e

Let's take the derivative of a^x, with respect to x.

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h} = \Big(\lim_{h\to 0}\frac{a^h-1}{h}\Big)a^x$$

Notice that the limit in this last expression can be rewritten:

$$\lim_{h\to 0}\frac{a^h-1}{h} = \lim_{h\to 0}\frac{a^{0+h}-a^0}{h} = \Big(\frac{d}{dx}a^x\Big)\Big|_{x=0}.$$

Thus, the derivative of a^x with respect to x is

$$\frac{d}{dx}a^x = \Big(\frac{d}{dx}a^x\Big)\Big|_{x=0}a^x.$$

In order to obtain a more useful expression for $\frac{d}{dx}a^x$, we must evaluate the expression $\Big(\frac{d}{dx}a^x\Big)\Big|_{x=0}$.

A natural way to do this is to attempt to find the a for which $\Big(\frac{d}{dx}a^x\Big)\Big|_{x=0} = 1$. (Does such an a even exist? The answer is yes, but showing so is quite technical). For the sake of having concrete notation, we will *define* e to be the number such that $\Big(\frac{d}{dx}e^x\Big)\Big|_{x=0} = 1$. Then

$$\frac{d}{dx}e^x = \Big(\frac{d}{dx}e^x\Big)\Big|_{x=0}e^x = 1\cdot e^x = e^x.$$

Now, we use *change of base for exponential functions* to use the fact that $\frac{d}{dx}e^x = e^x$ to write $\frac{d}{dx}a^x$. In general, for any a, b, we have $a^x = b^{\log_b(a^x)} = b^{\log_b(a)x}$. That is, for any a, b, we have $a^x = b^{cx}$ (where $c = \log_b(a)$).

We have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\log_e(a)x} = \log_e(a)e^{\log_e(a)x}$$

by the chain rule. Doing some more algebra gives

$$\log_e(a)e^{\log_e(a)x} = \log_e(a)e^{\log_e(a^x)} = \log_e(a)a^x.$$

Of course, we use the notation $\ln(x) := \log_e(x)$, so

$$\frac{d}{dx}a^x = \ln(a)a^x.$$

Sidenote: using the above definition of e, one can prove that $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$.