Impact of Counterweight Mass, Initial Arm Angle, and Wheels on Trebuchet Projectile Firing Speed

Ross Grogan-Kaylor, Sebastian Vander Ploeg Fallon, and James Yan
Carleton College[*]
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We attempted to optimize the destructive capabilities of a trebuchet by theoretically and experimentally examining how changing the mass of the counterweight, the initial angle of the arm, and the presence of wheels impacted the speed of the trebuchet's projectile. After using a non-conservative force to capture the bending of the trebuchet arm during firing, we were able to closely match our theoretical and experimental results. These results supported the ideas that an increased counterweight mass, an increased initial arm angle from the horizontal, or the addition of wheels cause an increase in final projectile speed.

I. INTRODUCTION

A common type of catapult consists of a beam across a pivot point. The person using the catapult applies a force on one end of the beam, causing a projectile at the other end of the beam to rotate around the pivot point and fly towards a target at high speeds. These catapults come in two types: the traction type where the operator employs many men to pull on the end of the beam, and the counterweight type where a mass is attached to the end of the beam so that gravity provides the force powering the machine.[1] These counterweight type catapults were first developed in the late 1100s.[1,2,3,4] Through the early middle ages, they had an advantage over cannon in that they could be easily assembled.[1,3]

There appears to be consensus about the typical construction of these counterweight type catapults. It is important to construct the beam so that it does not flex or snap.[1,2,3] Also, the optimal ratio of the distance from the long arm to the pivot divided by the distance from the short arm to the pivot appears to be 5:1 or 6:1.[1,3] However, for smaller machines a ratio of 2:1 or 3:1 may also work well.[1]

Although there are some common parameters for the construction of counterweight type catapults, these catapults can be optimized by attaching a hanging counterweight instead of a fixed counterweight. This allows the counterweight to fall in a straighter path down, and therefore transfer more potential energy to the missile.[3] The hanging counterweight also slows down the rotation of the arm after firing the projectile, leading to a shorter time between shots.[5] The best distance between the pivot from which the counterweight hangs to the counterweight appears to be half of the shorter arm length.[3] We shall follow the convention of referring to this particular type of catapult with a hanging counterweight as a trebuchet.[2]

A second optimization is to attach a sling. A trebuchet without a sling is difficult to optimize for military use because the arm has such a large moment of inertia and the length of the throwing arm is too small.[1] The sling increases the effective length of the throwing arm, and with it the range of the trebuchet.[4] Adding a sling also dramatically increases the engine and kinetic efficiency of the trebuchet.[2]

In terms of other optimizations, one study claimed that adding wheels to the trebuchet may increase performance because it would allow the counterweight to fall straighter.[2] In addition, another study indicated that increasing counterweight mass should increase range, and having an initial angle of 45 degrees between throwing arm and the horizontal may be optimal.[3] In our experiment, we attempted to test these claims by constructing a hanging counterweight trebuchet with wheels. We hypothesized that each of adding wheels, increasing the initial angle of the arm from the horizontal, and increasing the counterweight mass would yield greater projectile speed upon release.

II. MECHANICS

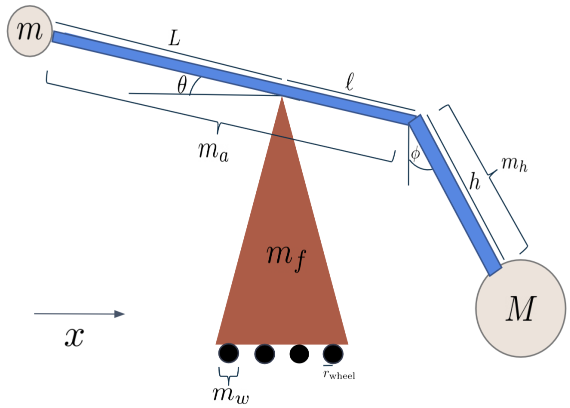

FIG. 1. Trebuchet design. Here, M is the mass of the counterweight, m is the mass of the projectile and projectile throwing carriage, m_h is the mass of the arm connecting the counterweight to the throwing arm, m_a is the mass of the throwing arm, m_f is the mass of the trebuchet frame, m_w is the mass of a wheel, h is the length of the arm connecting the counterweight to the throwing arm, l is the distance between the pivot on the frame and the pivot for the counterweight, L is the distance between the pivot on the frame and the projectile, r_{wheel} is the radius of the wheel, x is the distance between the frame and the origin, θ is the angle between the throwing arm and the horizontal, and ϕ is the angle between the arm connected to the counterweight and the vertical.

We chose to model this system using the Lagrangian formulation of mechanics. Using the quantities defined in FIG. 1, we found that the Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m_x\dot{x}^2 + \frac{1}{2}I_\theta\dot{\theta}^2 + \frac{1}{2}I_\phi\dot{\phi}^2 \\ &+ (mL - M\ell + m_a\Big(\frac{L-\ell}{2}\Big) - m_h\ell)\dot{x}\dot{\theta}\sin(\theta) \\ &+ (M + \frac{1}{2}m_h)h\cos(\phi)\dot{x}\dot{\phi} \\ &- \ell h\sin(\theta+\phi)\dot{\theta}\dot{\phi}\Big(M + \frac{1}{2}mh\Big) \\ &- g\sin(\theta)\Big(mL + \frac{1}{2}m_a(L-\ell) - m_h\ell - M\ell\Big) \\ &+ gh\cos(\phi)\Big(\frac{1}{2}m_h + M\Big),\end{aligned} \tag{1}$$

where

$$\begin{aligned}m_x &= m + M + m_a + 8m_{\text{w}} + 8\frac{I_{\text{wheel}}}{r_{\text{wheel}}^2} + m_{\text{frame}} \\ I_\theta &= mL^2 + M\ell^2 + m_h\ell^2 + \frac{1}{3}m_a\Big(\frac{L^3+\ell^3}{L+\ell}\Big) \\ I_\phi &= (M + \frac{1}{3}m_h)h^2.\end{aligned}$$

Here, I_{wheel} is the moment of inertia of a wheel about its central axis; m_x is the mass of the entire trebuchet plus a term coming from the rotational kinetic energy of the wheels; I_θ is the effective moment of inertia of the two arms, the projectile with its carriage, and the counterweight about the axis at the pivot of the beam; I_ϕ is the moment of inertia of the counterweight arm and the counterweight about the axis at the end point of the throwing arm where the counterweight is attached. Here, we approximated both the counterweight and the projectile with its carriage as point masses. The equations of motion for the generalized coordinates q_i, which in this case were x, θ, and ϕ, were then given by the standard Euler-Lagrange equations

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big) - \frac{\partial\mathcal{L}}{\partial q_i} = 0. \tag{2}$$

For our specific trebuchet, the relevant masses and lengths were as follows:

	Physical Values
m (g)	16.4 $\pm$ 0.1
m_a (g)	81.7 $\pm$ 0.1
m_h (g)	0
M (g)	264.4 $\pm$ 0.1
m_{w} (g)	8.15$\pm$ 0.01
m_{f} (g)	151.1 $\pm$ 0.1
L (cm)	40.7 $\pm$ 0.3
l (cm)	21.0 $\pm$ 0.3
h (cm)	8.7 $\pm$ 0.1
r_{wheel} (cm)	1.55 $\pm$ .05

TABLE I. Physical data concerning our trebuchet. All variables refer to the dimensions defined in Figure 1.

For the purposes of computation, we took the wheels to be cylinders of uniform density with the familiar moment of inertia about their center of mass of

$$I_{\text{wheel}} = \frac{1}{2}m_{\text{w}}r_{\text{wheel}}^2. \tag{3}$$

III. EXPERIMENTAL DATA

Our first test was a control trial where we kept the wheels on the trebuchet, set the counterweight mass at $M = (264.4 \pm 0.1)$g, and started the trebuchet arm at a sizeable displacement from the horizontal, (-0.533 ± 0.00) rad. In the second test, we varied the counterweight mass M from (264.4 ± 0.1)g to (238.6 ± 0.1)g; in the third test we intentionally set the initial angle between the arm and the horizontal to nearly zero, specifically $\theta_i = (-0.025 \pm 0.002)$ rad; and in the fourth test we removed the wheels.

For each trial, we fired the trebuchets and used the program Tracker to measure the initial angle θ_i the arm made with the horizontal when we released the counterweight, as well as the angle θ_f of the arm and the speed v_f of the projectile when the projectile fired. To compute θ_i and θ_f, we centered the coordinate axes at the trebuchet's pivot point, determined the position (x, y) of the mass on the arm, and computed $\theta_{i,f} = \arctan(\frac{y}{x})$ at the appropriate time in the video. The final speed v_f was computed by using Tracker; see Tracker's documentation for more details.

To reiterate, for each experiment, we measured θ_i, θ_f, and v_f. Our experimental results are summarized in the following table:

Trial	M (g)	θ_i (rad)	θ_f (rad)	v_f ($\frac{\text{m}}{\text{s}}$)
1	264.4 ± 0.1	-0.533 ± 0.002	1.399 ± 0.002	3.5 ± 0.5
2	238.6 ± 0.1	-0.468 ± 0.002	1.140 ± 0.002	3.3 ± 0.4
3	264.4 ± 0.1	-0.025 ± 0.002	1.568 ± 0.002	2.6 ± 0.4
4	264.4 ± 0.1	-1.102 ± 0.002	1.318 ± 0.002	3.3 ± 0.4

TABLE II. Physical data from trebuchet tests. Here, θ_i is initial value of θ right before the counterweight was released, θ_f is the angle of the arm when the projectile was released, M is the mass of the counterweight. Trial 1 is the control. In trial 2, M was decreased. In trial 3, we used a low value of θ_i. (Note: while the initial angle θ_i varies across all trials, our intent was to hold θ_i constant across the trials 1, 2, 4, and to intentionally pick θ_i close to zero for trial 3). In trial 4, we removed the wheels; wheels were used in all previous trials.

IV. DISCUSSION

We numerically solved the Euler-Lagrange equations from our theoretical model so we could see how our theoretical predictions matched our experimental results. Our predicted final projectile speeds, using the parameters from each test, are as follows:

Trial	Theoretical Final Speed ($\frac{\text{m}}{\text{s}}$)
1	2.656 ± 0.007
2	2.991± 0.007
3	3.114± 0.007
4	2.637± 0.007

TABLE III. Predicted projectile speed when projectile leaves trebuchet arm. We quote uncertainty because the speed is computed given knowledge of the experimentally determined release angle.

These theoretical final speeds in TABLE III support the hypothesis that the addition of wheels should theoretically increase the final speed of the trebuchet because the predicted final speed of the control trial, trial 1, is greater than the predicted final speed of the trial without wheels, trial 4. However, these theoretically predicted projectile final speeds suggest that the trebuchet with a lower counterweight mass, trial 2, and the trebuchet with a smaller initial arm angle from the horizontal, trial 3, should have higher final speeds than trial 1, our control.

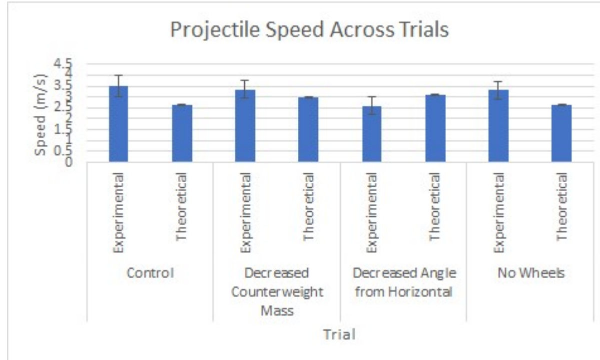

FIG. 2. Comparison of theoretical final projectile speeds and experimental data. Here, the control trial corresponds to trial 1, the Decreased Counterweight Mass trial corresponds to trial 2, the Decreased Angle from Horizontal trial corresponds to trial 3, and the No Wheels trial corresponds to trial 4.

There are several takeaways from FIG. 2. The experimentally measured value of the projectile's final speed for the control trial, trial 1, was greater than the other experimentally measured final speeds which supports our hypothesis. In addition, the error in the final measured speed for the control trial does not capture the the final measured speed for decreased angle from horizontal, trial 3. This means that we can definitively say that the final speed from the control trial was greater than the final speed from reducing the angle the arm made with the horizontal. However, the error in the final speed measured in the control trial does capture the experimentally measured final speeds for the decreased counterweight mass trial, trial 2, and the trial without wheels, trial 4. This means that we were unable to experimentally show that removing wheels or decreasing the counterweight mass has an effect on the final speed of the projectile. As we can also see from FIG. 2, only the final speed from trial 2 captures the theoretical final speed within error.

Trial	Final Speed Signed % Error
1	31.8
2	10.3
3	-16.5
4	25.1

TABLE IV. Percent error between the observed projectile speeds and theoretical projectile speeds.

It is also important to note in FIG. 2 that the experimentally determined projectile speeds were much greater than those theoretically predicted in three of the four cases and, as we see from TABLE IV, the errors in the theoretical predictions were large.

We can attempt to explain these discrepancies by appealing to the fact that our model did not incorporate non-conservative forces such as friction, and perhaps more importantly, bending in the throwing arm. During the experimental trials, the arm did bend quite significantly. This is problematic because the bent arm was actually farther ahead on the trajectory than the measured angle implies. Therefore, the counterweight would have dropped farther, giving more energy to the projectile. To correct for this, we proposed a non-conservative force to model the bending of the throwing arm which we would incorporate into the Euler-Lagrange equation for θ. The non-conservative force would have dimensions of torque and would be positive to cause an increase in $\ddot{\theta}$. This would model the bending by increasing $\dot{\theta}$ to correspond to the true speed of the arm. We assumed that the only variables which could determine this bending were the mass of the arm m_a, the mass of the projectile and carriage m, the length of the throwing arm L, and θ and its derivatives. If the arm is at rest, then we expect the arm to experience no bending. Therefore, the non-conservative force should be proportional to exclusively $\dot{\theta}$ and none of θ's other derivatives. To get the mass dimension of the torque, we need to multiply $L^2\dot{\theta}^2$ by either m or m_a. There are two ways to arrive at dimensions of torque, so in general, we can write our non-conservative force as a linear combination of the two ways to get dimensions of torque:

$$Q_{\text{bending}} = (c_1 m + c_2 m_a)L^2\dot{\theta}^2. \tag{4}$$

We would expect masses farther from the pivot point to contribute more to the bending of the arm. Therefore, we expect $c_1 > c_2$. If this force acts at the center of mass of the arm and the projectile carriage, then since the center of mass of the arm is half of the way to the projectile carriage, we can take $4c_2 = c_1$. This leaves us with a generalized force of the form

$$Q_{\text{bending}} = \alpha(m + \frac{1}{4}m_a)L^2\dot{\theta}^2. \tag{5}$$

where α is the proportionality constant.

The addition of this non-conservative force changes the Euler-Lagrange equation for $\dot{\theta}$ to

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \alpha(m + \frac{1}{4}m_a)L^2\dot{\theta}^2. \tag{6}$$

The Euler-Lagrange equations for x and ϕ are unchanged.

We compared the theoretical predictions and experimental predictions for the projectile speed for different values of α, with a step size of 0.001, across all of our trials. The best α values we found are given in the following table:

Trial	Best α
1	0.173
2	0.127
3	0.373
4	0.173

TABLE V. Best values of α for each trial. We tested all α between 0 and 5 inclusive with a step size of 0.001.

The α values in TABLE V are still in the same general neighborhood, so this gives us some confidence that the same bending force acts in all of the trials. This is promising because in our experimental trials, we did not vary any of the parameters Q_{bending} depends on. If we average α over all four trials, then we find $\alpha = 0.2115$. If the same bending force acted over all four trials, then we would expect using this averaged value of α to provide significantly better theoretical predictions for all trials. The theoretical predictions from taking $\alpha = 0.2115$ for all trials are summarized in the following table.

Trial	Theoretical Prediction ($\frac{\text{m}}{\text{s}}$)	Final Speed Signed % Error
1	3.652± 0.007	-4.53
2	3.583± 0.007	-6.76
3	2.981± 0.007	-12.8
4	3.448± 0.007	-4.01

TABLE VI. Final speed theoretical predictions and percent error between the experimental result and theoretical prediction for $\alpha = 0.2115$. We quote uncertainty because the speed is computed given knowledge of the experimentally determined release angle.

As can be seen from comparing TABLE VI and TABLE IV, the inclusion of this non-conservative force modeling the bending of the arm reduced all percent errors.

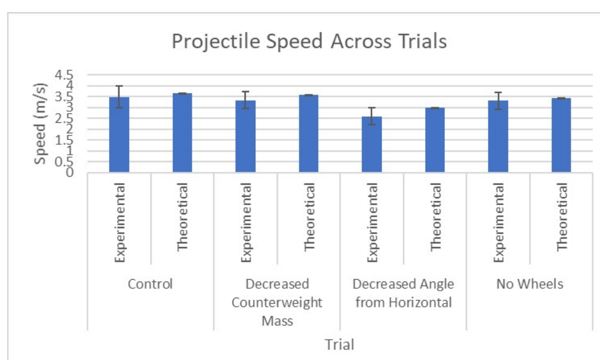

FIG. 3. Comparison of new theoretical final speeds and experimental data. Here, the control trial corresponds to trial 1, the Decreased Counterweight Mass trial corresponds to trial 2, the Decreased Angle from Horizontal trial corresponds to trial 3, and the No Wheels trial corresponds to trial 4.

In addition, as we can see from FIG 3, all of the new theoretically predicted speeds are within uncertainty of our experimental data. Therefore, this nonconservative force we proposed seems to well model the bending of our trebuchet arm.

V. CONCLUSION

After including the force modeling the bending in the arm, and taking a proportionality value for our nonconservative force of 0.2115, the error in the experimental values capture the theoretical results for all experiments. We originally hypothesized that increasing counterweight mass, increasing the initial angle between the arm and the horizontal, and adding wheels would increase projectile speed. Our experimental and theoretical results support all of our hypotheses, with the experimental results giving the most support to the idea that increasing the initial angle between the arm and the horizontal will increase projectile speed.

VI. ACKNOWLEDGEMENTS

We would like to thank Jay for reviewing our paper.

Appendix A: How We Contributed to the Project

Ross Grogan-Kaylor was the code keeper for the group. He worked with Sebastian and James to construct the frame of the trebuchet. After Sebastian derived the Lagrangian and transformation equations, he double-checked these equations and typed them up in LATEX. With regards to experimentation, Ross took slow-motion videos of the trebuchet in action and processed the data alongside Sebastian in the Tracker software program. As far as code keeper duties go, Ross wrote most of the Mathematica code. This included writing the core modules (such as `simulateTrebuchetHelper`, `simulateTrebuchet`, `findBest`α) and the code that looped over all simulations to produce the results of this paper. He also propagated error on the theoretical release speeds of the projectile, as these "theoretical" speeds depended to a small degree on experimental measurements. Ross discussed and "fought" with Sebastian on the theory of the nonconservative generalized force. For the presentation, Ross prepared the slides relating to the nonconservative bending force, and made the diagram of the trebuchet.

Sebastian Vander Ploeg Fallon was the paper coordinator for the group. He constructed the arm and counterweight for the trebuchet as well as helping out on constructing the frame. He also computed the Lagrangian and transformation equations between Cartesian and generalized coordinates by hand so that we could enter them into Mathematica. He also helped to run trebuchet testing by holding the trebuchet in position, going to the Tracker talk so he could teach his teammates, and doing error propogation on the experimental data later. He wrote a lot of the paper, and he came up with the generalized bending force to account for the sizeable difference between out theoretical and experimental predictions for projectile speed. For the presentation, he wrote several slides, and produced the graphs.

James Yan was the trebuchet builder. He led the group in designing the trebuchet. He constructed the frame and base of the first trebuchet and organized testing. He double checked the Lagrangian and transformation equations after Sebastian derived them. He helped the slow-motion videos of the trebuchet by handling the trebuchet and launching the projectile. He found the literature for coming up with optimizations and historical references. He made most of the slides. This included summing up the paper and main results and creating figures. After Sebastian had decided to introduce the nonconservative force, he brought up skepticism and checked the correctness and eventually supported the idea. He started the work of error propagation and checked the error propagation code at the end.

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* groganr@carleton.edu, vanders@carleton.edu, yanj@carleton.edu

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