

$e$

Let's take the derivative of  $a^x$ , with respect to  $x$ .

$$\frac{d}{dx}a^x = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} = \left( \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right) a^x$$

Notice that the limit in this last expression can be rewritten:

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \lim_{h \rightarrow 0} \frac{a^{0+h} - a^0}{h} = \left( \frac{d}{dx}a^x \right) \Big|_{x=0}.$$

Thus, the derivative of  $a^x$  with respect to  $x$  is

$$\frac{d}{dx}a^x = \left( \frac{d}{dx}a^x \right) \Big|_{x=0} a^x.$$

In order to obtain a more useful expression for  $\frac{d}{dx}a^x$ , we must evaluate the expression  $\left( \frac{d}{dx}a^x \right) \Big|_{x=0}$ .

A natural way to do this is to attempt to find the  $a$  for which  $\left( \frac{d}{dx}a^x \right) \Big|_{x=0} = 1$ . (Does such an  $a$  even exist? The answer is yes, but showing so is quite technical). For the sake of having concrete notation, we will *define*  $e$  to be the number such that  $\left( \frac{d}{dx}e^x \right) \Big|_{x=0} = 1$ . Then

$$\frac{d}{dx}e^x = \left( \frac{d}{dx}e^x \right) \Big|_{x=0} e^x = 1 \cdot e^x = e^x.$$

Now, we use *change of base for exponential functions* to use the fact that  $\frac{d}{dx}e^x = e^x$  to write  $\frac{d}{dx}a^x$ . In general, for any  $a, b$ , we have  $a^x = b^{\log_b(a^x)} = b^{\log_b(a)x}$ . That is, for any  $a, b$ , we have  $a^x = b^{cx}$  (where  $c = \log_b(a)$ ).

We have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\log_e(a)x} = \log_e(a)e^{\log_e(a)x}$$

by the chain rule. Doing some more algebra gives

$$\log_e(a)e^{\log_e(a)x} = \log_e(a)e^{\log_e(a^x)} = \log_e(a)a^x.$$

Of course, we use the notation  $\ln(x) := \log_e(x)$ , so

$$\frac{d}{dx}a^x = \ln(a)a^x.$$

Sidenote: using the above definition of  $e$ , one can prove that  $e = \lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n$ .