y)$ be the Green's function for the operator $\Delta u^{(1)} + (2k)^2$ as in Section 1. Let

$$T\colon C\left(\overline{\Omega}\right)^2 \to C\left(\overline{\Omega}\right)^2$$

be given by

$$T\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + \begin{pmatrix} -8\pi k^2 \int_\Omega G(k; x, y)\,\eta(y)\,u^{(1)}(y)\,u^{(2)}(y)\,dy \\ -4\pi (2k)^2 \int_\Omega G(2k; x, y)\,\eta(y)\left(u^{(1)}\right)^2(y)\,dy \end{pmatrix}. \tag{16}$$

If we can prove that eq. (16) satisfies the hypothesis of Theorem 13 for some ball $B \subseteq C\left(\overline{\Omega}\right)^2$, then we will obtain a unique $u = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} \in C\left(\overline{\Omega}\right)^2$ that satisfies eq. (1), solving our PDEs.

We begin by making precise some of our statements in the introduction. Define the norm $\|\cdot\|$ on $C\left(\overline{\Omega}\right)^2$ to be

$$\left\| \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} \right\| := \|u^{(1)}\|_{C(\overline{\Omega})} + \|u^{(2)}\|_{C(\overline{\Omega})}.$$

Let

$$G\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} := \begin{pmatrix} -8\pi k^2 \int_\Omega G(k; x, y)\,\eta(y)\,u^{(1)}(y)\,u^{(2)}(y)\,dy \\ -4\pi (2k)^2 \int_\Omega G(2k; x, y)\,\eta(y)\left(u^{(1)}\right)^2(y)\,dy \end{pmatrix}. \tag{17}$$

Then, define the operator

$$T\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} = \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} + G\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}.$$

To bound the norm of G in eq. (17), we define

$$\mu_1 := 8\pi k^2 \sup_{x\in\Omega} \int_\Omega |G(k; x, y)|\,dy$$
$$\mu_2 := 16\pi k^2 \sup_{x\in\Omega} \int_\Omega |G(2k; x, y)|\,dy$$

and let $\mu := \max\{\mu_1, \mu_2\}$. Now, we have

$$\left\| -8\pi k^2 \int_\Omega G(x, y)\,\eta(y)\,u^{(1)}(y)\,u^{(2)}(y)\,dy \right\| \leq \mu\|\eta\|\|u^{(1)}\|\|u^{(2)}\|$$
$$\left\| -4\pi (2k)^2 \int_\Omega G(x, y)\,\eta(y)\left(u^{(1)}\right)^2(y)\,dy \right\| \leq \mu\|\eta\|\|u^{(1)}\|^2.$$

Remember that our goal is to show that T satisfies Theorem 13. We begin by giving conditions under which T is a contraction.

Lemma 14. *Let $f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}$ and $g = \begin{pmatrix} g^{(1)} \\ g^{(2)} \end{pmatrix}$. Let such that $\|f\|, \|g\| \leq R$. Then,*

$$\|Tf - Tg\| \leq q\|f - g\|$$

where

$$q = 2R\mu\|\eta\|.$$

Proof. Notice that

$$T(f) - T(g) = \begin{pmatrix} -8\pi k^2 \int_\Omega G(x,y)\,\eta(y)\left(f^{(1)}(y)\,f^{(2)}(y) - g^{(1)}(y)\,g^{(2)}(y)\right)dy \\ -8\pi k^2 \int_\Omega G(x,y)\,\eta(y)\left(\left(f^{(1)}\right)^2(y) - \left(g^{(1)}\right)^2(y)\right)dy \end{pmatrix}$$

so

$$\begin{aligned}
\|T(f) - T(g)\| &\le \mu\|\eta\|\|f^{(1)}f^{(2)} - g^{(1)}g^{(2)}\| + \mu\|\eta\|\|\left(f^{(1)}\right)^2 - \left(g^{(1)}\right)^2\| \\
&\le \mu\|\eta\|\left[\|\left(f^{(1)} - g^{(1)}\right)f^{(2)} + \left(f^{(2)} - g^{(2)}\right)g^{(1)}\| + \|f^{(1)} + g^{(1)}\|\|f^{(1)} - g^{(1)}\|\right] \\
&\le \mu\|\eta\|\left[\|\left(f^{(1)} - g^{(1)}\right)\|\|f^{(2)}\| + \|\left(f^{(2)} - g^{(2)}\right)\|\|g^{(1)}\| + \left(\|f^{(1)}\| + \|g^{(1)}\|\right)\|f^{(1)} - g^{(1)}\|\right] \\
&= \mu\|\eta\|\left[\left(\|f^{(1)}\| + \|f^{(2)}\| + \|g^{(1)}\|\right)\|f^{(1)} - g^{(1)}\| + \|g^{(1)}\|\|f^{(2)} - g^{(2)}\|\right] \\
&< \mu\|\eta\|\left[2R\|f^{(1)} - g^{(1)}\| + R\|f^{(2)} - g^{(2)}\|\right] \\
&< \mu\|\eta\|\left[2R\|f^{(1)} - g^{(1)}\| + 2R\|f^{(2)} - g^{(2)}\|\right] \\
&= 2R\mu\|\eta\|\left[\|f^{(1)} - g^{(1)}\| + \|f^{(2)} - g^{(2)}\|\right] \\
&= 2R\mu\|\eta\|\|f - g\|.
\end{aligned}$$

□

The next lemma gives a ball that T maps into itself (a ball $B \subseteq C\left(\overline{\Omega}\right)^2$ for which $T(B) \subseteq B$).

Lemma 15. *Let $r > 0$ and let $B(u_0, r)$ be the ball of radius r about $u_0 := \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix}$ in $C^0\left(\overline{\Omega}\right)^2$. Define $R := \|u_0\| + r$. Then if*

$$\mu R^2\|\eta\| < r,$$

T maps $B(u_0, r)$ into itself.

Proof. Let $f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix} \in B(u_0, r)$. Then,

$$\begin{aligned}
\|T(f) - u_0\| &= \left\|G\begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}\right\| \\
&\le \mu\|\eta\|\|f^{(1)}\|\|f^{(2)}\| + \mu\|\eta\|\|f^{(1)}\|^2 \\
&= \mu\|\eta\|\left(\|f^{(1)}\|\|f^{(2)}\| + \|f^{(1)}\|^2\right) \\
&= \mu\|\eta\|\|f^{(1)}\|\left(\|f^{(2)}\| + \|f^{(1)}\|\right) \\
&< \mu\|\eta\|\|f^{(1)}\|R \\
&< \mu\|\eta\|R^2.
\end{aligned}$$

□

Since Lemma 14 gives conditions for T to be a contraction and Lemma 15 gives a ball that T maps into itself, if we can find $R > 0$ that satisfies both lemmas simultaneously, we will have satisfied all hypothesis in Theorem 13 and solve our problem.

Proposition 16. *If*

$$\|\eta\| \le \frac{1}{4\mu\|u_0\|}, \tag{18}$$

then T has a unique fixed point in the ball $B(u_0, \|u_0\|)$.

Proof. First, $r = \|u_0\| \implies R = 2\|u_0\|$. Thus,

$$\begin{aligned}
\mu\|\eta\|2R &\leq \mu\|\eta\|4\|u_0\| \\
&< \mu 4\|u_0\| \cdot \frac{1}{\mu 4\|u_0\|} \\
&= 1.
\end{aligned}$$

And so we have satisfied Lemma 14. Now,

$$\begin{aligned}
\mu\|\eta\|R^2 &\leq \mu\|\eta\|\left(2\|u_0\|\right)^2 \\
&< \mu\frac{1}{4\|u_0\|} \cdot 4\|u_0\|^2 \\
&= \|u_0\|.
\end{aligned}$$

Thus Lemma 15 is also satisfied which implies that T has a unique fix point on $B\left(u_0, \|u_0\|\right)$. $\square$

Remark. Note that with Proposition 16, we have guaranteed the existence and uniqueness of solutions for any data (not just small input data).

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