

e

Let's take the derivative of a^x , with respect to x .

$$\frac{d}{dx}a^x = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} = \left(\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right) a^x$$

Notice that the limit in this last expression can be rewritten:

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \lim_{h \rightarrow 0} \frac{a^{0+h} - a^0}{h} = \left(\frac{d}{dx}a^x \right) \Big|_{x=0}.$$

Thus, the derivative of a^x with respect to x is

$$\frac{d}{dx}a^x = \left(\frac{d}{dx}a^x \right) \Big|_{x=0} a^x.$$

In order to obtain a more useful expression for $\frac{d}{dx}a^x$, we must evaluate the expression $\left(\frac{d}{dx}a^x \right) \Big|_{x=0}$.

A natural way to do this is to attempt to find the a for which $\left(\frac{d}{dx}a^x \right) \Big|_{x=0} = 1$. (Does such an a even exist? The answer is yes, but showing so is quite technical). For the sake of having concrete notation, we will *define* e to be the number such that $\left(\frac{d}{dx}e^x \right) \Big|_{x=0} = 1$. Then

$$\frac{d}{dx}e^x = \left(\frac{d}{dx}e^x \right) \Big|_{x=0} e^x = 1 \cdot e^x = e^x.$$

Now, we use *change of base for exponential functions* to use the fact that $\frac{d}{dx}e^x = e^x$ to write $\frac{d}{dx}a^x$. In general, for any a, b , we have $a^x = b^{\log_b(a^x)} = b^{\log_b(a)x}$. That is, for any a, b , we have $a^x = b^{cx}$ (where $c = \log_b(a)$).

We have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\log_e(a)x} = \log_e(a)e^{\log_e(a)x}$$

by the chain rule. Doing some more algebra gives

$$\log_e(a)e^{\log_e(a)x} = \log_e(a)e^{\log_e(a^x)} = \log_e(a)a^x.$$

Of course, we use the notation $\ln(x) := \log_e(x)$, so

$$\frac{d}{dx}a^x = \ln(a)a^x.$$

Sidenote: using the above definition of e , one can prove that $e = \lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n$.