Phys 111 Lecture #14 Math Notes

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1 C&J 7.3

The lead female character in the movie *Diamonds Are Forever* is standing at the edge of an offshore oil rig. As she fires a gun, she is driven back over the edge and into the sea. Suppose the mass of a bullet is $1.0 \times 10^{-2}\ kg$ and its velocity is $+720\ m/s$. Her mass (including the gun) is $51\ kg$.

a. What recoil velocity does she acquire in response to a single shot from a stationary position, assuming that no external force keeps her in place?

b. Under the same assumption, what would be her recoil velocity if, instead, she shoots a blank cartridge that ejects a mass of $5.0 \times 10^{-4}\ kg$ at a velocity of $+720\ m/s$?

1.1 Answer Slide Information

Find v_2 given $\vec{\mathbf{p}}_o = 0$, $v_1 = 720\ m/s$, $m_t = 51.0\ kg$ plus quantities for each case where $m_t = m_1 + m_2$.

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f = 0$$

$$m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 = 0$$

$$\vec{\mathbf{v}}_2 = -\frac{m_1}{m_2}\,\vec{\mathbf{v}}_1$$

a. $m_1 = 1.0 \times 10^{-2}\ kg$, $m_2 = 50.99\ kg \quad \rightarrow \quad \mathrm{v}_2 = 141\ mm/s$

b. $m_1 = 5.0 \times 10^{-4}\ kg$, $m_2 = 50.9995\ kg \quad \rightarrow \quad \mathrm{v}_2 = 7\ mm/s$

1.2 Math with Explanation

Initially the system, the lady holding the gun with the bullet in it, is at rest

$$\vec{\mathbf{p}}_o = 0.$$

Then there is an internal explosion in the gun so the system of interest has conserved momentum

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f$$

such that the system is now two objects moving with equal and opposite momentum:

$$\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 = 0$$

$$\vec{\mathbf{p}}_2 = -\vec{\mathbf{p}}_1$$

Recall that linear momenta are always quantified with inertia and velocity

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$$

Therefore

$$\vec{\mathbf{p}}_2 = -\vec{\mathbf{p}}_1$$

becomes

$$m_2\vec{\mathbf{v}}_2 = -m_1\vec{\mathbf{v}}_1$$

and we can relate the magnitudes exactly so we can get the speed of the second object in relation to the two masses and the first speed

$$v_2 = \frac{m_1 v_1}{m_2}.$$

We can use the equation above as is to find the second speeds for the two different mass ratios given or we can rewrite the division to put emphasis on the mass ratio

$$v_2 = \frac{m_1}{m_2} v_1.$$

Please perform the arithmetic as stated the first way, but either order we calculate the two different speeds when using a regular bullet or a blank that fits in the gun:

$$v_{2a} = \left(\frac{0.01}{50.99}\right)(720\ m/s)$$

$$v_{2b} = \left(\frac{0.0005}{50.9995}\right)(720\ m/s)$$

The final answers are

$$v_{2a} = 0.141\ m/s = 141\ mm/s$$

$$v_{2b} = 0.007\ m/s = 7\ mm/s$$

2 C&J 7.17

A 2.3 kg cart is rolling across a friction-less, horizontal track toward a 1.5 kg cart that is held initially at rest. The carts are loaded with strong magnets that cause them to attract one another. Thus, the speed of each cart increases. At a certain instant before the carts collide, the first cart's velocity is $+4.5\ m/s$, and the second cart's velocity is $-1.9\ m/s$.

a. What is the total momentum of the system of the two carts at this instant?

b. What was the velocity of the first cart when the second cart was still at rest?

2.1 Answer Slide

$m_1 = 2.3\ kg$, $m_2 = 1.5\ kg$, $v_{2o} = 0\ m/s$, $v_{1f} = 4.5\ m/s$, $v_{2f} = -1.9\ m/s$.

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f$$

$$\vec{\mathbf{p}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2$$

a. $\vec{\mathbf{p}}_f = +7.5\ kg \cdot m/s$

b. $\vec{\mathbf{v}}_{1o} = +3.26\ m/s$

2.2 Math with Explanation

We are told that our "initial" state consists of one car moving and the other stationary, but we are not told the initial speed of the first car. We are however informed of two magnets providing a mutual attraction between the two cars. The cars would feel equal magnitude, but oppositely directed forces so the magnetic forces are internal forces causing a net zero impulse if the two cars are the system of interest. At least up to the moment right before they collide.

$$\vec{\mathbf{p}}_o = m_1\vec{\mathbf{v}}_{1o} + m_2\vec{\mathbf{v}}_{2o}$$

$$\vec{\mathbf{p}}_o = m_1\vec{\mathbf{v}}_{1o}$$

$$\vec{\mathbf{J}} = 0$$

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f$$

$$\vec{\mathbf{p}}_f = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f}$$

We are told the final speeds and directions indicated by $+$ and $-$ signs.

$$\vec{\mathbf{p}}_f = (2.3\ kg)(+4.5\ m/s) + (1.5\ kg)(-1.9\ m/s)$$

Thus the initial and final momentum for this interaction can be quantified to be

$$\vec{\mathbf{p}}_f = \vec{\mathbf{p}}_o = +7.5\ kg \cdot m/s.$$

We can directly relate the initial momentum factors to this amount with

$$m_1 \vec{\mathbf{v}}_{1o} = +7.5\ kg \cdot m/s$$

and conclude the magnitude and direction simultaneously

$$\vec{\mathbf{v}}_{1o} = \frac{\vec{\mathbf{p}}_f}{m_1}$$

$$\vec{\mathbf{v}}_{1o} = \frac{+7.5\ kg \cdot m/s}{2.3\ kg}$$

$$\vec{\mathbf{v}}_{1o} = +3.3\ m/s$$

3 C&J 7.21

A two-stage rocket moves in space at a constant velocity of 4900 m/s. The two stages are then separated by a small explosive charge placed between them. Immediately after the explosion the velocity of the 1200 kg upper stage is 5700 m/s in the same direction as before the explosion. What is the velocity (magnitude and direction) of the 2400 kg lower stage after the explosion?

3.1 Answer Slide

Find $\vec{\mathbf{v}}_{lf}$ if $m_u = 1200\ kg$, $m_l = 2400\ kg$, $\vec{\mathbf{v}}_o = +4900\ m/s$, $\vec{\mathbf{v}}_{uf} = +5700\ m/s$.

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f$$

$$(m_u + m_l)\vec{\mathbf{v}}_o = m_u \vec{\mathbf{v}}_{uf} + m_l \vec{\mathbf{v}}_{lf}$$

$$m_l \vec{\mathbf{v}}_{lf} = (m_u + m_l)\vec{\mathbf{v}}_o - m_u \vec{\mathbf{v}}_{uf}$$

$$\vec{\mathbf{v}}_{lf} = \frac{(m_u + m_l)\vec{\mathbf{v}}_o - m_u \vec{\mathbf{v}}_{uf}}{m_l}$$

$$\vec{\mathbf{v}}_{lf} = +4500\ m/s$$

3.2 Math with Explanation

Just like with the first problem there is an internal explosion causing two pieces of one object to come apart, but this time there is a nonzero initial momentum. Let's call the initial direction of the rocket the positive direction.

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f$$

$$\vec{\mathbf{p}}_o = (m_1 + m_2)\vec{\mathbf{v}}_o$$

$$\vec{\mathbf{p}}_o = (3600\ kg)(+4900\ m/s)$$

$$\vec{\mathbf{p}}_o = 1.76 \times 10^7\ kg \cdot m/s$$

Let's keep our naming scheme simple and label the final pieces 1 and 2 because they were attached as an initial piece moving, but now they are 2 separate moving pieces

$$\vec{\mathbf{p}}_f = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2.$$

This problem can be thought of as either a half-level or a full level increase in difficulty from the first problem. From conservation of momentum we can solve for the momentum of the second piece:

$$\vec{\mathbf{p}}_2 = \vec{\mathbf{p}}_o - \vec{\mathbf{p}}_1$$

$$m_2\vec{\mathbf{v}}_2 = (m_1 + m_2)\vec{\mathbf{v}}_o - m_1\vec{\mathbf{v}}_1.$$

Now let's solve for the velocity of the second piece

$$\vec{\mathbf{v}}_2 = \frac{(m_1 + m_2)\vec{\mathbf{v}}_o - m_1\vec{\mathbf{v}}_1}{m_2}$$

$$\vec{\mathbf{v}}_2 = \frac{(+1.76 \times 10^7\ kg \cdot m/s) - (1200\ kg)(+5700\ m/s)}{(2400\ kg)}$$

$$\vec{\mathbf{v}}_2 = \frac{10.8 \times 10^6}{2400}\ m/s$$

$$\vec{\mathbf{v}}_2 = +4500\ m/s$$

So you might have expected the bottom/lower piece to be going backwards, but keep in mind that it can be quite difficult to completely reverse the momentum of an object, especially if the mass of that object is the largest mass.

4 C&J 7.25

By accident, a large plate is dropped and breaks into three pieces. The pieces fly apart parallel to the floor. As the plate falls, its momentum has only a vertical component and no component parallel to the floor. After the collision, the component of the total momentum parallel to the floor must remain zero, since the net external force acting on the plate has no component parallel to the floor. Using the data shown in the drawing, find the masses of pieces 1 and 2.

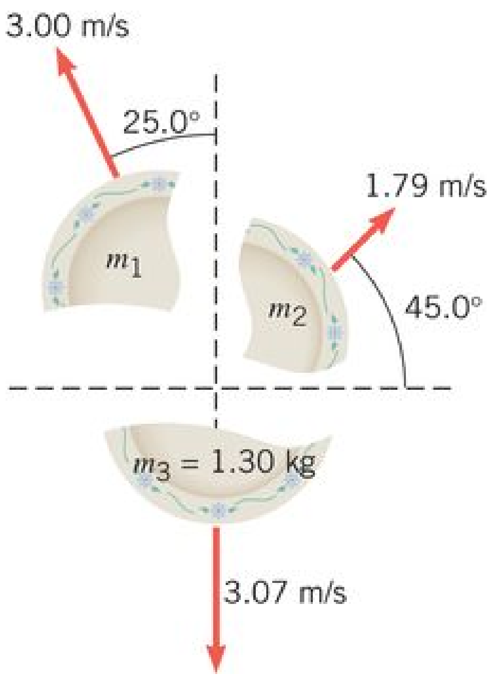

Figure 1: Conservation of Momentum in 2D

4.1 Answer Slide

C&J 6.4.35 $m_3 = 1.30\ kg$, $\vec{\mathbf{v}}_3 = -3.07\ m/s\ \hat{\mathbf{y}}$, $\vec{\mathbf{v}}_2 = 1.79\ m/s\ 45°$ *CCW from +x*, $\vec{\mathbf{v}}_1 = 3.00\ m/s\ 25°$ *CCW from +y*. Find m_1 and m_2.

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f = 0$$

$$p_{1x} = p_{2x}$$

$$p_{1y} + p_{2y} = p_{3y}$$

$$m_2 = \frac{v_{1x}}{v_{2x}}\, m_1$$

$$m_1 v_{1y} + m_2 v_{2y} = m_3 v_3$$

Answer: $m_1 = 1.00\ kg$, $m_2 = 1.00\ kg$

4.2 Math with Explanation

Here we have an example of needing two sets of momentum equations due to the two dimensional (2D) nature of the "internal-force explosion" that occurred when this plate broke. In this case, the nice thing is that the momentum in the two dimensions of interest is zero.

$$\vec{\mathbf{p}}_o = 0$$

$$\vec{\mathbf{p}}_o = \vec{\mathbf{p}}_f$$

$$\vec{\mathbf{p}}_f = p_x\,\hat{\mathbf{x}} + p_y\,\hat{\mathbf{y}}$$

The equation directly above this sentence is a fancy way of taking the condensed equation and breaking or decomposing the momentum into an x-component and a y-component to simplify how to handle a complicated system. Since each component is conserved and equal to zero we can rewrite our single vector equation as two component equations:

$$p_x = 0$$

$$p_y = 0.$$

We need to keep in mind that we simply double the initial set up. Usually we need to solve one equation to relate the two unknown quantities and substitute that relationship into the second equation to solve for one of the unknowns first. Then we use our result for the first unknown in the substitution relationship to solve for the second unknown. Our unknowns in this case are mass 1 (m_1) and mass 2 (m_2). If we use a standard x-y coordinate system for the image given we can now write in our plus and minus signs to indicate direction:

$$p_x = m_2 v_{2x} - m_1 v_{1x}$$

$$p_y = m_1 v_{1y} + m_2 v_{2y} - m_3 v_3.$$

Please note that the third piece has a momentum solely in one direction so we include all of its momentum in only one of the equations.

$$m_2 v_{2x} - m_1 v_{1x} = 0$$

$$m_1 v_{1y} + m_2 v_{2y} - m_3 v_3 = 0.$$

In general, draw velocity triangles to avoid making a mistake for which trig function is used for x and y components. For our velocities we use the given angles and speeds with the following component equations

$$v_{1x} = v_1 sin(\theta_1)$$

$$v_{1y} = v_1 cos(\theta_1)$$

$$v_{2x} = v_2 cos(\theta_2)$$

$$v_{2y} = v_2 sin(\theta_2)$$

and after plugging in the speeds, $v_1 = 3.00\, m/s$ and $v_2 = 1.79\, m/s$, as well as the angles, $\theta_1 = 25°$ and $\theta_2 = 45°$, we get

$$v_{1x} = 1.27\, m/s$$

$$v_{1y} = 2.72\, m/s$$

$$v_{2x} = 1.27\, m/s$$

$$v_{2y} = 1.27\, m/s.$$

Remember that our goal is to use these values in the following equations relating momentum components

$$m_2 v_{2x} - m_1 v_{1x} = 0$$

$$m_1 v_{1y} + m_2 v_{2y} - m_3 v_3 = 0.$$

From the x component set we get

$$m_2 v_{2x} = m_1 v_{1x}$$

$$m_2 = \frac{m_1 v_{1x}}{v_{2x}}$$

and once again I want to consider the given ratio

$$m_2 = \frac{v_{1x}}{v_{2x}} m_1.$$

Please note that the two x speeds are the same and so we can deduce that the two unknown masses should be equal.

$$m_2 = \frac{1.27}{1.27} m_1$$

$$m_2 = m_1.$$

Let's use this relationship in the second equation to actually calculate the mass values. From the y component equation we get the relationship

$$m_1 v_{1y} + m_2 v_{2y} = m_3 v_3$$

and with our mass relationship we can simplify it to

$$m_1 v_{1y} + m_1 v_{2y} = m_3 v_3$$

$$(v_{1y} + v_{2y}) m_1 = m_3 v_3$$

and when solved for mass 1 we write

$$m_1 = \frac{m_3 v_3}{(v_{1y} + v_{2y})}$$

$$m_1 = \frac{v_3}{(v_{1y} + v_{2y})} m_3.$$

Plugging in given values we get

$$m_1 = \frac{3.07}{2.72 + 1.27}(1.30\ kg)$$

$$m_1 = \frac{3.07}{3.99}(1.30 kg)$$

$$m_1 = (0.769)(1.30)\ kg$$

$$m_1 = 1.00\ kg$$

$$m_2 = 1.00\ kg.$$

5 Summary

I hope by now it's clear that we are utilizing the same algebra that is covered in Precalculus or in Algebra 2 and Trigonometry courses. The caveat is that we using it for quantitative analysis which means we need to be a little more thoughtful about why we are using equations and what information we can deduce from said equations. One of the goals for this course is to provide all of you with an opportunity to better know and understand mathematics by applying it to quantify relationships for physical phenomena.

In summary, provide a physical reason for an equation, justify choices for directions and components, be aware of a couple good ways to write/perform the required math, and don't forget to think about the physics we are using the math for.