1: Differential Equations

One of the strengths of calculus is its ability to describe real-world phenomena. We have seen hints of this in our discussion of the applications of derivatives and integrals in the previous chapters. The process of formulating an equation or multiple equations to describe a physical phenomenon is called *mathematical modeling.* As a simple example, populations of bacteria are often described as "growing exponentially." Looking in a biology text, we might see $P(t) = P_0e^{kt}$, where $P(t)$ is the bacteria population at time t, P_0 is the initial population at time $t = 0$, and the constant k describes how quickly the population grows. This equation for exponential growth arises from the assumption that the population of bacteria grows at a rate proportional to its size. Recalling that the derivative gives the rate of change of a function, we can describe the growth assumption precisely using the equation $P' = kP$. This equation is called a *differential equation*, and is the subject of the current chapter.

1.1 Graphical and Numerical Solutions to Differential Equations

In section **??**, we were introduced to the idea of a differential equation. Given a function $y = f(x)$, we defined a *differential equation* as an equation involving y, x, and derivatives of y. We explored the simple differential equation $y' = 2x$, and saw that a *solution* to a differential equation is simply a function that satisfies the differential equation.

Introduction and Terminology

Definition 1 Differential Equation

Given a function $y = f(x)$, a **differential equation** is an equation relating x, y, and derivatives of y.

- The variable x is called the **independent variable**.
- The variable y is called the **dependent variable**.
- The **order** of the differential equation is the order of the highest derivative of y.

Let us return to the simple differential equation

$$y' = 2x.$$

To find a solution, we must find a function whose derivative is $2x$. In other words, we seek an antiderivative of $2x$. The function

$$y = x^2$$

is an antiderivative of $2x$, and solves the differential equation. So do the functions

$$y = x^2 + 1$$

and

$$y = x^2 - 2346.$$

We call the function

$$y = x^2 + C,$$

with C an arbitrary constant of integration, the *general solution* to the differential equation.

In order to specify the value of the integration constant C, we require additional information. For example, if we know that $y(1) = 3$, it follows that $C = 2$. This additional information is called an *initial condition.*

Definition 2 Initial Value Problem

A differential equation paired with an initial condition (or initial conditions) is called an **initial value problem.**

The solution to an initial value problem is called a **particular solution**. A particular solution does not include arbitrary constants.

The family of solutions to a differential that encompasses all possible solutions is called the **general solution** to the differential equation. A general solution includes one or more arbitrary constants. The particular solution to an initial value problem is one specific member in the family of solutions.

Example 1 A simple first-order differential equation

Solve the differential equation $y' = 2y$.

Solution The solution is a function y such that differentiation yields twice the original function. Unlike our starting example, finding the solution

Notes:

here does not involve computing an antiderivative. Notice that "integrating both sides" would yield the result $y = \int 2y\,dx$, which is not useful. Without knowledge of the function y, we can't compute the indefinite integral. Later sections will explore systematic ways to find analytic solution to simple differential equations. For now, a bit of though might let us guess the solution

$$y = e^{2x}.$$

Notice that application of the chain rule yields $y' = 2e^{2x} = 2y$. Another solution is given by

$$y = -3e^{2x}.$$

In fact

$$y = Ce^{2x},$$

where C is any constant, is the *general solution* to the differential equation because $y' = 2Ce^{2x} = 2y$.

If we are provided with a single initial condition, say $y(0) = \frac{3}{2}$, we can identify $C = \frac{3}{2}$ so that

$$y = \frac{3}{2}e^{2x}$$

is the *particular solution* to the initial value problem

$$y' = 2y, \text{ with } y(0) = \frac{3}{2}.$$

Figure 1.1 shows various members of the general solution to the differential equation $y' = 2y$. Each C value yields a different member of the family, and a different function. We emphasize the particular solution corresponding to the initial condition $y(0) = \frac{3}{2}$.

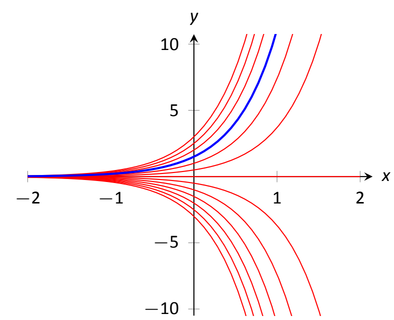

Figure 1.1: A representation of some of the members of general solution to the differential equation $y' = 2y$, including the particular solution to the initial value problem with $y(0) = \dfrac{3}{2}$, from example 1

Example 2 **A second-order differential equation**

Solve the differential equation $y'' + 9y = 0$.

Solution We seek a function such that two derivatives returns negative 9 multiplied by the original function. Both $\sin(3x)$ and $\cos(3x)$ have this feature. The general solution to the differential equation is given by

$$y = C_1 \sin(3x) + C_2 \cos(3x),$$

where C_1 and C_2 are arbitrary constants. To fully specify a particular solution, we require two additional conditions. For example, the initial conditions $y(0) = 1$ and $y'(0) = 3$ yield $C_1 = C_2 = 1$.

Notes:

The differential equation in example 2 is second order because the equation involves a second derivative. In general, the number of initial conditions required to specify a particular solution depends on the order of the differential equation. For the remainder of the chapter, we restrict our attention to first order differential equations and first order initial value problems.

Example 3 **Verifying a solution to the differential equation**
Which of the following is a solution to the differential equation

$$y' + \frac{y}{x} - \sqrt{y} = 0?$$

a) $y = C\left(1 + \ln x\right)^2$ b) $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ c) $y = Ce^{-3x} + \sqrt{\sin x}$

Solution Verifying a solution to a differential equation is simply an exercise in differentiation and simplification. We substitute each potential solution into the differential equation to see if it satisfies the equation.

a) Testing the potential solution $y = C\left(1 + \ln x\right)^2$:

Differentiating, we have $y' = \dfrac{2C(1 + \ln x)}{x}$. Substituting into the differential equation,

$$\begin{aligned}
&\frac{2C(1+\ln x)}{x} + \frac{C(1+\ln x)^2}{x} - \sqrt{C}(1+\ln x) \\
&= (1+\ln x)\left(\frac{2C}{x} + \frac{C(1+\ln x)}{x} - \sqrt{C}\right) \\
&\neq 0.
\end{aligned}$$

Since it doesn't satisfy the differential equation, $y = C(1+\ln x)^2$ is *not* a solution.

b) Testing the potential solution $y = \left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)^2$:

Differentiating, we have $y' = 2\left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)\left(\dfrac{1}{3} - \dfrac{C}{2x^{3/2}}\right)$. Substituting into the differential equation,

$$\begin{aligned}
&2\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{1}{3} - \frac{C}{2x^{3/2}}\right) + \frac{1}{x}\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2 - \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right) \\
&= \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{2}{3} - \frac{C}{x^{3/2}} + \frac{1}{3} + \frac{C}{x^{3/2}} - 1\right) \\
&= 0.
\end{aligned}$$

Notes:

Thus $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ *is* a solution to the differential equation.

c) Testing the potential solution $y = Ce^{-3x} + \sqrt{\sin x}$:

Differentiating, $y' = -3Ce^{-3x} + \frac{\cos x}{2\sqrt{\sin x}}$. Substituting into the differential equation,

$$-3Ce^{-3x} + \frac{\cos x}{2\sqrt{\sin x}} + \frac{Ce^{-3x} + \sqrt{\sin x}}{x} - \sqrt{Ce^{-3x} + \sqrt{\sin x}} \neq 0.$$

The function $y = Ce^{-3x} + \sqrt{\sin x}$ is *not* a solution to the differential equation.

Example 4 Verifying a Solution to a Differential Equation

Verify that $x^2 + y^2 = Cy$ is a solution to $y' = \frac{2xy}{x^2 - y^2}$.

Solution The solution in this example is called an *implicit solution.* That means the dependent variable y is a function of x, but has not been explicitly solved for. Verifying the solution still involves differentiation, but we must take the derivatives implicitly. Differentiating, we have

$$2x + 2yy' = Cy'.$$

Solving for y', we have

$$y' = \frac{2x}{C - 2y}.$$

From the solution, we know that $C = \frac{x^2 + y^2}{y}$. Then

$$\begin{aligned} y &= \frac{2x}{\frac{x^2 + y^2}{y} - 2y} \\ &= \frac{2xy}{x^2 + y^2 - 2y^2} \\ &= \frac{2xy}{x^2 - y^2}. \end{aligned}$$

We have verified that $x^2 + y^2 = Cy$ is a solution to $y' = \frac{2xy}{x^2 - y^2}$.

Graphical Solutions to Differential Equations

The solutions to the differential equations we have explored so far are called *analytic solutions.* We have found exact forms for the functions that solve the

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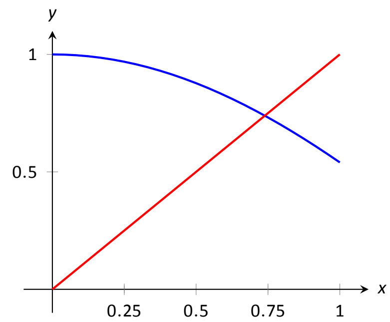

Figure 1.2: Graphically finding an approximate solution to $\cos x = x$.

differential equations. Many times a differential equation will have a solution, but it is difficult or impossible to find the solution analytically. This is analogous to algebraic equations. The algebraic equation $x^2 + 3x - 1 = 0$ has two real solutions that can be found analytically by using the quadratic formula. The equation $\cos x = x$ has one real solution, but we can't find it analytically. As shown in figure 1.2, we can find an approximate solution graphically by plotting $\cos x$ and x and observing the x-value of the intersection. We can similarly use graphical tools to understand the qualitative behavior of solutions to a first order-differential equation.

Consider the first-order differential equation

$$y' = f(x, y).$$

The function f could be any function of the two variables x and y. Written in this way, we can think of the function f as providing a formula to find the slope of a solution at a given point in the xy-plane. In other words, suppose a solution to the differential equation passes through the point (x_0, y_0). At the point (x_0, y_0), the slope of the solution curve will be $f(x_0, y_0)$. Since this calculation of the slope is possible at any point (x, y) where the function $f(x, y)$ is defined, we can produce a plot called a *slope field* (or direction field) that shows the slope of a solution at any point in the xy-plane where the solution is defined. Further, this process can be done purely by working with the differential equation itself. In other words, we can draw a slope field and use it to determine the qualitative behavior of solutions to a differential equation without having to solve the differential equation.

Definition 3 Slope Field

A **slope field** for a first-order differential equation $y' = f(x, y)$ is a plot in the xy-plane made up of short line segments or arrows. For each point (x_0, y_0) where $f(x, y)$ is defined, the slope of the line segment is given by $f(x_0, y_0)$. Plots of solutions to a differential equation are tangent to the line segments in the slope field.

Example 5 Finding a slope field

Find a slope field for the differential equation $y' = x + y$.

Solution Because the function $f(x, y) = x + y$ is defined for all points (x, y), every point in the xy-plane has an associated line segment. It is not practical to draw an entire slope field by hand, but many tools exist for drawing slope fields on a computer. Here, we explicitly calculate and plot a few of the line segments in the slope field.

Notes:

- The slope of the line segment at $(0,0)$ is $f(0,0) = 0+0 = 0$.
- The slope of the line segment at $(1,1)$ is $f(1,1) = 1+1 = 2$.
- The slope of the line segment at $(1,-1)$ is $f(1,-1) = 1-1 = 0$.
- The slope of the line segment at $(-2,3)$ is $f(-2,-1) = -2-1 = -3$.

Though it is possible to continue this process to sketch a slope field, we usually use a computer to make the drawing. Most popular computer algebra systems can draw slope fields. There are also various online tools that can make the drawings. The slope field for $y' = x+y$ is shown in figure 1.3.

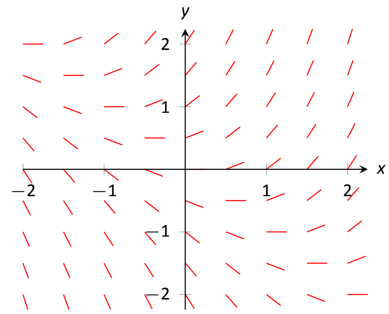

Figure 1.3: Slope field for $y' = x+y$ from example 5.

Example 6 **Finding a graphical solution to an initial value problem**
Find a graphical solution to the initial value problem $y' = x+y$, with $y(1) = -1$.

Solution The solution to the initial value problem should be a continuous smooth curve. Using the slope field, we can draw of a sketch of the solution using the following two criteria:

1. The solution must pass through the point $(1,-1)$.
2. When the solution passes through a point (x_0, y_0) it must be tangent to the line segment at (x_0, y_0).

Essentially, we sketch a solution to the initial value problem by starting at the point $(1,-1)$ and "following the lines" in either direction. A sketch of the solution is shown in figure 1.4.

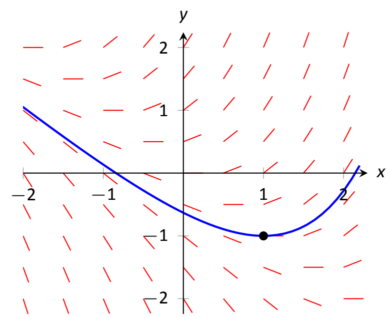

Figure 1.4: Solution to the initial value problem $y' = x+y$, with $y(1) = -1$ from example 6

Example 7 **Using a slope field to predict long term behavior**
Use the slope field for the differential equation $y' = y(1-y)$, shown in figure 1.5, to predict long term behavior of solutions to the equation.

Solution This differential equation, called the *logistic differential equation*, often appears in population biology to describe the size of a population. For that reason, we use t (time) as the independent variable instead of x. We also often restrict attention to non-negative y-values because negative values correspond to a negative population.

Looking at the slope field in figure 1.5, we can predict long term behavior for a given initial condition.

- If the initial y-value is negative ($y(0) < 0$), the solution curve must pass though the point $(0, y(0))$ and follow the slope field. We expect the solution y to become more and more negative as time increases. Note that this result is not physically relevant when considering a population.

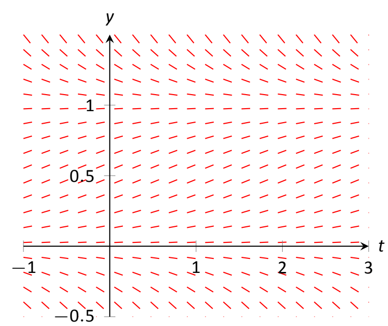

Figure 1.5: Slope field for the logistic differential equation $y' = y(1-y)$ from example 7.

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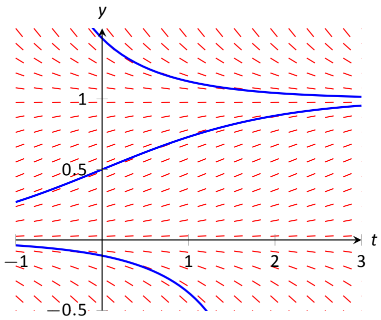

Figure 1.6: Slope field for the logistic differential equation $y' = y(1-y)$ from example 7 with a few representative solution curves.

- If the initial y-value is greater than 0 but less than 1, we expect the solution y to increase and level off at $y = 1$.
- If the initial y-value is greater than 1, we expect the solution y to decrease and level off at $y = 1$.

The slope field for the logistic differential equation, along with representative solution curves, is shown in figure 1.6. Notice that any solution curve with positive initial value will tend towards the value $y = 1$. We call this the *carrying capacity.*

Numerical Solutions to Differential Equations: Euler's Method

While the slope field is an effective way to understand the qualitative behavior of solutions to a differential equation, it is difficult to use a slope field to make quantitative predictions. For example, if we have the slope field for the differential equation $y' = x + y$ from example 5 along with the initial condition $y(0) = 1$, we can understand the qualitative behavior of the solution to the initial value problem, but will struggle to predict a specific value, $y(2)$ for example, with any degree of confidence. The most straight forward way to predict $y(2)$ is to find the analytic solution to the the initial value problem and evaluate it at $x = 2$. Unfortunately, we have already mentioned that it is impossible to find analytic solutions to many differential equations. In the absence of an analytic solution, a *numerical solution* can serve as an effective tool to make quantitative predictions about the solution to an initial value problem.

There are many techniques for computing numerical solutions to initial value problems. A course in numerical analysis will discuss various techniques along with their strengths and weaknesses. The simplest technique is called *Euler's Method* (pronounced "oil-er," not "you-ler"). Consider the first-order initial value problem

$$y' = f(x, y), \text{ with } y(x_0) = y_0.$$

Using the definition of the derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

If we remove the limit but restrict h to be "small," we have

$$y'(x) \approx \frac{y(x+h) - y(x)}{h},$$

so that

$$f(x, y) \approx \frac{y(x+h) - y(x)}{h},$$

Notes:

because $y' = f(x, y)$ according to the differential equation. Rearranging terms,

$$y(x + h) \approx y(x) + hf(x, y).$$

This statement says that if we know the solution (y-value) to the initial value problem for some given x-value, we can find an approximation for the solution at the value $x + h$ by taking our y-value and adding h times the function f evaluated at the x and y values. Euler's method uses the initial condition of an initial value problem as the starting point, and then uses the above idea to find approximate values for the solution y at later x-values. The algorithm is summarized in key idea 1.

Key Idea 1 Euler's Method

Consider the initial value problem

$$y' = f(x, y) \text{ with } y(x_0) = y_0.$$

Let h be a small positive number and N be an integer.

1. For $i = 0, 1, 2, \ldots, N$, define
$$x_i = x_0 + ih.$$
2. The value y_0 is given by the initial condition.
For $i = 0, 1, 2, \ldots, N - 1$, define
$$y_{i+1} = y_i + hf(x_i, y_i).$$

This process yields a sequence of $N+1$ points (x_i, y_i) for $i = 0, 1, 2, \ldots, N$, where (x_i, y_i) is an approximation for $(x_i, y(x_i))$.

Let's practice Euler's Method using a few concrete examples.

Example 8 Using Euler's Method 1

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.5$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.5$, it takes $N = 2$ steps to get to $x = 2$. Using steps 1 and

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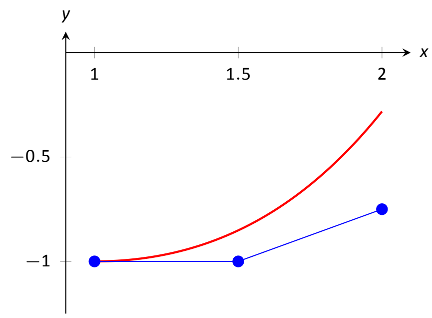

Figure 1.7: Euler's Method approximation to $y' = x + y$ with $y(1) = -1$ from example 8, along with the analytical solution to the initial value problem.

2 from the Euler's Method algorithm,

x_0	$= 1$	y_0	$= -1$
x_1	$= x_0 + h$ $= 1 + 0.5$ $= 1.5$	y_1	$= y_0 + hf(x_0, y_0)$ $= -1 + 0.5(1 - 1)$ $= -1$
x_2	$= x_0 + 2h$ $= 1 + 2(0.5)$ $= 2$	y_2	$= y_1 + hf(x_1, y_1)$ $= -1 + 0.5(1.5 - 1)$ $= -0.75$

Using Euler's method, we find the approximate $y(2) \approx -0.75$.

To help visualize the Euler's method approximation, these three points (connected by line segments) are plotted along with the analytical solution to the initial value problem in figure 1.7.

Let's repeat the previous example using a smaller h-value.

Example 9 Using Euler's Method 2

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.25$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.25$, it takes $N = 4$ steps to get to $x = 2$. Using steps 1 and 2 from the Euler's Method algorithm (and rounding to 4 decimal points), we have

x_0	$= 1$	y_0	$= -1$
x_1	$= 1.25$	y_1	$= -1 + 0.25(1 - 1)$ $= -1$
x_2	$= 1.5$	y_2	$= -1 + 0.25(1.25 - 1)$ $= -0.9375$
x_3	$= 1.75$	y_3	$= -0.9375 + 0.25(1.5 - 0.9375)$ $= -0.7969$
x_4	$= 2$	y_4	$= -0.7969 + 0.25(1.75 - 0.7969)$ $= -0.5586$

Using Euler's method, we find the approximate $y(2) \approx -0.5584$.

These five points, along with the points from example 8 and the analytic solution, are plotted in figure 1.8.

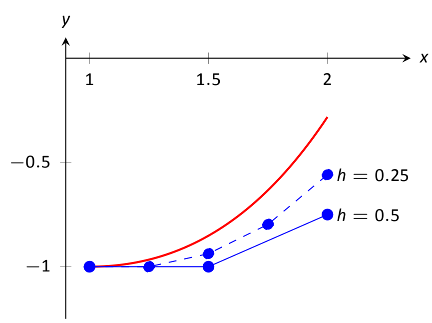

Figure 1.8: Euler's Method approximations to $y' = x + y$ with $y(1) = -1$ from examples 8 and 9, along with the analytical solution.

Using the results from examples 8 and 9, we can make a few observations about Euler's method. First, the Euler approximation gets successively worse as we get farther from the initial condition. This is because Euler's method involves two sources of error. The first comes from the fact that we're using a positive

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h-value in the derivative approximation instead of using a limit as h approaches zero. Essentially, we're using a linear approximation to the solution y (similar to the process described in section **??** on differentials.) This error is often called the *local truncation error.* The second source of error comes from the fact that every step in Euler's method uses the result of the previous step. That means we're using an approximate y-value to approximate the next y-value. Doing this repeatedly causes the errors to build on each other. This second type of error is often called the *propagated* or *accumulated error.* A second observation is that the Euler approximation is more accurate for smaller h-values. This accuracy comes at a cost, though. Example 9 is more accurate than example 8, but takes twice as many computations. In general, numerical algorithms (even when performed by a computer program) require striking a balance between a desired level of accuracy and the amount of computational effort we are willing to undertake.

Let's do one final example of Euler's Method.

Example 10 Using Euler's Method 3

Find an approximation for the solution to the logistic differential equation $y' = y(1-y)$ with $y(0) = 0.25$, for $0 \le y \le 4$. Use $N = 10$ steps.

Solution The logistic differential equation is what is called an *autonomous equation*. An autonomous differential equation has no explicit dependence on the independent variable (t in this case). This has no real effect on the application of Euler's method other than the fact that the function $f(t, y)$ is really just a function of y. To take steps in the y variable, we use

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + hy_i(1 - y_i).$$

Using $N = 10$ steps requires $h = \dfrac{4-0}{10} = 0.4$. Implementing Euler's Method, we have

x_0	$= 0$	y_0	$= 0.25$
x_1	$= 0.4$	y_1	$= 0.25 + 0.4(0.25)(1 - 0.25)$ $= 0.325$
x_2	$= 0.8$	y_2	$= 0.325 + 0.4(0.325)(1 - 0.325)$ $= 0.41275$
x_3	$= 1.2$	y_3	$= 0.41275 + 0.4(0.41275)(1 - 0.41275)$ $= 0.50970$
x_4	$= 1.6$	y_4	$= 0.50970 + 0.4(0.50970)(1 - 0.50970)$ $= 0.60966$
x_5	$= 2.0$ $=$	y_5	$= 0.60966 + 0.4(0.60966)(1 - 0.60966)$ $= 0.70485$

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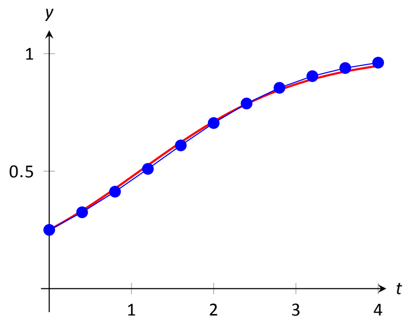


Figure 1.9: Euler's Method approximation to $y' = y(1-y)$ with $y(0) = 0.25$ from example 10, along with the analytical solution.

x_6	$= 2.4$	y_6	$= 0.70485 + 0.4(0.70485)(1 - 0.70485)$ $= 0.78806$
x_7	$= 2.8$	y_7	$= 0.78806 + 0.4(0.78806)(1 - 0.78806)$ $= 0.85487$
x_8	$= 3.2$	y_8	$= 0.85487 + 0.4(0.85487)(1 - 0.85487)$ $= 0.90450$
x_9	$= 3.6$	y_9	$= 0.90450 + 0.4(0.90450)(1 - 0.90450)$ $= 0.93905$
x_{10}	$= 4.0$	y_{10}	$= 0.93905 + 0.4(0.93905)(1 - 0.93905)$ $= 0.96194$

These 11 points, along with the the analytic solution, are plotted in figure 1.9.

The study of differential equations is a natural extension of the study of derivatives and integrals. The equations themselves involve derivatives, and methods to find analytic solutions often involve finding antiderivatives. In this section, we focus on graphical and numerical techniques to understand solutions to differential equations. We restrict our examples to relatively simple initial value problems that permit analytic solution to the equations, but should remember that this is only for comparison purposes. In reality, many differential equations, even some that appear straight forward, do not have solutions we can find analytically. Even so, we can use the techniques presented in this section to understand the behavior of solutions. In the next two sections, we explore two techniques to find analytic solutions to two different classes of differential equations.

Notes:

Exercises 1.1

Terms and Concepts

1. In your own words, what is an initial value problem, and how is it different than a differential equation?

2. In your own words, describe what it means for a function to be a solution to a differential equation.

3. How can we verify that a function is a solution to a differential equation?

4. Describe the difference between a particular solution and a general solution.

5. Why might we use a graphical or numerical technique to study solutions to a differential equation instead of simply solving the differential equation to find an analytic solution?

6. Describe the considerations that should be made when choosing an h value to use in a numerical method like Euler's Method.

Problems

In Exercises 7 – 10, verify that the given function is a solution to the differential equation or initial value problem.

7. $y = Ce^{-6x^2}$; $y' = -12xy$.

8. $y = x\sin x$; $y' - x\cos x = (x^2+1)\sin x - xy$, with $y(\pi) = 0$.

9. $2x^2 - y^2 = C$; $yy' - 2x = 0$

10. $y = xe^x$; $y'' - 2y' + y = 0$

In Exercises 11 – 12, verify that the given function is a solution to the differential equation and find the C value required to make the function satisfy the initial condition.

11. $y = 4e^{3x}\sin x + Ce^{3x}$; $y' - 3y = 4e^{3x}\cos x$, with $y(0) = 2$

12. $y(x^2 + y) = C$; $2xy + (x^2 + 2y)y' = 0$, with $y(1) = 2$

In Exercises 13 – 16, sketch a slope field for the given differential equation.

13. $y' = y - x$

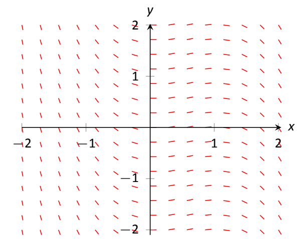
14. $y' = \dfrac{x}{2y}$

15. $y' = \sin(\pi y)$

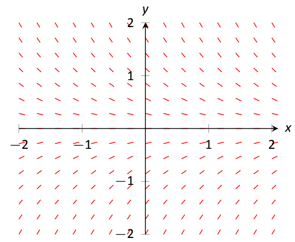
16. $y' = \frac{y}{4}$

In Exercises 17 – 20, match the slope field with the appropriate differential equation.

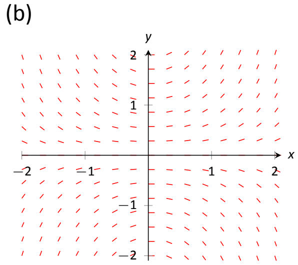
(a)



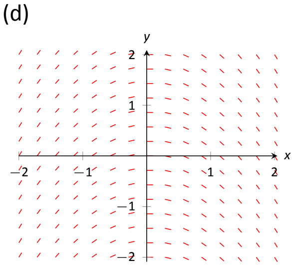
(c)



(b)



(d)



17. $y' = xy$

18. $y' = -y$

19. $y' = -x$

20. $y' = x(1 - x)$

In Exercises 21 – 24, sketch the slope field for the differential equation, and use it to draw a sketch of the solution to the initial value problem.

21. $y' = \dfrac{y}{x} - y$, with $y(0.5) = 1$.

22. $y' = y\sin x$, with $y(0) = 1$.

23. $y' = y^2 - 3y + 2$, with $y(0) = 2$.

24. $y' = -\dfrac{xy}{1 + x^2}$, with $y(0) = 1$.

In Exercises 25 – 28, use Euler's Method to make a table of values that approximates the solution to the initial value problem on the given interval. Use the specified h or N value.

25. $y' = x + 2y$
    $y(0) = 1$
    interval: $[0, 1]$
    $h = 0.25$

26. $y' = xe^{-y}$
    $y(0) = 1$
    interval: $[0, 0.5]$
    $N = 5$

27. $y' = y + \sin x$
    $y(0) = 2$
    interval: $[0, 1]$
    $h = 0.2$

28. $y' = e^{x-y}$
    $y(0) = 0$
    interval: $[0, 2]$
    $h = 0.5$

In Exercises 29 – 30, use the provided solution $y(x)$ and Euler's Method with the $h = 0.2$ and $h = 0.1$ to complete the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
$y(x)$						
$h = 0.2$						
$h = 0.1$						

29. $y' = xy^2$
    $y(0) = 1$
    solution: $y(x) = \dfrac{2}{1 - x^2}$

30. $y' = xe^{x^2} + \dfrac{1}{2}xy$
    $y(0) = \dfrac{1}{2}$
    solution: $y(x) = \dfrac{1}{2}(x^2 + 1)e^{x^2}$

1.2 Separable Differential Equations

Similar to algebraic equations, there are specific techniques that can be used to solve specific types of differential equations. In algebra, we can use the quadratic formula to solve a quadratic equation, but not a linear or cubic equation. In the same way, techniques that can be used for a specific type of differential equation are ineffective for a differential equation of a different type. In this section, we describe and practice a technique to solve a class of differential equations called *separable equations.*

Definition 4 Separable Differential Equation

A **separable differential equation** is one that can be written in the form

$$n(y)\frac{dy}{dx} = m(x),$$

where n is a function that depends only on the dependent variable y, and m is a function that depends only on the independent variable x.

Below, we show a few examples of separable differential equations, along with similar looking equations that are not separable.

Separable	Not Separable
1. $\dfrac{dy}{dx} = x^2y$	1. $\dfrac{dy}{dx} = x^2 + y$
2. $y\sqrt{y^2-5}\dfrac{dy}{dx} - \sin x \cos x = 0$	2. $y\sqrt{y^2-1}\dfrac{dy}{dx} - \sin x \cos y = 0$
3. $\dfrac{dy}{dx} = \dfrac{(x^2+1)e^y}{y}$	3. $\dfrac{dy}{dx} = \dfrac{(xy+1)e^y}{y}$

Notice that a separable equation requires that the functions of the dependent and independent variables be multiplied, not added (like example 1 of the not separable column). An alternate definition of a separable differential equation states that an equation is separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

for some functions f and g.

Notes:

Separation of Variables

Let's find a formal solution to the separable equation

$$n(y)\frac{dy}{dx} = m(x).$$

Since the functions on the left and right hand sides of the equation are equal, their antiderivatives should be equal up to an arbitrary constant of integration. That is

$$\int n(y)\frac{dy}{dx}\,dx = \int m(x)\,dx + C.$$

Though the integral on the left may look a bit strange, recall that y itself is a function of x. Consider the substitution $u = y(x)$. The differential is $du = \frac{dy}{dx}\,dx$. Using this substitution, the above equation becomes

$$\int n(u)\,du = \int m(x)\,dx + C.$$

Let $N(u)$ and $M(x)$ be antiderivatives of $n(u)$ and $m(x)$, respectively. Then

$$N(u) = M(x) + C.$$

Since $u = y(x)$, this is

$$N(y) = M(x) + C.$$

This relationship between y and x is an implicit form of the solution to the differential equation. Sometimes (but not always) it is possible to solve for y to find an explicit version of the solution.

Though the technique outlined above is formally correct, what we did essentially amounts to integrating the function n with respect to its variable and integrating the function m with respect to its variable. The informal way to solve a separable equation is to treat the derivative $\frac{dy}{dx}$ as if it were a fraction. The separated form of the equation is

$$n(y)\,dy = m(x)\,dx.$$

To solve, we integrate the left hand side with respect to y and the right hand side with respect to x and add a constant of integration. As long as we are able to find the antiderivatives, we can find an implicit form for the solution. Sometimes we are able to solve for y in the implicit solution to find an explicit form of the solution to the differential equation. We practice the technique by solving the three differential equations listed in the separable column above, and conclude by revisiting and finding the general solution to the logistic differential equation from section 1.1

Notes:

Example 11 Solving a Separable Differential Equation

Find the general solution to the differential equation $y' = x^2y$.

Solution Using the informal solution method outlined above, we treat $\frac{dy}{dx}$ as a fraction, and write the separated form of the differential equation as

$$\frac{dy}{y} = x^2dx.$$

Integrating the left hand side of the equation with respect to y and the right hand side of the equation with respect to x yields

$$\ln|y| = \frac{1}{3}x^3 + C.$$

This is an implicit form of the solution to the differential equation. Solving for y yields an explicit form for the solution. Exponentiating both sides, we have

$$|y| = e^{\frac{x^3}{3}+C} = e^{\frac{x^3}{3}}e^C.$$

This solution is a bit problematic. First, the absolute value makes the solution difficult to understand. The second issue comes from our desire to find the *general solution.* Recall that a general solution includes all possible solutions to the differential equation. In other words, for any given initial condition, the general solution must include the solution to that specific initial value problem. We can often satisfy any given initial condition by choosing an appropriate C value. When solving separable equations, though, it is possible to lose solutions that have the form $y =$ constant. Notice that $y = 0$ solves the differential equation, but it is not possible to choose a finite C to make our solution look like $y = 0$. Our solution cannot solve the initial value problem $\frac{dy}{dx} = x^2y$, with $y(a) = 0$ (where a is any value). Thus, we haven't actually found a general solution to the problem. We can clean up the solution and recover the missing solution with a bit of clever thought.

Recall the formal definition of the absolute value: $|y| = y$ if $y \geq 0$ and $|y| = -y$ if $y < 0$. Our solution is either $y = e^Ce^{\frac{x^3}{3}}$ or $y = -e^Ce^{\frac{x^3}{3}}$. Further, note that C is constant, so e^C is also constant. If we write our solution as $y = Ce^{\frac{x^3}{3}}$, and allow C to take on both positive and negative values, we incorporate both cases of the absolute value. Finally, if we allow C to be zero, we recover the missing solution discussed above. The best way to express the general solution to our differential equation is

$$y = Ce^{\frac{x^3}{3}}.$$

Note: The indefinite integrals $\int \frac{dy}{y}\,dy$ and $\int x^2\,dx$ both produce arbitrary constants. Since both constants are arbitrary, we combine them into a single constant of integration.

Note: Missing constant solutions can't always be recovered by cleverly redefining the arbitrary constant. The differential equation $y' = y^2 - 1$ is an example of this fact. Both $y = 1$ and $y = -1$ are constant solutions to this differential equation. Separation of variables yields a solution where $y = 1$ can be attained by choosing an appropriate C value, but $y = -1$ can't. The general solution is the set containing the solution produced by separation of variables *and* the missing solution $y = -1$. We should always be careful to look for missing constant solutions when seeking the general solution to a separable differential equation.

Notes:

Example 12 **Solving a Separable Initial Value Problem**

Solve the initial value problem $(y\sqrt{y^2-5})y' - \sin x \cos x = 0$, with $y(0) = -3$.

Solution We first put the differential equation in separated form

$$y\sqrt{y^2-5}\,dy = \sin x \cos x\,dx.$$

The indefinite integral $\displaystyle\int y\sqrt{y^2-5}\,dy$ requires the substitution $u = y^2 - 5$. Using this substitute yields the antiderivative $\frac{1}{3}(y^2-5)^{3/2}$. The indefinite integral $\displaystyle\int \sin x \cos x\,dx$ requires the substitution $u = \sin x$. Using this substitution yields the antiderivative $\frac{1}{2}\sin^2 x$. Thus, we have an implicit form of the solution to the differential equation given by

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + C.$$

The initial condition says that y should be -3 when x is 0, or

$$\frac{1}{3}((-3)^2-5)^{3/2} = \frac{1}{2}\sin^2 0 + C.$$

This is $C = 8/3$, yielding the particular solution to the initial value problem

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + \frac{8}{3}.$$

Example 13 **Solving a Separable Differential Equation**

Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{(x^2+1)e^y}{y}$.

Solution We start by observing that there are no constant solutions to this differential equation because there are no constant y values that make the right hand side of the equation identically zero. Thus, we need not worry about losing solutions during the separation of variables process. The separated form of the equation is given by

$$ye^{-y}\,dy = (x^2+1)\,dx.$$

The antiderivative of the left hand side requires integration by parts. Evaluating both indefinite integrals yields the implicit solution

$$-(y+1)e^{-y} = \frac{1}{3}x^3 + x + C.$$

Since we cannot solve for y, we cannot find an explicit form of the solution.

Notes:

Example 14 **Solving the Logistic Differential Equation**

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right)$

Solution We looked at a slope field for this equation in section 1.1 in the specific case of $k = M = 1$. Here, we use separation of variables to find an analytic solution to the more general equation. Notice that the independent variable t does not explicitly appear in the differential equation. We mentioned that an equation of this type is called *autonomous*. All autonomous differential equation are separable.

We start by making the observation that both $y = 0$ and $y = M$ are constant solutions to the differential equation. We must check that these solutions are not lost during the separation of variables process. The separated form of the equation is

$$\frac{1}{y\left(1 - \frac{y}{M}\right)}\,dy = k\,dt.$$

The antiderivative of the left hand side of the equation can be found by making use of partial fractions. Using the techniques discussed in section **??**, we write

$$y\left(1 - \frac{y}{M}\right) = \frac{1}{y} + \frac{1}{M - y}.$$

Then an implicit form of the solution is given by

$$\ln|y| - \ln|M - y| = kt + C.$$

Combining the logarithms,

$$\ln\left|\frac{y}{M - y}\right| = kt + C.$$

Similarly to example 11, we can write

$$\frac{y}{M - y} = Ce^{kt}.$$

Letting C take on positive values or negative values incorporates both cases of the absolute value. This is another implicit form of the solution. Solving for y gives the explicit form

$$y = \frac{M}{1 + be^{-kt}},$$

where b is an arbitrary constant. Notice that $b = 0$ recovers the constant solution $y = M$. The constant solution $y = 0$ cannot be produced with a finite b value, and has been lost. The general solution the logistic differential equation is the set containing $y = \dfrac{M}{1 + be^{-kt}}$ and $y = 0$.

Note: Solving for y initially yields the explicit solution $y = \dfrac{CMe^{kt}}{1 + Cekt}$. Dividing numerator and denominator by Ce^{kt} and defining $b = 1/C$ yields the commonly presented form of the solution given in example 14.

Notes:

Exercises 1.2

Problems

In Exercises 1 – 4, decide whether the differential equation is separable or not separable. If the equation is separable. write it in separated form.

1. $y' = y^2 - y$

2. $xy' + x^2y = \dfrac{\sin x}{x - y}$

3. $(y+3)y' + (\ln x)y' - x \sin y = (y+3)\ln x$

4. $y' - x^2 \cos y + y = \cos y - x^2 y$

In Exercises 5 – 12, find the general solution to the separable differential equation. Be sure to check for missing constant solutions.

5. $y' + 1 - y^2 = 0$

6. $y' = y - 2$

7. $xy' = 4y$

8. $yy' = 4x$

9. $e^x yy' = e^{-y} + e^{-2x-y}$

10. $(x^2+1)y' = \dfrac{x}{y-1}$

11. $y' = \dfrac{x\sqrt{1-4y^2}}{x^4 + 2x^2 + 2}$

12. $(e^x + e^{-x})y' = y^2$

In Exercises 13 – 20, find the particular solution to the separable initial value problem.

13. $y' = \dfrac{\sin x}{\cos y}$, with $y(0) = \dfrac{\pi}{2}$

14. $y' = \dfrac{x^2}{1 - y^2}$, with $y(0) = 1$

15. $y' = \dfrac{2x}{y + x^2 y}$, with $y(0) = -4$

16. $x + ye^{-x}y' = 0$, with $y(0) = -2$

17. $y' = \dfrac{x \ln(x^2+1)}{y - 1}$, with $y(0) = 2$

18. $\sqrt{1-x^2}\, y' - \dfrac{\arcsin x}{y\cos(y^2)} = 0$, with $y(0) = \sqrt{\dfrac{7\pi}{6}}$

19. $y' = (\cos^2 x)(\cos^2 2y)$, with $y(0) = 0$

20. $y' = \dfrac{y^2\sqrt{1-y^2}}{x}$, with $y(0) = 1$

1.3 First Order Linear Differential Equations

In the previous section, we explored a specific techique to solve a specific type of differential equation; a separable differential equation. In this section, we develop and practice a technique to solve a type of differential equation called a *first order linear* differential equation.

Recall than a linear algebraic equation in one variable is one that can be written $ax + b = 0$, where a and b are real numbers. Notice that the variable x appears to the first power. The equations $\sqrt{x} + 1 = 0$ and $\sin(x) - 3x = 0$ are both nonlinear. A linear differential equation is one in which the dependent variable and its derivatives appear only to the first power. We focus on first order equations, which involve first (but not higher order) derivatives of the dependent variable.

Definition 5 First Order Linear Differential Equation

A **first order linear differential equation** is a differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

where p and q are arbitrary functions of the independent variable x.

Example 15 Classifying Differential Equations

Classify each differential equation as first order linear, separable, both, or neither.

(a) $y' = xy$

(b) $y' = e^y + 3x$

(c) $y' - (\cos x)y = \cos x$

(d) $yy' - 3xy = 4\ln x$

Solution (a) Both. We identify $p(x) = -x$ and $q(x) = 0$. The separated form of the equation is $\dfrac{dy}{y} = x\,dx$.

(b) Neither. The e^y term makes the equation nonlinear. Because of the addition, it is not possible to write the equation in separated form.

(c) First order linear. We identify $p(x) = -\cos x$ and $q(x) = \cos x$. The equation cannot be written in separated form.

(d) Neither. Notice that dividing by y results in the nonlinear term $\dfrac{4\ln x}{y}$. It is not possible to write the equation in separated form.

Notes:

Notice that linearity depends on the dependent variable y, not the independent variable x. The functions $p(x)$ and $q(x)$ need not be linear, as demonstrated in part (c) of example 15. Neither $\cos x$ nor $\sin x$ are linear functions of x, but the differential equation is still linear.

Solving First Order Linear Equations

We motivate the solution technique by way of an observation and an example. We first observe that the expression $\frac{d}{dx}(xy)$ can be expanded via the product rule and implicit differentiation to the expression $x\frac{dy}{dx}+y$. Now we look at an example. Consider the first order linear differential equation

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{\sin x\cos x}{x}.$$

If we multiply both sides of the differential equation by x and use our observation, we see that the differential equation can be written

$$\frac{d}{dx}(xy)=\sin x\cos x.$$

We can now integrate both sides of the differential equation with respect to x. On th left, the antiderivative of the derivative is simply the function xy. Using the substitution $u=\sin x$ on the right results in the implicit solution

$$xy=\frac{1}{2}\sin^2 x+C.$$

Note: In the examples in the previous section, we performed operations on the arbitrary constant C, but still called the result C. The justification is that the result after the operation is *still* an arbitrary contant. Here, we divide C by x, so the result depends explicitly on the independent variable x. Since C/x is *not* contant, we can't just call it C.

Solving for y yields the explicit solution

$$y=\frac{\sin^2 x}{2x}+\frac{C}{x}.$$

As motivated by the problem we just solved, the basic idea behind solving first order linear differential equations is to multiply both sides of the differential equation by a function, called an *integrating factor*, that makes the left hand side of the equation look like an expanded product rule. We then condense the left hand side into the derivative of a product and integrate both sides. An obvious question is how to find the integrating factor.

Consider the first order linear equation

$$\frac{dy}{dx}+p(x)y=q(x).$$

Notes:

Let's call the integrating factor $\mu(x)$. We multiply both sides of the differential equation by $\mu(x)$ to get

$$\mu(x)\left(\frac{dy}{dx}+p(x)y\right)=\mu(x)q(x).$$

Our goal is to choose $\mu(x)$ so that the left hand side of the differential equation looks like the result of a product rule. The left hand side of the equation is

$$\mu(x)\frac{dy}{dx}+\mu(x)p(x)y.$$

Using the product rule and implicit differentiation,

$$\frac{d}{dx}\big(\mu(x)y\big)=\frac{d\mu}{dx}y+\mu(x)\frac{dy}{dx}.$$

Equating these two gives

$$\frac{d\mu}{dx}y+\mu(x)\frac{dy}{dx}=\mu(x)\frac{dy}{dx}+\mu(x)p(x)y,$$

or

$$\frac{d\mu}{dx}=\mu(x)p(x).$$

In order for the integrating factor $\mu(x)$ to perform its job, it must solve the differential equation above. But that differential equation is separable, so we can solve it. The separated form is

$$\frac{d\mu}{\mu}=p(x)\,dx.$$

Integrating,

$$\ln\mu=\int p(x)\,dx,$$

or

$$\mu(x)=e^{\int p(x)\,dx}.$$

Note: Following the steps outlined in the previous section, we should technically end up with $\mu(x)=Ce^{\int p(x)\,dx}$, where C is an arbitrary constant. Because we multiply both sides of the differential equation by $\mu(x)$, the arbitrary constant cancels, and we omit it when finding the integrating factor.

If $\mu(x)$ is chosen this way, after multiplying by $\mu(x)$, we can always write the differential equation in the form

$$\frac{d}{dx}\big(\mu(x)y\big)=\mu(x)q(x).$$

Integrating and solving for y, the explicit solution is

$$y=\frac{1}{\mu(x)}\int\big(\mu(x)q(x)\big)\,dx.$$

Notes:

Though this formula can be used to write down the solution to a first order linear equation, we shy away from simply memorizing a formula. The process is lost, and it's easy to forget the formula. Rather, we always always follow the steps outlined in key idea 2 when solving equations of this type.

Key Idea 2 Solving First Order Linear Equations

1. Write the differential equation in the form
$$\frac{dy}{dx} + p(x)y = q(x).$$
2. Compute the integrating factor
$$\mu(x) = e^{\int p(x)\,dx}.$$
3. Multiply both sides of the differential equation by $\mu(x)$, and condense the left hand side to get
$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$
4. Integrate both sides of the differential equation with respect to x, taking care to remember the arbitrary constant.
5. Solve for y to find the explicit solution to the differential equation.

Let's practice the process by solving the two first order linear differential equations from example 15.

Example 16 Solving a First Order Linear Equation
Find the general solution to $y' = xy$.

Solution We solve by following the steps in key idea 2. Unlike the process for solving separable equations, we need not worry about losing constant solutions. The answer we find *will* be the general solution to the differential equation. We first write the equation in the form

$$\frac{dy}{dx} - xy = 0.$$

Notes:

By identifying $p(x) = -x$, we can compute the integrating factor

$$\mu(x) = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}.$$

Multiplying both side of the differential equation by $\mu(x)$, we have

$$e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right) = 0.$$

The left hand side of the differential equation condenses to yield

$$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right) = 0.$$

We integrate both sides with respect to x to find the implicit solution

$$e^{-\frac{1}{2}x^2}y = C,$$

or the explicit solution

$$y = Ce^{\frac{1}{2}x^2}.$$

Note: The step where the left hand side of the differential equation condenses to the derivative of a product can feel a bit magical. The reality is that we choose $\mu(x)$ so that we can get exactly this condensing behavior. It's not magic, it's math! If you're still skeptical, try using the product rule and implicit differentiation to evaluate $\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right)$, and verify that it becomes $e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right)$.

Example 17 **Solving a First Order Linear Equation**

Find the general solution to $y' - (\cos x)y = \cos x$.

Solution The differential equation is already in the correct form. The integrating factor is given by

$$\mu(x) = e^{-\int \cos x\,dx} = e^{-\sin x}.$$

Multiplying both sides of the equation by the integrating factor and condensing,

$$\frac{d}{dx}\left(e^{-\sin x}y\right) = (\cos x)e^{-\sin x}$$

Using the substitution $u = -\sin x$, we can integrate to find the implicit solution

$$e^{-\sin x}y = -e^{-\sin x} + C.$$

The explicit form of the general solution is

$$y = -1 + Ce^{\sin x}.$$

We continue our practice by finding the particular solution to an initial value problem.

Notes:

Example 18 **Solving a First Order Linear Initial Value Problem**

Solve the initial value problem $xy' - y = x^3 \ln x$, with $y(1) = 0$.

Solution We first divide by x to get

$$\frac{dy}{dx} - \frac{1}{x}y = x^2 \ln x.$$

The integrating factor is given by

$$\begin{aligned}\mu(x) &= e^{-\int \frac{1}{x}\,dx}\\ &= e^{-\ln x}\\ &= e^{\ln x^{-1}}\\ &= x^{-1}.\end{aligned}$$

Multiplying both sides of the differential equation by the integrating factor and condensing the left hand side, we have

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x \ln x.$$

Using integrating by parts to find the antiderivative of $x \ln x$, we find the implicit solution

$$\frac{y}{x} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.$$

Solving for y, the explicit solution is

$$y = \frac{1}{2}x^3 \ln x - \frac{1}{4}x^3 + Cx.$$

The initial condition $y(1) = 0$ yields $C = 1/4$. The solution to the initial value problem is

$$y = \frac{1}{2}x^3 \ln x - \frac{1}{4}x^3 + \frac{1}{4}x.$$

Differential equations are a valuable tool for exploring various physical problems. This process of using equations to describe real world situations is called mathematical modeling, and is the topic of the next section. The last two examples in this section begin our discussion of mathematical modeling.

Example 19 **A Falling Object Without Air Resistance**

Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object assuming no air resistance.

Notes:

Solution The basic physical law at play is Newton's second law,

$$\text{mass} \times \text{acceleration} = \text{the sum of the forces.}$$

Using the fact that acceleration is the derivative of velocity, mass $\times$ acceleration can be writting mv'. In the absence of air resistance, the only force of interest is the force due to gravity. This force is approximately constant, and is given by mg, where g is the gravitational constant. The word equation above can be written as the differential equation

$$m\frac{dv}{dt} = mg.$$

Because g is constant, this differential equation is simply an integration problem, and we find

$$v = gt + C.$$

Since $v = C$ with $t = 0$, we see that the arbitrary constant here corresponds to the initial vertical velocity of the object.

The process of mathematical modeling does not stop simply because we have found an answer. We must examine the answer to see how well it can describe real world observations. In the previous example, the answer may be somewhat useful for short times, but intuition tells us that something is missing. Our answer says that a falling object's velocity will increase linearly as a function of time, but we know that a falling object does not speed up indefinitely. In order to more fully describe real world behavior, our mathematical model must be revised.

Example 20 **A Falling Object with Air Resistance**

Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object, taking air resistance into account.

Solution We still begin with Newon's second law, but now we assume that the forces in the object come both from gravity and from air resistance. The gravitational force is still given by mg. For air resistance, we assume the force is related to the velocity of the object. A simple way to describe this assumption might be kv^p, where k is a proportionality constant and p is a positive real number. The value k depends on various factors such as the density of the object, surface area of the object, and density of the air. The value p affects how changes in the velocity affect the force. Taken together, a function of the form kv^p is often called a *power law*. The differential equation for the velocity is given by

$$m\frac{dv}{dt} = mg - kv^p.$$

Notes:

(Notice that the force from air resistance opposes motion, and points in the opposite direction as the force from gravity.) This differential equation is separable, and can be written in the separated form

$$\frac{m}{mg - kv^p}\,dv = dt.$$

For arbitrary positive p, the integration is difficult, making this problem hard to solve analytically. In the case that $p = 1$, the differential equation becomes linear, and is easy to solve either using either separation of variables or integrating factor techniques. We assume $p = 1$, and proceed with an integrating factor so we can continue practicing the process. Writing

$$\frac{dv}{dt} + \frac{k}{m}v = g,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{k}{m}\,dt} = e^{\frac{k}{m}t}.$$

Then

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = ge^{\frac{k}{m}t},$$

so

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C,$$

or

$$v = \frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

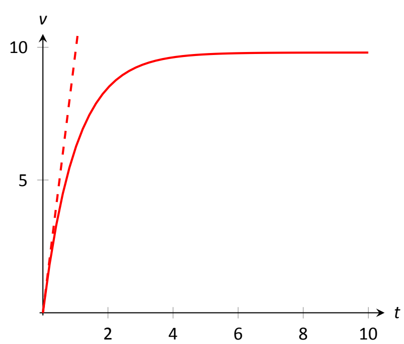


Figure 1.10: The velocity function from examples 19 (dashed) and 20 (solid) under the assumption that $v(0) = 0$, with $g = 9.8, m = 1$ and $k = 1$.

In the solution above, the exponential term decays as time increases, causing the velocity to approach the constant value mg/k in the limit as t approaches infinity. This value is called the *terminal velocity*. If we assume a zero initial velocity (the object is dropped, not thrown from the plane), the velocities from examples 19 and 20 are given by $v = gt$ and $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$, respectively. These two functions are shown in figure 1.10, with $g = 9.8, m = 1$, and $k = 1$. Notice that the two curves agree well for short times, but have dramatically different behaviors as t increases. Part of the art in mathematical modeling is deciding on the level of detail required to answer the question of interest. If we are only interested in the initial behavior of the falling object, the simple model in example 19 may be sufficient. If we are interested in the longer term behavior of the object, the simple model is not sufficient, and we should consider a more complicated model.

Notes:

Exercises 1.3

Problems

In Exercises 1 – 8, Find the general solution to the first order linear differential equation.

1. $y' = 2y - 3$
2. $x^2y' + xy = 1$
3. $x^2y' - xy = 1$
4. $xy' + 4y = x^3 - x$
5. $(\cos^2 x \sin x)y' + (\cos^3 x)y = 1$
6. $\dfrac{y'}{x} = 1 - 2y$
7. $x^3y' - 3x^3y = x^4e^{2x}$
8. $y' + y = 5\sin(2x)$

In Exercises 9 – 16, Find the particular solution to the initial value problem.

9. $y' = y + 2xe^x, \qquad y(0) = 2$
10. $xy' + 2y = x^2 - x + 1, \qquad y(1) = 1$
11. $xy' + (x+2)y = x, \qquad y(1) = 0$
12. $y' + 2y = 0, \qquad y(0) = 3$
13. $(x+1)y' + (x+2)y = 2xe^{-x}, \qquad y(0) = 1$
14. $(\cos x)y' + (\sin x)y = 1, \qquad y(0) = -3$
15. $(x^2 - 1)y' + 2y = (x+1)^2, \qquad y(0) = 2$
16. $xy' - 2y = \dfrac{x^3}{1+x^2}, \qquad y(1) = 0$

In Exercises 17 – 20, classify the differential equation as separable, first order linear, or both, and solve the initial value problem using an appropriate method.

17. $y' = y + yx^2, \qquad y(0) = -5$
18. $xe^yy' = x^2 \sin x, \qquad y(0) = 0$
19. $(x-1)y' + y = x^2 - 1, \qquad y(0) = 2$
20. $y' = y^2 + y - 2, \qquad y(0) = 1$

In Exercises 21 – 22, draw a slope field for the differential equation. Use the slope field to predict the behavior of the solution to the initial value problem for large *x* values. Solve the initial value problem, and verify your prediction.

21. $y' = x - y, \qquad y(0) = 0$
22. $(X+1)y' + y = \dfrac{1}{x+1}, \qquad y(0) = 2$

1.4 Modeling with Differential Equations

In the first three sections of this chapter, we focused on the basic ideas behind differential equations and the mechanics of solving certain types of differential equations. We have only hinted at their practical use. In this section, we use differential equations for mathematical modeling, the process of using equations to describe real world processes. We explore a few different mathematical models with the goal of gaining an introduction to this large field of applied mathematics.

Models Involving Proportional Change

Some of the simplest differential equation models involve one quantity that changes at a rate proportional to another quantity. In the introduction to this chapter, we consider a population that grows at a rate proportional to the current population. The words in this assumption can be directly translated into a differential equation as shown below.

is proportional to

$$\frac{dp}{dt} = kp$$

The rate of change of the population the population.

Figure 1.11: Translating words into a differential equation.

There are some key ideas that can be helpful when translating words into a differential equation. Any time we see something about rates or changes, we should think about derivatives. The word “is” usually corresponds to an equal sign in the equation. The words “proportional to” mean we have a constant multiplied by something.

The differential equation in figure 1.11 is easily solved using separation of variables. We find

$$p = Ce^{kt}.$$

Notice that we need values for both C and k before we can use this formula to predict population size. We require information about the population at two different times in order to fully determine the population model.

Example 21 Bacterial Growth

Suppose a population of *e-coli* bacteria grows at a rate proportional to the current population. If an initial popluation of 200 bacteria has grown to 1600 three hours later, find a function for the size of the population at time t, and use it to predict when the population size will reach 10,000.

Notes:

Solution We already know that the population at time t is given by $p = Ce^{kt}$ for some C and k. The information about the initial size of the population means that $p(0) = 200$. Thus $C = 200$. Our knowledge of the population size after three hours allows us to solve for k via the equation

$$1600 = 200e^{3k}.$$

Solving this exponential equation yields $k = \ln(8)/3 \approx 0.6931$. The popluation at time t is given by

$$p = 200e^{(\ln(8)/3)t}.$$

Solving

$$10000 = 200e^{(\ln(8)/3)t}$$

yields $t = (3\ln 50)/\ln 8 \approx 5.644$. The population is predicted to reach 10,000 bacteria in slightly more than five and a half hours.

Another example of porportional change is **Newton's Law of Cooling.** The laws of thermodynamics state that heat flows from areas of high temperature to areas of lower temperature. A simple example is a hot object that cools down when placed in a cool room. Newton's Law of Cooling is the simple assumption that the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the ambient temperature of the room. If T is the temperature of the object, and A is the constant ambient temperature, Newton's Law of Cooling can be expressed as the differential equation

$$\frac{dT}{dt} = k(A - T.)$$

This differential equation is both linear and separable. The separated form is

$$\frac{1}{A - T}\,dT = k\,dt.$$

Then an implicit definition of the temperature is given by

$$-\ln|A - T| = kt + C.$$

If we solve for T, we find the explicit temperature

$$T = A - Ce^{-kt}.$$

Though we didn't show the steps, the explicit solution involves the typical process of renaming the constant $\pm e^{-C}$ as C, and allowing C to be positive, negative, or zero to account for both cases of the absolution value and to catch the constant solution $T = A$. Notice that the temperature of the object approaches the ambient temperature in the limit as $t \to \infty$.

Note: The equation $\dfrac{dT}{dt} = k(T - A)$ is also a valid representation of Newton's Law of Cooling. Intuition tells us that T will increase if T is less than A and decrease if T is greater then A. The form we use in the text follows this intuition with a positive k value. The form above will require that k take on a negative value. In the end, both forms result in the same general solution.

Notes:

Example 22 **Hot Coffee**

A freshly brewed cup of coffee is set on the counter and has a temperature of 200° fahrenheit. After 3 minutes, it has cooled to 190°, but is still too hot to drink. If the room is 72° and the coffee cools according to Newton's Law of Cooling, how long will the impatient coffee drinker have to wait until the coffee has cooled to 165°?

Solution Since we have already solved the differential equation for Newton's Law of Cooling, we can immediately use the function

$$T = A - Ce^{-kt}.$$

Since the room is 72°, we know $A = 72$. The initial temperature is 200°, which means $C = -128$. At this point, we have

$$T = 72 + 128e^{-kt}$$

The information about the coffee cooling to 190° in 3 minutes leads to the equation

$$190 = 72 + 128e^{-3k}.$$

solving the exponential equation for k, we have

$$k = -\frac{1}{3}\ln\left(\frac{59}{64}\right) \approx 0.0271.$$

Finally, we finish the problem by solving the exponential equation

$$165 = 72 + 128e^{\frac{1}{3}\ln\left(\frac{59}{64}\right)t}.$$

The coffee drinker must wait $t = \dfrac{3\ln\left(\frac{93}{128}\right)}{\ln\left(\frac{59}{64}\right)} \approx 11.78$ minutes.

We finish our discussion of models of proportional change by exploring three different models of disease spread through a population. In all of the models, we let y denote the proportion of the population that is sick ($0 \leq y \leq 1$). We assume a proportion of 0.05 is initially sick and that a proportion of 0.1 is sick 1 week later.

Example 23 **Disease Spread 1**

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the popoulation that is sick at time t.

Notes:

Solution The assumption here seems to have some merit because it matches our intuition that a disease should spread more rapidly when more individuals are sick. The differential equation is simply

$$\frac{dy}{dt} = ky,$$

with solution

$$y = Ce^{kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ lead to $C = 0.05$ a and $k = \ln 2$, so the function is

$$y = 0.05e^{(\ln 2)t}.$$

We should point out a glaring problem with this model. The variable y is a proportion and should take on values between 0 and 1, but the function $y = 0.05e^{2t}$ grows without bound. After $t \approx 4.32$ weeks, y exceeds 1, and the model ceases to make physical sense.

Example 24 **Disease Spread 2**

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are not sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the popoulation that is sick at time t

Solution The intuition behind the assumption here is that a disease can only spread if there are individuals who are susceptible to the infection. As fewer and fewer people are able to be infected, the disease spread should slow down. Since y is proportion of the population that is sick, $1 - y$ is the proportion who are not sick, and the differential equation is

$$\frac{dy}{dt} = k(1 - y).$$

Though the context is quite different, the differential equation is identical to the differential equation for Newton's Law of Cooling, with $A = 1$. The solution is

$$y = 1 - Ce^{-kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $C = 0.95$ and $k = -\ln\left(\frac{18}{19}\right) \approx 0.0541$, so the final function is

$$y = 1 - .95e^{\ln\left(\frac{18}{19}\right)t}.$$

Notice that this function approches $y = 1$ in the limit as $t \to \infty$, and does not suffer from the non-physical behavior described in example 23.

Notes:

In example 23, we assumed disease spread depends on the number of infected individuals. In example 24, we assumed disease spread depends on the number of susceptible individuals who are able to become infected. In reality, we would expect many diseases to require the interaction of both infected on susceptible individuals in order to spread. One of the simplest ways to model this required interaction is to assume disease spread depends on the product of the proportions of infected and uninfected individuals. This assumption is often called the *law of mass action.*

Example 25 Disease Spread 3

Suppose a disease spreads through a population at a rate proportional to the product of the number of infected and uninfected individuals. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the population that is sick at time t

Solution The differential equation is

$$\frac{dy}{dt} = ky(1-y).$$

This is exactly the logistic equation with $M = 1$. We solve this differential equation in example 14, and find

$$y = \frac{1}{1+be^{-kt}}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $b = 19$ and $k = -\ln\left(\frac{9}{19}\right) \approx 0.7472$. The final function is

$$y = \frac{1}{1+19e^{\ln\left(\frac{9}{19}\right)t}}.$$

Based on the three different assumptions about the rate of disease spread explored in the last three examples, we now have three different functions giving the proportion of a population that is sick at time t. Each of the three functions meets the conditions $y(0) = 0.05$ and $y(1) = 0.1$. The three functions are shown in figure 1.12. Notice that the logistic function mimics specific parts of the functions from examples 23 and 24. We see in figure 1.12 (a) that the logistic and exponential functions are virtually indistinguishable for small t values. When there are few infected individuals and lots of susceptible individuals, the spread of a disease is largely determined by the number of sick people. The logistic curve captures this feature, and is "almost exponential" early on. In figure 1.12, we see that the logistic curve leaves the exponential curve from example 23 and approaches the curve from example 24. This result implies that when most

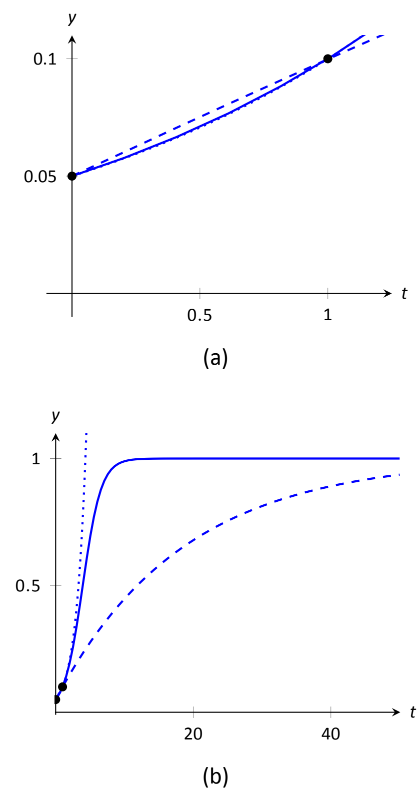


Figure 1.12: Plots of the functions from example 23 (dotted), example 24 (dashed), and example 25 (solid).

Notes:

of the population is sick, the spread of the disease is largely dependent on the number of susceptible individuals. Though there are much more sophisticated mathematical models describing the spread of infections, we could argue that the logistic model presented in this example is the "best" of the three.

Rate-in Rate-out Problems

One of the classic ways to build a mathematical model involves tracking the way the amount of something can change. Consider a box with a specific type of stuff inside. The amount of stuff of the specific type in the box can only change in three ways; we can add more stuff to the box, we can remove some of the stuff from the box, or some of the stuff can change into stuff of a different type. In the examples that follow, we assume stuff doesn't change type, so we only need to keep track of stuff coming into the box and stuff leaving the box. To derive a differential equation, we track rates:

$$\text{rate of change of some quantity} = \text{rate in} - \text{rate out}.$$

Though we stick to relatively simple examples, this basic idea can be used to derive some very important differential equations in mathematics and physics.

The examples to follow involve tracking the amount of a chemical in solution. We assume liquid containing some chemical flows into a container at some rate. That liquid mixes instantaneously with the liquid already in the container. Then the liquid from the container flows out at some (potentially different) rate.

Note: The assumption about instantaneous mixing, though not physically accurate, leads to a differential equation we have hope of solving. In reality, the amount of chemical at a specific location in the container depends both on the location and how long we have been waiting. This dependence on both space and time leads to a type of differential equation called a *partial differential equation.* Differential equations of this type are more interesting, but significantly harder to study. Instantaneous mixing removes any spatial dependence from the problem, and leaves us with an *ordinary differential equation*.

Example 26 Equal Flow Rates

Suppose a 10 liter bucket has 5 liters of salt solution in it. The initial concentration of the salt solution is 1 g/L. A salt solution with concentration 3 g/L flows into the bucket at a rate of 2 L/min. Suppose the salt solution mixes instantaneously with the solution already in the bucket and that the mixed solution from the bucket flows out at a rate of 2 L/min. Find a function that gives the amount of salt in the bucket at time t.

Solution We use the rate in - rate out setup described above. The quantity here is the amount (in grams) of salt in the bucket at time t. Let y denote the amount of salt. In words, the differential equation is given by

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}.$$

Thinking in terms of units can help fill in the details of the differential equation. Since y has units of grams, the left hand side of the equation has units g/min. Both terms on the right hand side must have these same units. Notice that the product of a concentration (with units g/L) and a flow rate (with units L/min)

Notes:

results in a quantity with units g/min. Both terms on the right hand side of the equation will include a concentration multiplied by a flow rate.

For the rate in, we multiply the inflow concentration by the rate that fluid is flowing into the bucket. This is $(3)(2) = 6$ g/min.

The rate out is more complicated. The flow rate is still 2 L/min, meaning that the overall volume of the fluid in the bucket is the constant 5 L. The salt concentration in the bucket is not constant though, meaning that the outflow concentration is not constant. In particular, the outflow concentration is *not* the constant 1 g/L. This is simply the initial concentration. To find the concentration at any time, we need the amount of salt in the bucket at that time and the volume of liquid in the bucket at that time. The volume of liquid is the constant 5 L, and the amount of salt is given by the dependent variable y. Thus, the outflow concentration is $y/5$, yielding a rate out given by $\left(\frac{y}{5}\right)(2) = \frac{2y}{5}$.

The differential equation we wish to solve is given by

$$\frac{dy}{dt} = 6 - \frac{2y}{5}.$$

To furnish an initial condition, we must convert the initial salt concentration into an initial amount of salt. This is $(1)(5) = 5$ g, so $y(0) = 5$ is our initial condition.

Our differential equation is both separable and linear. We solve using separation of variables. The separated form of the differential equation is

$$\frac{5}{30 - 2y}\,dy = dt.$$

Integrating, yields the implicit solution

$$-\frac{5}{2}\ln|30 - 2y| = t + C.$$

Solving for y (end redefining the arbitrary constant C as necessary) yields the explicit solution

$$y = 15 + Ce^{-\frac{2}{5}t}.$$

The initial condition $y(0) = 5$ means that $C = -10$ so that

$$y = 15 - 10e^{-\frac{2}{5}t}$$

Is the particular solution to our initial value problem.

This function is plotted in figure 1.13. Notice that in the limit as $t \to \infty$, y approaches 15. This corresponds to a bucket concentration of $15/5 = 3$ g/L. It should not be surprising that salt concentration inside the tank will move to match the inflow salt concentration.

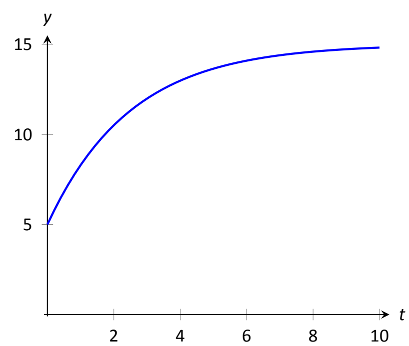


Figure 1.13: Salt concentration at time t, from example 26.

Example 27 Unequal Flow Rates

Suppose the setup is identical to the setup in example 26 except that now liquid

Notes:

flows out of the bucket at a rate of 1 L/min. Find a function that gives the amount of salt in the bucket at time t. What is the salt concentration when the solution ceases to be valid?

Solution Because the inflow and outflow rates no longer match, the volume of liquid in the bucket is not the constant 5 L. In general, we can find the volume of liquid via the equation

$$\text{volume} = \text{initial volume} + (\text{inflow rate - outflow rate})t.$$

In this example, the volume at time t is $5 + t$. Because the total volume of the bucket is only 10 L, it follows that our solution will only be valid for $0 \leq t \leq 5$. At that point it is no longer possible to have liquid flow into a the bucket at a rate of 2 L/min and out of the bucket at a rate of 1 L/min.

To update the differential equation, we must modify the rate out. Since the volume is $5+t$, the concentration at time t is given by $\frac{y}{5+t}$. Thus for rate out, we must use $\left(\frac{y}{5+t}\right)(1)$. The initial value problem is

$$\frac{dy}{dt} = 6 - \frac{y}{5+t}, \text{ with } y(0) = 5.$$

Unlike example 26, where we had equal flow rates, this differential equation is no longer separable. We must proceed with an integrating factor. Writing the differential equation in the form

$$\frac{dy}{dt} + \frac{1}{5+t}y = 6,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{1}{5+t}\,dt} = e^{\ln(5+t)} = 5 + t.$$

Then

$$\frac{d}{dt}\big((5+t)y\big) = 6(5+t),$$

yielding the implicit solution

$$(5+t)y = 30t + 3t^2 + C.$$

The initial condition $y(0) = 5$ implies $C = 25$, so the explicit solution to our initial value problem is given by

$$y = \frac{3t^2 + 30t + 25}{5+t}.$$

Notes:

This solution ceases to be valid at $t = 5$. At that time, there are 25 g of salt in the tank. The volume of liquid is 10 L, resulting in a salt concentration of 2.5 g/L.

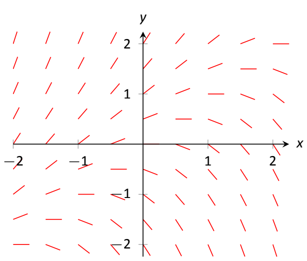
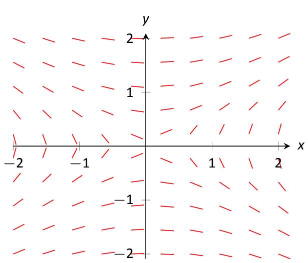
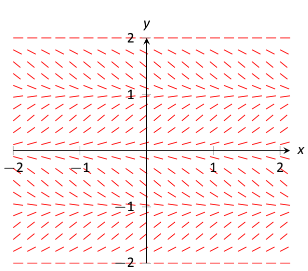
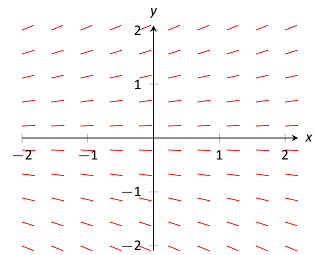
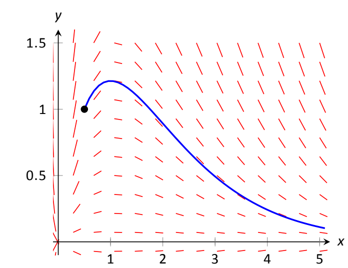
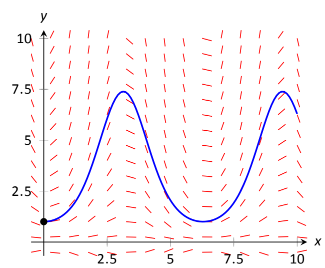
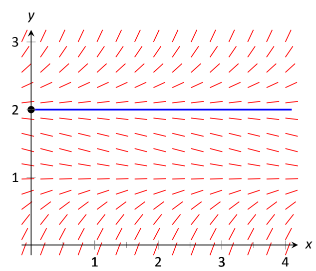
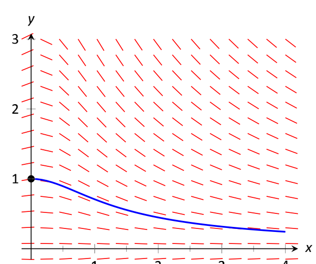
Differential equations are powerful tools that can be used to help describe the world around us. Though relatively simple in concept, the ideas of proportional change and matching rates can serve as building blocks in the development of more sophisticated mathematical models. As we saw in this section, some simple mathematical models can be solved analytically using the techniques developed in this chapter. Most more sophicticated mathematical models don't allow for analytic solutions. Even so, there are an array of graphical and numerical techniques that can be used to analyze the model to make predictions and infer information about real world phenomenon.

Notes:

A: Solutions To Selected Problems

Chapter 1

Section 1.1

1. An initial value problems is a differential equation that is paired with one or more initial conditions. A differential equation is simply the equation without the initial conditions.
2. Answers will vary.
3. Substitute the proposed function into the differential equation, and show the the statement is satisfied.
4. A particular solution is one specifica member of a family of solutions, and has no arbitrary constants. A general solution is a family of solutions, includes all possible solutions to the differential equation, and typically includes one or more arbitrary constants.
5. Many differential equations are impossible to solve analytically.
6. A smaller h value leads to a numerical solution that is closer to the true solution, but decreasing the h value leads to more computational effort.
7. Answers will vary.
8. Answers will vary.
9. Answers will vary.
10. Answers will vary.
11. $C = 2$
12. $C = 6$
13. 
14. 
15. 
16. 
17. b
18. c
19. d
20. a
21. 
22. 
23. 
24. 
25.

x_i	y_i
0.00	1.0000
0.25	1.5000
0.50	2.3125
0.75	3.5938
1.00	5.5781

26.

x_i	y_i
0.0	1.0000
0.1	1.0000
0.2	1.0037
0.3	1.0110
0.4	1.0219
0.5	1.0363

27.

x_i	y_i
0.0	2.0000
0.2	2.4000
0.4	2.9197
0.6	3.5816
0.8	4.4108
1.0	5.4364

28.

x_i	y_i
0.0	0.0000
0.5	0.5000
1.0	1.8591
1.5	10.5824
2.0	88378.1190

29.

x_i	$y(x)$	h=0.2	$h = 0.1$
0.0	1.0000	1.0000	1.0000
0.2	1.0204	1.0000	1.0100
0.4	1.0870	1.0400	1.0623
0.6	1.2195	1.1265	1.1687
0.8	1.4706	1.2788	1.3601
1.0	2.0000	1.5405	1.7129

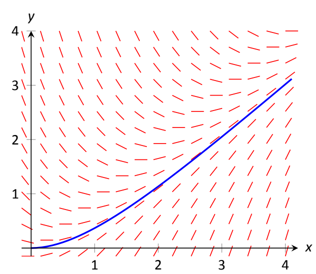
30.

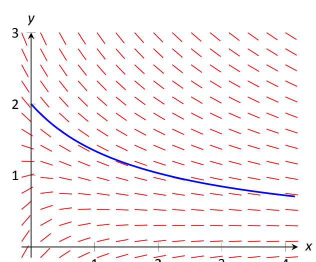
x_i	$y(x)$	$h = 0.2$	$h = 0.1$
0.0	0.5000	0.5000	0.5000
0.2	0.5412	0.5000	0.5201
0.4	0.6806	0.5816	0.6282
0.6	0.9747	0.7686	0.8622
0.8	1.5551	1.1250	1.3132
1.0	2.7183	1.7885	2.1788

Section 1.2

1. Separable. $\dfrac{1}{y^2-y}\,dy = dx$
2. Not separable.
3. Not separable.
4. Separable. $\dfrac{1}{\cos y - y}\,dy = (x^2+1)\,dx$
5. $\left\{y = \dfrac{1+Ce^{2x}}{1-Ce^{2x}}, y = -1\right\}$
6. $y = 2 + Ce^x$
7. $y = Cx^4$
8. $y^2 - 4x^2 = C$
9. $(y-1)e^y = -e^{-x} - \dfrac{1}{3}e^{-3x} + C$
10. $(y-1)^2 = \ln(x^2+1) + C$
11. $\left\{\arcsin 2y - \arctan(x^2+1) = C, y = \pm\dfrac{1}{2}\right\}$
12. $\left\{y = \dfrac{1}{C - \arctan x}, y = 0\right\}$
13. $\sin y + \cos x = 2$
14. $-x^3 + 3y - y^3 = 2$
15. $\frac{1}{2}y^2 - \ln(1+x^2) = 8$
16. $y^2 + 2xe^x - 2e^x = 2$
17. $\dfrac{1}{2}y^2 - y = \dfrac{1}{2}\left((x^2+1)\ln(x^2+1) - (x^2+1)\right) + \dfrac{1}{2}$
18. $\sin(y^2) - (\arcsin x)^2 = -\frac{1}{2}$
19. $2\tan 2y = 2x + \sin 2x$
20. $x = exp\left(-\dfrac{\sqrt{1-y^2}}{y}\right)$

Section 1.3

1. $y = \dfrac{3}{2} + Ce^{2x}$
2. $y = \dfrac{\ln|x| + C}{x}$
3. $y = -\dfrac{1}{2x} + Cx$
4. $y = \dfrac{x^3}{7} - \dfrac{x}{5} + \dfrac{C}{x^4}$
5. $y = \sec x + C(\csc x)$
6. $y = \dfrac{1}{2} + Ce^{-x^2}$
7. $y = Ce^{3x} - (x+1)e^{2x}$
8. $y = sin(2x) - 2\cos(2x) + Ce^{-x}$
9. $y = (x^2+2)e^x$
10. $y = \dfrac{1}{4}x^2 - \dfrac{1}{3}x + \dfrac{1}{2} + \dfrac{7}{12x^2}$
11. $y = 1 - \dfrac{2}{x} + \dfrac{2-e^{1-x}}{x^2}$
12. $y = 3e^{-2x}$
13. $y = \dfrac{x^2+1}{x+1}e^{-x}$
14. $y = \sin x - 3\cos x$
15. $y = \dfrac{(x-2)(x+1)}{x-1}$
16. $y = x^2\left(\arctan x - \dfrac{\pi}{4}\right)$
17. Both; $y = -5e^{x+\frac{1}{3}x^3}$
18. separable; $e^y = \sin x - x\cos x + 1$
19. linear; $y = \dfrac{x^3-3x-6}{3(x-1)}$
20. separable; $y = 1$
21. 

The solution will increase and begin to follow the line $y = x - 1$.
$y = x - 1 + e^{-x}$
22. 

The solution will decrease and approach $y = 0$.
$y = \dfrac{2 + \ln(x+1)}{x+1}$

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