APEX CALCULUS

Version 3.0

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Contents

Preface

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three–volume series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.apexcalculus.com`.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactive the interactivity, one can right-click and choose the “Disable Content” option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

1: DIFFERENTIAL EQUATIONS

One of the strengths of calculus is its ability to describe real-world phenomena. We have seen hints of this in our discussion of the applications of derivatives and integrals in the previous chapters. The process of formulating an equation or multiple equations to describe a physical phenomenon is called *mathematical modeling.* As a simple example, populations of bacteria are often described as "growing exponentially." Looking in a biology text, we might see $P(t) = P_0e^{kt}$, where $P(t)$ is the bacteria population at time t, P_0 is the initial population at time $t = 0$, and the constant k describes how quickly the population grows. This equation for exponential growth arises from the assumption that the population of bacteria grows at a rate proportional to its size. Recalling that the derivative gives the rate of change of a function, we can describe the growth assumption precisely using the equation $P' = kP$. This equation is called a *differential equation*, and is the subject of the current chapter.

1.1 Graphical and Numerical Solutions to Differential Equations

In section **??**, we were introduced to the idea of a differential equation. Given a function $y = f(x)$, we defined a *differential equation* as an equation involving y, x, and derivatives of y. We explored the simple differential equation $y' = 2x$, and saw that a *solution* to a differential equation is simply a function that satisfies the differential equation.

Introduction and Terminology

Definition 1 Differential Equation

Given a function $y = f(x)$, a **differential equation** is an equation relating x, y, and derivatives of y.

- The variable x is called the **independent variable**.
- The variable y is called the **dependent variable**.
- The **order** of the differential equation is the order of the highest derivative of y.

Let us return to the simple differential equation

$$y' = 2x.$$

To find a solution, we must find a function whose derivative is $2x$. In other words, we seek an antiderivative of $2x$. The function

$$y = x^2$$

is an antiderivative of $2x$, and solves the differential equation. So do the functions

$$y = x^2 + 1$$

and

$$y = x^2 - 2346.$$

We call the function

$$y = x^2 + C,$$

with C an arbitrary constant of integration, the *general solution* to the differential equation.

In order to specify the value of the integration constant C, we require additional information. For example, if we know that $y(1) = 3$, it follows that $C = 2$. This additional information is called an *initial condition.*

Definition 2 Initial Value Problem

A differential equation paired with an initial condition (or initial conditions) is called an **initial value problem.**

The solution to an initial value problem is called a **particular solution**. A particular solution does not include arbitrary constants.

The solution to a differential that encompasses all possible solutions is called the **general solution** to the differential equation. A general solution includes one or more arbitrary constants.

Example 1 A simple first-order differential equation

Solve the differential equation $y' = 2y$.

SOLUTION The solution is a function y such that differentiation yields twice the original function. Unlike our starting example, finding the solution

Notes:

here does not involve computing an antiderivative. Notice that "integrating both sides" would yield the result $y = \int 2y\,dx$, which is not useful. Without knowledge of the function y, we can't compute the indefinite integral. Later sections will explore systematic ways to find analytic solution to simple differential equations. For now, a bit of though might let us guess the solution

$$y = e^{2x}.$$

Notice that application of the chain rule yields $y' = 2e^{2x} = 2y$. Another solution is given by

$$y = -3e^{2x}.$$

In fact

$$y = Ce^{2x},$$

where C is any constant, is the *general solution* to the differential equation because $y' = 2Ce^{2x} = 2y$.

If we are provided with a single initial condition, say $y(0) = 3$, we can identify $C = 3$ so that

$$y = 3e^{2x}$$

is the *particular solution* to the initial value problem

$$y' = 2y, \text{ with } y(0) = 3.$$

Example 2 A second-order differential equation

Solve the differential equation $y'' + 9y = 0$.

Solution We seek a function such that two derivatives returns negative 9 multiplied by the original function. Both $\sin(3x)$ and $\cos(3x)$ have this feature. The general solution to the differential equation is given by

$$y = C_1 \sin(3x) + C_2 \cos(3x),$$

where C_1 and C_2 are arbitrary constants. To fully specify a particular solution, we require two additional conditions. For example, the initial conditions $y(0) = 1$ and $y'(0) = 3$ yield $C_1 = C_2 = 1$.

The differential equation in example 2 is second order because the equation involves a second derivative. In general, the number of initial conditions required to specify a particular solution depends on the order of the differential equation. For the remainder of the chapter, we restrict our attention to first order differential equations and first order initial value problems.

Notes:

Example 3 **Verifying a solution to the differential equation**

Which of the following is a solution to the differential equation

$$y' + \frac{y}{x} - \sqrt{y} = 0?$$

a) $y = C\left(1+\ln x\right)^2$ b) $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ c) $y = Ce^{-3x} + \sqrt{\sin x}$

Solution Verifying a solution to a differential equation is simply an exercise in differentiation and simplification. We substitute each potential solution into the differential equation to see if it satisfies the equation.

a) Testing the potential solution $y = C\left(1+\ln x\right)^2$:

Differentiating, we have $y' = \dfrac{2C(1+\ln x)}{x}$. Substituting into the differential equation,

$$\begin{aligned}
&\frac{2C(1+\ln x)}{x} + \frac{C(1+\ln x)^2}{x} - \sqrt{C}(1+\ln x)\\
&= (1+\ln x)\left(\frac{2C}{x} + \frac{C(1+\ln x)}{x} - \sqrt{C}\right)\\
&\neq 0.
\end{aligned}$$

Since it doesn't satisfy the differential equation, $y = C(1+\ln x)^2$ is *not* a solution.

b) Testing the potential solution $y = \left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)^2$:

Differentiating, we have $y' = 2\left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)\left(\dfrac{1}{3} - \dfrac{C}{2x^{3/2}}\right)$. Substituting into the differential equation,

$$\begin{aligned}
&2\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{1}{3} - \frac{C}{2x^{3/2}}\right) + \frac{1}{x}\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2 - \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\\
&= \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{2}{3} - \frac{C}{x^{3/2}} + \frac{1}{3} + \frac{C}{x^{3/2}} - 1\right)\\
&= 0.
\end{aligned}$$

Thus $y = \left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)^2$ *is* a solution to the differential equation.

c) Testing the potential solution $y = Ce^{-3x} + \sqrt{\sin x}$:

Notes:

Differentiating, $y' = -3Ce^{-3x} + \dfrac{\cos x}{2\sqrt{\sin x}}$. Substituting into the differential equation,

$$-3Ce^{-3x} + \frac{\cos x}{2\sqrt{\sin x}} + \frac{Ce^{-3x} + \sqrt{\sin x}}{x} - \sqrt{Ce^{-3x} + \sqrt{\sin x}} \neq 0.$$

The function $y = Ce^{-3x} + \sqrt{\sin x}$ is *not* a solution to the differential equation.

Example 4 Verifying a Solution to a Differential Equation

Verify that $x^2 + y^2 = Cy$ is a solution to $y' = \dfrac{2xy}{x^2 - y^2}$.

Solution The solution in this example is called an *implicit solution.* That means the dependent variable y is a function of x, but has not been explicitly solved for. Verifying the solution still involves differentiation, but we must take the derivatives implicitly. Differentiating, we have

$$2x + 2yy' = Cy'.$$

Solving for y', we have

$$y' = \frac{2x}{C - 2y}.$$

From the solution, we know that $C = \dfrac{x^2 + y^2}{y}$. Then

$$\begin{aligned} y &= \frac{2x}{\dfrac{x^2 + y^2}{y} - 2y} \\ &= \frac{2xy}{x^2 + y^2 - 2y^2} \\ &= \frac{2xy}{x^2 - y^2}. \end{aligned}$$

We have verified that $x^2 + y^2 = Cy$ is a solution to $y' = \dfrac{2xy}{x^2 - y^2}$.

Graphical Solutions to Differential Equations

The solutions to the differential equations we have explored so far are called *analytic solutions.* We have found exact forms for the functions that solve the differential equations. Many times a differential equation will have a solution, but it is difficult or impossible to find the solution analytically. This is analogous to algebraic equations. The algebraic equation $x^2 + 3x - 1 = 0$ has two

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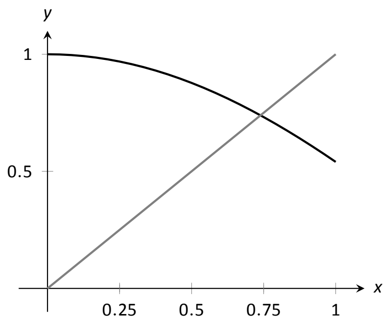

Figure 1.1: Graphically finding an approximate solution to $\cos x = x$.

real solutions that can be found analytically by using the quadratic formula. The equation $\cos x = x$ has one real solution, but we can't find it analytically. As shown in figure 1.1, we can find an approximate solution graphically by plotting $\cos x$ and x and observing the x-value of the intersection. We can similarly use graphical tools to understand the qualitative behavior of solutions to a first order-differential equation.

Consider the first-order differential equation

$$y' = f(x, y).$$

The function f could be any function of the two variables x and y. Written in this way, we can think of the function f as providing a formula to find the slope of a solution at a given point in the xy-plane. In other words, suppose a solution to the differential equation passes through the point (x_0, y_0). At the point (x_0, y_0), the slope of the solution curve will be $f(x_0, y_0)$. Since this calculation of the slope is possible at any point (x, y) where the function $f(x, y)$ is defined, we can produce a plot called a *slope field* that shows the slope of a solution at any point in the xy-plane where the solution is defined. Further, this process can be done purely by working with the differential equation itself. In other words, we can draw a slope field and use it to determine the qualitative behavior of solutions to a differential equation without having to solve the differential equation.

Definition 3 Slope Field

A **slope field** for a first-order differential equation $y' = f(x, y)$ is a plot in the xy-plane made up of short line segments or arrows. For each point (x_0, y_0) where $f(x, y)$ is defined, the slope of the line segment is given by $f(x_0, y_0)$. Plots of solutions to a differential equation are tangent to the line segments in the slope field.

Example 5 Finding a slope field

Find a slope field for the differential equation $y' = x + y$.

Solution Because the function $f(x, y) = x + y$ is defined for all points (x, y), every point in the xy-plane has an associated line segment. It is not practical to draw an entire slope field by hand, but many tools exist for drawing slope fields on a computer. Here, we explicitly calculate and plot a few of the line segments in the slope field.

- The slope of the line segment at $(0, 0)$ is $f(0, 0) = 0 + 0 = 0$.

Notes:

- The slope of the line segment at $(1, 1)$ is $f(1, 1) = 1 + 1 = 2$.
- The slope of the line segment at $(1, -1)$ is $f(1, -1) = 1 - 1 = 0$.
- The slope of the line segment at $(-2, 3)$ is $f(-2, -1) = -2 - 1 = -3$.

Continuing the above process and plotting the line segments with appropriate slopes results in the slope field shown in figure 1.2.

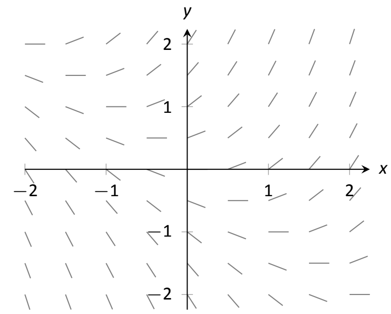

Figure 1.2: Slope field for $y' = x + y$ from example 5.

Example 6 **Finding a graphical solution to an initial value problem**
Find a graphical solution to the initial value problem $y' = x + y$, with $y(1) = -1$.

Solution The solution to the initial value problem should be a continuous smooth curve. Using the slope field, we can draw of a sketch of the solution using the following two criteria:

1. The solution must pass through the point $(1, -1)$.
2. When the solution passes through a point (x_0, y_0) it must be tangent to the line segment at (x_0, y_0).

Essentially, we sketch a solution to the initial value problem by starting at the point $(1, -1)$ and "following the lines" in either direction. A sketch of the solution is shown in figure 1.3.

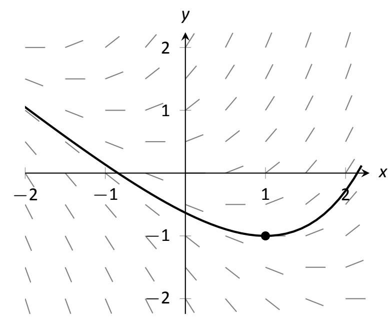

Figure 1.3: Solution to the initial value problem $y' = x + y$, with $y(1) = -1$ from example 6

Example 7 **Using a slope field to predict long term behavior**
Use the slope field for the differential equation $y' = y(1 - y)$, shown in figure 1.4, to predict long term behavior of solutions to the equation.

Solution This differential equation, called the *logistic differential equation*, often appears in population biology to describe the size of a population. For that reason, we use t (time) as the independent variable instead of x. We also often restrict attention to non-negative y-values because negative values correspond to a negative population.

Looking at the slope field in figure 1.4, we can predict long term behavior for a given initial condition.

- If the initial y-value is negative ($y(0) < 0$), the solution curve must pass though the point $(0, y(0))$ and follow the slope field. We expect the solution y to become more and more negative as time increases. Note that this result is not physically relevant when considering a population.
- If the initial y-value is greater than 0 but less than 1, we expect the solution y to increase and level off at $y = 1$.

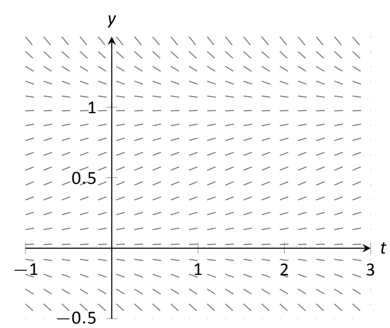

Figure 1.4: Slope field for the logistic differential equation $y' = y(1 - y)$ from example 7.

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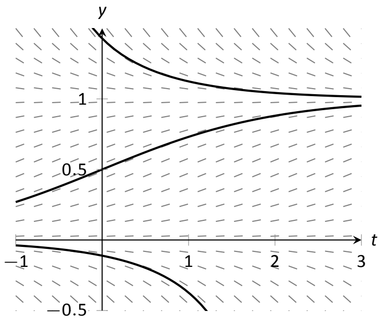


Figure 1.5: Slope field for the logistic differential equation $y' = y(1 - y)$ from example 7 with a few representative solution curves.

- If the initial y-value is greater than 1, we expect the solution y to decrease and level off at $y = 1$.

The slope field for the logistic differential equation, along with representative solution curves, is shown in figure 1.5. Notice that any solution curve with positive initial value will tend towards the value $y = 1$. We call this the *carrying capacity.*

Numerical Solutions to Differential Equations: Euler's Method

While the slope field is an effective way to understand the qualitative behavior of solutions to a differential equation, it is difficult to use a slope field to make quantitative predictions. For example, if we have the slope field for the differential equation $y' = x + y$ from example 5 along with the initial condition $y(0) = 1$, we can understand the qualitative behavior of the solution to the initial value problem, but will struggle to predict a specific value, $y(2)$ for example, with any degree of confidence. The most straight forward way to predict $y(2)$ is to find the analytic solution to the the initial value problem and evaluate it at $x = 2$. Unfortunately, we have already mentioned that it is impossible to find analytic solutions to many differential equations. In the absence of an analytic solution, a *numerical solution* can serve as an effective tool to make quantitative predictions about the solution to an initial value problem.

There are many techniques for computing numerical solutions to initial value problems. A course in numerical analysis will discuss various techniques along with their strengths and weaknesses. The simplest technique is called *Euler's Method* (pronounced "oil-er," not "you-ler"). Consider the first-order initial value problem

$$y' = f(x, y), \text{ with } y(x_0) = y_0.$$

Using the definition of the derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

If we remove the limit but restrict h to be "small," we have

$$y'(x) \approx \frac{y(x+h) - y(x)}{h},$$

so that

$$f(x, y) \approx \frac{y(x+h) - y(x)}{h},$$

because $y' = f(x, y)$ according to the differential equation. Rearranging terms,

$$y(x+h) \approx y(x) + hf(x, y).$$

Notes:

This statement says that if we know the solution (y-value) to the initial value problem for some given x-value, we can find an approximation for the solution at the value $x+h$ by taking our y-value and adding h times the function f evaluated at the x and y values. Euler's method uses the initial condition of an initial value problem as the starting point, and then uses the above idea to find approximate values for the solution y at later x-values. The algorithm is summarized in key idea 1.

Key Idea 1 Euler's Method

Consider the initial value problem

$$y' = f(x, y) \text{ with } y(x_0) = y_0.$$

Let h be a small positive number and N be an integer.

1. For $i = 0, 1, 2, \ldots, N$, define
$$x_i = x_0 + ih.$$
2. The value y_0 is given by the initial condition.
For $i = 0, 1, 2, \ldots, N-1$, define
$$y_{i+1} = y_i + hf(x_i, y_i).$$

This process yields a sequence of $N+1$ points (x_i, y_i) for $i = 0, 1, 2, \ldots, N$, where (x_i, y_i) is an approximation for $(x_i, y(x_i))$.

Let's practice Euler's Method using a few concrete examples.

Example 8 Using Euler's Method 1

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.5$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.5$, it takes $N = 2$ steps to get to $x = 2$. Using steps 1 and

Notes:

2 from the Euler's Method algorithm,

$x_0 = 1$	$y_0 = -1$
$x_1 = x_0 + h$ $= 1 + 0.5$ $= 1.5$	$y_1 = y_0 + hf(x_0, y_0)$ $= -1 + 0.5(1 - 1)$ $= -1$
$x_2 = x_0 + 2h$ $= 1 + 2(0.5)$ $= 2$	$y_2 = y_1 + hf(x_1, y_1)$ $= -1 + 0.5(1.5 - 1)$ $= -0.75$

Using Euler's method, we find the approximate $y(2) \approx -0.75$.

To help visualize the Euler's method approximation, these three points (connected by line segments) are plotted along with the analytical solution to the initial value problem in figure 1.6.

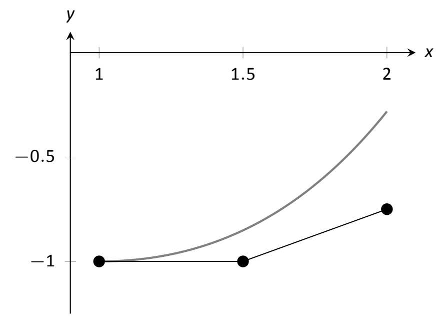


Figure 1.6: Euler's Method approximation to $y' = x + y$ with $y(1) = -1$ from example 8, along with the analytical solution to the initial value problem.

Let's repeat the previous example using a smaller h-value.

Example 9 **Using Euler's Method 2**

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.25$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.25$, it takes $N = 4$ steps to get to $x = 2$. Using steps 1 and 2 from the Euler's Method algorithm (and rounding to 4 decimal points), we have

$x_0 = 1$	$y_0 = -1$
$x_1 = 1.25$	$y_1 = -1 + 0.25(1 - 1)$ $= -1$
$x_2 = 1.5$	$y_2 = -1 + 0.25(1.25 - 1)$ $= -0.9375$
$x_3 = 1.75$	$y_3 = -0.9375 + 0.25(1.5 - 0.9375)$ $= -0.7969$
$x_4 = 2$	$y_4 = -0.7969 + 0.25(1.75 - 0.7969)$ $= -0.5586$

Using Euler's method, we find the approximate $y(2) \approx -0.5584$.

These five points, along with the points from example 8 and the analytic solution, are plotted in figure 1.7.

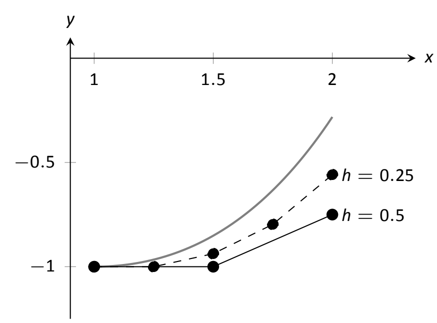


Figure 1.7: Euler's Method approximations to $y' = x + y$ with $y(1) = -1$ from examples 8 and 9, along with the analytical solution.

Using the results from examples 8 and 9, we can make a few observations about Euler's method. First, the Euler approximation gets successively worse as we get farther from the initial condition. This is because Euler's method involves two sources of error. The first comes from the fact that we're using a positive

Notes:

h-value in the derivative approximation instead of using a limit as h approaches zero. Essentially, we're using a linear approximation to the solution y (similar to the process described in section **??** on differentials.) This error is often called the *local truncation error.* The second source of error comes from the fact that every step in Euler's method uses the result of the previous step. That means we're using an approximate y-value to approximate the next y-value. Doing this repeatedly causes the errors to build on each other. This second type of error is often called the *propagated* or *accumulated error.* A second observation is that the Euler approximation is more accurate for smaller h-values. This accuracy comes at a cost, though. Example 9 is more accurate than example 8, but takes twice as many computations. In general, numerical algorithms (even when performed by a computer program) require striking a balance between a desired level of accuracy and the amount of computational effort we are willing to undertake.

Let's do one final example of Euler's Method.

Example 10 Using Euler's Method 3

Find an approximation for the solution to the logistic differential equation $y' = y(1-y)$ with $y(0) = 0.25$, for $0 \leq y \leq 4$. Use $N = 10$ steps.

Solution The logistic differential equation is what is called an *autonomous equation*. An autonomous differential equation has no explicit dependence on the independent variable (t in this case). This has no real effect on the application of Euler's method other than the fact that the function $f(t, y)$ is really just a function of y. To take steps in the y variable, we use

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + hy_i(1 - y_i).$$

Using $N = 10$ steps requires $h = \dfrac{4-0}{10} = 0.4$. Implementing Euler's Method, we have

x_0	$= 0$	y_0	$= 0.25$
x_1	$= 0.4$	y_1	$= 0.25 + 0.4(0.25)(1 - 0.25)$ $= 0.325$
x_2	$= 0.8$	y_2	$= 0.325 + 0.4(0.325)(1 - 0.325)$ $= 0.41275$
x_3	$= 1.2$	y_3	$= 0.41275 + 0.4(0.41275)(1 - 0.41275)$ $= 0.50970$
x_4	$= 1.6$	y_4	$= 0.50970 + 0.4(0.50970)(1 - 0.50970)$ $= 0.60966$
x_5	$= 2.0$ $=$	y_5	$= 0.60966 + 0.4(0.60966)(1 - 0.60966)$ $= 0.70485$

Notes:

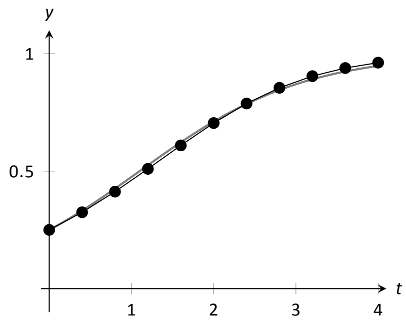


Figure 1.8: Euler's Method approximation to $y' = y(1-y)$ with $y(0) = 0.25$ from example 10, along with the analytical solution.

x_6	$= 2.4$	y_6	$= 0.70485 + 0.4(0.70485)(1 - 0.70485)$ $= 0.78806$
x_7	$= 2.8$	y_7	$= 0.78806 + 0.4(0.78806)(1 - 0.78806)$ $= 0.85487$
x_8	$= 3.2$	y_8	$= 0.85487 + 0.4(0.85487)(1 - 0.85487)$ $= 0.90450$
x_9	$= 3.6$	y_9	$= 0.90450 + 0.4(0.90450)(1 - 0.90450)$ $= 0.93905$
x_{10}	$= 4.0$	y_{10}	$= 0.93905 + 0.4(0.93905)(1 - 0.93905)$ $= 0.96194$

These 11 points, along with the the analytic solution, are plotted in figure 1.8.

The study of differential equations is a natural extension of the study of derivatives and integrals. The equations themselves involve derivatives, and methods to find analytic solutions often involve finding antiderivatives. In this section, we focus on graphical and numerical techniques to understand solutions to differential equations. We restrict our examples to relatively simple initial value problems that permit analytic solution to the equations, but should remember that this is only for comparison purposes. In reality, many differential equations, even some that appear straight forward, do not have solutions we can find analytically. Even so, we can use the techniques presented in this section to understand the behavior of solutions. In the next two sections, we explore two techniques to find analytic solutions to two different classes of differential equations.

Notes:

Exercises 1.1

Terms and Concepts

1. In your own words, what is an initial value problem, and how is it different than a differential equation?

2. In your own words, describe what it means for a function to be a solution to a differential equation.

3. How can we verify that a function is a solution to a differential equation?

4. Describe the difference between a particular solution and a general solution.

5. Why might we use a graphical or numerical technique to study solutions to a differential equation instead of simply solving the differential equation to find an analytic solution?

Problems

In Exercises 6 – 8, verify that the given function is a solution to the differential equation or initial value problem.

6. $y = Ce^{-6x^2}$; $y' = -12xy$.

7. $y = x\sin x$; $y' - x\cos x = (x^2+1)\sin x - xy$, with $y(\pi) = 0$.

8. $2x^2 - y^2 = C$; $yy' - 2x = 0$

In Exercises 9 – 10, verify that the given function is a solution to the differential equation and find the C value required to make the function satisfy the initial condition.

9. $y = 4e^{3x}\sin x + Ce^{3x}$; $y' - 3y = 4e^{3x}\cos x$, with $y(0) = 2$

10. $y(x^2 + y) = C$; $2xy + (x^2 + 2y)y' = 0$, with $y(1) = 2$

In Exercises 11 – 13, sketch a slope field for the given differential equation.

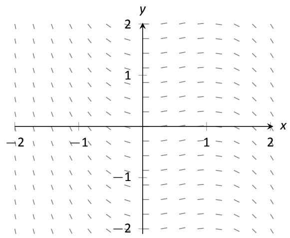
11. $y' = y - x$

12. $y' = \dfrac{x}{2y}$

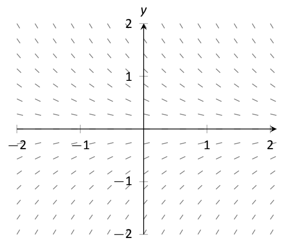
13. $y' = \sin(\pi y)$

In Exercises 14 – 17, match the slope field with the appropriate differential equation.

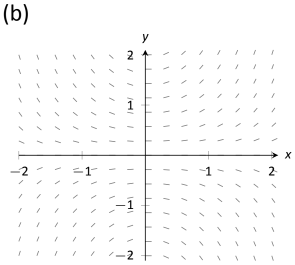
(a)



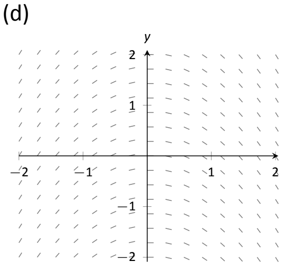
(c)



(b)



(d)



14. $\dfrac{dy}{dx} = xy$

15. $\dfrac{dy}{dx} = -y$

16. $\dfrac{dy}{dx} = -x$

17. $\dfrac{dy}{dx} = x(1 - x)$

In Exercises 18 – 20, sketch the flope field for the differential equation, and use it to draw a sketch of the solution to the initial value problem.

18. $\dfrac{dy}{dx} = \dfrac{y}{x} - y$, with $y(0.5) = 1$.

19. $\dfrac{dy}{dx} = y\sin x$, with $y(0) = 1$.

20. $\dfrac{dy}{dx} = y^2 - 3y + 2$, with $y(0) = 2$.

In Exercises 21 – 25, use Euler's Method to make a table of values that approximates the solution to the initial value problem on the given interval. Use the specified h or N value.

21. $\dfrac{dy}{dx} = x + 2y$
$y(0) = 1$
interval: $[0, 1]$
$h = 0.25$

22. $\dfrac{dy}{dx} = xe^{-y}$
$y(0) = 1$
interval: $[0, 0.5]$
$N = 5$

23. $\dfrac{dy}{dx} = y + \sin x$
$y(0) = 2$
interval: $[0, 1]$
$h = 0.2$

24. $\dfrac{dy}{dx} = e^{x-y}$
$y(0) = 0$
interval: $[0, 2]$
$h = 0.5$

25. $\dfrac{dy}{dx} = \dfrac{1}{x} - y \ln x$
$y(1) = 1$
interval: $[1, 2]$
$N = 5$

In Exercises 26 – 27, use the provided solution $y(x)$ and Euler's Method with the $h = 0.2$ and $h = 0.1$ to complete the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
$y(x)$						
$h = 0.2$						
$h = 0.1$						

26. $\dfrac{dy}{dx} = xy^2$
$y(0) = 1$
solution: $y(x) = \dfrac{2}{1 - x^2}$

27. $\dfrac{dy}{dx} = xe^{x^2} + \dfrac{1}{2}xy$
$y(0) = \frac{1}{2}$
solution: $y(x) = \dfrac{1}{2}(x^2 + 1)e^{x^2}$

1.2 Separable Differential Equations

Similar to algebraic equations, there are specific techniques that can be used to solve specific types of differential equations. In algebra, we can use the quadratic formula to solve a quadratic equation, but not a linear or cubic equation. In the same way, techniques that can be used for a specific type of differential equation are ineffective for a differential equation of a different type. In this section, we describe and practice a technique to solve a class of differential equations called *separable equations.*

Definition 4 Separable Differential Equation

A **separable differential equation** is one that can be written in the form

$$n(y)\frac{dy}{dx} = m(x),$$

where n is a function that depends only on the dependent variable y, and m is a function that depends only on the independent variable x.

Below, we show a few examples of separable differential equations, along with similar looking equations that are not separable.

Separable	Not Separable
1. $\frac{dy}{dx} = x^2 y$	1. $\frac{dy}{dx} = x^2 + y$
2. $y\sqrt{y^2-5}\frac{dy}{dx} - \sin x \cos x = 0$	2. $y\sqrt{y^2-1}\frac{dy}{dx} - \sin x \cos y = 0$
3. $\frac{dy}{dx} = \frac{(x^2+1)e^y}{y}$	3. $\frac{dy}{dx} = \frac{(xy+1)e^y}{y}$

Notice that a separable equation requires that the functions of the dependent and independent variables be multiplied, not added (like example 1 of the not separable column). An alternate definition of a separable differential equation states that an equation is separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

for some functions f and g.

Notes:

Separation of Variables

Let's find a formal solution to the separable equation

$$n(y)\frac{dy}{dx} = m(x).$$

Since the functions on the left and right hand sides of the equation are equal, their antiderivatives should be equal up to an arbitrary constant of integration. That is

$$\int n(y)\frac{dy}{dx}\,dx = \int m(x)\,dx + C.$$

Though the integral on the left may look a bit strange, recall that y itself is a function of x. Consider the substitution $u = y(x)$. The differential is $du = \frac{dy}{dx}\,dx$. Using this substitution, the above equation becomes

$$\int n(u)\,du = \int m(x)\,dx + C.$$

Let $N(u)$ and $M(x)$ be antiderivatives of $n(u)$ and $m(x)$, respectively. Then

$$N(u) = M(x) + C.$$

Since $u = y(x)$, this is

$$N(y) = M(x) + C.$$

This relationship between y and x is an implicit form of the solution to the differential equation. Sometimes (but not always) it is possible to solve for y to find an explicit version of the solution.

Though the technique outlined above is formally correct, what we did essentially amounts to integrating the function n with respect to its variable and integrating the function m with respect to its variable. The informal way to solve a separable equation is to treat the derivative $\frac{dy}{dx}$ as if it were a fraction. The separated form of the equation is

$$n(y)\,dy = m(x)\,dx.$$

To solve, we integrate the left hand side with respect to y and the right hand side with respect to x and add a constant of integration. As long as we are able to find the antiderivatives, we can find an implicit form for the solution. Sometimes we are able to solve for y in the implicit solution to find an explicit form of the solution to the differential equation. We practice the technique by solving the three differential equations listed in the separable column above, and conclude by revisiting and finding the general solution to the logistic differential equation from section 1.1

Notes:

Example 11 **Solving a Separable Differential Equation**

Find the general solution to the differential equation $\frac{dy}{dx} = x^2y$.

Solution Using the informal solution method outlined above, we treat $\frac{dy}{dx}$ as a fraction, and write the separated form of the differential equation as

$$\frac{dy}{y} = x^2 dx.$$

Integrating the left hand side of the equation with respect to y and the right hand side of the equation with respect to x yields

$$\ln|y| = \frac{1}{3}x^3 + C.$$

This is an implicit form of the solution to the differential equation. Solving for y yields an explicit form for the solution. Exponentiating both sides, we have

$$|y| = e^{\frac{x^3}{3}+C} = e^{\frac{x^3}{3}}e^C.$$

This solution is a bit problematic. First, the absolute value makes the solution difficult to understand. The second issue comes from our desire to find the *general solution.* Recall that a general solution includes all possible solutions to the differential equation. In other words, for any given initial condition, the general solution must include the solution to that specific initial value problem. We can often satisfy any given initial condition by choosing an appropriate C value. When solving separable equations, though, it is possible to lose solutions that have the form $y =$ constant. Notice that $y = 0$ solves the differential equation, but it is not possible to choose a finite C to make our solution look like $y = 0$. Our solution cannot solve the initial value problem $\frac{dy}{dx} = x^2y$, with $y(a) = 0$ (where a is any value). Thus, we haven't actually found a general solution to the problem. We can clean up the solution and recover the missing solution with a bit of clever thought.

Recall the formal definition of the absolute value: $|y| = y$ if $y \geq 0$ and $|y| = -y$ if $y < 0$. Our solution is either $y = e^Ce^{\frac{x^3}{3}}$ or $y = -e^Ce^{\frac{x^3}{3}}$. Further, note that C is constant, so e^C is also constant. If we write our solution as $y = Ce^{\frac{x^3}{3}}$, and allow C to take on both positive and negative values, we incorporate both cases of the absolute value. Finally, if we allow C to be zero, we recover the missing solution discussed above. The best way to express the general solution to our differential equation is

$$y = Ce^{\frac{x^3}{3}}.$$

Note: The indefinite integrals $\int \frac{dy}{y}\,dy$ and $\int x^2\,dx$ both produce arbitrary constants. Since both constants are arbitrary, we combine them into a single constant of integration.

Note: Missing constant solutions can't always be recovered by cleverly redefining the arbitrary constant. The differential equation $\frac{dy}{dx} = y^2 - 1$ is an example of this fact. Both $y = 1$ and $y = -1$ are constant solutions to this differential equation. Separation of variables yields a solution where $y = 1$ can be attained by choosing an appropriate C value, but $y = -1$ can't. The general solution is the set containing the solution produced by separation of variables *and* the missing solution $y = -1$. We should always be careful to look for missing constant solutions when seeking the general solution to a separable differential equation.

Notes:

Example 12 **Solving a Separable Initial Value Problem**

Solve the initial value problem $y\sqrt{y^2-5}\,\dfrac{dy}{dx} - \sin x \cos x = 0$, with $y(0) = -3$.

Solution We first put the differential equation in separated form

$$y\sqrt{y^2-5}\,dy = \sin x \cos x\,dx.$$

The indefinite integral $\displaystyle\int y\sqrt{y^2-5}\,dy$ requires the substitution $u = y^2 - 5$. Using this substitute yields the antiderivative $\dfrac{1}{3}(y^2-5)^{3/2}$. The indefinite integral $\displaystyle\int \sin x \cos x\,dx$ requires the substitution $u = \sin x$. Using this substitution yields the antiderivative $\dfrac{1}{2}\sin^2 x$. Thus, we have an implicit form of the solution to the differential equation given by

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + C.$$

The initial condition says that y should be -3 when x is 0, or

$$\frac{1}{3}((-3)^2-5)^{3/2} = \frac{1}{2}\sin^2 0 + C.$$

This is $C = 8/3$, yielding the particular solution to the initial value problem

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + \frac{8}{3}.$$

Example 13 **Solving a Separable Differential Equation**

Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{(x^2+1)e^y}{y}$.

Solution We start by observing that there are no constant solutions to this differential equation because there are no constant y values that make the right hand side of the equation identically zero. Thus, we need not worry about losing solutions during the separation of variables process. The separated form of the equation is given by

$$ye^{-y}\,dy = (x^2+1)\,dx.$$

The antiderivative of the left hand side requires integration by parts. Evaluating both indefinite integrals yields the implicit solution

$$-(y+1)e^{-y} = \frac{1}{3}x^3 + x + C.$$

Since we cannot solve for y, we cannot find an explicit form of the solution.

Notes:

Example 14 **Solving the Logistic Differential Equation**

Solve the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$

Solution We looked at a slope field for this equation in section 1.1 in the specific case of $k = M = 1$. Here, we use separation of variables to find an analytic solution to the more general equation. Notice that the independent variable t does not explicitly appear in the differential equation. We mentioned that an equation of this type is called *autonomous*. All autonomous differential equation are separable.

We start by making the observation that both $y = 0$ and $y = M$ are constant solutions to the differential equation. We must check that these solutions are not lost during the separation of variables process. The separated form of the equation is

$$\frac{1}{y\left(1 - \frac{y}{M}\right)}\,dy = k\,dt.$$

The antiderivative of the left hand side of the equation can be found by making use of partial fractions. Using the techniques discussed in section **??**, we write

$$y\left(1 - \frac{y}{M}\right) = \frac{1}{y} + \frac{1}{M - y}.$$

Then an implicit form of the solution is given by

$$\ln|y| - \ln|M - y| = kt + C.$$

Combining the logarithms,

$$\ln\left|\frac{y}{M - y}\right| = kt + C.$$

Similarly to example 11, we can write

$$\frac{y}{M - y} = Ce^{kt}.$$

Letting C take on positive values or negative values incorporates both cases of the absolute value. This is another implicit form of the solution. Solving for y gives the explicit form

$$y = \frac{M}{1 + be^{-kt}},$$

where b is an arbitrary constant. Notice that $b = 0$ recovers the constant solution $y = M$. The constant solution $y = 0$ cannot be produced with a finite b value, and has been lost. The general solution the logistic differential equation is the set containing $y = \frac{M}{1 + be^{-kt}}$ and $y = 0$.

Note: Solving for y initially yields the explicit solution $y = \frac{CMe^{kt}}{1 + Cekt}$. Dividing numerator and denominator by Ce^{kt} and defining $b = 1/C$ yields the commonly presented form of the solution given in example 14.

Notes:

1.3 First Order Linear Differential Equations

In the previous section, we explored a specific techique to solve a specific type of differential equation; a separable differential equation. In this section, we develop and practice a technique to solve a type of differential equation called a *first order linear* differential equation.

Recall than an algebraic equation in one variable is one that can be written $ax + b = 0$, where a and b are real numbers. Notice that the variable x appears to the first power. The equations $\sqrt{x} + 1 = 0$ and $\sin(x) - 3x = 0$ are both nonlinear. A linear differential equation is one in which the dependent variable and its derivatives appear only to the first power. We focus on first order equations, which first (but not higher order) derivatives of the independent variable.

Definition 5 First Order Linear Differential Equation

A **first order linear differential equation** is a differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

where p and q are arbitrary functions of the independent variable x.

Example 15 Classifying Differential Equations

Classify the followind differential equations as first order linear, separable, both, or neither.

(a) $\dfrac{dy}{dx} = xy$

(b) $\dfrac{dy}{dx} = e^y + 3x$

(c) $\dfrac{dy}{dx} - (\cos x)y = \cos x$

(d) $y\dfrac{dy}{dx} - 3xy = 4\ln x$

Solution (a) Both. We identify $p(x) = -x$ and $q(x) = 0$. The separated form of the equation is $\dfrac{dy}{y} = x\,dx$.

(b) Neither. The e^y term makes the equation nonlinear. Because of the addition, it is not possible to write the equation in separated form.

(c) First order linear. We identify $p(x) = -\cos x$ and $q(x) = \sin x$. The equation cannot be written in separated form.

(d) Neither. Notice that dividing by y results in the nonlinear term $\dfrac{4\ln x}{y}$. It is not possible to write the equation in separated form.

Notes:

Notice that linearity depends on the dependent variable y, not the independent variable x. The functions $p(x)$ and $q(x)$ need not be linear as demonstrated in part (c) of example 15. Neither $\cos x$ nor $\sin x$ are linear functions of x, but the differential equation is still linear.

Solving First Order Linear Equations

We motivate the solution technique by way of an observation and an example. We first observe that the expression $\frac{d}{dx}(xy)$ can be expanded via the product rule and implicit differentiation to the expression $x\frac{dy}{dx}+y$. Now we look at an example. Consider the first order linear differential equation

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{\sin x\cos x}{x}.$$

If we multiply both sides of the differential equation by x and use our observation, we see that the differential equation can be written

$$\frac{d}{dx}(xy)=\sin x\cos x.$$

Integrating both sides of the equation with respect to x and using the substitution $u=\sin x$ on the right results in the implicit solution

$$xy=\frac{1}{2}\sin^2 x+C.$$

Solving for y yields the explicit solution

$$y=\frac{\sin^2 x}{2x}+\frac{C}{x}.$$

Note: In the examples in the previous section, we performed operations on the arbitrary constant C, but still called the result C. The justification is that the result after the operation is *still* an arbitrary contant. Here, we divide C by x, so the result depends explicitly on the independent variable x. Since C/x is *not* contant, we can't just call it C.

As motivated by the problem we just solved, the basic idea behing solving first order linear differential equations is to multiply both sides of the differential equation by a function, called an *integrating factor*, that makes the left hand side of the equation look like an expanded product rule. We then condense the left hand side into the derivative of a product and integrate both sides. An obvious question is how to find the integrating factor.

Consider the first order linear equation

$$\frac{dy}{dx}+p(x)y=q(x).$$

Let's call the integrating factor $\mu(x)$. We multiply both sides of the differential equation by $\mu(x)$ to get

$$\mu(x)\left(\frac{dy}{dx}+p(x)y\right)=\mu(x)q(x).$$

Notes:

Our goal is to choose $\mu(x)$ so that the left hand side of the differential equation looks like the result of a product rule. The left hand side of the equation is

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y,$$

and using the product rule and implicit differentiation,

$$\frac{d}{dx}\big(\mu(x)y\big) = \frac{d\mu}{dx}y + \mu(x)\frac{dy}{dx}.$$

Equating these two gives

$$\frac{d\mu}{dx}y + \mu(x)\frac{dy}{dx} = \mu(x)\frac{dy}{dx} + \mu(x)p(x)y,$$

or

$$\frac{d\mu}{dx} = \mu(x)p(x).$$

In order for the integrating factor $\mu(x)$ to perform its job, it must solve the differential equation above. But that differential equation is separable, so we can solve it. The separated form is

$$\frac{d\mu}{\mu} = p(x)\,dx.$$

Integrating,

$$\ln\mu = \int p(x)\,dx,$$

or

$$\mu(x) = e^{\int p(x)\,dx}.$$

Note: Following the steps outlined in the previous section, we should technically end up with $\mu(x) = Ce^{\int p(x)\,dx}$, where C is an arbitrary constant. Because we multiply both sides of the differential equation by $\mu(x)$, the arbitrary constant cancels, and we omit it when finding the integrating factor.

If $\mu(x)$ is chosen this way, the differential equation can be written in the form

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$

Integrating and solving for y, the explicit solution is

$$y = \frac{1}{\mu(x)}\int \big(\mu(x)q(x)\big)\,dx.$$

Though this formula can be used to write down the solution to a first order linear equation, the process is lost by simply memorizing a formula, and always always follow the steps outlined in key idea 2 when solving equations of this type.

Notes:

Key Idea 2 Solving First Order Linear Equations

1. Write the differential equation in the form
$$\frac{dy}{dx} + p(x)y = q(x).$$
2. Compute the integrating factor
$$\mu(x) = e^{\int p(x)\,dx}.$$
3. Multiply both sides of the differential equation by $\mu(x)$, and condense the left hand side to get
$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$
4. Integrate both sides of the differential equation with respect to x, taking care to remember the arbitrary constant.
5. Solve for y to find the explicit solution to the differential equation.

Let's practice the process by solving the two first order linear differential equations from example 15.

Example 16 Solving a First Order Linear Equation

Find the general solution to $\dfrac{dy}{dx} = xy$.

Solution We solve by following the steps in key idea 2. Unlike the process for solving separable equations, we need not worry about losing constant solutions. The answer we find *will* be the general solution to the differential equation. We first write the equation in the form
$$\frac{dy}{dx} - xy = 0.$$
By identifying $p(x) = -x$, we can compute the integrating factor
$$\mu(x) = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}.$$
Multiplying both side of the differential equation by $\mu(x)$, we have
$$e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right) = 0.$$

Notes:

Note: The step where the left hand side of the differential equation condenses to the derivative of a product can feel a bit magical. The reality is that we choose $\mu(x)$ so that we can get exactly this condensing behavior. It's not magic, it's math! If you're still skeptical, try using the product rule and implicit differentiation to evaluate $\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right)$, and verify that it becomes $e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx}-xy\right)$.

The left hand side of the differential equation condenses to yield

$$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right)=0.$$

We integrate both sides with respect to x to find

$$e^{-\frac{1}{2}x^2}y=C,$$

or the explicit solution

$$y=Ce^{-\frac{1}{2}x^2}.$$

Example 17 **Solving a First Order Linear Equation**

Find the general solution to $\frac{dy}{dx}-(\cos x)y=\cos x$.

Solution The differential equation is already in the correct form. The integrating factor is given by

$$\mu(x)=e^{-\int\cos x\,dx}=e^{-\sin x}.$$

Multiplying both sides of the equation by the integrating factor and condensing,

$$\frac{d}{dx}\left(e^{-\sin x}y\right)=(\cos x)e^{-\sin x}$$

Using the substitution $u=-\sin x$, we can integrate to find the implicit solution

$$e^{-\sin x}y=-e^{-\sin x}+C.$$

The explicit form of the general solution is

$$y=-1+Ce^{\sin x}.$$

Example 18 **Solving a First Order Linear Initial Value Problem**

Solve the initial value problem $x\frac{dy}{dx}-y=x^3\ln x$, with $y(1)=0$.

Solution We first divide by x to get

$$\frac{dy}{dx}-\frac{1}{x}y=x^2\ln x.$$

Notes:

The integrating factor is given by

$$\begin{aligned} \mu(x) &= e^{-\int \frac{1}{x}\,dx} \\ &= e^{-\ln x} \\ &= e^{\ln x^{-1}} \\ &= x^{-1}. \end{aligned}$$

Multiplying both sides of the differential equation by the integrating factor and condensing the left hand side, we have

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x\ln x.$$

Using integrating by parts to fidnd the antiderivative of $x\ln x$, we find the implicit solution

$$\frac{y}{x} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

Solving for y, the explicit solution is

$$y = \frac{1}{2}x^3\ln x - \frac{1}{4}x^3 + Cx.$$

The initial condition $y(1) = 0$ yields $C = 1/4$. The solution to the initial value problem is

$$y = \frac{1}{2}x^3\ln x - \frac{1}{4}x^3 + \frac{1}{4}x.$$

Differential equations are a valuable took for exploring various problems in physics. This process of using equations to describe real world situations is called mathematical modeling, and is the topic of the next section. The last two examples in this section begin our discussion of mathematical modeling.

Example 19 A Falling Object Without Air Resistance

Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object assuming no air resistance.

SOLUTION The basic physical law at play is Newton's second law,

force = mass $\times$ acceleration.

Using the fact that acceleration is the derivative of velocity, mass $\times$ acceleration can be writting mv'. In the absence of air resistance, the only force of interest in

Notes:

the force due to gravity. This force is approximately constant, and is given by mg, where g is the gravitational constant. The word equation above can be written as the differential equation

$$m\frac{dv}{dt} = mg.$$

Because g is constant, this differential equation is simply an integration problem, and we find

$$v = gt + C.$$

Since $v = C$ with $t = 0$, we see that the arbitrary constant here corresponds to the initial vertical velocity of the object.

The process of mathematical modeling does not stop simply because we have found an answer. We must examine the answer to see how well it can describe real world observations. In the previous example, the answer may be somewhat useful for short times, but intuition tells us that something is missing. Our answer says that a falling object's velocity will increase linearly as a function of time, but we know that a falling object will eventually approach a maximum velocity. In order to describe this behaviour, our mathematical model must be revised.

Example 20 **A Falling Object with Air Resistance**

Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object, taking air resistance into account.

Solution We still begin with Newon's second law, but now we assume that the forces in the object come both from gravity and from air resistance. The gravitational force is still given by mg. For air resistance, we assume the force is related to the velocity of the object. A simple way to describe this assumption might be kv^p, where k is a proportionality constant and p is a positive real number. The differential equation for the velocity is given by

$$m\frac{dv}{dt} = mg - kv^p.$$

(Notice that the force from air resistance opposes motion, and points in the opposite direction as the force from gravity.) This differential equation is separable, and can be written in the separated form

$$\frac{m}{mg - kv^p}\,dv = dt.$$

For arbitrary positive p, the integration is difficult, making this problem hard to solve analytically. In the case that $p = 1$, the differential equation becomes

Notes:

linear, and is easy to solve either using the separation of variables or integrating factor techniques. We assume $p = 1$, and proceed with an integrating factor. Writing

$$\frac{dv}{dt} + \frac{k}{m}v = g,$$

we identify the integrating factor

$$\mu(t) = e^{\frac{k}{m}t}.$$

Then

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = ge^{\frac{k}{m}t},$$

so

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C,$$

or

$$v = \frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

In the solution above, the exponential term decays as time increases, causing the velocity to approch that constant value mg/k in the limit as t approaches infinity. This value is called the *terminal velocity*.

Notes:

A: Solutions To Selected Problems

Chapter 1

Section 1.1

1. An initial value problems is a differential equation that is paired with one or more initial conditions. A differential equation is simply the equation without the initial conditions.

2. Answers will vary.

3. Substitute the proposed function into the differential equation, and show the the statement is satisfied.

4. A particular solution has no arbitrary constants. A general solution includes all possible solutions to the differential equation, and typically includes one or more arbitrary constants.

5. Many differential equations are impossible to solve analytically.

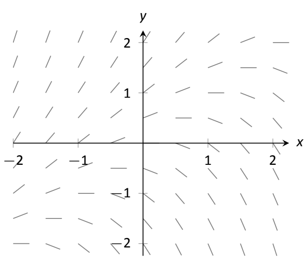
6. Answers will vary.

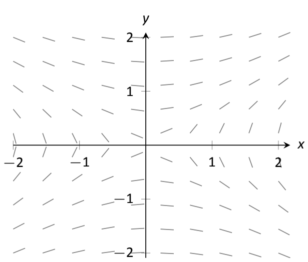
7. Answers will vary.

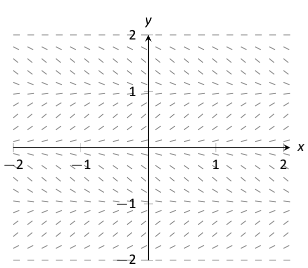
8. Answers will vary.

9. $C = 2$

10. $C = 6$

11. 

12. 

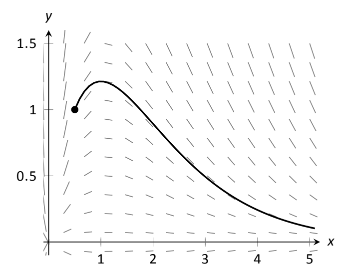
13. 

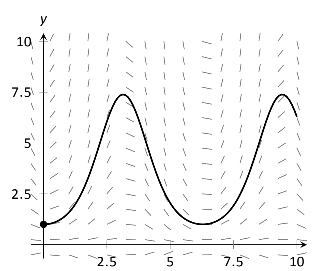
14. b

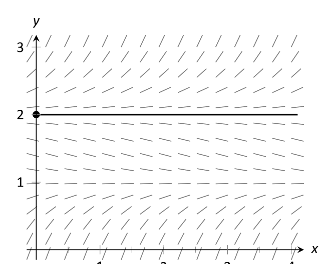
15. c

16. d

17. a

18. 

19. 

20. 

21.

x_i	y_i
0.00	1.0000
0.25	1.5000
0.50	2.3125
0.75	3.5938
1.00	5.5781

22.

x_i	y_i
0.0	1.0000
0.1	1.0000
0.2	1.0037
0.3	1.0110
0.4	1.0219
0.5	1.0363

23.

x_i	y_i
0.0	2.0000
0.2	2.4000
0.4	2.9197
0.6	3.5816
0.8	4.4108
1.0	5.4364

24.

x_i	y_i
0.0	0.0000
0.5	0.5000
1.0	1.8591
1.5	10.5824
2.0	88378.1190

25.

x_i	y_i
1.0	1.0000
1.2	1.2000
1.4	1.3229
1.6	1.3767
1.8	1.3723
2.0	1.3221

26.

x_i	$y(x)$	h=0.2	$h = 0.1$
0.0	1.0000	1.0000	1.0000
0.2	1.0204	1.0000	1.0100
0.4	1.0870	1.0400	1.0623
0.6	1.2195	1.1265	1.1687
0.8	1.4706	1.2788	1.3601
1.0	2.0000	1.5405	1.7129

27.

x_i	$y(x)$	$h = 0.2$	$h = 0.1$
0.0	0.5000	0.5000	0.5000
0.2	0.5412	0.5000	0.5201
0.4	0.6806	0.5816	0.6282
0.6	0.9747	0.7686	0.8622
0.8	1.5551	1.1250	1.3132
1.0	2.7183	1.7885	2.1788