APEX CALCULUS

Version 3.0

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PREFACE

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three–volume series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.apexcalculus.com`.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactive the interactivity, one can right-click and choose the “Disable Content” option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

1: INTEGRATION

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute areas, volumes, mass, force, pressure, work, and much more.

1.1 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 1 Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\, dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them.

We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

> **Theorem 1 Antiderivative Forms**
>
> Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that
> $$G(x) = F(x) + C.$$

Given a function f and one of its antiderivatives F, we know *all* antiderivatives of f have the form $F(x) + C$ for some constant C. Using Definition 1, we can say that

$$\int f(x)\, dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

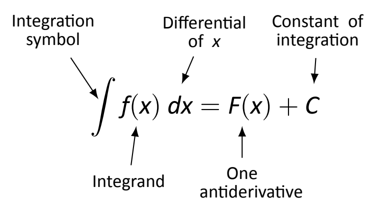


Figure 1.1: Understanding the indefinite integral notation.

Figure 1.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Example 1 Evaluating indefinite integrals

Evaluate $\displaystyle\int \sin x\, dx$.

Notes:

Solution We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x \, dx = -\cos x + C.$$

A commonly asked question is "What happened to the *dx*?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x \, dx$$

presents us with a differential, $dy = \sin x \, dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x \, dx$, rewrite

$$\int \sin x \, dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the *dx*?" with "It went away."

Let's practice once more before stating integration rules.

Example 2 **Evaluating indefinite integrals**

Evaluate $\displaystyle\int (3x^2 + 4x + 5) \, dx$.

Solution We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Notes:

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\ dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\ dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

Theorem **??** gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Notes:

Theorem 2 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	1. $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	2. $\int \big(f(x) \pm g(x)\big)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
3. $\frac{d}{dx}\big(C\big) = 0$	3. $\int 0\ dx = C$
4. $\frac{d}{dx}\big(x\big) = 1$	4. $\int 1\ dx = \int dx = x + C$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$	5. $\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$	6. $\int \cos x\ dx = \sin x + C$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$	7. $\int \sin x\ dx = -\cos x + C$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	8. $\int \sec^2 x\ dx = \tan x + C$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$	9. $\int \csc x \cot x\ dx = -\csc x + C$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	10. $\int \sec x \tan x\ dx = \sec x + C$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	11. $\int \csc^2 x\ dx = -\cot x + C$
12. $\frac{d}{dx}\big(e^x\big) = e^x$	12. $\int e^x\ dx = e^x + C$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$	13. $\int a^x\ dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	14. $\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$

We highlight a few important points from Theorem 2:

- Rule #1 states $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

  $$\int 5\cos x\ dx = 5 \cdot \int \cos x\ dx = 5 \cdot (\sin x + C) = 5\sin x + C.$$

  In the last step we can consider the constant as also being multiplied by

Notes:

5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 2. So:

$$\begin{aligned}\int (3x^2+4x+5)\,dx &= \int 3x^2\,dx + \int 4x\,dx + \int 5\,dx\\ &= 3\int x^2\,dx + 4\int x\,dx + \int 5\,dx\\ &= 3\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C\\ &= x^3+2x^2+5x+C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\,dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In Section **??** we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinite antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

Notes:

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 3 **Solving initial value problems**

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\,dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 4 **Solving initial value problems**

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\,dt = \int \cos t\,dt = \sin t + C = f'(t).$$

Notes:

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\,dt = \int (\sin t + 3)\,dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a position function given a velocity function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 1.4, we will see how areas and antiderivatives are closely tied together.

Notes:

Exercises 1.1

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the ________ things in the ________ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a ________ function.

7. The antiderivative of an acceleration function is a ________ function.

Problems

In Exercises 8 – 26, evaluate the given indefinite integral.

8. $\int 3x^3\,dx$

9. $\int x^8\,dx$

10. $\int (10x^2 - 2)\,dx$

11. $\int dt$

12. $\int 1\,ds$

13. $\int \frac{1}{3t^2}\,dt$

14. $\int \frac{3}{t^2}\,dt$

15. $\int \frac{1}{\sqrt{x}}\,dx$

16. $\int \sec^2\theta\,d\theta$

17. $\int \sin\theta\,d\theta$

18. $\int (\sec x\tan x + \csc x\cot x)\,dx$

19. $\int 5e^{\theta}\,d\theta$

20. $\int 3^t\,dt$

21. $\int \frac{5^t}{2}\,dt$

22. $\int (2t+3)^2\,dt$

23. $\int (t^2+3)(t^3-2t)\,dt$

24. $\int x^2x^3\,dx$

25. $\int e^{\pi}\,dx$

26. $\int a\,dx$

27. This problem investigates why Theorem 2 states that $\int \frac{1}{x}\,dx = \ln|x| + C$.

    (a) What is the domain of $y = \ln x$?

    (b) Find $\frac{d}{dx}\big(\ln x\big)$.

    (c) What is the domain of $y = \ln(-x)$?

    (d) Find $\frac{d}{dx}\big(\ln(-x)\big)$.

    (e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\int \frac{1}{x}\,dx$ that takes these different domains into account, and explain your answer.

In Exercises 28 – 38, find $f(x)$ described by the given initial value problem.

28. $f'(x) = \sin x$ and $f(0) = 2$

29. $f'(x) = 5e^x$ and $f(0) = 10$

30. $f'(x) = 4x^3 - 3x^2$ and $f(-1) = 9$

31. $f'(x) = \sec^2 x$ and $f(\pi/4) = 5$

32. $f'(x) = 7^x$ and $f(2) = 1$

33. $f''(x) = 5$ and $f'(0) = 7, f(0) = 3$

34. $f''(x) = 7x$ and $f'(1) = -1, f(1) = 10$

35. $f''(x) = 5e^x$ and $f'(0) = 3, f(0) = 5$

36. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2, f(\pi) = 4$

37. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5, f(0) = 0$

38. $f''(x) = 0$ and $f'(1) = 3, f(1) = 1$

Review

39. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

40. Given $y = x^2 e^x \cos x$, find dy.

1.2 The Definite Integral

We start with an easy problem. An object travels in a straight line at a constant velocity of 5 ft/s for 10 seconds. How far away from its starting point is the object?

We approach this problem with the familiar "Distance $=$ Rate $\times$ Time" equation. In this case, Distance = 5ft/s $\times$ 10s $=$ 50 feet.

It is interesting to note that this solution of 50 feet can be represented graphically. Consider Figure 1.2, where the constant velocity of 5ft/s is graphed on the axes. Shading the area under the line from $t = 0$ to $t = 10$ gives a rectangle with an area of 50 square units; when one considers the units of the axes, we can say this area represents 50 ft.

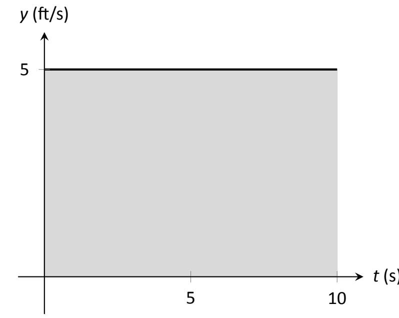


Figure 1.2: The area under a constant velocity function corresponds to distance traveled.

Now consider a slightly harder situation (and not particularly realistic): an object travels in a straight line with a constant velocity of 5ft/s for 10 seconds, then instantly reverses course at a rate of 2ft/s for 4 seconds. (Since the object is traveling in the opposite direction when reversing course, we say the velocity is a constant -2ft/s.) How far away from the starting point is the object – what is its *displacement*?

Here we use "Distance $=$ Rate$_1$ $\times$ Time$_1$ + Rate$_2$ $\times$ Time$_2$," which is

$$\text{Distance} \quad = 5 \cdot 10 + (-2) \cdot 4 = 42 \text{ ft.}$$

Hence the object is 42 feet from its starting location.

We can again depict this situation graphically. In Figure 1.3 we have the velocities graphed as straight lines on $[0, 10]$ and $[10, 14]$, respectively. The displacement of the object is

"Area above the *t*–axis $\quad - \quad$ Area below the *t*–axis,"

which is easy to calculate as $50 - 8 = 42$ feet.

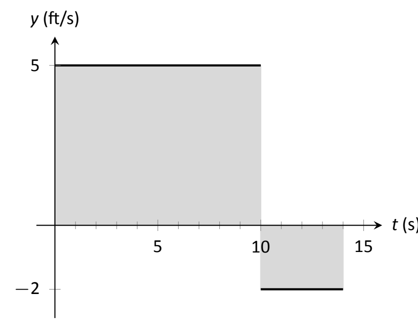


Figure 1.3: The total displacement is the area above the *t*–axis minus the area below the *t*–axis.

Now consider a more difficult problem.

Example 5 Finding position using velocity

The velocity of an object moving straight up/down under the acceleration of gravity is given as $v(t) = -32t+48$, where time t is given in seconds and velocity is in ft/s. When $t = 0$, the object had a height of 0 ft.

1. What was the initial velocity of the object?
2. What was the maximum height of the object?
3. What was the height of the object at time $t = 2$?

Solution It is straightforward to find the initial velocity; at time $t = 0$, $v(0) = -32 \cdot 0 + 48 = 48$ ft/s.

Notes:

To answer questions about the height of the object, we need to find the object's position function $s(t)$. This is an initial value problem, which we studied in the previous section. We are told the initial height is 0, i.e., $s(0) = 0$. We know $s'(t) = v(t) = -32t + 48$. To find s, we find the indefinite integral of $v(t)$:

$$\int v(t)\,dt = \int (-32t + 48)\,dt = -16t^2 + 48t + C = s(t).$$

Since $s(0) = 0$, we conclude that $C = 0$ and $s(t) = -16t^2 + 48t$.

To find the maximum height of the object, we need to find the maximum of s. Recalling our work finding extreme values, we find the critical points of s by setting its derivative equal to 0 and solving for t:

$$s'(t) = -32t + 48 = 0 \quad \Rightarrow \quad t = 48/32 = 1.5\text{s}.$$

(Notice how we ended up just finding when the velocity was 0ft/s!) The first derivative test shows this is a maximum, so the maximum height of the object is found at

$$s(1.5) = -16(1.5)^2 + 48(1.5) = 36\text{ft}.$$

The height at time $t = 2$ is now straightforward to compute: it is $s(2) = 32$ft.

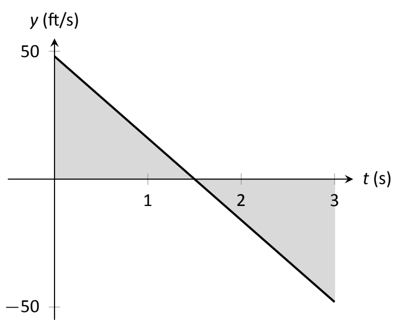


Figure 1.4: A graph of $v(t) = -32t + 48$; the shaded areas help determine displacement.

While we have answered all three questions, let's look at them again graphically, using the concepts of area that we explored earlier.

Figure 1.4 shows a graph of $v(t)$ on axes from $t = 0$ to $t = 3$. It is again straightforward to find $v(0)$. How can we use the graph to find the maximum height of the object?

Recall how in our previous work that the displacement of the object (in this case, its height) was found as the area under the velocity curve, as shaded in the figure. Moreover, the area between the curve and the t–axis that is below the t–axis counted as "negative" area. That is, it represents the object coming back toward its starting position. So to find the maximum distance from the starting point – the maximum height – we find the area under the velocity line that is above the t–axis, i.e., from $t = 0$ to $t = 1.5$. This region is a triangle; its area is

$$\text{Area} = \frac{1}{2}\text{Base} \times \text{Height} = \frac{1}{2} \times 1.5\text{s} \times 48\text{ft/s} = 36\text{ft},$$

which matches our previous calculation of the maximum height.

Finally, to find the height of the object at time $t = 2$ we calculate the total signed area under the velocity function from $t = 0$ to $t = 2$. This signed area is equal to $s(2)$, the displacement (i.e., signed distance) from the starting position at $t = 0$ to the position at time $t = 2$. That is,

$$\text{Displacement = Area above the } t\text{–axis} - \text{Area below } t\text{–axis}.$$

Notes:

The regions are triangles, and we find

$$\text{Displacement} = \frac{1}{2}(1.5\text{s})(48\text{ft/s}) - \frac{1}{2}(.5\text{s})(16\text{ft/s}) = 32\text{ft}.$$

This also matches our previous calculation of the height at $t = 2$.

Notice how we answered each question in this example in two ways. Our first method was to manipulate equations using our understanding of antiderivatives and derivatives. Our second method was geometric: we answered questions looking at a graph and finding the areas of certain regions of this graph.

The above example does not *prove* a relationship between area under a velocity function and displacement, but it does imply a relationship exists. Section 1.4 will fully establish fact that the area under a velocity function is displacement.

Given a graph of a function $y = f(x)$, we will find that there is great use in computing the area between the curve $y = f(x)$ and the x-axis. Because of this, we need to define some terms.

Definition 2 The Definite Integral, Total Signed Area

Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area from $x = a$ to $x = b$ under f** is:

(area under f and above the x–axis on $[a, b]$) $-$ (area above f and under the x–axis on $[a, b]$).

The **definite integral of f on $[a, b]$** is the total signed area of f on $[a, b]$, denoted

$$\int_a^b f(x)\, dx,$$

where a and b are the **bounds of integration.**

By our definition, the definite integral gives the "signed area under f." We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under f" or, more commonly, "the area under the curve."

The previous section introduced the indefinite integral, which related to antiderivatives. We have now defined the definite integral, which relates to areas under a function. The two are very much related, as we'll see when we learn the Fundamental Theorem of Calculus in Section 1.4. Recall that earlier we said that the "$\int$" symbol was an "elongated S" that represented finding a "sum." In the context of the definite integral, this notation makes a bit more sense, as we are adding up areas under the function f.

Notes:

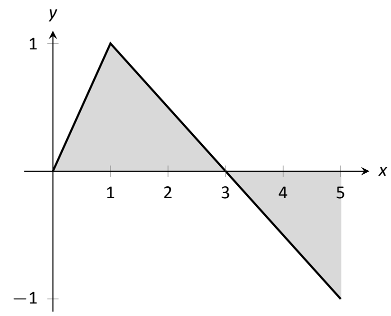


Figure 1.5: A graph of $f(x)$ in Example 6.

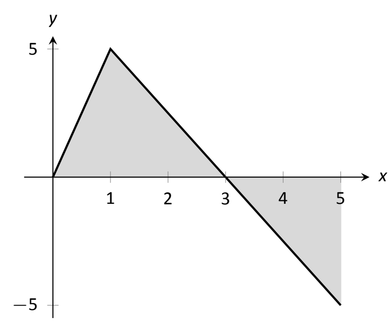


Figure 1.6: A graph of $5f$ in Example 6. (Yes, it looks just like the graph of f in Figure 1.5, just with a different y-scale.)

We practice using this notation.

Example 6 Evaluating definite integrals

Consider the function f given in Figure 1.5.

Find:

1. $\displaystyle\int_0^3 f(x)\,dx$
2. $\displaystyle\int_3^5 f(x)\,dx$
3. $\displaystyle\int_0^5 f(x)\,dx$
4. $\displaystyle\int_0^3 5f(x)\,dx$
5. $\displaystyle\int_1^1 f(x)\,dx$

Solution

1. $\int_0^3 f(x)\,dx$ is the area under f on the interval $[0,3]$. This region is a triangle, so the area is $\int_0^3 f(x)\,dx = \frac{1}{2}(3)(1) = 1.5$.

2. $\int_3^5 f(x)\,dx$ represents the area of the triangle found under the x–axis on $[3,5]$. The area is $\frac{1}{2}(2)(1) = 1$; since it is found *under* the x–axis, this is "negative area." Therefore $\int_3^5 f(x)\,dx = -1$.

3. $\int_0^5 f(x)\,dx$ is the total signed area under f on $[0,5]$. This is $1.5+(-1) = 0.5$.

4. $\int_0^3 5f(x)\,dx$ is the area under $5f$ on $[0,3]$. This is sketched in Figure 1.6. Again, the region is a triangle, with height 5 times that of the height of the original triangle. Thus the area is $\int_0^3 5f(x)\,dx = 15/2 = 7.5$.

5. $\int_1^1 f(x)\,dx$ is the area under f on the "interval" $[1,1]$. This describes a line segment, not a region; it has no width. Therefore the area is 0.

This example illustrates some of the properties of the definite integral, given here.

Notes:

Theorem 3 Properties of the Definite Integral

Let f and g be defined on a closed interval I that contains the values a, b and c, and let k be a constant. The following hold:

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
3. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\displaystyle\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

We give a brief justification of Theorem 3 here.

1. As demonstrated in Example 6, there is no "area under the curve" when the region has no width; hence this definite integral is 0.
2. This states that total area is the sum of the areas of subregions. It is easily considered when we let $a < b < c$. We can break the interval $[a, c]$ into two subintervals, $[a, b]$ and $[b, c]$. The total area over $[a, c]$ is the area over $[a, b]$ plus the area over $[b, c]$.

   It is important to note that this still holds true even if $a < b < c$ is not true. We discuss this in the next point.
3. This property can be viewed a merely a convention to make other properties work well. (Later we will see how this property has a justification all its own, not necessarily in support of other properties.) Suppose $b < a < c$. The discussion from the previous point clearly justifies

$$\int_b^a f(x)\,dx + \int_a^c f(x)\,dx = \int_b^c f(x)\,dx. \tag{1.1}$$

   However, we still claim that, as originally stated,

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx. \tag{1.2}$$

Notes:

How do Equations (1.1) and (1.2) relate? Start with Equation (1.1):

$$\int_b^a f(x)\,dx + \int_a^c f(x)\,dx = \int_b^c f(x)\,dx$$
$$\int_a^c f(x)\,dx = -\int_b^a f(x)\,dx + \int_b^c f(x)\,dx$$

Property (3) justifies changing the sign and switching the bounds of integration on the $-\int_b^a f(x)\,dx$ term; when this is done, Equations (1.1) and (1.2) are equivalent.

The conclusion is this: by adopting the convention of Property (3), Property (2) holds no matter the order of a, b and c. Again, in the next section we will see another justification for this property.

4,5. Each of these may be non–intuitive. Property (5) states that when one scales a function by, for instance, 7, the area of the enclosed region also is scaled by a factor of 7. Both Properties (4) and (5) can be proved using geometry. The details are not complicated but are not discussed here.

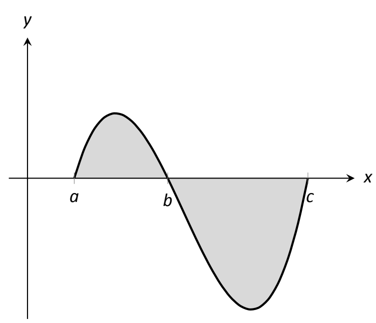


Figure 1.7: A graph of a function in Example 7.

Example 7 **Evaluating definite integrals using Theorem 3.**

Consider the graph of a function $f(x)$ shown in Figure 1.7. Answer the following:

1. Which value is greater: $\int_a^b f(x)\,dx$ or $\int_b^c f(x)\,dx$?
2. Is $\int_a^c f(x)\,dx$ greater or less than 0?
3. Which value is greater: $\int_a^b f(x)\,dx$ or $\int_c^b f(x)\,dx$?

Solution

1. $\int_a^b f(x)\,dx$ has a positive value (since the area is above the x–axis) whereas $\int_b^c f(x)\,dx$ has a negative value. Hence $\int_a^b f(x)\,dx$ is bigger.
2. $\int_a^c f(x)\,dx$ is the total signed area under f between $x = a$ and $x = c$. Since the region below the x–axis looks to be larger than the region above, we conclude that the definite integral has a value less than 0.
3. Note how the second integral has the bounds "reversed." Therefore $\int_c^b f(x)\,dx$ represents a positive number, greater than the area described by the first definite integral. Hence $\int_c^b f(x)\,dx$ is greater.

Notes:

The area definition of the definite integral allows us to use geometry compute the definite integral of some simple functions.

Example 8 **Evaluating definite integrals using geometry**

Evaluate the following definite integrals:

$$1. \int_{-2}^{5} (2x-4)\,dx \qquad 2. \int_{-3}^{3} \sqrt{9-x^2}\,dx.$$

Solution

1. It is useful to sketch the function in the integrand, as shown in Figure 1.8(a). We see we need to compute the areas of two regions, which we have labeled R_1 and R_2. Both are triangles, so the area computation is straightforward:

$$R_1 : \frac{1}{2}(4)(8) = 16 \qquad R_2 : \frac{1}{2}(3)6 = 9.$$

Region R_1 lies under the x–axis, hence it is counted as negative area (we can think of the triangle's height as being "-8"), so

$$\int_{-2}^{5} (2x-4)\,dx = -16 + 9 = -7.$$

2. Recognize that the integrand of this definite integral describes a half circle, as sketched in Figure 1.8(b), with radius 3. Thus the area is:

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{1}{2}\pi r^2 = \frac{9}{2}\pi.$$

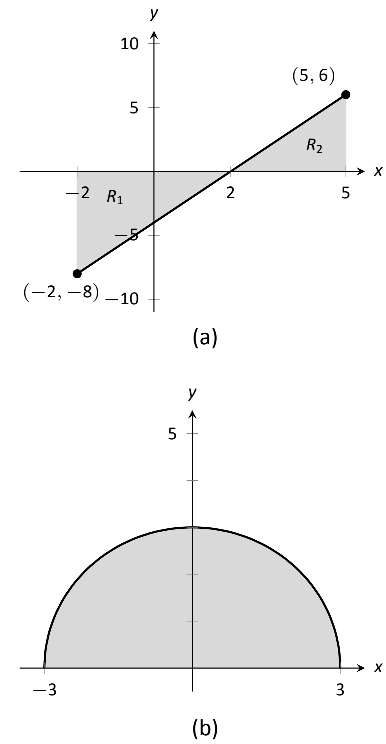


Figure 1.8: A graph of $f(x) = 2x - 4$ in (a) and $f(x) = \sqrt{9-x^2}$ in (b), from Example 8.

Example 9 **Understanding motion given velocity**

Consider the graph of a velocity function of an object moving in a straight line, given in Figure 1.9, where the numbers in the given regions gives the area of that region. Assume that the definite integral of a velocity function gives displacement. Find the maximum speed of the object and its maximum displacement from its starting position.

Solution Since the graph gives velocity, finding the maximum speed is simple: it looks to be 15ft/s.

At time $t = 0$, the displacement is 0; the object is at its starting position. At time $t = a$, the object has moved backward 11 feet. Between times $t = a$ and

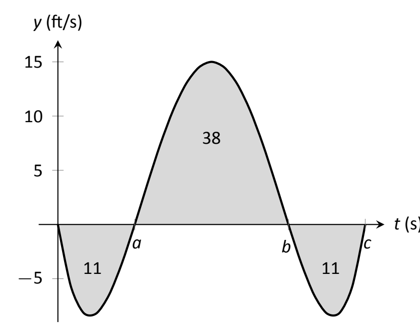


Figure 1.9: A graph of a velocity in Example 9.

Notes:

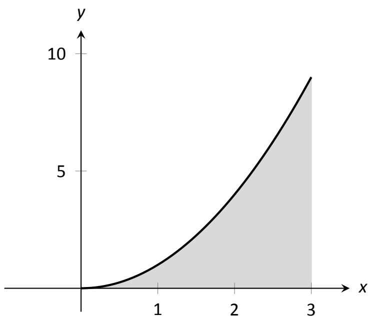


Figure 1.10: What is the area below $y = x^2$ on $[0, 3]$? The region is not a usual geometric shape.

$t = b$, the object moves forward 38 feet, bringing it into a position 27 feet forward of its starting position. From $t = b$ to $t = c$ the object is moving backwards again, hence its maximum displacement is 27 feet from its starting position.

In our examples, we have either found the areas of regions that have nice geometric shapes (such as rectangles, triangles and circles) or the areas were given to us. Consider Figure 1.10, where a region below $y = x^2$ is shaded. What is its area? The function $y = x^2$ is relatively simple, yet the shape it defines has an area that is not simple to find geometrically.

In the next section we will explore how to find the areas of such regions.

Notes:

Exercises 1.2

Terms and Concepts

1. What is "total signed area"?

2. What is "displacement"?

3. What is $\int_3^3 \sin x \, dx$?

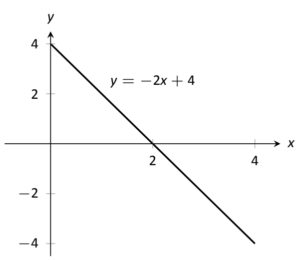
4. Give a single definite integral that has the same value as

$$\int_0^1 (2x+3)\,dx + \int_1^2 (2x+3)\,dx.$$

Problems

In Exercises 5 – 9, a graph of a function $f(x)$ is given. Using the geometry of the graph, evaluate the definite integrals.

5.



(a) $\int_0^1 (-2x+4)\,dx$

(b) $\int_0^2 (-2x+4)\,dx$

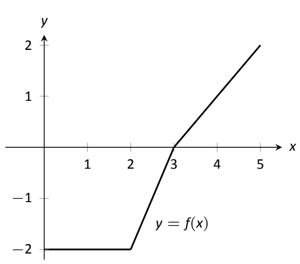
(c) $\int_0^3 (-2x+4)\,dx$

(d) $\int_1^3 (-2x+4)\,dx$

(e) $\int_2^4 (-2x+4)\,dx$

(f) $\int_0^1 (-6x+12)\,dx$

6.



(a) $\int_0^2 f(x)\,dx$

(b) $\int_0^3 f(x)\,dx$

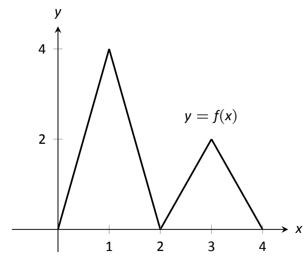
(c) $\int_0^5 f(x)\,dx$

(d) $\int_2^5 f(x)\,dx$

(e) $\int_5^3 f(x)\,dx$

(f) $\int_0^3 -2f(x)\,dx$

7.



(a) $\int_0^2 f(x)\,dx$

(b) $\int_2^4 f(x)\,dx$

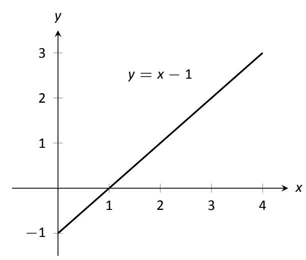
(c) $\int_2^4 2f(x)\,dx$

(d) $\int_0^1 4x\,dx$

(e) $\int_2^3 (2x-4)\,dx$

(f) $\int_2^3 (4x-8)\,dx$

8.



(a) $\int_0^1 (x-1)\,dx$

(b) $\int_0^2 (x-1)\,dx$

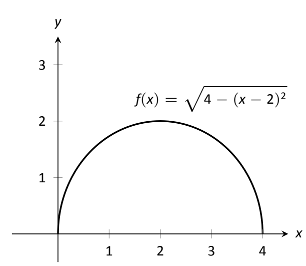
(c) $\int_0^3 (x-1)\,dx$

(d) $\int_2^3 (x-1)\,dx$

(e) $\int_1^4 (x-1)\,dx$

(f) $\int_1^4 \big((x-1)+1\big)\,dx$

9.



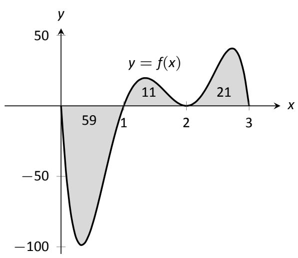
(a) $\int_0^2 f(x)\,dx$

(b) $\int_2^4 f(x)\,dx$

(c) $\int_0^4 f(x)\,dx$

(d) $\int_0^4 5f(x)\,dx$

In Exercises 10 – 13, a graph of a function $f(x)$ is given; the numbers inside the shaded regions give the area of that region. Evaluate the definite integrals using this area information.

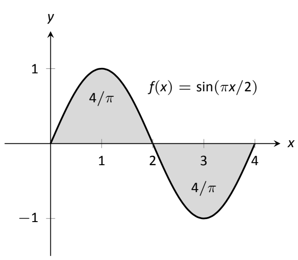
10. 

(a) $\displaystyle\int_0^1 f(x)\,dx$

(b) $\displaystyle\int_0^2 f(x)\,dx$

(c) $\displaystyle\int_0^3 f(x)\,dx$

(d) $\displaystyle\int_1^2 -3f(x)\,dx$

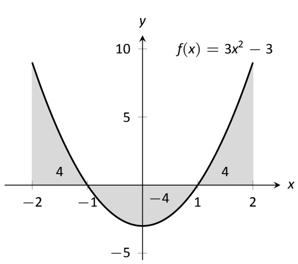
11. 

(a) $\displaystyle\int_0^2 f(x)\,dx$

(b) $\displaystyle\int_2^4 f(x)\,dx$

(c) $\displaystyle\int_0^4 f(x)\,dx$

(d) $\displaystyle\int_0^1 f(x)\,dx$

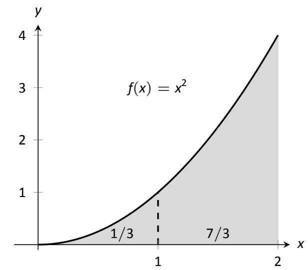
12. 

(a) $\displaystyle\int_{-2}^{-1} f(x)\,dx$

(b) $\displaystyle\int_1^2 f(x)\,dx$

(c) $\displaystyle\int_{-1}^1 f(x)\,dx$

(d) $\displaystyle\int_0^1 f(x)\,dx$

13. 

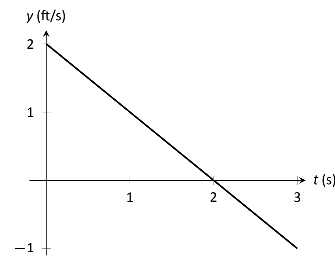
(a) $\displaystyle\int_0^2 5x^2\,dx$

(b) $\displaystyle\int_0^2 (x^2+3)\,dx$

(c) $\displaystyle\int_1^3 (x-1)^2\,dx$

(d) $\displaystyle\int_2^4 \left((x-2)^2+5\right)\,dx$

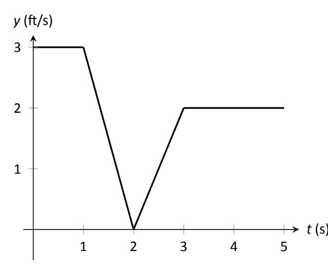
In Exercises 14 – 15, a graph of the velocity function of an object moving in a straight line is given. Answer the questions based on that graph.

14. 

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0,3]$?

15. 

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0,5]$?

16. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 64$, where t is in seconds, from a height of 48 feet.

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) When does the maximum displacement occur?

(d) When will the object reach a height of 0? (Hint: find when the displacement is -48ft.)

17. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 96$, where t is in seconds, from a height of 64 feet.

    (a) What is the object's initial velocity?

    (b) When is the object's displacement 0?

    (c) How long does it take for the object to return to its initial height?

    (d) When will the object reach a height of 210 feet?

In Exercises 18 – 21, let

- $\int_0^2 f(x)\,dx = 5$,
- $\int_0^3 f(x)\,dx = 7$,
- $\int_0^2 g(x)\,dx = -3$, **and**
- $\int_2^3 g(x)\,dx = 5$.

Use these values to evaluate the given definite integrals.

18. $\int_0^2 \big(f(x) + g(x)\big)\,dx$

19. $\int_0^3 \big(f(x) - g(x)\big)\,dx$

20. $\int_2^3 \big(3f(x) + 2g(x)\big)\,dx$

21. Find values for a and b such that $\int_0^3 \big(af(x) + bg(x)\big)\,dx = 0$

In Exercises 22 – 25, let

- $\int_0^3 s(t)\,dt = 10$,
- $\int_3^5 s(t)\,dt = 8$,
- $\int_3^5 r(t)\,dt = -1$, **and**
- $\int_0^5 r(t)\,dt = 11$.

Use these values to evaluate the given definite integrals.

22. $\int_0^3 \big(s(t) + r(t)\big)\,dt$

23. $\int_5^0 \big(s(t) - r(t)\big)\,dt$

24. $\int_3^3 \big(\pi s(t) - 7r(t)\big)\,dt$

25. Find values for a and b such that $\int_0^5 \big(ar(t) + bs(t)\big)\,dt = 0$

Review

In Exercises 26 – 29, evaluate the given indefinite integral.

26. $\int \big(x^3 - 2x^2 + 7x - 9\big)\,dx$

27. $\int \big(\sin x - \cos x + \sec^2 x\big)\,dx$

28. $\int \big(\sqrt[3]{t} + \frac{1}{t^2} + 2^t\big)\,dt$

29. $\int \left(\frac{1}{x} - \csc x \cot x\right)\,dx$

1.3 Riemann Sums

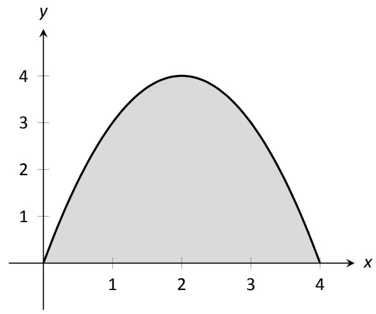


Figure 1.11: A graph of $f(x) = 4x - x^2$. What is the area of the shaded region?

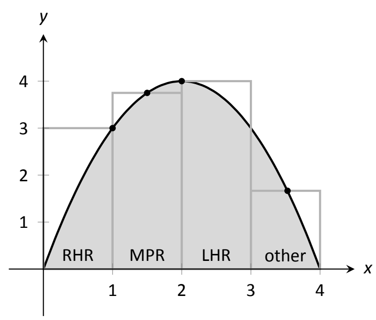


Figure 1.12: Approximating $\int_0^4 (4x - x^2)\,dx$ using rectangles. The heights of the rectangles are determined using different rules.

In the previous section we defined the definite integral of a function on $[a, b]$ to be the signed area between the curve and the x–axis. Some areas were simple to compute; we ended the section with a region whose area was not simple to compute. In this section we develop a technique to find such areas.

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is exactly what we will do here.

Consider the region given in Figure 1.11, which is the area under $y = 4x - x^2$ on $[0, 4]$. What is the signed area of this region – i.e., what is $\int_0^4 (4x - x^2)\,dx$?

We start by approximating. We can surround the region with a rectangle with height and width of 4 and find the area is approximately 16 square units. This is obviously an *over–approximation*; we are including area in the rectangle that is not under the parabola.

We have an approximation of the area, using one rectangle. How can we refine our approximation to make it better? The key to this section is this answer: *use more rectangles.*

Let's use 4 rectangles of equal width of 1. This *partitions* the interval $[0, 4]$ into 4 *subintervals*, $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the **Left Hand Rule**, the **Right Hand Rule**, and the **Midpoint Rule**. The **Left Hand Rule** says to evaluate the function at the left–hand endpoint of the subinterval and make the rectangle that height. In Figure 1.12, the rectangle drawn on the interval $[2, 3]$ has height determined by the Left Hand Rule; it has a height of $f(2)$. (The rectangle is labeled "LHR.")

The **Right Hand Rule** says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In the figure, the rectangle drawn on $[0, 1]$ is drawn using $f(1)$ as its height; this rectangle is labeled "RHR.".

The **Midpoint Rule** says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. The rectangle drawn on $[1, 2]$ was made using the Midpoint Rule, with a height of $f(1.5)$. That rectangle is labeled "MPR."

These are the three most common rules for determining the heights of approximating rectangles, but one is not forced to use one of these three methods. The rectangle on $[3, 4]$ has a height of approximately $f(3.53)$, very close to the Midpoint Rule. It was chosen so that the area of the rectangle is *exactly* the area of the region under f on $[3, 4]$. (Later you'll be able to figure how to do this, too.)

The following example will approximate the value of $\int_0^4 (4x - x^2)\,dx$ using

Notes:

these rules.

Example 10 **Using the Left Hand, Right Hand and Midpoint Rules**
Approximate the value of $\int_0^4 (4x - x^2)\, dx$ using the Left Hand Rule, the Right Hand Rule, and the Midpoint Rule, using 4 equally spaced subintervals.

Solution We break the interval $[0, 4]$ into four subintervals as before. In Figure 1.13 we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule. (The areas of the rectangles are given in each figure.)
Note how in the first subinterval, $[0, 1]$, the rectangle has height $f(0) = 0$. We add up the areas of each rectangle (height$\times$ width) for our Left Hand Rule approximation:

$$f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 =$$
$$0 + 3 + 4 + 3 = 10.$$

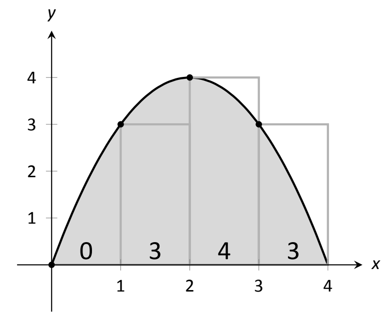


Figure 1.13: Approximating $\int_0^4 (4x - x^2)\, dx$ using the Left Hand Rule in Example 10.

Figure 1.14 shows 4 rectangles drawn under f using the Right Hand Rule; note how the $[3, 4]$ subinterval has a rectangle of height 0.
In this example, these rectangle seem to be the mirror image of those found in Figure 1.13. (This is because of the symmetry of our shaded region.) Our approximation gives the same answer as before, though calculated a different way:

$$f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 =$$
$$3 + 4 + 3 + 0 = 10.$$

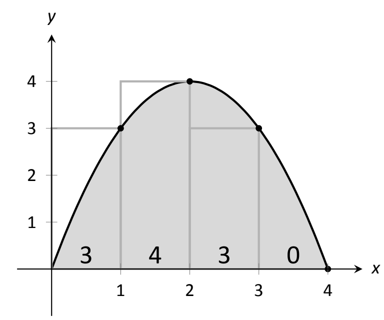


Figure 1.14: Approximating $\int_0^4 (4x - x^2)\, dx$ using the Right Hand Rule in Example 10.

Figure 1.15 shows 4 rectangles drawn under f using the Midpoint Rule.
This gives an approximation of $\int_0^4 (4x - x^2)\, dx$ as:

$$f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 =$$
$$1.75 + 3.75 + 3.75 + 1.75 = 11.$$

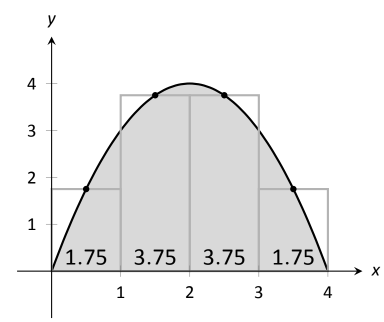


Figure 1.15: Approximating $\int_0^4 (4x - x^2)\, dx$ using the Midpoint Rule in Example 10.

Our three methods provide two approximations of $\int_0^4 (4x - x^2)\, dx$: 10 and 11.

Summation Notation

It is hard to tell at this moment which is a better approximation: 10 or 11? We can continue to refine our approximation by using more rectangles. The notation can become unwieldy, though, as we add up longer and longer lists of numbers. We introduce **summation notation** to ameliorate this problem.

Notes:

Suppose we wish to add up a list of numbers a_1, a_2, a_3, ..., a_9. Instead of writing

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9,$$

we use summation notation and write

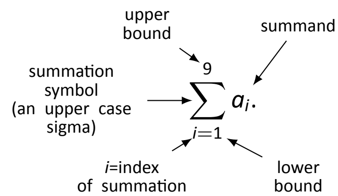


Figure 1.16: Understanding summation notation.

The upper case sigma represents the term "sum." The index of summation in this example is i; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds.

Let's practice using this notation.

Example 11 Using summation notation

Let the numbers $\{a_i\}$ be defined as $a_i = 2i - 1$ for integers i, where $i \geq 1$. So $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, etc. (The output is the positive odd integers). Evaluate the following summations:

1. $\displaystyle\sum_{i=1}^{6} a_i$ 2. $\displaystyle\sum_{i=3}^{7} (3a_i - 4)$ 3. $\displaystyle\sum_{i=1}^{4} (a_i)^2$

Solution

1.
$$\begin{aligned}\sum_{i=1}^{6} a_i &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= 1 + 3 + 5 + 7 + 9 + 11 \\ &= 36.\end{aligned}$$

2. Note the starting value is different than 1:

$$\begin{aligned}\sum_{i=3}^{7} a_i &= (3a_3 - 4) + (3a_4 - 4) + (3a_5 - 4) + (3a_6 - 4) + (3a_7 - 4) \\ &= 11 + 17 + 23 + 29 + 35 \\ &= 115.\end{aligned}$$

Notes:

3.

$$\sum_{i=1}^{4}(a_i)^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2$$
$$= 1^2 + 3^2 + 5^2 + 7^2$$
$$= 84$$

It might seem odd to stress a new, concise way of writing summations only to write each term out as we add them up. It is. The following theorem gives some of the properties of summations that allow us to work with them without writing individual terms. Examples will follow.

Theorem 4 Properties of Summations

1. $\displaystyle\sum_{i=1}^{n} c = c \cdot n$, where c is a constant.
2. $\displaystyle\sum_{i=m}^{n}(a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$
3. $\displaystyle\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$
4. $\displaystyle\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$
5. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 12 Evaluating summations using Theorem 4

Revisit Example 11 and, using Theorem 4, evaluate

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6}(2i-1).$$

Notes:

Solution

$$\begin{aligned}
\sum_{i=1}^{6}(2i-1) &= \sum_{i=1}^{6}2i - \sum_{i=1}^{6}(1)\\
&= \left(2\sum_{i=1}^{6}i\right) - 6\\
&= 2\frac{6(6+1)}{2} - 6\\
&= 42 - 6 = 36
\end{aligned}$$

We obtained the same answer without writing out all six terms. When dealing with small sizes of n, it may be faster to write the terms out by hand. However, Theorem 4 is incredibly important when dealing with large sums as we'll soon see.

Riemann Sums

Consider again $\int_0^4 (4x - x^2)\ dx$. We will approximate this definite integral using 16 equally spaced subintervals and the Right Hand Rule in Example 13. Before doing so, it will pay to do some careful preparation.

Figure 1.17 shows a number line of $[0, 4]$ divided into 16 equally spaced subintervals. We denote 0 as x_1; we have marked the values of x_5, x_9, x_{13} and x_{17}. We could mark them all, but the figure would get crowded. While it is easy to figure that $x_{10} = 2.25$, in general, we want a method of determining the value of x_i without consulting the figure. Consider:

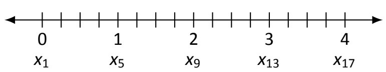


Figure 1.17: Dividing $[0, 4]$ into 16 equally spaced subintervals.

$$x_i = \underbrace{x_1}_{\text{starting value}} + \overbrace{(i-1)}^{\text{number of subintervals between } x_1 \text{ and } x_i}\underbrace{\Delta x}_{\text{subinterval size}}$$

So $x_{10} = x_1 + 9(4/16) = 2.25$.

If we had partitioned $[0, 4]$ into 100 equally spaced subintervals, each subinterval would have length $\Delta x = 4/100 = 0.04$. We could compute x_{32} as

$$x_{32} = x_1 + 31(4/100) = 1.24.$$

(That was far faster than creating a sketch first.)

Notes:

Given any subdivision of $[0,4]$, the first subinterval is $[x_1,x_2]$; the second is $[x_2,x_3]$; the i^{th} subinterval is $[x_i,x_{i+1}]$.

When using the Left Hand Rule, the height of the i^{th} rectangle will be $f(x_i)$.

When using the Right Hand Rule, the height of the i^{th} rectangle will be $f(x_{i+1})$.

When using the Midpoint Rule, the height of the i^{th} rectangle will be $f\left(\dfrac{x_i+x_{i+1}}{2}\right)$.

Thus approximating $\int_0^4(4x-x^2)\,dx$ with 16 equally spaced subintervals can be expressed as follows, where $\Delta x=4/16=1/4$:

Left Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_i)\Delta x$

Right Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_{i+1})\Delta x$

Midpoint Rule: $\displaystyle\sum_{i=1}^{16} f\left(\frac{x_i+x_{i+1}}{2}\right)\Delta x$

We use these formulas in the next two examples. The following example lets us practice using the Right Hand Rule and the summation formulas introduced in Theorem 4.

Example 13 Approximating definite integrals using sums

Approximate $\int_0^4(4x-x^2)\,dx$ using the Right Hand Rule and summation formulas with 16 and 1000 equally spaced intervals.

Solution Using the formula derived before, using 16 equally spaced intervals and the Right Hand Rule, we can approximate the definite integral as

$$\sum_{i=1}^{16} f(x_{i+1})\Delta x.$$

We have $\Delta x=4/16=0.25$. Since $x_i=0+(i-1)\Delta x$, we have

$$\begin{aligned} x_{i+1} &= 0+\big((i+1)-1\big)\Delta x \\ &= i\Delta x \end{aligned}$$

Notes:

Using the summation formulas, consider:

$$
\begin{aligned}
\int_0^4 (4x - x^2)\,dx &\approx \sum_{i=1}^{16} f(x_{i+1})\Delta x \\
&= \sum_{i=1}^{16} f(i\Delta x)\Delta x \\
&= \sum_{i=1}^{16} \big(4i\Delta x - (i\Delta x)^2\big)\Delta x \\
&= \sum_{i=1}^{16} (4i\Delta x^2 - i^2\Delta x^3) \\
&= (4\Delta x^2)\sum_{i=1}^{16} i - \Delta x^3 \sum_{i=1}^{16} i^2 \qquad (1.3) \\
&= (4\Delta x^2)\frac{16\cdot 17}{2} - \Delta x^3\frac{16(17)(33)}{6} \\
&= 4\cdot 0.25^2\cdot 136 - 0.25^3\cdot 1496 \\
&= 10.625
\end{aligned}
$$

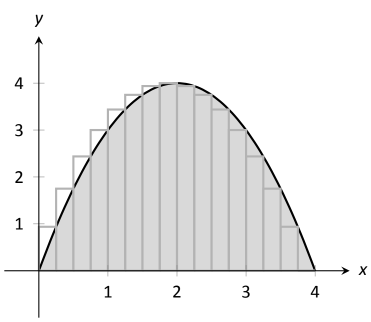


Figure 1.18: Approximating $\int_0^4 (4x - x^2)\,dx$ with the Right Hand Rule and 16 evenly spaced subintervals.

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 1.18 the function and the 16 rectangles are graphed. While some rectangles over–approximate the area, other under–approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

Notice Equation (1.3); by changing the 16's to 1,000's (and appropriately changing the value of Δx), we can use that equation to sum up 1000 rectangles! We do so here, skipping from the original summand to the equivalent of Equation (1.3) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$
\begin{aligned}
\int_0^4 (4x - x^2)\,dx &\approx \sum_{i=1}^{1000} f(x_{i+1})\Delta x \\
&= (4\Delta x^2)\sum_{i=1}^{1000} i - \Delta x^3 \sum_{i=1}^{1000} i^2 \\
&= (4\Delta x^2)\frac{1000\cdot 1001}{2} - \Delta x^3\frac{1000(1001)(2001)}{6} \\
&= 4\cdot 0.004^2\cdot 500500 - 0.004^3\cdot 333,833,500 \\
&= 10.666656
\end{aligned}
$$

Notes:

Using many, many rectangles, we have a likely good approximation of $\int_0^4 (4x - x^2)\Delta x$. That is,

$$\int_0^4 (4x - x^2)\, dx \approx 10.666656.$$

Before the above example, we stated what the summations for the Left Hand, Right Hand and Midpoint Rules looked like. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as Δx, and
2. each rectangle's height is determined by evaluating f at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating f at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a, b]$ with subintervals that did not have the same size. We refer to the length of the first subinterval as Δx_1, the length of the second subinterval as Δx_2, and so on, giving the length of the i^{th} subinterval as Δx_i. Also, one could determine each rectangle's height by evaluating f at *any* point in the i^{th} subinterval. We refer to the point picked in the first subinterval as c_1, the point picked in the second subinterval as c_2, and so on, with c_i representing the point picked in the i^{th} subinterval. Thus the height of the i^{th} subinterval would be $f(c_i)$, and the area of the i^{th} rectangle would be $f(c_i)\Delta x_i$.

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

Definition 3 Riemann Sum

Let f be defined on the closed interval $[a, b]$ and let Δx be a partition of $[a, b]$, with

$$a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b.$$

Let Δx_i denote the length of the i^{th} subinterval $[x_i, x_{i+1}]$ and let c_i denote any value in the i^{th} subinterval.

The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is a **Riemann sum** of f on $[a, b]$.

Figure 1.19 shows the approximating rectangles of a Riemann sum of $\int_0^4 (4x - x^2)\, dx$. While the rectangles in this example do not approximate well the shaded area, they demonstrate that the subinterval widths may vary and the heights of the rectangles can be determined without following a particular rule.

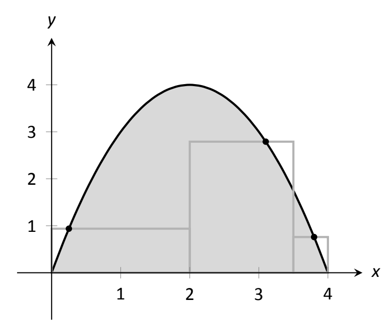


Figure 1.19: An example of a general Riemann sum to approximate $\int_0^4 (4x - x^2)\, dx$.

Notes:

"Usually" Riemann sums are calculated using one of the three methods we have introduced. The uniformity of construction makes computations easier. Before working another example, let's summarize some of what we have learned in a convenient way.

Key Idea 1 Riemann Sum Concepts

Consider $\displaystyle\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\Delta x_i$.

1. When the n subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.
2. The i^{th} term of the partition is $x_i = a + (i-1)\Delta x$. (This makes $x_{n+1} = b$.)
3. The Left Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$.
4. The Right Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x$.
5. The Midpoint Rule summation is: $\displaystyle\sum_{i=1}^{n} f\left(\frac{x_i + x_{x+1}}{2}\right)\Delta x$.

Let's do another example.

Example 14 Approximating definite integrals with sums

Approximate $\int_{-2}^{3}(5x+2)\,dx$ using the Midpoint Rule and 10 equally spaced intervals.

Solution Following Key Idea 1, we have

$$\Delta x = \frac{3-(-2)}{10} = 1/2 \quad \text{and} \quad x_i = (-2) + (1/2)(i-1) = i/2 - 5/2.$$

As we are using the Midpoint Rule, we will also need x_{i+1} and $\dfrac{x_i + x_{i+1}}{2}$. Since $x_i = i/2 - 5/2$, $x_{i+1} = (i+1)/2 - 5/2 = i/2 - 2$. This gives

$$\frac{x_i + x_{i+1}}{2} = \frac{(i/2 - 5/2) + (i/2 - 2)}{2} = \frac{i - 9/2}{2} = i/2 - 9/4.$$

Notes:

We now construct the Riemann sum and compute its value using summation formulas.

$$\begin{aligned}
\int_{-2}^{3}(5x+2)\,dx &\approx \sum_{i=1}^{10} f\left(\frac{x_i+x_{i+1}}{2}\right)\Delta x\\
&= \sum_{i=1}^{10} f(i/2-9/4)\Delta x\\
&= \sum_{i=1}^{10} \big(5(i/2-9/4)+2\big)\Delta x\\
&= \Delta x\sum_{i=1}^{10}\left[\left(\frac{5}{2}\right)i-\frac{37}{4}\right]\\
&= \Delta x\left(\frac{5}{2}\sum_{i=1}^{10}(i)-\sum_{i=1}^{10}\left(\frac{37}{4}\right)\right)\\
&= \frac{1}{2}\left(\frac{5}{2}\cdot\frac{10(11)}{2}-10\cdot\frac{37}{4}\right)\\
&= \frac{45}{2}=22.5
\end{aligned}$$

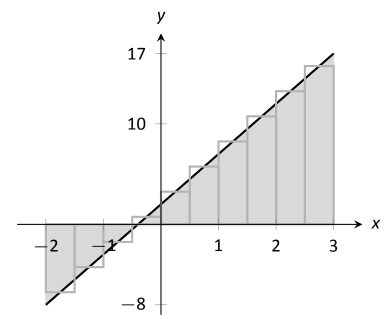


Figure 1.20: Approximating $\int_{-2}^{3}(5x+2)\,dx$ using the Midpoint Rule and 10 evenly spaced subintervals in Example 14.

Note the graph of $f(x) = 5x + 2$ in Figure 1.20. The regions whose area is computed by the definite integral are triangles, meaning we can find the exact answer without summation techniques. We find that the exact answer is indeed 22.5. One of the strengths of the Midpoint Rule is that often each rectangle includes area that should not be counted, but misses other area that should. When the partition size is small, these two amounts are about equal and these errors almost "cancel each other out." In this example, since our function is a line, these errors are exactly equal and they do cancel each other out, giving us the exact answer.

Note too that when the function is negative, the rectangles have a "negative" height. When we compute the area of the rectangle, we use $f(c_i)\Delta x$; when f is negative, the area is counted as negative.

Notice in the previous example that while we used 10 equally spaced intervals, the number "10" didn't play a big role in the calculations until the very end. Mathematicians love to abstract ideas; let's approximate the area of another region using n subintervals, where we do not specify a value of n until the very end.

Example 15 Approximating definite integrals with a formula, using sums

Revisit $\int_0^4(4x-x^2)\,dx$ yet again. Approximate this definite integral using the Right

Notes:

Hand Rule with n equally spaced subintervals.

Solution Using Key Idea 1, we know $\Delta x = \frac{4-0}{n} = 4/n$. We also find $x_i = 0 + \Delta x(i-1) = 4(i-1)/n$. The Right Hand Rule uses x_{i+1}, which is $x_{i+1} = 4i/n$.

We construct the Right Hand Rule Riemann sum as follows. Be sure to follow each step carefully. If you get stuck, and do not understand how one line proceeds to the next, you may skip to the result and consider how this result is used. You should come back, though, and work through each step for full understanding.

$$\begin{aligned}
\int_0^4 (4x-x^2)\,dx &\approx \sum_{i=1}^{n} f(x_{i+1})\Delta x \\
&= \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\Delta x \\
&= \sum_{i=1}^{n} \left[4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right]\Delta x \\
&= \sum_{i=1}^{n} \left(\frac{16\Delta x}{n}\right) i - \sum_{i=1}^{n}\left(\frac{16\Delta x}{n^2}\right) i^2 \\
&= \left(\frac{16\Delta x}{n}\right)\sum_{i=1}^{n} i - \left(\frac{16\Delta x}{n^2}\right)\sum_{i=1}^{n} i^2 \\
&= \left(\frac{16\Delta x}{n}\right)\cdot\frac{n(n+1)}{2} - \left(\frac{16\Delta x}{n^2}\right)\frac{n(n+1)(2n+1)}{6} \qquad \left(\begin{matrix}\text{recall}\\ \Delta x = 4/n\end{matrix}\right) \\
&= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} \qquad \text{(now simplify)} \\
&= \frac{32}{3}\left(1-\frac{1}{n^2}\right)
\end{aligned}$$

The result is an amazing, easy to use formula. To approximate the definite integral with 10 equally spaced subintervals and the Right Hand Rule, set $n = 10$ and compute

$$\int_0^4 (4x-x^2)\,dx \approx \frac{32}{3}\left(1-\frac{1}{10^2}\right) = 10.56.$$

Recall how earlier we approximated the definite integral with 4 subintervals; with $n = 4$, the formula gives 10, our answer as before.

It is now easy to approximate the integral with 1,000,000 subintervals! Handheld calculators will round off the answer a bit prematurely giving an answer of

Notes:

10.66666667. (The actual answer is 10.666666666656.)

We now take an important leap. Up to this point, our mathematics has been limited to geometry and algebra (finding areas and manipulating expressions). Now we apply *calculus*. For any *finite n*, we know that

$$\int_0^4 (4x - x^2)\, dx \approx \frac{32}{3}\left(1 - \frac{1}{n^2}\right).$$

Both common sense and high–level mathematics tell us that as n gets large, the approximation gets better. In fact, if we take the *limit* as $n \to \infty$, we get the *exact area* described by $\int_0^4 (4x - x^2)\, dx$. That is,

$$\begin{aligned}\int_0^4 (4x - x^2)\, dx &= \lim_{n\to\infty} \frac{32}{3}\left(1 - \frac{1}{n^2}\right)\\ &= \frac{32}{3}\left(1 - 0\right)\\ &= \frac{32}{3} = 10.\overline{6}\end{aligned}$$

This is a fantastic result. By considering n equally–spaced subintervals, we obtained a formula for an approximation of the definite integral that involved our variable n. As n grows large – without bound – the error shrinks to zero and we obtain the exact area.

This section started with a fundamental calculus technique: make an approximation, refine the approximation to make it better, then use limits in the refining process to get an exact answer. That is precisely what we just did.

Let's practice this again.

Example 16 Approximating definite integrals with a formula, using sums

Find a formula that approximates $\int_{-1}^5 x^3\, dx$ using the Right Hand Rule and n equally spaced subintervals, then take the limit as $n \to \infty$ to find the exact area.

Solution Following Key Idea 1, we have $\Delta x = \frac{5-(-1)}{n} = 6/n$. We have $x_i = (-1) + (i-1)\Delta x$; as the Right Hand Rule uses x_{i+1}, we have $x_{i+1} = (-1) + i\Delta x$.

The Riemann sum corresponding to the Right Hand Rule is (followed by sim-

Notes:

plifications):

$$\begin{aligned}
\int_{-1}^{5} x^3\,dx &\approx \sum_{i=1}^{n} f(x_{i+1})\Delta x\\
&= \sum_{i=1}^{n} f(-1+i\Delta x)\Delta x\\
&= \sum_{i=1}^{n} (-1+i\Delta x)^3\Delta x\\
&= \sum_{i=1}^{n} \big((i\Delta x)^3 - 3(i\Delta x)^2 + 3i\Delta x - 1\big)\Delta x \quad \text{(now distribute } \Delta x\text{)}\\
&= \sum_{i=1}^{n} \big(i^3\Delta x^4 - 3i^2\Delta x^3 + 3i\Delta x^2 - \Delta x\big) \quad \text{(now split up summation)}\\
&= \Delta x^4\sum_{i=1}^{n} i^3 - 3\Delta x^3\sum_{i=1}^{n} i^2 + 3\Delta x^2\sum_{i=1}^{n} i - \sum_{i=1}^{n}\Delta x\\
&= \Delta x^4\left(\frac{n(n+1)}{2}\right)^2 - 3\Delta x^3\frac{n(n+1)(2n+1)}{6} + 3\Delta x^2\frac{n(n+1)}{2} - n\Delta x
\end{aligned}$$

(use $\Delta x = 6/n$)

$$= \frac{1296}{n^4}\cdot\frac{n^2(n+1)^2}{4} - 3\frac{216}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + 3\frac{36}{n^2}\frac{n(n+1)}{2} - 6$$

(now do a sizable amount of algebra to simplify)

$$= 156 + \frac{378}{n} + \frac{216}{n^2}$$

Once again, we have found a compact formula for approximating the definite integral with n equally spaced subintervals and the Right Hand Rule. Using 10 subintervals, we have an approximation of 195.96 (these rectangles are shown in Figure 1.21). Using $n = 100$ gives an approximation of 159.802.

Now find the exact answer using a limit:

$$\int_{-1}^{5} x^3\,dx = \lim_{n\to\infty}\left(156 + \frac{378}{n} + \frac{216}{n^2}\right) = 156.$$

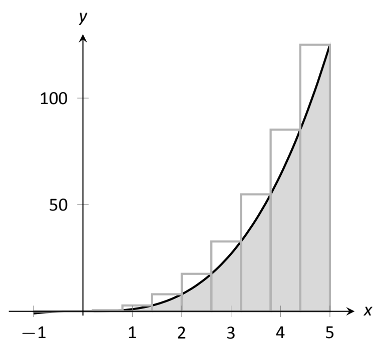


Figure 1.21: Approximating $\int_{-1}^{5} x^3\,dx$ using the Right Hand Rule and 10 evenly spaced subintervals.

Limits of Riemann Sums

We have used limits to evaluate exactly given definite limits. Will this always work? We will show, given not–very–restrictive conditions, that yes, it will always work.

Notes:

The previous two examples demonstrated how an expression such as

$$\sum_{i=1}^{n} f(x_{i+1})\Delta x$$

can be rewritten as an expression explicitly involving n, such as $32/3(1-1/n^2)$.

Viewed in this manner, we can think of the summation as a function of n. An n value is given (where n is a positive integer), and the sum of areas of n equally spaced rectangles is returned, using the Left Hand, Right Hand, or Midpoint Rules.

Given a definite integral $\int_a^b f(x)\,dx$, let:

- $S_L(n) = \displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$, the sum of equally spaced rectangles formed using the Left Hand Rule,
- $S_R(n) = \displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x$, the sum of equally spaced rectangles formed using the Right Hand Rule, and
- $S_M(n) = \displaystyle\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$, the sum of equally spaced rectangles formed using the Midpoint Rule.

Recall the definition of a limit as $n \to \infty$: $\lim\limits_{n\to\infty} S_L(n) = K$ if, given any $\varepsilon > 0$, there exists $N > 0$ such that

$$|S_L(n) - K| < \varepsilon \quad \text{when} \quad n \geq N.$$

The following theorem states that we can use any of our three rules to find the exact value of a definite integral $\int_a^b f(x)\,dx$. It also goes two steps further. The theorem states that the height of each rectangle doesn't have to be determined following a specific rule, but could be $f(c_i)$, where c_i is any point in the i^{th} subinterval, as discussed before Riemann Sums where defined in Definition 3.

The theorem goes on to state that the rectangles do not need to be of the same width. Using the notation of Definition 3, let Δx_i denote the length of the i^{th} subinterval in a partition of $[a, b]$. Now let $||\Delta x||$ represent the length of the largest subinterval in the partition: that is, $||\Delta x||$ is the largest of all the Δx_i's. If $||\Delta x||$ is small, then $[a, b]$ must be partitioned into many subintervals, since all subintervals must have small lengths. "Taking the limit as $||\Delta x||$ goes to zero" implies that the number n of subintervals in the partition is growing to infinity,

Notes:

as the largest subinterval length is becoming arbitrarily small. We then interpret the expression

$$\lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

as "the limit of the sum of rectangles, where the width of each rectangle can be different but getting small, and the height of each rectangle is not necessarily determined by a particular rule." The theorem states that this Riemann Sum also gives the value of the definite integral of f over $[a, b]$.

Theorem 5 Definite Integrals and the Limit of Riemann Sums

Let f be continuous on the closed interval $[a, b]$ and let $S_L(n)$, $S_R(n)$ and $S_M(n)$ be defined as before. Then:

1. $\displaystyle \lim_{n\to\infty} S_L(n) = \lim_{n\to\infty} S_R(n) = \lim_{n\to\infty} S_M(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$
2. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx$, and
3. $\displaystyle \lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx.$

We summarize what we have learned over the past few sections here.

- Knowing the "area under the curve" can be useful. One common example is: the area under a velocity curve is displacement.
- We have defined the definite integral, $\int_a^b f(x)\,dx$, to be the signed area under f on the interval $[a, b]$.
- While we can approximate a definite integral many ways, we have focused on using rectangles whose heights can be determined using: the Left Hand Rule, the Right Hand Rule and the Midpoint Rule.
- Sums of rectangles of this type are called Riemann sums.
- The exact value of the definite integral can be computed using the limit of a Riemann sum. We generally use one of the above methods as it makes the algebra simpler.

Notes:

We first learned of derivatives through limits then learned rules that made the process simpler. We know of a way to evaluate a definite integral using limits; in the next section we will see how the Fundamental Theorem of Calculus makes the process simpler. The key feature of this theorem is its connection between the indefinite integral and the definite integral.

Notes:

Exercises 1.3

Terms and Concepts

1. A fundamental calculus technique is to use _________ to refine approximations to get an exact answer.

2. What is the upper bound in the summation $\sum_{i=7}^{14}(48i - 201)$?

3. This section approximates definite integrals using what geometric shape?

4. T/F: A sum using the Right Hand Rule is an example of a Riemann Sum.

Problems

In Exercises 5 – 11, write out each term of the summation and compute the sum.

5. $\sum_{i=2}^{4} i^2$

6. $\sum_{i=-1}^{3} (4i - 2)$

7. $\sum_{i=-2}^{2} \sin(\pi i/2)$

8. $\sum_{i=1}^{5} \frac{1}{i}$

9. $\sum_{i=1}^{6} (-1)^i i$

10. $\sum_{i=1}^{4} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

11. $\sum_{i=0}^{5} (-1)^i \cos(\pi i)$

In Exercises 12 – 15, write each sum in summation notation.

12. $3 + 6 + 9 + 12 + 15$

13. $-1 + 0 + 3 + 8 + 15 + 24 + 35 + 48 + 63$

14. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

15. $1 - e + e^2 - e^3 + e^4$

In Exercises 16 – 22, evaluate the summation using Theorem 4.

16. $\sum_{i=1}^{25} i$

17. $\sum_{i=1}^{10} (3i^2 - 2i)$

18. $\sum_{i=1}^{15} (2i^3 - 10)$

19. $\sum_{i=1}^{10} (-4i^3 + 10i^2 - 7i + 11)$

20. $\sum_{i=1}^{10} (i^3 - 3i^2 + 2i + 7)$

21. $1 + 2 + 3 + \ldots + 99 + 100$

22. $1 + 4 + 9 + \ldots + 361 + 400$

Theorem 4 states

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i \text{, so}$$

$$\sum_{i=k+1}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{k} a_i .$$

Use this fact, along with other parts of Theorem 4, to evaluate the summations given in Exercises 23 – 26.

23. $\sum_{i=11}^{20} i$

24. $\sum_{i=16}^{25} i^3$

25. $\sum_{i=7}^{12} 4$

26. $\sum_{i=5}^{10} 4i^3$

In Exercises 27 – 32, a definite integral $\int_a^b f(x)\,dx$ is given.

(a) Graph $f(x)$ on $[a, b]$.

(b) Add to the sketch rectangles using the provided rule.

(c) Approximate $\int_a^b f(x)\,dx$ by summing the areas of the rectangles.

27. $\int_{-3}^{3} x^2\,dx$, with 6 rectangles using the Left Hand Rule.

28. $\int_{0}^{2} (5-x^2)\,dx$, with 4 rectangles using the Midpoint Rule.

29. $\int_{0}^{\pi} \sin x\,dx$, with 6 rectangles using the Right Hand Rule.

30. $\int_{0}^{3} 2^x\,dx$, with 5 rectangles using the Left Hand Rule.

31. $\int_{1}^{2} \ln x\,dx$, with 3 rectangles using the Midpoint Rule.

32. $\int_{1}^{9} \frac{1}{x}\,dx$, with 4 rectangles using the Right Hand Rule.

In Exercises 33 – 38, a definite integral $\int_a^b f(x)\,dx$ is given. As demonstrated in Examples 15 and 16, do the following.

(a) Find a formula to approximate $\int_a^b f(x)\,dx$ using n subintervals and the provided rule.

(b) Evaluate the formula using $n = 10$, 100 and $1,000$.

(c) Find the limit of the formula, as $n \to \infty$, to find the exact value of $\int_a^b f(x)\,dx$.

33. $\int_{0}^{1} x^3\,dx$, using the Right Hand Rule.

34. $\int_{-1}^{1} 3x^2\,dx$, using the Left Hand Rule.

35. $\int_{-1}^{3} (3x-1)\,dx$, using the Midpoint Rule.

36. $\int_{1}^{4} (2x^2-3)\,dx$, using the Left Hand Rule.

37. $\int_{-10}^{10} (5-x)\,dx$, using the Right Hand Rule.

38. $\int_{0}^{1} (x^3-x^2)\,dx$, using the Right Hand Rule.

Review

In Exercises 39 – 44, find an antiderivative of the given function.

39. $f(x) = 5\sec^2 x$

40. $f(x) = \frac{7}{x}$

41. $g(t) = 4t^5 - 5t^3 + 8$

42. $g(t) = 5 \cdot 8^t$

43. $g(t) = \cos t + \sin t$

44. $f(x) = \frac{1}{\sqrt{x}}$

1.4 The Fundamental Theorem of Calculus

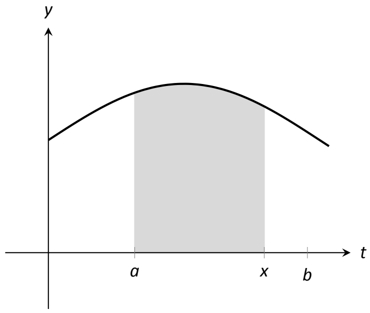


Figure 1.22: The area of the shaded region is $F(x) = \int_a^x f(t)\,dt$.

Let $f(t)$ be a continuous function defined on $[a, b]$. The definite integral $\int_a^b f(x)\,dx$ is the "area under f" on $[a, b]$. We can turn this concept into a function by letting the upper (or lower) bound vary.

Let $F(x) = \int_a^x f(t)\,dt$. It computes the area under f on $[a, x]$ as illustrated in Figure 1.22. We can study this function using our knowledge of the definite integral. For instance, $F(a) = 0$ since $\int_a^a f(t)\,dt = 0$.

We can also apply calculus ideas to $F(x)$; in particular, we can compute its derivative. While this may seem like an innocuous thing to do, it has far–reaching implications, as demonstrated by the fact that the result is given as an important theorem.

Theorem 6 The Fundamental Theorem of Calculus, Part 1

Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\,dt$. Then F is a differentiable function on (a, b), and

$$F'(x) = f(x).$$

Initially this seems simple, as demonstrated in the following example.

Example 17 Using the Fundamental Theorem of Calculus, Part 1

Let $F(x) = \displaystyle\int_{-5}^{x} (t^2 + \sin t)\,dt$. What is $F'(x)$?

Solution Using the Fundamental Theorem of Calculus, we have $F'(x) = x^2 + \sin x$.

This simple example reveals something incredible: $F(x)$ is an antiderivative of $x^2 + \sin x$! Therefore, $F(x) = \frac{1}{3}x^3 - \cos x + C$ for some value of C. (We can find C, but generally we do not care. We know that $F(-5) = 0$, which allows us to compute C. In this case, $C = \cos(-5) + \frac{125}{3}$.)

We have done more than found a complicated way of computing an antiderivative. Consider a function f defined on an open interval containing a, b and c. Suppose we want to compute $\int_a^b f(t)\,dt$. First, let $F(x) = \int_c^x f(t)\,dt$. Using

Notes:

the properties of the definite integral found in Theorem 3, we know

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^c f(t)\,dt + \int_c^b f(t)\,dt\\ &= -\int_c^a f(t)\,dt + \int_c^b f(t)\,dt\\ &= -F(a) + F(b)\\ &= F(b) - F(a).\end{aligned}$$

We now see how indefinite integrals and definite integrals are related: we can evaluate a definite integral using antiderivatives! This is the second part of the Fundamental Theorem of Calculus.

Theorem 7 The Fundamental Theorem of Calculus, Part 2

Let f be continuous on $[a, b]$ and let F be *any* antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Example 18 Using the Fundamental Theorem of Calculus, Part 2

We spent a great deal of time in the previous section studying $\int_0^4 (4x - x^2)\,dx$. Using the Fundamental Theorem of Calculus, evaluate this definite integral.

Solution We need an antiderivative of $f(x) = 4x - x^2$. All antiderivatives of f have the form $F(x) = 2x^2 - \frac{1}{3}x^3 + C$; for simplicity, choose $C = 0$.

The Fundamental Theorem of Calculus states

$$\int_0^4 (4x - x^2)\,dx = F(4) - F(0) = \big(2(4)^2 - \frac{1}{3}4^3\big) - \big(0 - 0\big) = 32 - \frac{64}{3} = 32/3.$$

This is the same answer we obtained using limits in the previous section, just with much less work.

Notation: A special notation is often used in the process of evaluating definite integrals using the Fundamental Theorem of Calculus. Instead of explicitly writing $F(b) - F(a)$, the notation $F(x)\Big|_a^b$ is used. Thus the solution to Example 18 would be written as:

$$\int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{1}{3}x^3\right)\Bigg|_0^4 = \big(2(4)^2 - \frac{1}{3}4^3\big) - \big(0 - 0\big) = 32/3.$$

Notes:

The Constant C: *Any* antiderivative $F(x)$ can be chosen when using the Fundamental Theorem of Calculus to evaluate a definite integral, meaning any value of C can be picked. The constant *always* cancels out of the expression when evaluating $F(b) - F(a)$, so it does not matter what value is picked. This being the case, we might as well let $C = 0$.

Example 19 **Using the Fundamental Theorem of Calculus, Part 2**

Evaluate the following definite integrals.

1. $\displaystyle\int_{-2}^{2} x^3\,dx$ 2. $\displaystyle\int_{0}^{\pi} \sin x\,dx$ 3. $\displaystyle\int_{0}^{5} e^t\,dt$ 4. $\displaystyle\int_{4}^{9} \sqrt{u}\,du$ 5. $\displaystyle\int_{1}^{5} 2\,dx$

Solution

1. $\displaystyle\int_{-2}^{2} x^3\,dx = \frac{1}{4}x^4\Big|_{-2}^{2} = \left(\frac{1}{4}2^4\right) - \left(\frac{1}{4}(-2)^4\right) = 0.$

2. $\displaystyle\int_{0}^{\pi} \sin x\,dx = -\cos x\Big|_{0}^{\pi} = -\cos\pi - \big(-\cos 0\big) = 1 + 1 = 2.$

   (This is interesting; it says that the area under one "hump" of a sine curve is 2.)

3. $\displaystyle\int_{0}^{5} e^t\,dt = e^t\Big|_{0}^{5} = e^5 - e^0 = e^5 - 1 \approx 147.41.$

4. $\displaystyle\int_{4}^{9} \sqrt{u}\,du = \int_{4}^{9} u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_{4}^{9} = \frac{2}{3}\left(9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = \frac{2}{3}\big(27 - 8\big) = \frac{38}{3}.$

5. $\displaystyle\int_{1}^{5} 2\,dx = 2x\Big|_{1}^{5} = 2(5) - 2 = 2(5 - 1) = 8.$

   This integral is interesting; the integrand is a constant function, hence we are finding the area of a rectangle with width $(5 - 1) = 4$ and height 2. Notice how the evaluation of the definite integral led to $2(4) = 8$.

   In general, if c is a constant, then $\int_a^b c\,dx = c(b - a)$.

Understanding Motion with the Fundamental Theorem of Calculus

We established, starting with Key Idea **??**, that the derivative of a position function is a velocity function, and the derivative of a velocity function is an acceleration function. Now consider definite integrals of velocity and acceleration functions. Specifically, if $v(t)$ is a velocity function, what does $\displaystyle\int_a^b v(t)\,dt$ mean?

Notes:

The Fundamental Theorem of Calculus states that

$$\int_a^b v(t)\,dt = V(b) - V(a),$$

where $V(t)$ is any antiderivative of $v(t)$. Since $v(t)$ is a velocity function, $V(t)$ must be a position function, and $V(b) - V(a)$ measures a change in position, or **displacement**.

Example 20 **Finding displacement**

A ball is thrown straight up with velocity given by $v(t) = -32t + 20$ft/s, where t is measured in seconds. Find, and interpret, $\displaystyle\int_0^1 v(t)\,dt$.

SOLUTION Using the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_0^1 v(t)\,dt &= \int_0^1 (-32t + 20)\,dt\\ &= -16t^2 + 20t\Big|_0^1\\ &= 4.\end{aligned}$$

Thus if a ball is thrown straight up into the air with velocity $v(t) = -32t + 20$, the height of the ball, 1 second later, will be 4 feet above the initial height. (Note that the ball has *traveled* much farther. It has gone up to its peak and is falling down, but the difference between its height at $t = 0$ and $t = 1$ is 4ft.)

Integrating a rate of change function gives total change. Velocity is the rate of position change; integrating velocity gives the total change of position, i.e., displacement.

Integrating a speed function gives a similar, though different, result. Speed is also the rate of position change, but does not account for direction. So integrating a speed function gives total change of position, without the possibility of "negative position change." Hence the integral of a speed function gives *distance traveled.*

As acceleration is the rate of velocity change, integrating an acceleration function gives total change in velocity. We do not have a simple term for this analogous to displacement. If $a(t) = 5$miles/h^2 and t is measured in hours, then

$$\int_0^3 a(t)\,dt = 15$$

means the velocity has increased by 15m/h from $t = 0$ to $t = 3$.

Notes:

The Fundamental Theorem of Calculus and the Chain Rule

Part 1 of the Fundamental Theorem of Calculus (FTC) states that given $F(x) = \int_a^x f(t)\, dt$, $F'(x) = f(x)$. Using other notation, $\frac{d}{dx}\big(F(x)\big) = f(x)$. While we have just practiced evaluating definite integrals, sometimes finding antiderivatives is impossible and we need to rely on other techniques to approximate the value of a definite integral. Functions written as $F(x) = \int_a^x f(t)\, dt$ are useful in such situations.

It may be of further use to compose such a function with another. As an example, we may compose $F(x)$ with $g(x)$ to get

$$F\big(g(x)\big) = \int_a^{g(x)} f(t)\, dt.$$

What is the derivative of such a function? The Chain Rule can be employed to state

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x).$$

An example will help us understand this.

Example 21 **The FTC, Part 1, and the Chain Rule**

Find the derivative of $F(x) = \displaystyle\int_2^{x^2} \ln t\, dt$.

Solution We can view $F(x)$ as being the function $G(x) = \displaystyle\int_2^x \ln t\, dt$ composed with $g(x) = x^2$; that is, $F(x) = G\big(g(x)\big)$. The Fundamental Theorem of Calculus states that $G'(x) = \ln x$. The Chain Rule gives us

$$\begin{aligned} F'(x) &= G'\big(g(x)\big)g'(x) \\ &= \ln(g(x))g'(x) \\ &= \ln(x^2)2x \\ &= 2x\ln x^2 \end{aligned}$$

Normally, the steps defining $G(x)$ and $g(x)$ are skipped.

Practice this once more.

Example 22 **The FTC, Part 1, and the Chain Rule**

Find the derivative of $F(x) = \displaystyle\int_{\cos x}^{5} t^3\, dt$.

Notes:

Solution Note that $F(x) = -\int_5^{\cos x} t^3\, dt$. Viewed this way, the derivative of F is straightforward:

$$F'(x) = \sin x \cos^3 x.$$

Area Between Curves

Consider continuous functions $f(x)$ and $g(x)$ defined on $[a, b]$, where $f(x) \geq g(x)$ for all x in $[a, b]$, as demonstrated in Figure 1.23. What is the area of the shaded region bounded by the two curves over $[a, b]$?

The area can be found by recognizing that this area is "the area under f – the area under g." Using mathematical notation, the area is

$$\int_a^b f(x)\, dx - \int_a^b g(x)\, dx.$$

Properties of the definite integral allow us to simplify this expression to

$$\int_a^b \big(f(x) - g(x)\big)\, dx.$$

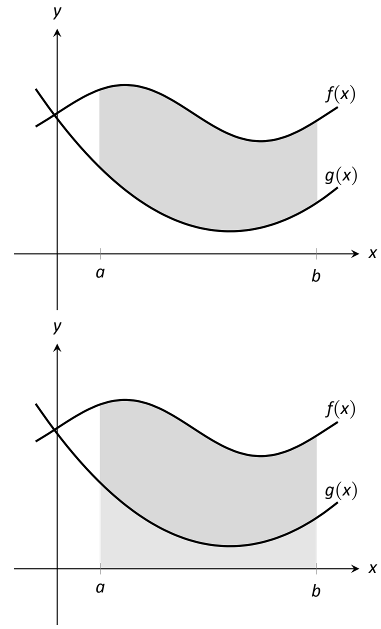


Figure 1.23: Finding the area bounded by two functions on an interval; it is found by subtracting the area under g from the area under f.

> **Theorem 8 Area Between Curves**
>
> Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is
>
> $$\int_a^b \big(f(x) - g(x)\big)\, dx.$$

Example 23 Finding area between curves

Find the area of the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$.

Solution It will help to sketch these two functions, as done in Figure 1.24. The region whose area we seek is completely bounded by these two functions; they seem to intersect at $x = -1$ and $x = 3$. To check, set $x^2 + x - 5 =$

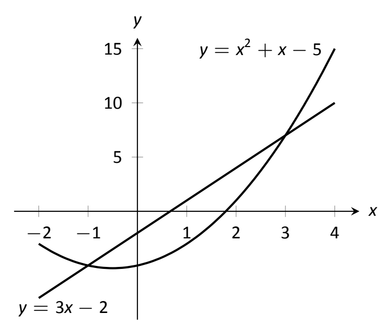


Figure 1.24: Sketching the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$ in Example 23.

Notes:

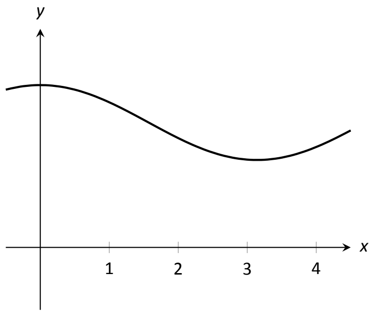
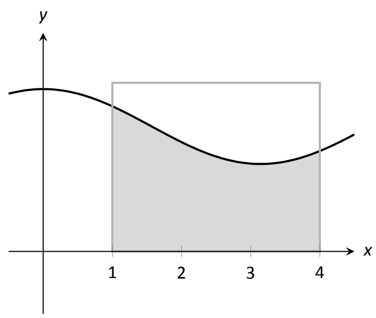
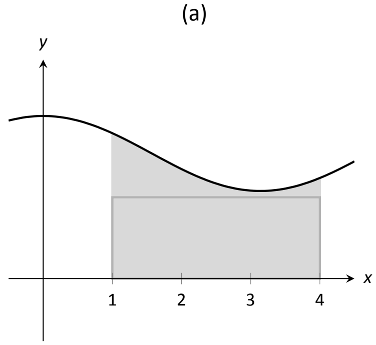


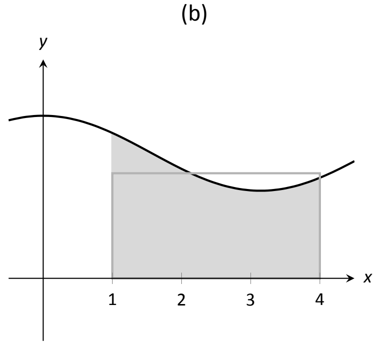
Figure 1.25: A graph of a function f to introduce the Mean Value Theorem.



(a)



(b)



(c)

Figure 1.26: Differently sized rectangles give upper and lower bounds on $\int_1^4 f(x)\,dx$; the last rectangle matches the area exactly.

$3x-2$ and solve for x:

$$\begin{aligned} x^2+x-5&=3x-2\\ (x^2+x-5)-(3x-2)&=0\\ x^2-2x-3&=0\\ (x-3)(x+1)&=0\\ x&=-1,\ 3. \end{aligned}$$

Following Theorem 8, the area is

$$\begin{aligned} \int_{-1}^{3}\big(3x-2-(x^2+x-5)\big)\,dx &= \int_{-1}^{3}(-x^2+2x+3)\,dx\\ &=\left(-\frac{1}{3}x^3+x^2+3x\right)\Bigg|_{-1}^{3}\\ &=-\frac{1}{3}(27)+9+9-\left(\frac{1}{3}+1-3\right)\\ &=10\frac{2}{3}=10.\overline{6} \end{aligned}$$

The Mean Value Theorem and Average Value

Consider the graph of a function f in Figure 1.25 and the area defined by $\int_1^4 f(x)\,dx$. Three rectangles are drawn in Figure 1.26; in (a), the height of the rectangle is greater than f on $[1,4]$, hence the area of this rectangle is is greater than $\int_0^4 f(x)\,dx$.

In (b), the height of the rectangle is smaller than f on $[1,4]$, hence the area of this rectangle is less than $\int_1^4 f(x)\,dx$.

Finally, in (c) the height of the rectangle is such that the area of the rectangle is *exactly* that of $\int_0^4 f(x)\,dx$. Since rectangles that are "too big", as in (a), and rectangles that are "too little," as in (b), give areas greater/lesser than $\int_1^4 f(x)\,dx$, it makes sense that there is a rectangle, whose top intersects $f(x)$ somewhere on $[1,4]$, whose area is *exactly* that of the definite integral.

We state this idea formally in a theorem.

Notes:

Theorem 9 The Mean Value Theorem of Integration

Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

This is an *existential* statement; c exists, but we do not provide a method of finding it. Theorem 9 is directly connected to the Mean Value Theorem of Differentiation, given as Theorem **??**; we leave it to the reader to see how.

We demonstrate the principles involved in this version of the Mean Value Theorem in the following example.

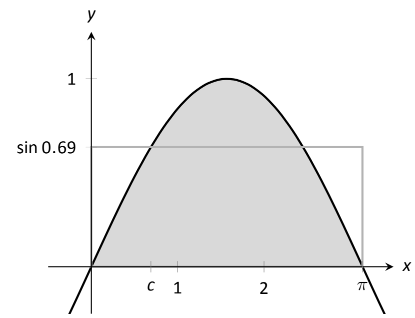


Figure 1.27: A graph of $y = \sin x$ on $[0, \pi]$ and the rectangle guaranteed by the Mean Value Theorem.

Example 24 Using the Mean Value Theorem

Consider $\int_0^\pi \sin x\,dx$. Find a value c guaranteed by the Mean Value Theorem.

Solution We first need to evaluate $\int_0^\pi \sin x\,dx$. (This was previously done in Example 19.)

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2.$$

Thus we seek a value c in $[0, \pi]$ such that $\pi \sin c = 2$.

$$\pi \sin c = 2 \;\Rightarrow\; \sin c = 2/\pi \;\Rightarrow\; c = \arcsin(2/\pi) \approx 0.69.$$

In Figure 1.27 $\sin x$ is sketched along with a rectangle with height $\sin(0.69)$. The area of the rectangle is the same as the area under $\sin x$ on $[0, \pi]$.

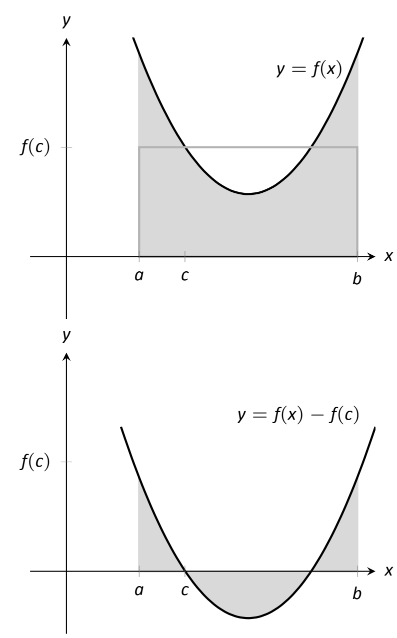


Figure 1.28: On top, a graph of $y = f(x)$ and the rectangle guaranteed by the Mean Value Theorem. Below, $y = f(x)$ is shifted down by $f(c)$; the resulting "area under the curve" is 0.

Let f be a function on $[a, b]$ with c such that $f(c)(b-a) = \int_a^b f(x)\,dx$. Consider $\int_a^b \big(f(x) - f(c)\big)\,dx$:

$$\begin{aligned}\int_a^b \big(f(x) - f(c)\big)\,dx &= \int_a^b f(x) - \int_a^b f(c)\,dx \\ &= f(c)(b-a) - f(c)(b-a) \\ &= 0.\end{aligned}$$

When $f(x)$ is shifted by $-f(c)$, the amount of area under f above the x–axis on $[a, b]$ is the same as the amount of area below the x–axis above f; see Figure 1.28 for an illustration of this. In this sense, we can say that $f(c)$ is the *average value* of f on $[a, b]$.

Notes:

The value $f(c)$ is the average value in another sense. First, recognize that the Mean Value Theorem can be rewritten as

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

for some value of c in $[a,b]$. Next, partition the interval $[a,b]$ into n equally spaced subintervals, $a = x_1 < x_2 < \ldots < x_{n+1} = b$ and choose any c_i in $[x_i, x_{i+1}]$. The average of the numbers $f(c_1)$, $f(c_2)$, ..., $f(c_n)$ is:

$$\frac{1}{n}\Big(f(c_1)+f(c_2)+\ldots+f(c_n)\Big) = \frac{1}{n}\sum_{i=1}^{n} f(c_i).$$

Multiply this last expression by 1 in the form of $\frac{(b-a)}{(b-a)}$:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f(c_i) &= \sum_{i=1}^{n} f(c_i)\frac{1}{n}\\ &= \sum_{i=1}^{n} f(c_i)\frac{1}{n}\frac{(b-a)}{(b-a)}\\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\frac{b-a}{n}\\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x \quad \text{(where } \Delta x = (b-a)/n\text{)}\end{aligned}$$

Now take the limit as $n\to\infty$:

$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x \quad = \quad \frac{1}{b-a}\int_a^b f(x)\,dx \quad = \quad f(c).$$

This tells us this: when we evaluate f at n (somewhat) equally spaced points in $[a,b]$, the average value of these samples is $f(c)$ as $n\to\infty$.

This leads us to a definition.

Definition 4 The Average Value of f on $[a,b]$

Let f be continuous on $[a,b]$. The **average value of f on** $[a,b]$ is $f(c)$, where c is a value in $[a,b]$ guaranteed by the Mean Value Theorem. I.e.,

$$\text{Average Value of } f \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Notes:

An application of this definition is given in the following example.

Example 25 **Finding the average value of a function**

An object moves back and forth along a straight line with a velocity given by $v(t) = (t-1)^2$ on $[0,3]$, where t is measured in seconds and $v(t)$ is measured in ft/s.

What is the average velocity of the object?

Solution By our definition, the average velocity is:

$$\frac{1}{3-0}\int_0^3 (t-1)^2\,dt = \frac{1}{3}\int_0^3 \left(t^2-2t+1\right)\,dt = \frac{1}{3}\left(\frac{1}{3}t^3 - t^2 + t\right)\Big|_0^3 = 1 \text{ ft/s}.$$

We can understand the above example through a simpler situation. Suppose you drove 100 miles in 2 hours. What was your average speed? The answer is simple: displacement/time = 100 miles/2 hours = 50 mph.

What was the displacement of the object in Example 25? We calculate this by integrating its velocity function: $\int_0^3 (t-1)^2\,dt = 3$ ft. Its final position was 3 feet from its initial position after 3 seconds: its average velocity was 1 ft/s.

This section has laid the groundwork for a lot of great mathematics to follow. The most important lesson is this: definite integrals can be evaluated using antiderivatives. Since the previous section established that definite integrals are the limit of Riemann sums, we can later create Riemann sums to approximate values other than "area under the curve," convert the sums to definite integrals, then evaluate these using the Fundamental Theorem of Calculus. This will allow us to compute the work done by a variable force, the volume of certain solids, the arc length of curves, and more.

The downside is this: generally speaking, computing antiderivatives is much more difficult than computing derivatives. The next chapter is devoted to techniques of finding antiderivatives so that a wide variety of definite integrals can be evaluated. Before that, the next section explores techniques of approximating the value of definite integrals beyond using the Left Hand, Right Hand and Midpoint Rules.

Notes:

Exercises 1.4

Terms and Concepts

1. How are definite and indefinite integrals related?

2. What constant of integration is most commonly used when evaluating definite integrals?

3. T/F: If f is a continuous function, then $F(x) = \int_a^x f(t)\,dt$ is also a continuous function.

4. The definite integral can be used to find "the area under a curve." Give two other uses for definite integrals.

Problems

In Exercises 5 – 28, evaluate the definite integral.

5. $\int_1^3 (3x^2 - 2x + 1)\,dx$

6. $\int_0^4 (x-1)^2\,dx$

7. $\int_{-1}^1 (x^3 - x^5)\,dx$

8. $\int_{\pi/2}^{\pi} \cos x\,dx$

9. $\int_0^{\pi/4} \sec^2 x\,dx$

10. $\int_1^e \frac{1}{x}\,dx$

11. $\int_{-1}^1 5^x\,dx$

12. $\int_{-2}^{-1} (4 - 2x^3)\,dx$

13. $\int_0^{\pi} (2\cos x - 2\sin x)\,dx$

14. $\int_1^3 e^x\,dx$

15. $\int_0^4 \sqrt{t}\,dt$

16. $\int_9^{25} \frac{1}{\sqrt{t}}\,dt$

17. $\int_1^8 \sqrt[3]{x}\,dx$

18. $\int_1^2 \frac{1}{x}\,dx$

19. $\int_1^2 \frac{1}{x^2}\,dx$

20. $\int_1^2 \frac{1}{x^3}\,dx$

21. $\int_0^1 x\,dx$

22. $\int_0^1 x^2\,dx$

23. $\int_0^1 x^3\,dx$

24. $\int_0^1 x^{100}\,dx$

25. $\int_{-4}^4 dx$

26. $\int_{-10}^{-5} 3\,dx$

27. $\int_{-2}^2 0\,dx$

28. $\int_{\pi/6}^{\pi/3} \csc x \cot x\,dx$

29. Explain why:

    (a) $\int_{-1}^1 x^n\,dx = 0$, when n is a positive, odd integer, and

    (b) $\int_{-1}^1 x^n\,dx = 2\int_0^1 x^n\,dx$ when n is a positive, even integer.

In Exercises 30 – 33, find a value c guaranteed by the Mean Value Theorem.

30. $\int_0^2 x^2\,dx$

31. $\int_{-2}^2 x^2\,dx$

32. $\int_0^1 e^x\,dx$

33. $\int_0^{16} \sqrt{x}\,dx$

In Exercises 34 – 39, find the average value of the function on the given interval.

34. $f(x) = \sin x$ on $[0, \pi/2]$

35. $y = \sin x$ on $[0, \pi]$

36. $y = x$ on $[0, 4]$

37. $y = x^2$ on $[0, 4]$

38. $y = x^3$ on $[0, 4]$

39. $g(t) = 1/t$ on $[1, e]$

In Exercises 40 – 44, a velocity function of an object moving along a straight line is given. Find the displacement of the object over the given time interval.

40. $v(t) = -32t + 20$ft/s on $[0, 5]$

41. $v(t) = -32t + 200$ft/s on $[0, 10]$

42. $v(t) = 2^t$mph on $[-1, 1]$

43. $v(t) = \cos t$ ft/s on $[0, 3\pi/2]$

44. $v(t) = \sqrt[4]{t}$ ft/s on $[0, 16]$

In Exercises 45 – 48, an acceleration function of an object moving along a straight line is given. Find the change of the object's velocity over the given time interval.

45. $a(t) = -32$ft/s^2 on $[0, 2]$

46. $a(t) = 10$ft/s^2 on $[0, 5]$

47. $a(t) = t$ ft/s^2 on $[0, 2]$

48. $a(t) = \cos t$ ft/s^2 on $[0, \pi]$

In Exercises 49 – 52, sketch the given functions and find the area of the enclosed region.

49. $y = 2x$, $y = 5x$, and $x = 3$.

50. $y = -x + 1$, $y = 3x + 6$, $x = 2$ and $x = -1$.

51. $y = x^2 - 2x + 5$, $y = 5x - 5$.

52. $y = 2x^2 + 2x - 5$, $y = x^2 + 3x + 7$.

In Exercises 53 – 56, find $F'(x)$.

53. $F(x) = \displaystyle\int_2^{x^3+x} \frac{1}{t}\,dt$

54. $F(x) = \displaystyle\int_{x^3}^{0} t^3\,dt$

55. $F(x) = \displaystyle\int_{x}^{x^2} (t+2)\,dt$

56. $F(x) = \displaystyle\int_{\ln x}^{e^x} \sin t\,dt$

1.5 Numerical Integration

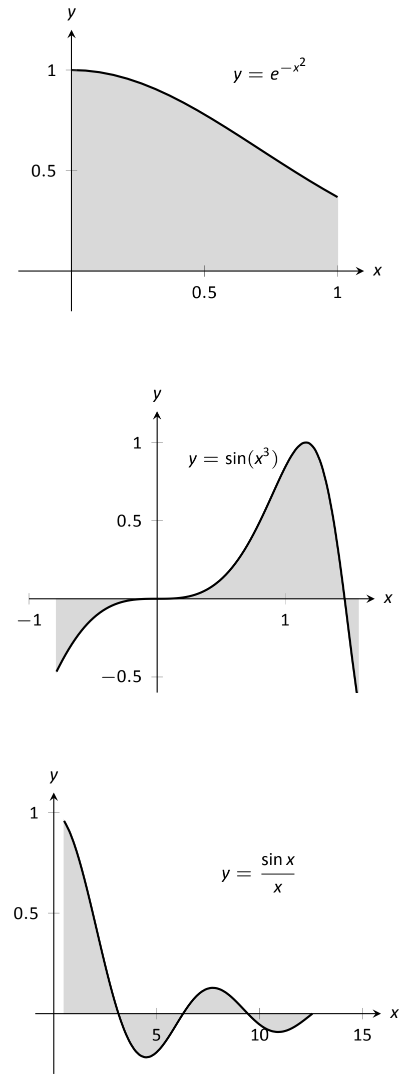


Figure 1.29: Graphically representing three definite integrals that cannot be evaluated using antiderivatives.

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute the antiderivative of the integrand. The second case is when we actually do not know the integrand, but only its value when evaluated at certain points.

An **elementary function** is any function that is a combination of polynomials, n^{th} roots, rational, exponential, logarithmic and trigonometric functions. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{-x^2}, \quad \sin(x^3) \quad \text{and} \quad \frac{\sin x}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,dx$.

This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,dx, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx, \quad \text{and} \quad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,dx,$$

as pictured in Figure 1.29.

The Left and Right Hand Rule Methods

In Section 1.3 we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a,b]$. We wish to approximate $\displaystyle\int_a^b f(x)\,dx$. We partition $[a,b]$ into n equally spaced subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. The endpoints of these

Notes:

subintervals are labeled as

$$x_1 = a,\ x_2 = a + \Delta x,\ x_3 = a + 2\Delta x,\ \ldots,\ x_i = a + (i-1)\Delta x,\ \ldots,\ x_{n+1} = b.$$

Key Idea 1 states that to use the Left Hand Rule we use the summation $\sum_{i=1}^{n} f(x_i)\Delta x$ and to use the Right Hand Rule we use $\sum_{i=1}^{n} f(x_{i+1})\Delta x$. We review the use of these rules in the context of examples.

Example 26 **Approximating definite integrals with rectangles**

Approximate $\int_0^1 e^{-x^2}\,dx$ using the Left and Right Hand Rules with 5 equally spaced subintervals.

Solution We begin by partitioning the interval $[0,1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

$$x_1 = 0,\ x_2 = 0.2,\ x_3 = 0.4,\ x_4 = 0.6,\ x_5 = 0.8,\ \text{and}\ x_6 = 1.$$

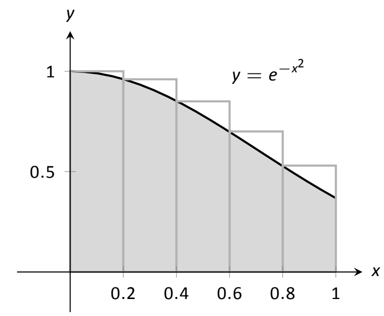
Using the Left Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\big)\Delta x\\ &= \big(f(0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)\big)\Delta x\\ &\approx \big(1+0.961+0.852+0.698+0.527\big)(0.2)\\ &\approx 0.808.\end{aligned}$$

Using the Right Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_{i+1})\Delta x &= \big(f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)\big)\Delta x\\ &= \big(f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\big)\Delta x\\ &\approx \big(0.961+0.852+0.698+0.527+0.368\big)(0.2)\\ &\approx 0.681.\end{aligned}$$

Figure 1.30 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an under approximation. To get a better approximation, we could use more rectangles, as we did in



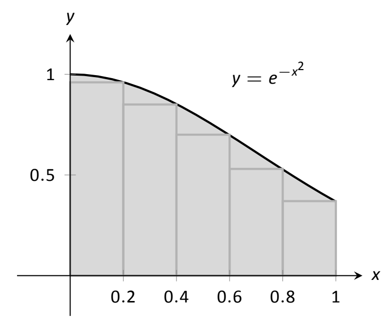
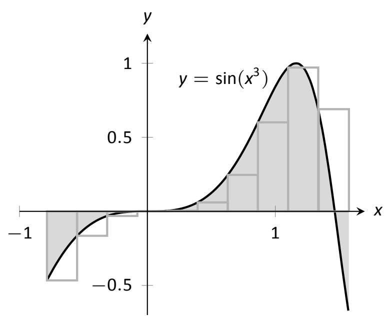


Figure 1.30: Approximating $\int_0^1 e^{-x^2}\,dx$ in Example 26.

Notes:

x_i	Exact	Approx.	$\sin(x_i^3)$
x_1	$-\pi/4$	-0.785	-0.466
x_2	$-7\pi/40$	-0.550	-0.165
x_3	$-\pi/10$	-0.314	-0.031
x_4	$-\pi/40$	-0.0785	0
x_5	$\pi/20$	0.157	0.004
x_6	$\pi/8$	0.393	0.061
x_7	$\pi/5$	0.628	0.246
x_8	$11\pi/40$	0.864	0.601
x_9	$7\pi/20$	1.10	0.971
x_{10}	$17\pi/40$	1.34	0.690
x_{11}	$\pi/2$	1.57	-0.670

Figure 1.31: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 27.



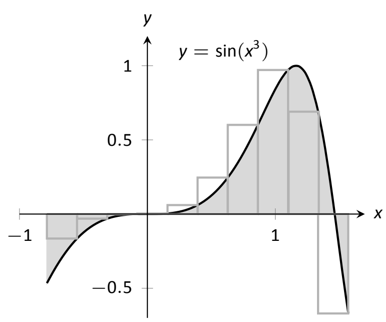


Figure 1.32: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 27.

Section 1.3. We could also average the Left and Right Hand Rule results together, giving

$$\frac{0.808 + 0.681}{2} = 0.7445.$$

The actual answer, accurate to 4 places after the decimal, is 0.7468, showing our average is a good approximation.

Example 27 Approximating definite integrals with rectangles

Approximate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Left and Right Hand Rules with 10 equally spaced subintervals.

Solution We begin by finding Δx:

$$\frac{b-a}{n} = \frac{\pi/2 - (-\pi/4)}{10} = \frac{3\pi}{40} \approx 0.236.$$

It is useful to write out the endpoints of the subintervals in a table; in Figure 1.31, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin(x_i^3)$ and multiplies the sum by Δx; the Right Hand Rule sums the last 10 values of $\sin(x_i^3)$ and multiplies by Δx. Therefore we have:

Left Hand Rule: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.91)(0.236) = 0.451.$

Right Hand Rule: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.71)(0.236) = 0.404.$

Average of the Left and Right Hand Rules: 0.4275.

The actual answer, accurate to 3 places after the decimal, is 0.460. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 1.32. It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation.

The Trapezoidal Rule

In Example 26 we approximated the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ with 5 rectangles of equal width. Figure 1.30 shows the rectangles used in the Left and Right Hand

Notes:

Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids.* In Figure 1.33, we show the region under $f(x) = e^{-x^2}$ on $[0, 1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lies on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\,dx$. (In fact, these trapezoids seem to give a *great* approximation of the area!)

The formula for the area of a trapezoid is given in Figure 1.34. We approximate $\int_0^1 e^{-x^2}\,dx$ with these trapezoids in the following example.

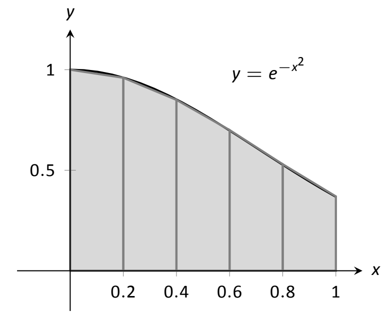


Figure 1.33: Approximating $\int_0^1 e^{-x^2}\,dx$ using 5 trapezoids of equal widths.

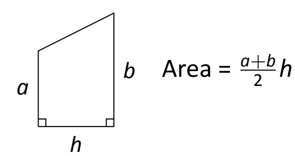


Figure 1.34: The area of a trapezoid.

x_i	$e^{-x_i^2}$
0	1
0.2	0.961
0.4	0.852
0.6	0.698
0.8	0.527
1	0.368

Figure 1.35: A table of values of e^{-x^2}.

Example 28 Approximating definite integrals using trapezoids

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution To compute the areas of the 5 trapezoids in Figure 1.33, it will again be useful to create a table of values as shown in Figure 1.35.

The leftmost trapezoid has legs of length 1 and 0.961 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1+0.961}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.961 and 0.852 and a height of 0.2. Thus its area is:

$$\frac{0.961+0.852}{2}(0.2) = 0.1813.$$

The sum of the areas of all 5 trapezoids is:

$$\frac{1+0.961}{2}(0.2) + \frac{0.961+0.852}{2}(0.2) + \frac{0.852+0.698}{2}(0.2) + \\ \frac{0.698+0.527}{2}(0.2) + \frac{0.527+0.368}{2}(0.2) = 0.7445.$$

We approximate $\displaystyle\int_0^1 e^{-x^2}\,dx \approx 0.7445$.

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.961)+(0.961+0.852)+(0.852+0.698)+(0.698+0.527)+(0.527+0.368)\Big].$$

Notes:

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1+2(0.961+0.852+0.698+0.527)+0.368\Big].$$

This is the heart of the **Trapezoidal Rule**, wherein a definite integral $\int_a^b f(x)\,dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints x_1, $x_2, \ldots, x_{n+1}$, we again have $\Delta x = \dfrac{b-a}{n}$. Thus:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i+1})}{2}\Delta x\\ &= \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_i)+f(x_{i+1})\big)\\ &= \frac{\Delta x}{2}\Big[f(x_1)+2\sum_{i=2}^{n} f(x_i)+f(x_{n+1})\Big].\end{aligned}$$

Example 29 **Using the Trapezoidal Rule**

Revisit Example 27 and approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Trapezoidal Rule and 10 equally spaced subintervals.

Solution We refer back to Figure 1.31 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \approx 0.236$. Thus we have:

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx &\approx \frac{0.236}{2}\Big[-0.466+2\Big(-0.165+(-0.031)+\ldots+0.69\Big)+(-0.67)\Big]\\ &= 0.4275.\end{aligned}$$

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively

Notes:

renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule.

How can we improve on the Trapezoidal Rule, apart from using more and more trapezoids? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule is not *really* about using rectangles to approximate area. Instead, it approximates a function *f* with constant functions on small subintervals and then computes the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function *f* with a linear function on a small subinterval, then computes the definite integral of this linear function. In both of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating *f* with a constant function and then with a linear function. What is next? A quadratic function. By approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process **Simpson's Rule**, named after Thomas Simpson (1710-1761), even though others had used this rule as much as 100 years prior.

Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same *x*–value).

Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose *x*–values are equally spaced and $x_1 < x_2 < x_3$. Let *f* be the quadratic function that goes through these three points. It is not hard to show that

$$\int_{x_1}^{x_3} f(x)\,dx = \frac{x_3 - x_1}{6}\big(y_1 + 4y_2 + y_3\big). \tag{1.4}$$

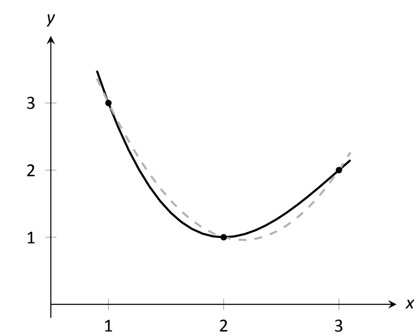


Figure 1.36: A graph of a function *f* and a parabola that approximates it well on $[1, 3]$.

Consider Figure 1.36. A function *f* goes through the 3 points shown and the parabola *g* that also goes through those points is graphed with a dashed line. Using our equation from above, we know exactly that

$$\int_1^3 g(x)\,dx = \frac{3-1}{6}\big(3 + 4(1) + 2\big) = 3.$$

Since *g* is a good approximation for *f* on $[1, 3]$, we can state that

$$\int_1^3 f(x)\,dx \approx 3.$$

Notes:

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\displaystyle\int_a^b f(x)\,dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b - a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (1.4) to compute the area under these parabolas. Adding up these areas gives the formula:

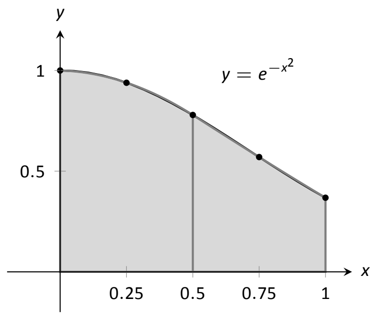
$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+\ldots+2f(x_{n-1})+4f(x_n)+f(x_{n+1})\Big].$$

Note how the coefficients of the terms in the summation have the pattern 1, 4, 2, 4, 2, 4, . . ., 2, 4, 1.

Let's demonstrate Simpson's Rule with a concrete example.

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)



(b)

Figure 1.37: A table of values to approximate $\int_0^1 e^{-x^2}\,dx$, along with a graph of the function.

Example 30 **Using Simpson's Rule**

Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Simpson's Rule and 4 equally spaced subintervals.

Solution We begin by making a table of values as we have in the past, as shown in Figure 1.37(a). Simpson's Rule states that

$$\int_0^1 e^{-x^2}\,dx \approx \frac{0.25}{3}\Big[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368\Big] = 0.7468\overline{3}.$$

Recall in Example 26 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5!

Figure 1.37(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating curves are nearly indistinguishable from the actual function.

x_i	$\sin(x_i^3)$
−0.785	−0.466
−0.550	−0.165
−0.314	−0.031
−0.0785	0
0.157	0.004
0.393	0.061
0.628	0.246
0.864	0.601
1.10	0.971
1.34	0.690
1.57	−0.670

Figure 1.38: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 31.

Example 31 **Using Simpson's Rule**

Approximate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using Simpson's Rule and 10 equally spaced intervals.

Solution Figure 1.38 shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \approx 0.236$.

Notes:

Simpson's Rule states that

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx &\approx \frac{0.236}{3}\Big[(-0.466)+4(-0.165)+2(-0.031)+\ldots\\ &\qquad \ldots+2(0.971)+4(0.69)+(-0.67)\Big]\\ &= 0.4701\end{aligned}$$

Recall that the actual value, accurate to 3 decimal places, is 0.460. Our approximation is within one 1/100th of the correct value. The graph in Figure 1.39 shows how closely the parabolas match the shape of the graph.

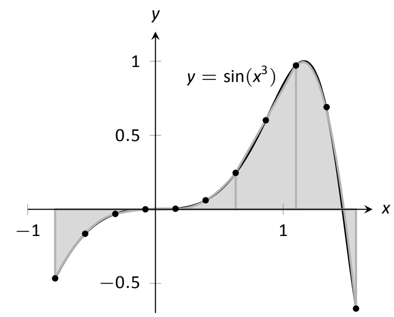


Figure 1.39: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example 31 with Simpson's Rule and 10 equally spaced intervals.

Summary and Error Analysis

We summarize the key concepts of this section thus far in the following Key Idea.

Key Idea 2 Numerical Integration

Let f be a continuous function on $[a,b]$, let n be a positive integer, and let $\Delta x = \dfrac{b-a}{n}$.
Set $x_1 = a$, $x_2 = a+\Delta x$, …, $x_i = a+(i-1)\Delta x$, $x_{n+1} = b$.
Consider $\displaystyle\int_a^b f(x)\,dx$.

Left Hand Rule: $\displaystyle\int_a^b f(x)\,dx \approx \Delta x\Big[f(x_1)+f(x_2)+\ldots+f(x_n)\Big].$

Right Hand Rule: $\displaystyle\int_a^b f(x)\,dx \approx \Delta x\Big[f(x_2)+f(x_3)+\ldots+f(x_{n+1})\Big].$

Trapezoidal Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_n)+f(x_{n+1})\Big].$

Simpson's Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_1)+4f(x_2)+2f(x_3)+\ldots+4f(x_n)+f(x_{n+1})\Big]$ (n even).

In our examples, we approximated the value of a definite integral using a given method then compared it to the "right" answer. This should have raised several questions in the reader's mind, such as:

1. How was the "right" answer computed?
2. If the right answer can be found, what is the point of approximating?
3. If there is value to approximating, how are we supposed to know if the approximation is any good?

Notes:

These are good questions, and their answers are educational. In the examples, *the* right answer was never computed. Rather, an approximation accurate to a certain number of places after the decimal was given. In Example 26, we do not know the *exact* answer, but we know it starts with 0.7468. These more accurate approximations were computed using numerical integration but with more precision (i.e., more subintervals and the help of a computer).

Since the exact answer cannot be found, approximation still has its place. How are we to tell if the approximation is any good?

"Trial and error" provides one way. Using technology, make an approximation with, say, 10, 100, and 200 subintervals. This likely will not take much time at all, and a trend should emerge. If a trend does not emerge, try using yet more subintervals. Keep in mind that trial and error is never foolproof; you might stumble upon a problem in which a trend will not emerge.

A second method is to use Error Analysis. While the details are beyond the scope of this text, there are some formulas that give *bounds* for how good your approximation will be. For instance, the formula might state that the approximation is within 0.1 of the correct answer. If the approximation is 1.58, then one knows that the correct answer is between 1.48 and 1.68. By using lots of subintervals, one can get an approximation as accurate as one likes. Theorem 10 states what these bounds are.

Theorem 10 Error Bounds in the Trapezoidal and Simpson's Rules

1. Let E_T be the error in approximating $\int_a^b f(x)\,dx$ using the Trapezoidal Rule.

   If f has a continuous 2nd derivative on $[a,b]$ and M is any upper bound of $\left|f''(x)\right|$ on $[a,b]$, then

   $$E_T \leq \frac{(b-a)^3}{12n^2}M.$$

2. Let E_S be the error in approximating $\int_a^b f(x)\,dx$ using Simpson's Rule.

   If f has a continuous 4th derivative on $[a,b]$ and M is any upper bound of $\left|f^{(4)}\right|$ on $[a,b]$, then

   $$E_S \leq \frac{(b-a)^5}{180n^4}M.$$

There are some key things to note about this theorem.

Notes:

1. The larger the interval, the larger the error. This should make sense intuitively.
2. The error shrinks as more subintervals are used (i.e., as n gets larger).
3. The error in Simpson's Rule has a term relating to the 4^{th} derivative of f. Consider a cubic polynomial: it's 4^{th} derivative is 0. Therefore, the error in approximating the definite integral of a cubic polynomial with Simpson's Rule is 0 – Simpson's Rule computes the exact answer!

We revisit Examples 28 and 30 and compute the error bounds using Theorem 10 in the following example.

Example 32 **Computing error bounds**

Find the error bounds when approximating $\displaystyle\int_0^1 e^{-x^2}\,dx$ using the Trapezoidal Rule and 5 subintervals, and using Simpson's Rule with 4 subintervals.

Solution

Trapezoidal Rule with $n = 5$:

We start by computing the 2^{nd} derivative of $f(x) = e^{-x^2}$:

$$f''(x) = e^{-x^2}(4x^2 - 2).$$

Figure 1.40 shows a graph of $f''(x)$ on $[0, 1]$. It is clear that the largest value of f'', in absolute value, is 2. Thus we let $M = 2$ and apply the error formula from Theorem 10.

$$E_T = \frac{(1-0)^3}{12 \cdot 5^2} \cdot 2 = 0.00\overline{6}.$$

Our error estimation formula states that our approximation of 0.7445 found in Example 28 is within 0.0067 of the correct answer, hence we know that

$$0.7445 - 0.0067 = .7378 \leq \int_0^1 e^{-x^2}\,dx \leq 0.7512 = 0.7445 + 0.0067.$$

We had earlier computed the exact answer, correct to 4 decimal places, to be 0.7468, affirming the validity of Theorem 10.

Simpson's Rule with $n = 4$:

We start by computing the 4^{th} derivative of $f(x) = e^{-x^2}$:

$$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12).$$

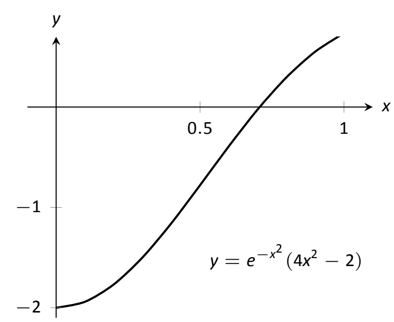


Figure 1.40: Graphing $f''(x)$ in Example 32 to help establish error bounds.

Notes:

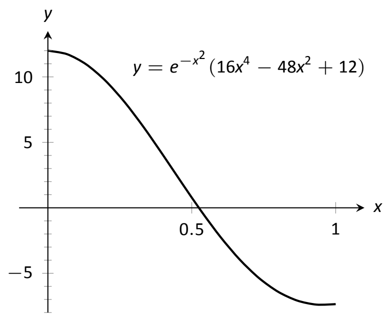


Figure 1.41: Graphing $f^{(4)}(x)$ in Example 32 to help establish error bounds.

Figure 1.41 shows a graph of $f^{(4)}(x)$ on $[0,1]$. It is clear that the largest value of $f^{(4)}$, in absolute value, is 12. Thus we let $M = 12$ and apply the error formula from Theorem 10.

$$E_s = \frac{(1-0)^5}{180 \cdot 4^4} \cdot 12 = 0.00026.$$

Our error estimation formula states that our approximation of $0.7468\overline{3}$ found in Example 30 is within 0.00026 of the correct answer, hence we know that

$$0.74683 - 0.00026 = .74657 \leq \int_0^1 e^{-x^2}\,dx \leq 0.74709 = 0.74683 + 0.00026.$$

Once again we affirm the validity of Theorem 10.

At the beginning of this section we mentioned two main situations where numerical integration was desirable. We have considered the case where an antiderivative of the integrand cannot be computed. We now investigate the situation where the integrand is not known. This is, in fact, the most widely used application of Numerical Integration methods. "Most of the time" we observe behavior but do not know "the" function that describes it. We instead collect data about the behavior and make approximations based off of this data. We demonstrate this in an example.

Time	Speed (mph)
0	0
1	25
2	22
3	19
4	39
5	0
6	43
7	59
8	54
9	51
10	43
11	35
12	40
13	43
14	30
15	0
16	0
17	28
18	40
19	42
20	40
21	39
22	40
23	23
24	0

Figure 1.42: Speed data collected at 30 second intervals for Example 33.

Example 33 Approximating distance traveled

One of the authors drove his daughter home from school while she recorded their speed every 30 seconds. The data is given in Figure 1.42. Approximate the distance they traveled.

Solution Recall that by integrating a speed function we get distance traveled. We have information about $v(t)$; we will use Simpson's Rule to approximate $\displaystyle\int_a^b v(t)\,dt$.

The most difficult aspect of this problem is converting the given data into the form we need it to be in. The speed is measured in miles per hour, whereas the time is measured in 30 second increments.

We need to compute $\Delta x = (b-a)/n$. Clearly, $n = 24$. What are a and b? Since we start at time $t = 0$, we have that $a = 0$. The final recorded time came after 24 periods of 30 seconds, which is 12 minutes or 1/5 of an hour. Thus we have

$$\Delta x = \frac{b-a}{n} = \frac{1/5 - 0}{24} = \frac{1}{120}; \quad \frac{\Delta x}{3} = \frac{1}{360}.$$

Notes:

Thus the distance traveled is approximately:

$$\begin{aligned}\int_0^{0.2} v(t)\,dt &\approx \frac{1}{360}\Big[f(x_1)+4f(x_2)+2f(x_3)+\cdots+4f(x_n)+f(x_{n+1})\Big]\\ &= \frac{1}{360}\Big[0+4\cdot 25+2\cdot 22+\cdots+2\cdot 40+4\cdot 23+0\Big]\\ &\approx 6.2167 \text{ miles.}\end{aligned}$$

We approximate the author drove 6.2 miles. (Because we are sure the reader wants to know, the author's odometer recorded the distance as about 6.05 miles.)

We started this chapter learning about antiderivatives and indefinite integrals. We then seemed to change focus by looking at areas between the graph of a function and the *x*-axis. We defined these areas as the definite integral of the function, using a notation very similar to the notation of the indefinite integral. The Fundamental Theorem of Calculus tied these two seemingly separate concepts together: we can find areas under a curve, i.e., we can evaluate a definite integral, using antiderivatives.

We ended the chapter by noting that antiderivatives are sometimes more than difficult to find: they are impossible. Therefore we developed numerical techniques that gave us good approximations of definite integrals.

We used the definite integral to compute areas, and also to compute displacements and distances traveled. There is far more we can do than that. In Chapter **??** we'll see more applications of the definite integral. Before that, in Chapter **??** we'll learn advanced techniques of integration, analogous to learning rules like the Product, Quotient and Chain Rules of differentiation.

Notes:

Exercises 1.5

Terms and Concepts

1. T/F: Simpson's Rule is a method of approximating antiderivatives.

2. What are the two basic situations where approximating the value of a definite integral is necessary?

3. Why are the Left and Right Hand Rules rarely used?

Problems

In Exercises 4 – 11, a definite integral is given.

(a) Approximate the definite integral with the Trapezoidal Rule and $n = 4$.

(b) Approximate the definite integral with Simpson's Rule and $n = 4$.

(c) Find the exact value of the integral.

4. $\displaystyle\int_{-1}^{1} x^2\,dx$

5. $\displaystyle\int_{0}^{10} 5x\,dx$

6. $\displaystyle\int_{0}^{\pi} \sin x\,dx$

7. $\displaystyle\int_{0}^{4} \sqrt{x}\,dx$

8. $\displaystyle\int_{0}^{3} (x^3 + 2x^2 - 5x + 7)\,dx$

9. $\displaystyle\int_{0}^{1} x^4\,dx$

10. $\displaystyle\int_{0}^{2\pi} \cos x\,dx$

11. $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2}\,dx$

In Exercises 12 – 19, approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 6$.

12. $\displaystyle\int_{0}^{1} \cos\left(x^2\right)\,dx$

13. $\displaystyle\int_{-1}^{1} e^{x^2}\,dx$

14. $\displaystyle\int_{0}^{5} \sqrt{x^2 + 1}\,dx$

15. $\displaystyle\int_{0}^{\pi} x\sin x\,dx$

16. $\displaystyle\int_{0}^{\pi/2} \sqrt{\cos x}\,dx$

17. $\displaystyle\int_{1}^{4} \ln x\,dx$

18. $\displaystyle\int_{-1}^{1} \frac{1}{\sin x + 2}\,dx$

19. $\displaystyle\int_{0}^{6} \frac{1}{\sin x + 2}\,dx$

In Exercises 20 – 23, find n such that the error in approximating the given definite integral is less than 0.0001 when using:

(a) the Trapezoidal Rule

(b) Simpson's Rule

20. $\displaystyle\int_{0}^{\pi} \sin x\,dx$

21. $\displaystyle\int_{1}^{4} \frac{1}{\sqrt{x}}\,dx$

22. $\displaystyle\int_{0}^{\pi} \cos\left(x^2\right)\,dx$

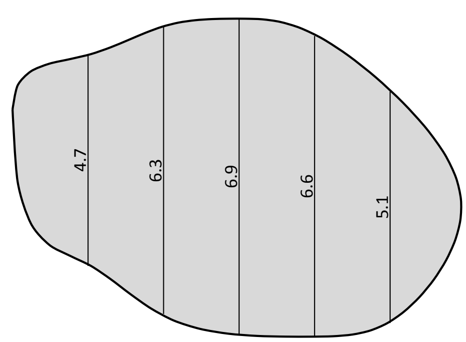
23. $\displaystyle\int_{0}^{5} x^4\,dx$

In Exercises 24 – 25, a region is given. Find the area of the region using Simpson's Rule:

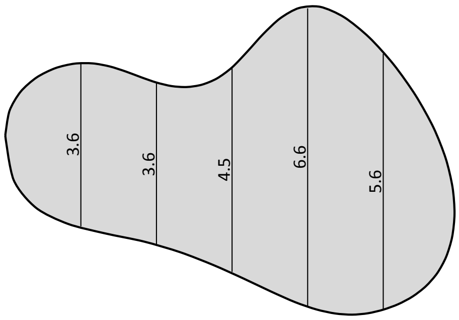
(a) where the measurements are in centimeters, taken in 1 cm increments, and

(b) where the measurements are in hundreds of yards, taken in 100 yd increments.

24.



25.



2: Differential Equations

One of the strengths of calculus is its ability to describe real-world phenomena. We have seen hints of this in our discussion of the applications of derivatives and integrals in the previous chapters. The process of formulating an equation or multiple equations to describe a physical phenomenon is called *mathematical modeling.* As a simple example, populations of bacteria are often described as "growing exponentially." Looking in a biology text, we might see $P(t) = P_0e^{kt}$, where $P(t)$ is the bacteria population at time t, P_0 is the initial population at time $t = 0$, and the constant k describes how quickly the population grows. This equation for exponential growth arises from the assumption that the population of bacteria grows at a rate proportional to its size. Recalling that the derivative gives the rate of change of a function, we can describe the growth assumption precisely using the equation $P' = kP$. This equation is called a *differential equation*, and is the subject of the current chapter.

2.1 Graphical and Numerical Solutions to Differential Equations

In section 1.1, we were introduced to the idea of a differential equation. Given a function $y = f(x)$, we defined a *differential equation* as an equation involving y, x, and derivatives of y. We explored the simple differential equation $y' = 2x$, and saw that a *solution* to a differential equation is simply a function that satisfies the differential equation.

Introduction and Terminology

Definition 5 Differential Equation

Given a function $y = f(x)$, a **differential equation** is an equation relating x, y, and derivatives of y.

- The variable x is called the **independent variable**.
- The variable y is called the **dependent variable**.
- The **order** of the differential equation is the order of the highest derivative of y.

Let us return to the simple differential equation

$$y' = 2x.$$

To find a solution, we must find a function whose derivative is $2x$. In other words, we seek an antiderivative of $2x$. The function

$$y = x^2$$

is an antiderivative of $2x$, and solves the differential equation. So do the functions

$$y = x^2 + 1$$

and

$$y = x^2 - 2346.$$

We call the function

$$y = x^2 + C,$$

with C an arbitrary constant of integration, the *general solution* to the differential equation.

In order to specify the value of the integration constant C, we require additional information. For example, if we know that $y(1) = 3$, it follows that $C = 2$. This additional information is called an *initial condition.*

Definition 6 Initial Value Problem

A differential equation paired with an initial condition (or initial conditions) is called an **initial value problem.**

The solution to an initial value problem is called a **particular solution** to the initial value problem.
The solution to a differential that encompasses all possible solutions is called the **general solution** to the differential equation.

Example 34 A simple first-order differential equation
Solve the differential equation $y' = 2y$.

Solution The solution is a function y such that differentiation yields twice the original function. Unlike our starting example, finding the solution here does not involve computing an antiderivative. Notice that "integrating both sides" would yield the result $y = \int 2y\,dx$, which is not useful. Without knowledge of the function y, we can't compute the indefinite integral. Later sections

Notes:

will explore systematic ways to find analytic solution to simple differential equations. For now, a bit of though might let us guess that solution

$$y = e^{2x}.$$

Notice that application of the chain rule yields $y' = 2e^{2x} = 2y$. Another solution is given by

$$y = -3e^{2x}.$$

In fact

$$y = Ce^{2x},$$

where C is any constant, is the *general solution* to the differential equation because $y' = 2Ce^{2x} = 2y$.

If we are provided with a single initial condition, say $y(0) = 3$, we can identify $C = 3$ so that

$$y = 3e^{2x}$$

is the *particular solution* to the initial value problem

$$y' = 2y, \text{ with } y(0) = 3.$$

Example 35 A second-order differential equation
Solve the differential equation $y'' + 9y = 0$.

Solution We seek a function such that two derivatives returns negative 9 multiplied by the original function. Both $\sin(3x)$ and $\cos(3x)$ have this feature. The general solution to the differential equation is given by

$$y = C_1 \sin(3x) + C_2 \cos(3x),$$

where C_1 and C_2 are arbitrary constants. To fully specify a particular solution, we require two additional conditions. For example, the initial conditions $y(0) = 1$ and $y'(0) = 3$ yield $C_1 = C_2 = 1$.

The differential equation in example 35 is second order because the equation involves a second derivative. In general, the number of initial conditions required to specify a particular solution depends on the order of the differential equation. For the remainder of the chapter, we restrict our attention to first order differential equations and first order initial value problems.

Example 36 Verifying a solution to the differential equation
Which of the following is a solution to the differential equation

$$y' + \frac{y}{x} - \sqrt{y} = 0?$$

Notes:

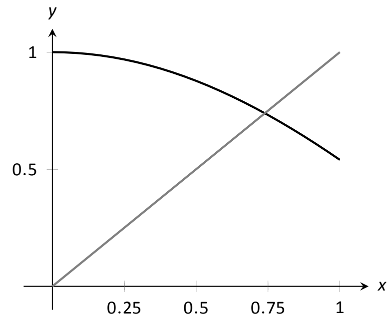


Figure 2.1: Graphically finding an approximate solution to $\cos x = x$.

a) $y = C\left(1 + \ln x\right)^2$ b) $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ c) $y = Ce^{-3x} + \sqrt{\sin x}$

Solution Verifying a solution to a differential equation is simply an exercise in differentiation and simplification. We substitute equation solution in the differential equation to see if it satisfies the equation.

a) Testing the potential solution $y = C\left(1 + \ln x\right)^2$:

Differentiating, we have $y' = \dfrac{2C(1 + \ln x)}{x}$. Substituting into the differential equation,

Graphical Solutions to Differential Equations

The solutions to the differential equations we have found so far are called *analytic solutions.* We have found exact forms for the functions that solve the differential equations. Many times a differential equation will have a solution, but it is difficult or impossible to find the solution analytically. This is analogous to algebraic equations. The algebraic equation $x^2 + 3x - 1 = 0$ has two real solutions that can be found analytically by using the quadratic formula. The equation $\cos x = x$ has one real solution, but we can find it analytically. As shown in figure 2.1, we can find an approximate solution graphically by plotting $\cos x$ and x and observing the x-value of the intersection. We can similarly use graphical tools to understand the qualitative behavior of solutions to a first order-differential equation.

Consider the first-order differential equation

$$y' = f(x, y).$$

The function f could be any function of the two variables x and y. Written in this way, we can think of the function f as providing a formula to find the slope of a solution at a given point in the xy-plane. In other words, suppose a solution to the differential equation passes through the point (x_0, y_0). Then, at the point (x_0, y_0) the slope of the solution curve will be $f(x_0, y_0)$. Since this calculation of the slope is possible at any point (x, y) where the function $f(x, y)$ is defined, we can produce a plot called a *slope field* that shows the slope of a solution at any point in the xy-plane where the solution is defined. Further, this process can be done purely by working with the differential equation itself. In other words, we can draw a slope field and use it to determine the qualitative behavior of solutions to a differential equation without having to solve the differential equation.

Notes:

Definition 7 Slope Field

A **slope field** for a first-order differential equation $y' = f(x, y)$ is a plot in the xy-plane made up of short line segments or arrows. For each point (x_0, y_0) where $f(x, y)$ is defined, the slope of the line segment is given by $f(x_0, y_0)$

Example 37 Finding a slope field

Find a slope field for the differential equation $y' = x + y$.

SOLUTION Because the function $f(x, y) = x + y$ is defined for all points (x, y), every point in the xy-plane has an associated line segment. It is not practical to draw an entire slope field by hand, but many tools exist for drawing slope fields on a computer. We will explicitly calculate and plot a few of the line segments in the slope field.

The slope of the line segment at $(0, 0)$ is given by $f(0, 0) = 0 + 0 = 0$.

The slope of the line segment at $(1, 1)$ is given by $f(1, 1) = 1 + 1 = 2$.

The slope of the line segment at $(1, -1)$ is given by $f(1, -1) = 1 - 1 = 0$.

The slope of the line segment at $(-2, 3)$ is given by $f(-2, 3) = -2 + 3 = 1$.

These four components of the slope field are shown in figure . The entire slope field for the differential equation is shown in figure .

Example 38 Finding a graphical solution to an initial value problem

Find a graphical solution to the initial value problem $y' = x + y$ with $y(1) = -1$.

SOLUTION The solution to the initial value problem should be a continuous smooth curve. Using the slope field, we can draw of a sketch of the solution using the following two criteria:

1. The solution must pass through the point $(1, -1)$.
2. When the solution passes through a point (x_0, y_0) it must be tangent to the line segment at (x_0, y_0).

Essentially, we sketch a solution to the initial value problem by starting at the point $(1, -1)$ and "following the lines" in either direction. A sketch of the solution is shown in figure .