APEX CALCULUS

Version 3.0

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Preface

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three–volume series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.apexcalculus.com`.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactive the interactivity, one can right-click and choose the "Disable Content" option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this "project from work" occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI's Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

1: LIMITS

Calculus means "a method of calculation or reasoning." When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17^{th} century, mathematicians and scientists were keenly aware of what they *could not do.* (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then "distance = rate $\times$ time." But what if the rate is not constant – can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was "a" calculus. However, as the power and importance of their discovery took hold, it became known to many as "the" calculus. Today, we generally shorten this to discuss "calculus."

The foundation of "the calculus" is the *limit.* It is a tool to describe a particular behavior of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make "finding limits" tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

1.1 An Introduction To Limits

We begin our study of *limits* by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $y = \frac{\sin x}{x}$. When x is near the value 1, what value (if any) is y near?

While our question is not precisely formed (what constitutes "near the value

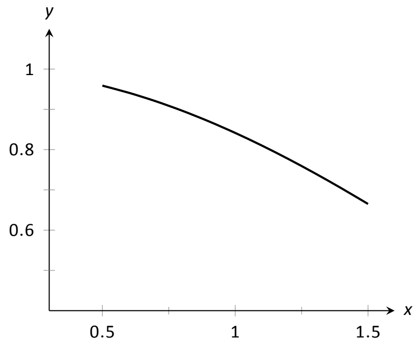


Figure 1.1: $\sin(x)/x$ near $x = 1$.

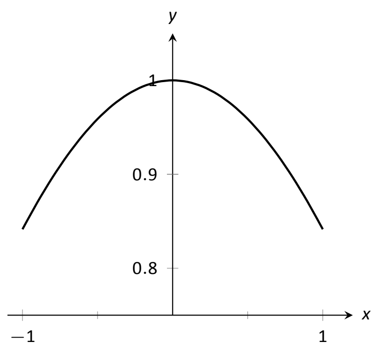


Figure 1.2: $\sin(x)/x$ near $x = 0$.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	**0.841471**
1.001	0.84117
1.01	0.838447
1.1	0.810189

Figure 1.3: Values of $\sin(x)/x$ with x near 1.

1"?), the answer does not seem difficult to find. One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure 1.1, where $y = \frac{\sin x}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin 1}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.

Consider this again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure 1.2, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \rightarrow \frac{\sin 0}{0} \rightarrow \text{“}\frac{0}{0}\text{”}.$$

The expression "0/0" has no value; it is *indeterminate.* Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. Let $y = f(x)$; that is, let y be a function of x for some function f. The expression "the limit of y as x approaches 1" describes a number, often referred to as L, that y nears as x nears 1. We write all this as

$$\lim_{x\to 1} y = \lim_{x\to 1} f(x) = L.$$

This is not a complete definition (that will come in the next section); this is a pseudo-definition that will allow us to explore the idea of a limit.

Above, where $f(x) = \sin(x)/x$, we approximated

$$\lim_{x\to 1} \frac{\sin x}{x} \approx 0.84 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} \approx 1.$$

(We *approximated* these limits, hence used the "$\approx$" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

Once we have the true definition of a limit, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

Consider again $\lim_{x\to 1} \sin(x)/x$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Figure 1.3.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is in bold to highlight the fact that when considering limits, we are *not* concerned

Notes:

with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

Now approximate $\lim_{x\to 0} \sin(x)/x$ numerically. We already approximated the value of this limit as 1 graphically in Figure 1.2. The table in Figure 1.4 shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row in bold again to stress that we are not concerned with the value of our function at $x = 0$, only on the behavior of the function *near* 0.

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	**not defined**
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Figure 1.4: Values of $\sin(x)/x$ with x near 0.

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0} \sin(x)/x$; that is,

$$\lim_{x\to 0} \sin(x)/x \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

Example 1 Approximating the value of a limit

Use graphical and numerical methods to approximate

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3}.$$

SOLUTION To graphically approximate the limit, graph

$$y = (x^2 - x - 6)/(6x^2 - 19x + 3)$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figures 1.5 and 1.6, respectively.

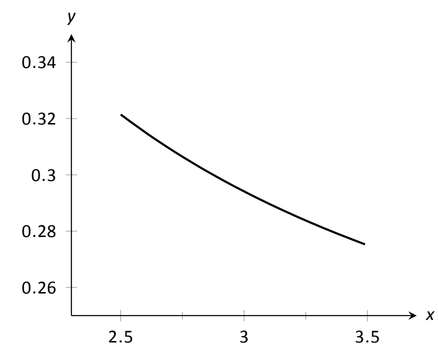


Figure 1.5: Graphically approximating a limit in Example 1.

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3} \approx 0.294.$$

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	**not defined**
3.001	0.294073
3.01	0.293669
3.1	0.289773

Figure 1.6: Numerically approximating a limit in Example 1.

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?

2. How many values of x in a table are "enough?" In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

Notes:

Graphs are useful since they give a visual understanding concerning the behavior of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically. Since graphing utilities are very accessible, it makes sense to make proper use of them.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example 1, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

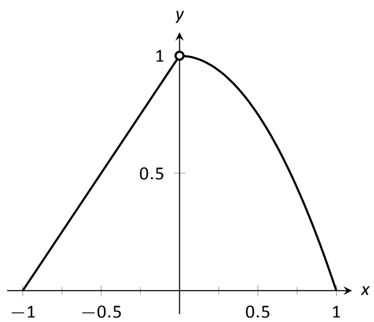


Figure 1.7: Graphically approximating a limit in Example 2.

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Figure 1.8: Numerically approximating a limit in Example 2.

Example 2 **Approximating the value of a limit**

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \left\{ \begin{array}{cc} x+1 & x < 0 \\ -x^2+1 & x > 0 \end{array} \right. .$$

Solution Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure 1.7 shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

The table shown in Figure 1.8 shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1.

Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot say $\lim_{x\to c} f(x) = L$ for some numbers L for all values of c, for there may not be a number that $f(x)$ is approaching. There are three ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.
2. The function may grow without upper or lower bound as x approaches c.
3. The function may oscillate as x approaches c.

Notes:

We'll explore each of these in turn.

Example 3 Different Values Approached From Left and Right

Explore why $\lim_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2 - 2x + 3 & x \le 1 \\ x & x > 1 \end{cases}.$$

Solution A graph of $f(x)$ around $x = 1$ and a table are given Figures 1.9 and 1.10, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behavior is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is not approaching one value as x approaches 1.

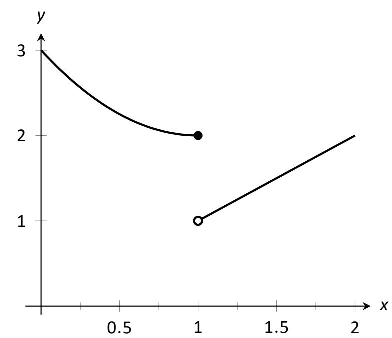


Figure 1.9: Observing no limit as $x \to 1$ in Example 3.

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Figure 1.10: Values of $f(x)$ near $x = 1$ in Example 3.

Example 4 The Function Grows Without Bound

Explore why $\lim_{x\to 1} 1/(x-1)^2$ does not exist.

Solution A graph and table of $f(x) = 1/(x-1)^2$ are given in Figures 1.11 and 1.12, respectively. Both show that as x approaches 1, $f(x)$ grows larger and larger.

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number}.$$

Since $f(x)$ is not approaching a single number, we conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

does not exist.

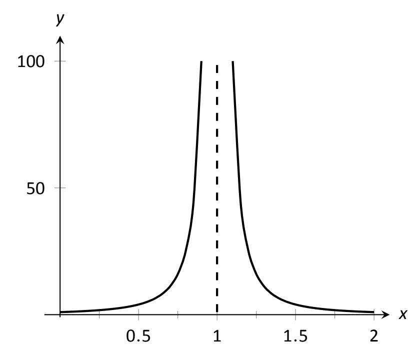


Figure 1.11: Observing no limit as $x \to 1$ in Example 4.

x	$f(x)$
0.9	100.
0.99	10000.
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	10000.
1.1	100.

Figure 1.12: Values of $f(x)$ near $x = 1$ in Example 4.

Example 5 The Function Oscillates

Explore why $\lim_{x\to 0} \sin(1/x)$ does not exist.

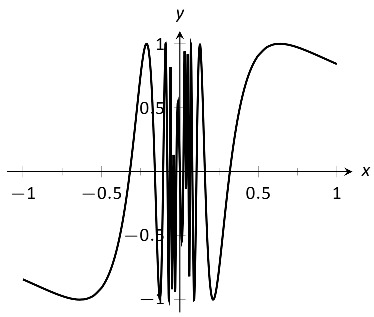
Solution Two graphs of $f(x) = \sin(1/x)$ are given in Figures 1.13. Figure 1.13(a) shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate

Notes:

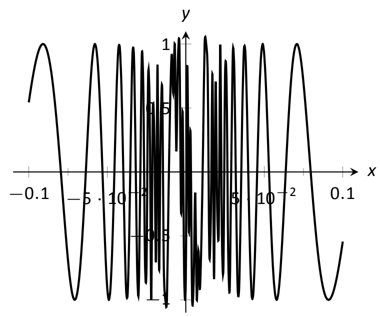
near $x = 0$. One might think that despite the oscillation, as x approaches 0, $f(x)$ approaches 0. However, Figure 1.13(b) zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in the table in Figure 1.13(c), we see $\sin(x)/x$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinite times! Because of this oscillation,

$\lim\limits_{x\to 0} \sin(1/x)$ does not exist.



(a)



(b)

x	$\sin(1/x)$
0.1	-0.544021
0.01	-0.506366
0.001	0.82688
0.0001	-0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	-0.349994
$1. \times 10^{-7}$	0.420548

(c)

Figure 1.13: Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example 5.

Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \quad \text{and} \quad f(5) = 20.$$

Since the particle traveled 10 feet in 4 seconds, we can say the particle's *average velocity* was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a *difference quotient*:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{10}{4} = 2.5\text{ft/s}.$$

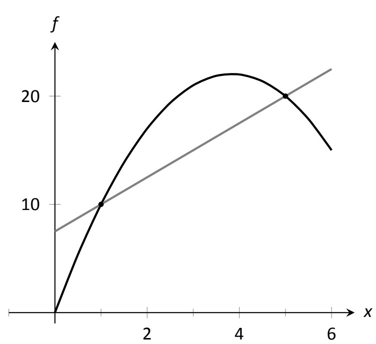


Figure 1.14: Interpreting a difference quotient as the slope of a secant line.

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This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing: given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure 1.14.

Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is, consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See Figure 1.15. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \ ?$$

As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Figure 1.16. The table gives us reason to assume the value of the limit is about 8.5.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

In the next section we give the formal definition of the limit and begin our study of finding limits analytically. In the following exercises, we continue our introduction and approximate the value of limits.

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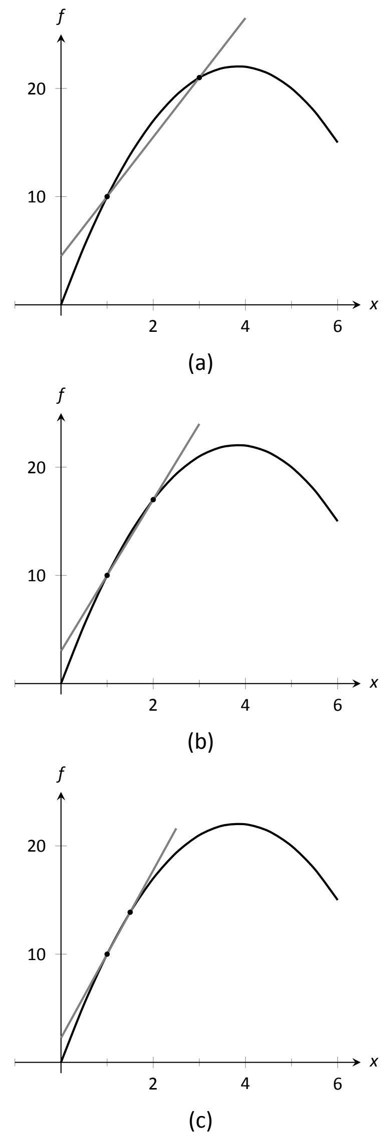


Figure 1.15: Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$).

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Figure 1.16: The difference quotient evaluated at values of h near 0.

Exercises 1.1

Terms and Concepts

1. In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?

2. An expression of the form $\frac{0}{0}$ is called ____.

3. T/F: The limit of $f(x)$ as x approaches 5 is $f(5)$.

4. Describe three situations where $\lim\limits_{x\to c} f(x)$ does not exist.

5. In your own words, what is a difference quotient?

Problems

In Exercises 6 – 16, approximate the given limits both numerically and graphically.

6. $\lim\limits_{x\to 1} x^2+3x-5$

7. $\lim\limits_{x\to 0} x^3-3x^2+x-5$

8. $\lim\limits_{x\to 0} \dfrac{x+1}{x^2+3x}$

9. $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$

10. $\lim\limits_{x\to -1} \dfrac{x^2+8x+7}{x^2+6x+5}$

11. $\lim\limits_{x\to 2} \dfrac{x^2+7x+10}{x^2-4x+4}$

12. $\lim\limits_{x\to 2} f(x)$, where

$$f(x)=\begin{cases} x+2 & x\le 2 \\ 3x-5 & x>2 \end{cases}.$$

13. $\lim\limits_{x\to 3} f(x)$, where

$$f(x)=\begin{cases} x^2-x+1 & x\le 3 \\ 2x+1 & x>3 \end{cases}.$$

14. $\lim\limits_{x\to 0} f(x)$, where

$$f(x)=\begin{cases} \cos x & x\le 0 \\ x^2+3x+1 & x>0 \end{cases}.$$

15. $\lim\limits_{x\to \pi/2} f(x)$, where

$$f(x)=\begin{cases} \sin x & x\le \pi/2 \\ \cos x & x>\pi/2 \end{cases}.$$

In Exercises 16 – 24, a function f and a value a are given. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$, using $h=\pm 0.1, \pm 0.01$.

16. $f(x)=-7x+2$, $\quad a=3$

17. $f(x)=9x+0.06$, $\quad a=-1$

18. $f(x)=x^2+3x-7$, $\quad a=1$

19. $f(x)=\dfrac{1}{x+1}$, $\quad a=2$

20. $f(x)=-4x^2+5x-1$, $\quad a=-3$

21. $f(x)=\ln x$, $\quad a=5$

22. $f(x)=\sin x$, $\quad a=\pi$

23. $f(x)=\cos x$, $\quad a=\pi$

1.2 Epsilon-Delta Definition of a Limit

This section introduces the formal definition of a limit. Many refer to this as "the epsilon–delta," definition, referring to the letters ε and δ of the Greek alphabet.

Before we give the actual definition, let's consider a few informal ways of describing a limit. Given a function $y = f(x)$ and an x-value, c, we say that "the limit of the function f, as x approaches c, is a value L":

1. if "y tends to L" as "x tends to c."

2. if "y approaches L" as "x approaches c."

3. if "y is near L" whenever "x is near c."

The problem with these definitions is that the words "tends," "approach," and especially "near" are not exact. In what way does the variable x tend to, or approach, c? How near do x and y have to be to c and L, respectively?

The definition we describe in this section comes from formalizing **3**. A quick restatement gets us closer to what we want:

3′. If x is within a certain *tolerance level* of c, then the corresponding value $y = f(x)$ is within a certain *tolerance level* of L.

The traditional notation for the x-tolerance is the lowercase Greek letter delta, or δ, and the y-tolerance is denoted by lowercase epsilon, or ε. One more rephrasing of **3′** nearly gets us to the actual definition:

3′′. If x is within δ units of c, then the corresponding value of y is within ε units of L.

We can write "x is within δ units of c" mathematically as

$$|x - c| < \delta, \qquad \text{which is equivalent to} \qquad c - \delta < x < c + \delta.$$

Letting the symbol "$\longrightarrow$" represent the word "implies," we can rewrite **3′′** as

$$|x-c| < \delta \longrightarrow |y-L| < \varepsilon \qquad \text{or} \qquad c-\delta < x < c+\delta \longrightarrow L-\varepsilon < y < L+\varepsilon.$$

The point is that δ and ε, being tolerances, can be any positive (but typically small) values. Finally, we have the formal definition of the limit with the notation seen in the previous section.

Notes:

Definition 1 The Limit of a Function f

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The **limit of $f(x)$, as x approaches c, is L**, denoted by

$$\lim_{x\to c} f(x) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

(Mathematicians often enjoy writing ideas without using any words. Here is the wordless definition of the limit:

$$\lim_{x\to c} f(x) = L \iff \forall\,\varepsilon > 0, \exists\,\delta > 0 \text{ s.t. } 0 < |x-c| < \delta \longrightarrow |f(x) - L| < \varepsilon.)$$

Note the order in which ε and δ are given. In the definition, the y-tolerance ε is given *first* and then the limit will exist **if** we can find an x-tolerance δ that works.

An example will help us understand this definition. Note that the explanation is long, but it will take one through all steps necessary to understand the ideas.

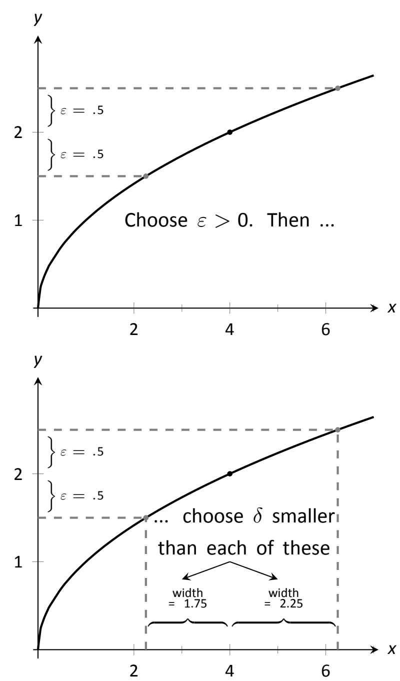
Example 6 Evaluating a limit using the definition

Show that $\displaystyle\lim_{x\to 4} \sqrt{x} = 2$.

Solution Before we use the formal definition, let's try some numerical tolerances. What if the y tolerance is 0.5, or $\varepsilon = 0.5$? How close to 4 does x have to be so that y is within 0.5 units of 2, i.e., $1.5 < y < 2.5$? In this case, we can proceed as follows:

$$\begin{aligned} 1.5 <\ & y < 2.5 \\ 1.5 <\ & \sqrt{x} < 2.5 \\ 1.5^2 <\ & x < 2.5^2 \\ 2.25 <\ & x < 6.25. \end{aligned}$$

So, what is the desired x tolerance? Remember, we want to find a symmetric interval of x values, namely $4 - \delta < x < 4 + \delta$. The lower bound of 2.25 is 1.75 units from 4; the upper bound of 6.25 is 2.25 units from 4. We need the smaller of these two distances; we must have $\delta \leq 1.75$. See Figure 1.17.



With $\varepsilon = 0.5$, we pick any $\delta < 1.75$.

Figure 1.17: Illustrating the $\varepsilon - \delta$ process.

Notes:

Given the y tolerance $\varepsilon = 0.5$, we have found an x tolerance, $\delta \leq 1.75$, such that whenever x is within δ units of 4, then y is within ε units of 2. That's what we were trying to find.

Let's try another value of ε.

What if the y tolerance is 0.01, i.e., $\varepsilon = 0.01$? How close to 4 does x have to be in order for y to be within 0.01 units of 2 (or $1.99 < y < 2.01$)? Again, we just square these values to get $1.99^2 < x < 2.01^2$, or

$$3.9601 < x < 4.0401.$$

What is the desired x tolerance? In this case we must have $\delta \leq 0.0399$, which is the minimum distance from 4 of the two bounds given above.

What we have so far: if $\varepsilon = 0.5$, then $\delta \leq 1.75$ and if $\varepsilon = 0.01$, then $\delta \leq 0.0399$. A pattern is not easy to see, so we switch to general ε try to determine δ symbolically. We start by assuming $y = \sqrt{x}$ is within ε units of 2:

$$\begin{aligned}
|y-2| &< \varepsilon & & \\
-\varepsilon < y-2 &< \varepsilon & & \text{(Definition of absolute value)}\\
-\varepsilon < \sqrt{x}-2 &< \varepsilon & & (y=\sqrt{x})\\
2-\varepsilon < \sqrt{x} &< 2+\varepsilon & & \text{(Add 2)}\\
(2-\varepsilon)^2 < x &< (2+\varepsilon)^2 & & \text{(Square all)}\\
4-4\varepsilon+\varepsilon^2 < x &< 4+4\varepsilon+\varepsilon^2 & & \text{(Expand)}\\
4-(4\varepsilon-\varepsilon^2) < x &< 4+(4\varepsilon+\varepsilon^2). & & \text{(Rewrite in the desired form)}
\end{aligned}$$

The "desired form" in the last step is "$4 -$ *something* $< x < 4 +$ *something*." Since we want this last interval to describe an x tolerance around 4, we have that either $\delta \leq 4\varepsilon - \varepsilon^2$ or $\delta \leq 4\varepsilon + \varepsilon^2$, whichever is smaller:

$$\delta \leq \min\{4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2\}.$$

Since $\varepsilon > 0$, the minimum is $\delta \leq 4\varepsilon - \varepsilon^2$. That's the formula: given an ε, set $\delta \leq 4\varepsilon - \varepsilon^2$.

We can check this for our previous values. If $\varepsilon = 0.5$, the formula gives $\delta \leq 4(0.5) - (0.5)^2 = 1.75$ and when $\varepsilon = 0.01$, the formula gives $\delta \leq 4(0.01) - (0.01)^2 = 0.399$.

So given any $\varepsilon > 0$, set $\delta \leq 4\varepsilon - \varepsilon^2$. Then if $|x - 4| < \delta$ (and $x \neq 4$), then $|f(x) - 2| < \varepsilon$, satisfying the definition of the limit. We have shown formally (and finally!) that $\lim\limits_{x\to 4} \sqrt{x} = 2$.

Notes:

The previous example was a little long in that we sampled a few specific cases of ε before handling the general case. Normally this is not done. The previous example is also a bit unsatisfying in that $\sqrt{4} = 2$; why work so hard to prove something so obvious? Many ε-δ proofs are long and difficult to do. In this section, we will focus on examples where the answer is, frankly, obvious, because the non–obvious examples are even harder. In the next section we will learn some theorems that allow us to evaluate limits *analytically*, that is, without using the ε-δ definition.

Example 7 Evaluating a limit using the definition

Show that $\lim\limits_{x\to 2} x^2 = 4$.

Solution Let's do this example symbolically from the start. Let $\varepsilon > 0$ be given; we want $|y-4| < \varepsilon$, i.e., $|x^2-4| < \varepsilon$. How do we find δ such that when $|x-2| < \delta$, we are guaranteed that $|x^2-4| < \varepsilon$?

This is a bit trickier than the previous example, but let's start by noticing that $|x^2-4| = |x-2| \cdot |x+2|$. Consider:

$$|x^2-4| < \varepsilon \longrightarrow |x-2| \cdot |x+2| < \varepsilon \longrightarrow |x-2| < \frac{\varepsilon}{|x+2|}. \tag{1.1}$$

Could we not set $\delta = \dfrac{\varepsilon}{|x+2|}$?

We are close to an answer, but the catch is that δ must be a *constant* value (so it can't contain x). There is a way to work around this, but we do have to make an assumption. Remember that ε is supposed to be a small number, which implies that δ will also be a small value. In particular, we can (probably) assume that $\delta < 1$. If this is true, then $|x-2| < \delta$ would imply that $|x-2| < 1$, giving $1 < x < 3$.

Now, back to the fraction $\dfrac{\varepsilon}{|x+2|}$. If $1 < x < 3$, then $3 < x+2 < 5$ (add 2 to all terms in the inequality). Taking reciprocals, we have

$$\begin{aligned} &\frac{1}{5} < \frac{1}{|x+2|} < \frac{1}{3} && \text{which implies}\\ &\frac{1}{5} < \frac{1}{|x+2|} && \text{which implies}\\ &\frac{\varepsilon}{5} < \frac{\varepsilon}{|x+2|}. && \end{aligned} \tag{1.2}$$

This suggests that we set $\delta \leq \dfrac{\varepsilon}{5}$. To see why, let consider what follows when we assume $|x-2| < \delta$:

Notes:

$$
\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \frac{\varepsilon}{5} && \text{(Our choice of } \delta\text{)} \\
|x-2|\cdot|x+2| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Multiply by } |x+2|\text{)} \\
|x^2-4| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Combine left side)} \\
|x^2-4| &< |x+2|\cdot\frac{\varepsilon}{5} < |x+2|\cdot\frac{\varepsilon}{|x+2|} = \varepsilon && \text{(Using (1.2) as long as } \delta < 1\text{)}
\end{aligned}
$$

We have arrived at $|x^2-4| < \varepsilon$ as desired. Note again, in order to make this happen we needed δ to first be less than 1. That is a safe assumption; we want ε to be arbitrarily small, forcing δ to also be small.

We have also picked δ to be smaller than "necessary." We could get by with a slightly larger δ, as shown in Figure 1.18. The dashed outer lines show the boundaries defined by our choice of ε. The dotted inner lines show the boundaries defined by setting $\delta = \varepsilon/5$. Note how these dotted lines are within the dashed lines. That is perfectly fine; by choosing x within the dotted lines we are guaranteed that $f(x)$ will be within ε of 4.

In summary, given $\varepsilon > 0$, set $\delta =\leq \varepsilon/5$. Then $|x-2| < \delta$ implies $|x^2-4| < \varepsilon$ (i.e. $|y-4| < \varepsilon$) as desired. This shows that $\lim\limits_{x\to 2} x^2 = 4$. Figure 1.18 gives a visualization of this; by restricting x to values within $\delta = \varepsilon/5$ of 2, we see that $f(x)$ is within ε of 4.

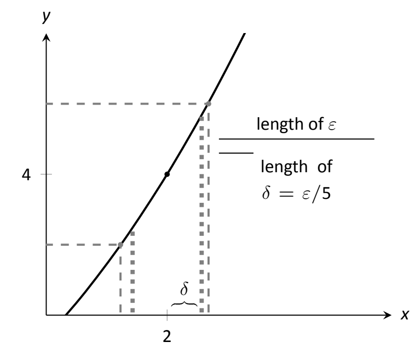


Figure 1.18: Choosing $\delta = \varepsilon/5$ in Example 7.

Make note of the general pattern exhibited in these last two examples. In some sense, each starts out "backwards." That is, while we want to

1. start with $|x-c| < \delta$ and conclude that
2. $|f(x)-L| < \varepsilon$,

we actually start by assuming

1. $|f(x)-L| < \varepsilon$, then perform some algebraic manipulations to give an inequality of the form
2. $|x-c| <$ *something*.

When we have properly done this, the *something* on the "greater than" side of the inequality becomes our δ. We can refer to this as the "scratch–work" phase of our proof. Once we have δ, we can formally start with $|x-c| < \delta$ and use algebraic manipulations to conclude that $|f(x)-L| < \varepsilon$, usually by using the same steps of our "scratch–work" in reverse order.

Notes:

We highlight this process in the following example.

Example 8 **Evaluating a limit using the definition**

Prove that $\lim\limits_{x\to 1} x^3 - 2x = -1$.

Solution We start our scratch–work by considering $|f(x) - (-1)| < \varepsilon$:

$$\begin{aligned} |f(x) - (-1)| &< \varepsilon \\ |x^3 - 2x + 1| &< \varepsilon && \text{(Now factor)} \\ |(x-1)(x^2 + x - 1)| &< \varepsilon \\ |x - 1| &< \frac{\varepsilon}{|x^2 + x - 1|}. && (1.3) \end{aligned}$$

We are at the phase of saying that $|x - 1| < $ *something*, where *something*$=$ $\varepsilon/|x^2 + x - 1|$. We want to turn that *something* into δ.

Since x is approaching 1, we are safe to assume that x is between 0 and 2. So

$$\begin{aligned} &0 < x < 2 \\ &0 < x^2 < 4. && \text{(squared each term)} \end{aligned}$$

Since $0 < x < 2$, we can add 0, x and 2, respectively, to each part of the inequality and maintain the inequality.

$$\begin{aligned} 0 < x^2 + x &< 6 \\ -1 < x^2 + x - 1 &< 5. && \text{(subtracted 1 from each part)} \end{aligned}$$

In Equation (1.3), we wanted $|x - 1| < \varepsilon/|x^2 + x - 1|$. The above shows that given any x in $[0, 2]$, we know that

$$\begin{aligned} x^2 + x - 1 &< 5 && \text{which implies that} \\ \frac{1}{5} &< \frac{1}{x^2 + x - 1} && \text{which implies that} \\ \frac{\varepsilon}{5} &< \frac{\varepsilon}{x^2 + x - 1}. && (1.4) \end{aligned}$$

So we set $\delta \leq \varepsilon/5$. This ends our scratch–work, and we begin the formal proof (which also helps us understand why this was a good choice of δ).

Given ε, let $\delta \leq \varepsilon/5$. We want to show that when $|x - 1| < \delta$, then $|(x^3 -$

Notes:

$2x) - (-1)| < \varepsilon$. We start with $|x - 1| < \delta$:

$$\begin{aligned}
|x-1| &< \delta \\
|x-1| &< \frac{\varepsilon}{5} \\
|x-1| &< \frac{\varepsilon}{5} < \frac{\varepsilon}{|x^2+x-1|} \quad \text{(for } x \text{ near 1, from Equation (1.4))} \\
|x-1| \cdot |x^2+x-1| &< \varepsilon \\
|x^3-2x+1| &< \varepsilon \\
|(x^3-2x)-(-1)| &< \varepsilon,
\end{aligned}$$

which is what we wanted to show. Thus $\lim\limits_{x\to 1} x^3 - 2x = -1$.

We illustrate evaluating limits once more.

Example 9 **Evaluating a limit using the definition**

Prove that $\lim\limits_{x\to 0} e^x = 1$.

Solution Symbolically, we want to take the equation $|e^x - 1| < \varepsilon$ and unravel it to the form $|x - 0| < \delta$. Here is our scratch–work:

$$\begin{aligned}
|e^x-1| &< \varepsilon \\
-\varepsilon < e^x - 1 &< \varepsilon \qquad \text{(Definition of absolute value)} \\
1-\varepsilon < e^x &< 1+\varepsilon \qquad \text{(Add 1)} \\
\ln(1-\varepsilon) < x &< \ln(1+\varepsilon) \qquad \text{(Take natural logs)}
\end{aligned}$$

Note: Recall $\ln 1 = 0$ and $\ln x < 0$ when $0 < x < 1$. So $\ln(1-\varepsilon) < 0$, hence we consider its absolute value.

Making the safe assumption that $\varepsilon < 1$ ensures the last inequality is valid (i.e., so that $\ln(1-\varepsilon)$ is defined). We can then set δ to be the minimum of $|\ln(1-\varepsilon)|$ and $\ln(1+\varepsilon)$; i.e.,

$$\delta = \min\{|\ln(1-\varepsilon)|, \ln(1+\varepsilon)\} = \ln(1+\varepsilon).$$

Now, we work through the actual the proof:

$$\begin{aligned}
|x-0| &< \delta \\
-\delta < x &< \delta \qquad \text{(Definition of absolute value)} \\
-\ln(1+\varepsilon) < x &< \ln(1+\varepsilon). \\
\ln(1-\varepsilon) < x &< \ln(1+\varepsilon). \qquad \text{(since } \ln(1-\varepsilon) < -\ln(1+\varepsilon))
\end{aligned}$$

Notes:

The above line is true by our choice of δ and by the fact that since $|\ln(1-\varepsilon)| > \ln(1+\varepsilon)$ and $\ln(1-\varepsilon) < 0$, we know $\ln(1-\varepsilon) < -\ln(1+\varepsilon)$.

$$1-\varepsilon < e^x < 1+\varepsilon \qquad \text{(Exponentiate)}$$
$$-\varepsilon < e^x - 1 < \varepsilon \qquad \text{(Subtract 1)}$$

In summary, given $\varepsilon > 0$, let $\delta = \ln(1+\varepsilon)$. Then $|x-0| < \delta$ implies $|e^x - 1| < \varepsilon$ as desired. We have shown that $\lim\limits_{x\to 0} e^x = 1$.

We note that we could actually show that $\lim_{x\to c} e^x = e^c$ for any constant c. We do this by factoring out e^c from both sides, leaving us to show $\lim_{x\to c} e^{x-c} = 1$ instead. By using the substitution $u = x-c$, this reduces to showing $\lim_{u\to 0} e^u = 1$ which we just did in the last example. As an added benefit, this shows that in fact the function $f(x) = e^x$ is *continuous* at all values of x, an important concept we will define in Section 1.5.

This formal definition of the limit is not an easy concept grasp. Our examples are actually "easy" examples, using "simple" functions like polynomials, square–roots and exponentials. It is very difficult to prove, using the techniques given above, that $\lim\limits_{x\to 0}(\sin x)/x = 1$, as we approximated in the previous section.

There is hope. The next section shows how one can evaluate complicated limits using certain basic limits as building blocks. While limits are an incredibly important part of calculus (and hence much of higher mathematics), rarely are limits evaluated using the definition. Rather, the techniques of the following section are employed.

Notes:

Exercises 1.2

Terms and Concepts

1. What is wrong with the following "definition" of a limit?

   > "The limit of $f(x)$, as x approaches a, is K" means that given any $\delta > 0$ there exists $\varepsilon > 0$ such that whenever $|f(x) - K| < \varepsilon$, we have $|x - a| < \delta$.

2. Which is given first in establishing a limit, the *x*–tolerance or the *y*–tolerance?

3. T/F: ε must always be positive.

4. T/F: δ must always be positive.

Problems

In Exercises 5 – 11, prove the given limit using an $\varepsilon - \delta$ proof.

5. $\lim\limits_{x \to 5} 3 - x = -2$

6. $\lim\limits_{x \to 3} x^2 - 3 = 6$

7. $\lim\limits_{x \to 4} x^2 + x - 5 = 15$

8. $\lim\limits_{x \to 2} x^3 - 1 = 7$

9. $\lim\limits_{x \to 2} 5 = 5$

10. $\lim\limits_{x \to 0} e^{2x} - 1 = 0$

11. $\lim\limits_{x \to 0} \sin x = 0$ (Hint: use the fact that $|\sin x| \leq |x|$, with equality only when $x = 0$.)

1.3 Finding Limits Analytically

In Section 1.1 we explored the concept of the limit without a strict definition, meaning we could only make approximations. In the previous section we gave the definition of the limit and demonstrated how to use it to verify our approximations were correct. Thus far, our method of finding a limit is 1) make a really good approximation either graphically or numerically, and 2) verify our approximation is correct using a ε-δ proof.

Recognizing that ε-δ proofs are cumbersome, this section gives a series of theorems which allow us to find limits much more quickly and intuitively.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x) + g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 1 Basic Limit Properties

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim\limits_{x\to c} b = b$
2. Identity $\lim\limits_{x\to c} x = c$
3. Sums/Differences: $\lim\limits_{x\to c}(f(x) \pm g(x)) = L \pm K$
4. Scalar Multiples: $\lim\limits_{x\to c} b \cdot f(x) = bL$
5. Products: $\lim\limits_{x\to c} f(x) \cdot g(x) = LK$
6. Quotients: $\lim\limits_{x\to c} f(x)/g(x) = L/K$, $(K \neq 0)$
7. Powers: $\lim\limits_{x\to c} f(x)^n = L^n$
8. Roots: $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
9. Compositions: Adjust our previously given limit situation to:

   $$\lim_{x\to c} f(x) = L,\ \lim_{x\to L} g(x) = K \text{ and } g(L) = K.$$

   Then $\lim\limits_{x\to c} g(f(x)) = K$.

Notes:

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L.

We apply the theorem to an example.

Example 10 **Using basic limit properties**

Let

$$\lim_{x\to 2} f(x) = 2, \quad \lim_{x\to 2} g(x) = 3 \quad \text{and} \quad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim_{x\to 2} \big(f(x) + g(x)\big)$
2. $\lim_{x\to 2} \big(5f(x) + g(x)^2\big)$
3. $\lim_{x\to 2} p(x)$

Solution

1. Using the Sum/Difference rule, we know that $\lim_{x\to 2} \big(f(x)+g(x)\big) = 2+3 = 5$.

2. Using the Scalar Multiple and Sum/Difference rules, we find that $\lim_{x\to 2} \big(5f(x)+g(x)^2\big) = 5\cdot 2 + 3^2 = 19$.

3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}\lim_{x\to 2} p(x) &= \lim_{x\to 2}(3x^2 - 5x + 7)\\ &= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7\\ &= 3\cdot 2^2 - 5\cdot 2 + 7\\ &= 9\end{aligned}$$

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 1. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Notes:

Theorem 2 Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim\limits_{x\to c} p(x) = p(c)$
2. $\lim\limits_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$, where $q(c) \neq 0$.

Example 11 Finding a limit of a rational function

Using Theorem 2, find

$$\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3}.$$

Solution Using Theorem 2, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3} &= \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

It was likely frustrating in Section 1.2 to do a lot of work to prove that

$$\lim_{x\to 2} x^2 = 4$$

as it seemed fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example 11.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behavior is particularly "nice" in terms of limits. In the next section, we will give a formal name to these functions that behave "nicely."

Theorem 3 Special Limits

Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim\limits_{x\to c} \sin x = \sin c$
2. $\lim\limits_{x\to c} \cos x = \cos c$
3. $\lim\limits_{x\to c} \tan x = \tan c$
4. $\lim\limits_{x\to c} \csc x = \csc c$
5. $\lim\limits_{x\to c} \sec x = \sec c$
6. $\lim\limits_{x\to c} \cot x = \cot c$
7. $\lim\limits_{x\to c} a^x = a^c$ $(a > 0)$
8. $\lim\limits_{x\to c} \ln x = \ln c$
9. $\lim\limits_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Notes:

Example 12 Evaluating limits analytically

Evaluate the following limits.

1. $\lim\limits_{x\to\pi} \cos x$
2. $\lim\limits_{x\to 3} (\sec^2 x - \tan^2 x)$
3. $\lim\limits_{x\to\pi/2} \cos x \sin x$
4. $\lim\limits_{x\to 1} e^{\ln x}$
5. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

Solution

1. This is a straightforward application of Theorem 3. $\lim\limits_{x\to\pi} \cos x = \cos\pi = -1$.
2. We can approach this in at least two ways. First, by directly applying Theorem 3, we have:
$$\lim_{x\to 3}(\sec^2 x - \tan^2 x) = \sec^2 3 - \tan^2 3.$$
Using the Pythagorean Theorem, this last expression is 1; therefore
$$\lim_{x\to 3}(\sec^2 x - \tan^2 x) = 1.$$
We can also use the Pythagorean Theorem from the start.
$$\lim_{x\to 3}(\sec^2 x - \tan^2 x) = \lim_{x\to 3} 1 = 1,$$
using the Constant limit rule. Either way, we find the limit is 1.
3. Applying the Product limit rule of Theorem 1 and Theorem 3 gives
$$\lim_{x\to\pi/2} \cos x \sin x = \cos(\pi/2)\sin(\pi/2) = 0\cdot 1 = 0.$$
4. Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln x} = x$ and evaluate $\lim\limits_{x\to 1} e^{\ln x} = \lim\limits_{x\to 1} x = 1$.
We can also use the limit Composition Rule of Theorem 1. Using Theorem 3, we have $\lim\limits_{x\to 1} \ln x = \ln 1 = 0$ and $\lim\limits_{x\to 0} e^x = e^0 = 1$, satisfying the conditions of the Composition Rule. Applying this rule,
$$\lim_{x\to 1} e^{\ln x} = \lim_{x\to 0} e^x = e^0 = 1.$$
Both approaches are valid, giving the same result.

Notes:

5. We encountered this limit in Section 1.1. Applying our theorems, we attempt to find the limit as
$$\lim_{x\to 0}\frac{\sin x}{x} \to \frac{\sin 0}{0} \to \text{“}\frac{0}{0}\text{”}.$$
This, of course, violates a condition of Theorem 1, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

The section could have been titled “Using Known Limits to Find Unknown Limits.” By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,
$$f(x) \le g(x) \le h(x).$$
If f and h have the same limit at c, and g is always “squeezed” between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

Theorem 4 Squeeze Theorem

Let f, g and h be functions on an open interval I containing c such that for all x in I,
$$f(x) \le g(x) \le h(x).$$
If
$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$
then
$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to “squeeze” the given function of which you are trying to evaluate a limit. However, that is generally the only place work is necessary; the theorem makes the “evaluating the limit part” very simple.

We use the Squeeze Theorem in the following example to finally prove that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Notes:

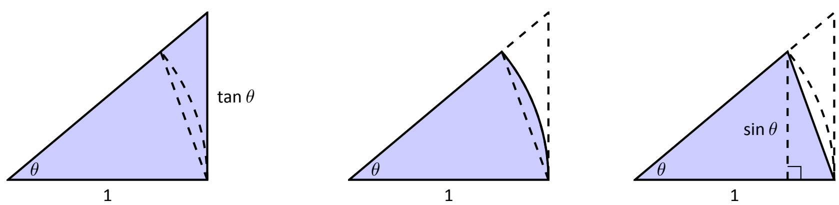
Example 13 Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Solution We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 1.19. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \le \theta \le \pi/2$. Later we will show that we can also consider $\theta \le 0$.)

Figure 1.19 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that



$$\frac{\tan\theta}{2} \quad \ge \quad \frac{\theta}{2} \quad \ge \quad \frac{\sin\theta}{2}$$

Multiply all terms by $\dfrac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta}\ge\frac{\theta}{\sin\theta}\ge 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta\le\frac{\sin\theta}{\theta}\le 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$.)

Now take limits.

$$\lim_{\theta\to 0}\cos\theta\le\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\le\lim_{\theta\to 0}1$$

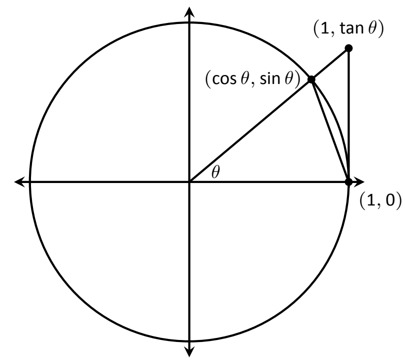


Figure 1.19: The unit circle and related triangles.

Notes:

$$\cos 0 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$$

Clearly this means that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Two notes about the previous example are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin x$ are approaching 0, but the *ratio* of x and $\sin x$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin x$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 5 Special Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
3. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
4. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos x$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos x$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

Notes:

Our final theorem for this section will be motivated by the following example.

Example 14 Using algebra to evaluate a limit

Evaluate the following limit:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}.$$

Solution We begin by attempting to apply Theorem 3 and substituting 1 for x in the quotient. This gives:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\frac{1^2-1}{1-1}=\text{“}\frac{0}{0}\text{”},$$

and indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure 1.20, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x=1$, but for all other x,

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}=\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}=x+1.$$

Clearly $\lim\limits_{x\to 1}x+1=2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except $x=1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2.$$

The key to the above example is that the functions $y=(x^2-1)/(x-1)$ and $y=x+1$ are identical except at $x=1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

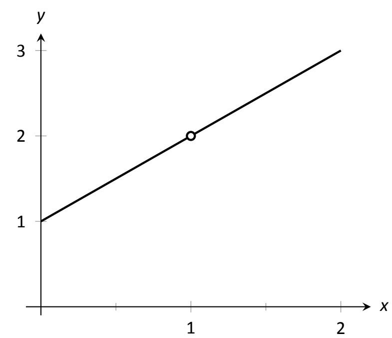


Figure 1.20: Graphing f in Example 14 to understand a limit.

Notes:

Theorem 6 Limits of Functions Equal At All But One Point

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim_{x\to c} \frac{g(x)}{f(x)}$ returns "0/0", then $(x - c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this term out, cancel, then apply Theorem 6. We demonstrate this once more.

Example 15 Evaluating a limit using Theorem 6

Evaluate $\lim_{x\to 3} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15}$.

Solution We begin by applying Theorem 3 and substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15} = \frac{(x-3)(x^2 + x - 2)}{(x-3)(2x^2 + 9x - 5)}.$$

We can cancel the $(x-3)$ terms as long as $x \neq 3$. Using Theorem 6 we conclude:

$$\begin{aligned}
\lim_{x\to 3} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15} &= \lim_{x\to 3} \frac{(x-3)(x^2 + x - 2)}{(x-3)(2x^2 + 9x - 5)} \\
&= \lim_{x\to 3} \frac{(x^2 + x - 2)}{(2x^2 + 9x - 5)} \\
&= \frac{10}{40} = \frac{1}{4}.
\end{aligned}$$

We end this section by revisiting a limit first seen in Section 1.1, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Notes:

Example 16 **Evaluating the limit of a difference quotient**

Let $f(x) = -1.5x^2 + 11.5x$; find $\displaystyle\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Solution Since f is a polynomial, our first attempt should be to employ Theorem 3 and substitute 0 for h. However, we see that this gives us "$0/0$." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{-1.5(1+h)^2 + 11.5(1+h) - \big(-1.5(1)^2 + 11.5(1)\big)}{h} \\
&= \lim_{h\to 0} \frac{-1.5(1+2h+h^2) + 11.5 + 11.5h - 10}{h} \\
&= \lim_{h\to 0} \frac{-1.5h^2 + 8.5h}{h} \\
&= \lim_{h\to 0} \frac{h(-1.5h+8.5)}{h} \\
&= \lim_{h\to 0} (-1.5h + 8.5) \quad \big(\text{using Theorem 6, as } h \neq 0\big) \\
&= 8.5 \quad \big(\text{using Theorem 3}\big)
\end{aligned}$$

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem 3; it states that many functions that we use regularly behave in a very nice, predictable way. In the next section we give a name to this nice behavior; we label such functions as *continuous.* Defining that term will require us to look again at what a limit is and what causes limits to not exist.

Notes:

Exercises 1.3

Terms and Concepts

1. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} b = b$.

2. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} x = c$.

3. What does the text mean when it says that certain functions' "behavior is 'nice' in terms of limits"? What, in particular, is "nice"?

4. Sketch a graph that visually demonstrates the Squeeze Theorem.

5. You are given the following information:

   (a) $\lim\limits_{x\to 1} f(x) = 0$

   (b) $\lim\limits_{x\to 1} g(x) = 0$

   (c) $\lim\limits_{x\to 1} f(x)/g(x) = 2$

   What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?

Problems

In Exercises 6 – 13, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim\limits_{x\to 9} f(x) = 6$, $\lim\limits_{x\to 6} f(x) = 9$, $f(9) = 6$
- $\lim\limits_{x\to 9} g(x) = 3$, $\lim\limits_{x\to 6} g(x) = 3$, $g(6) = 9$

6. $\lim\limits_{x\to 9} (f(x) + g(x))$

7. $\lim\limits_{x\to 9} (3f(x)/g(x))$

8. $\lim\limits_{x\to 9} \left(\dfrac{f(x) - 2g(x)}{g(x)}\right)$

9. $\lim\limits_{x\to 6} \left(\dfrac{f(x)}{3 - g(x)}\right)$

10. $\lim\limits_{x\to 9} g\big(f(x)\big)$

11. $\lim\limits_{x\to 6} f\big(g(x)\big)$

12. $\lim\limits_{x\to 6} g\big(f(f(x))\big)$

13. $\lim\limits_{x\to 6} f(x)g(x) - f^2(x) + g^2(x)$

In Exercises 14 – 17, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim\limits_{x\to 1} f(x) = 2$, $\lim\limits_{x\to 10} f(x) = 1$, $f(1) = 1/5$
- $\lim\limits_{x\to 1} g(x) = 0$, $\lim\limits_{x\to 10} g(x) = \pi$, $g(10) = \pi$

14. $\lim\limits_{x\to 1} f(x)^{g(x)}$

15. $\lim\limits_{x\to 10} \cos\big(g(x)\big)$

16. $\lim\limits_{x\to 1} f(x)g(x)$

17. $\lim\limits_{x\to 1} g\big(5f(x)\big)$

In Exercises 18 – 32, evaluate the given limit.

18. $\lim\limits_{x\to 3} x^2 - 3x + 7$

19. $\lim\limits_{x\to \pi} \left(\dfrac{x-3}{x-5}\right)^7$

20. $\lim\limits_{x\to \pi/4} \cos x \sin x$

21. $\lim\limits_{x\to 0} \ln x$

22. $\lim\limits_{x\to 3} 4^{x^3 - 8x}$

23. $\lim\limits_{x\to \pi/6} \csc x$

24. $\lim\limits_{x\to 0} \ln(1 + x)$

25. $\lim\limits_{x\to \pi} \dfrac{x^2 + 3x + 5}{5x^2 - 2x - 3}$

26. $\lim\limits_{x\to \pi} \dfrac{3x + 1}{1 - x}$

27. $\lim\limits_{x\to 6} \dfrac{x^2 - 4x - 12}{x^2 - 13x + 42}$

28. $\lim\limits_{x\to 0} \dfrac{x^2 + 2x}{x^2 - 2x}$

29. $\lim\limits_{x\to 2} \dfrac{x^2 + 6x - 16}{x^2 - 3x + 2}$

30. $\lim\limits_{x\to 2} \dfrac{x^2 - 10x + 16}{x^2 - x - 2}$

31. $\lim\limits_{x\to -2} \dfrac{x^2 - 5x - 14}{x^2 + 10x + 16}$

32. $\lim\limits_{x\to -1} \dfrac{x^2 + 9x + 8}{x^2 - 6x - 7}$

Use the Squeeze Theorem in Exercises 33 – 36, where appropriate, to evaluate the given limit.

33. $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

34. $\lim_{x\to 0} \sin x\cos\left(\frac{1}{x^2}\right)$

35. $\lim_{x\to 1} f(x)$, where $3x-2 \leq f(x) \leq x^3$.

36. $\lim_{x\to 3} f(x)$, where $6x-9 \leq f(x) \leq x^2$.

Exercises 37 – 41 challenge your understanding of limits but can be evaluated using the knowledge gained in this section.

37. $\lim_{x\to 0} \frac{\sin 3x}{x}$

38. $\lim_{x\to 0} \frac{\sin 5x}{8x}$

39. $\lim_{x\to 0} \frac{\ln(1+x)}{x}$

40. $\lim_{x\to 0} \frac{\sin x}{x}$, where x is measured in degrees, not radians.

41. Let $f(x) = 0$ and $g(x) = \frac{x}{x}$.

   (a) Show why $\lim_{x\to 2} f(x) = 0$.

   (b) Show why $\lim_{x\to 0} g(x) = 1$.

   (c) Show why $\lim_{x\to 2} g\big(f(x)\big)$ does not exist.

   (d) Show why the answer to part (c) does not violate the Composition Rule of Theorem 1.

1.4 One Sided Limits

We introduced the concept of a limit gently, approximating their values graphically and numerically. Next came the rigorous definition of the limit, along with an admittedly tedious method for evaluating them. The previous section gave us tools (which we call theorems) that allow us to compute limits with greater ease. Chief among the results were the facts that polynomials and rational, trigonometric, exponential and logarithmic functions (and their sums, products, etc.) all behave "nicely." In this section we rigorously define what we mean by "nicely."

In Section 1.1 we explored the three ways in which limits of functions failed to exist:

1. The function approached different values from the left and right,
2. The function grows without bound, and
3. The function oscillates.

In this section we explore in depth the concepts behind #1 by introducing the *one-sided limit*. We begin with formal definitions that are very similar to the definition of the limit given in Section 1.2, but the notation is slightly different and "$x \neq c$" is replaced with either "$x < c$" or "$x > c$."

Definition 2 One Sided Limits

Left-Hand Limit
Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The **limit of** $f(x)$**, as** x **approaches** c **from the left, is** L, or, **the left–hand limit of** f **at** c **is** L, denoted by

$$\lim_{x\to c^-} f(x) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x < c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Right-Hand Limit
Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The **limit of** $f(x)$**, as** x **approaches** c **from the right, is** L, or, **the right–hand limit of** f **at** c **is** L, denoted by

$$\lim_{x\to c^+} f(x) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x > c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Notes:

Practically speaking, when evaluating a left-hand limit, we consider only values of x "to the left of c," i.e., where $x < c$. The admittedly imperfect notation $x \to c^-$ is used to imply that we look at values of x to the left of c. The notation has nothing to do with positive or negative values of either x or c. A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of c, i.e., $x > c$. We can use the theorems from previous sections to help us evaluate these limits; we just restrict our view to one side of c.

We practice evaluating left and right-hand limits through a series of examples.

Example 17 **Evaluating one sided limits**

Let $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 3-x & 1 < x < 2 \end{cases}$, as shown in Figure 1.21. Find each of the following:

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x\to 2^-} f(x)$
8. $f(2)$

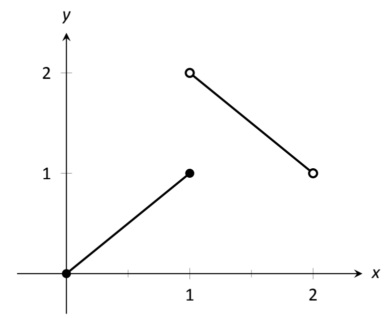


Figure 1.21: A graph of f in Example 17.

Solution For these problems, the visual aid of the graph is likely more effective in evaluating the limits than using f itself. Therefore we will refer often to the graph.

1. As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x\to 1^-} f(x) = 1$.
2. As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function. Therefore $\lim\limits_{x\to 1^+} f(x) = 2$.
3. *The* limit of f as x approaches 1 does not exist, as discussed in the first section. The function does not approach one particular value, but two different values from the left and the right.
4. Using the definition and by looking at the graph we see that $f(1) = 1$.
5. As x goes to 0 from the right, we see that $f(x)$ is also approaching 0. Therefore $\lim\limits_{x\to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.

Notes:

6. Using the definition and the graph, $f(0) = 0$.

7. As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x\to 2^-} f(x) = 1$.

8. The graph and the definition of the function show that $f(2)$ is not defined.

Note how the left and right-hand limits were different at $x = 1$. This, of course, causes *the* limit to not exist. The following theorem states what is fairly intuitive: *the* limit exists precisely when the left and right-hand limits are equal.

Theorem 7 Limits and One Sided Limits

Let f be a function defined on an open interval I containing c. Then

$$\lim_{x\to c} f(x) = L$$

if, and only if,

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

The phrase "if, and only if" means the two statements are *equivalent*: they are either both true or both false. If the limit equals L, then the left and right hand limits both equal L. If the limit is not equal to L, then at least one of the left and right-hand limits is not equal to L (it may not even exist).

One thing to consider in Examples 17 – 20 is that the value of the function may/may not be equal to the value(s) of its left/right-hand limits, even when these limits agree.

Example 18 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} 2-x & 0 < x < 1 \\ (x-2)^2 & 1 < x < 2 \end{cases}$, as shown in Figure 1.22. Evaluate the following.

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x\to 2^-} f(x)$
8. $f(2)$

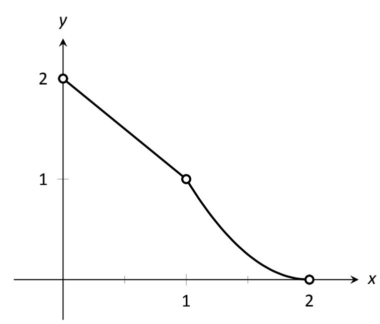


Figure 1.22: A graph of f from Example 18

Notes:

Solution Again we will evaluate each using both the definition of f and its graph.

1. As x approaches 1 from the left, we see that $f(x)$ approaches 1. Therefore $\lim\limits_{x\to1^-} f(x) = 1$.
2. As x approaches 1 from the right, we see that again $f(x)$ approaches 1. Therefore $\lim\limits_{x\to1+} f(x) = 1$.
3. *The* limit of f as x approaches 1 exists and is 1, as f approaches 1 from both the right and left. Therefore $\lim\limits_{x\to1} f(x) = 1$.
4. $f(1)$ is not defined. Note that 1 is not in the domain of f as defined by the problem, which is indicated on the graph by an open circle when $x = 1$.
5. As x goes to 0 from the right, $f(x)$ approaches 2. So $\lim\limits_{x\to0^+} f(x) = 2$.
6. $f(0)$ is not defined as 0 is not in the domain of f.
7. As x goes to 2 from the left, $f(x)$ approaches 0. So $\lim\limits_{x\to2^-} f(x) = 0$.
8. $f(2)$ is not defined as 2 is not in the domain of f.

Example 19 **Evaluating limits of a piecewise–defined function**

Let $f(x) = \begin{cases} (x-1)^2 & 0 \leq x \leq 2, x \neq 1 \\ 1 & x = 1 \end{cases}$, as shown in Figure 1.23. Evaluate the following.

1. $\lim\limits_{x\to1^-} f(x)$
2. $\lim\limits_{x\to1^+} f(x)$
3. $\lim\limits_{x\to1} f(x)$
4. $f(1)$

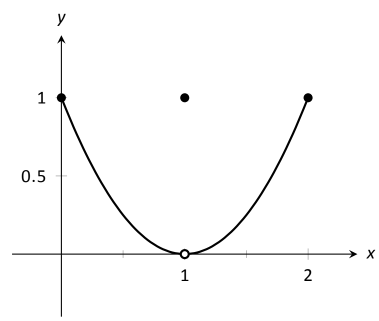


Figure 1.23: Graphing f in Example 19

Solution It is clear by looking at the graph that both the left and right-hand limits of f, as x approaches 1, is 0. Thus it is also clear that *the* limit is 0; i.e., $\lim\limits_{x\to1} f(x) = 0$. It is also clearly stated that $f(1) = 1$.

Example 20 **Evaluating limits of a piecewise–defined function**

Let $f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \end{cases}$, as shown in Figure 1.24. Evaluate the following.

Notes:

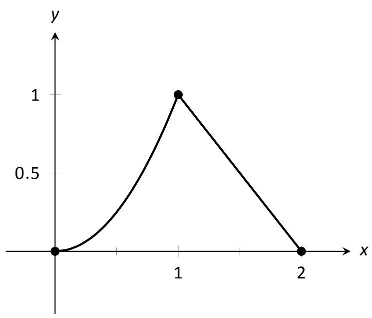


Figure 1.24: Graphing f in Example 20

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$

Solution It is clear from the definition of the function and its graph that all of the following are equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1) = 1.$$

In Examples 17 – 20 we were asked to find both $\lim\limits_{x\to 1} f(x)$ and $f(1)$. Consider the following table:

	$\lim\limits_{x\to 1} f(x)$	$f(1)$
Example 17	does not exist	1
Example 18	1	not defined
Example 19	0	1
Example 20	1	1

Only in Example 20 do both the function and the limit exist and agree. This seems "nice;" in fact, it seems "normal." This is in fact an important situation which we explore in the next section, entitled "Continuity." In short, a *continuous function* is one in which when a function approaches a value as $x \to c$ (i.e., when $\lim\limits_{x\to c} f(x) = L$), it actually *attains* that value at c. Such functions behave nicely as they are very predictable.

Notes:

Exercises 1.4

Terms and Concepts

1. What are the three ways in which a limit may fail to exist?

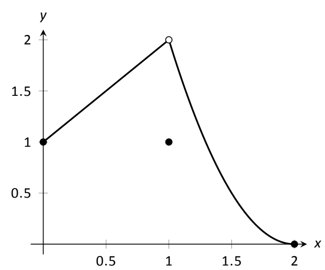
2. T/F: If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1} f(x) = 5$

3. T/F: If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1^+} f(x) = 5$

4. T/F: If $\lim\limits_{x\to 1} f(x) = 5$, then $\lim\limits_{x\to 1^-} f(x) = 5$

Problems

In Exercises 5 – 12, evaluate each expression using the given graph of $f(x)$.

5. 

(a) $\lim\limits_{x\to 1^-} f(x)$

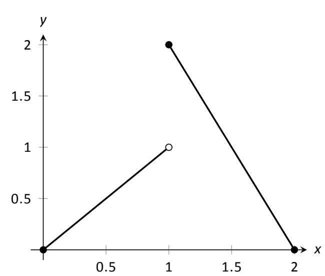
(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 0^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

6. 

(a) $\lim\limits_{x\to 1^-} f(x)$

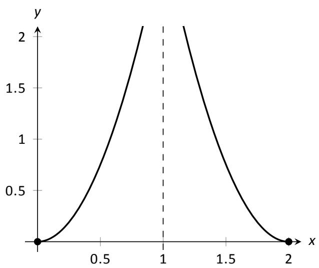
(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 2^+} f(x)$

7. 

(a) $\lim\limits_{x\to 1^-} f(x)$

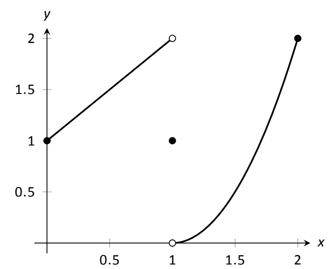
(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

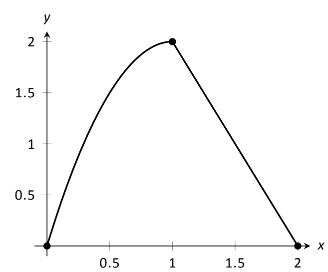
8. 

(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

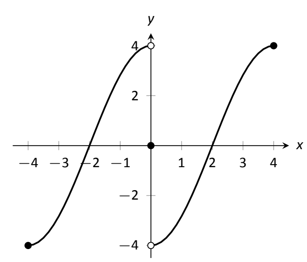
9. 

(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

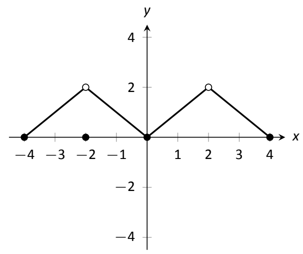
10. 

(a) $\lim\limits_{x\to 0^-} f(x)$

(b) $\lim\limits_{x\to 0^+} f(x)$

(c) $\lim\limits_{x\to 0} f(x)$

(d) $f(0)$

11. 

(a) $\lim_{x\to -2^-} f(x)$

(b) $\lim_{x\to -2^+} f(x)$

(c) $\lim_{x\to -2} f(x)$

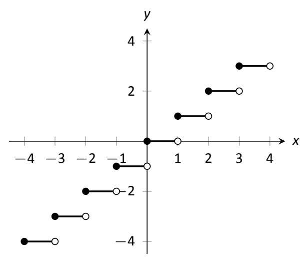
(d) $f(-2)$

(e) $\lim_{x\to 2^-} f(x)$

(f) $\lim_{x\to 2^+} f(x)$

(g) $\lim_{x\to 2} f(x)$

(h) $f(2)$

12. 

Let $-3 \le a \le 3$ be an integer.

(a) $\lim_{x\to a^-} f(x)$

(b) $\lim_{x\to a^+} f(x)$

(c) $\lim_{x\to a} f(x)$

(d) $f(a)$

In Exercises 13 – 21, evaluate the given limits of the piecewise defined functions *f*.

13. $f(x) = \begin{cases} x+1 & x \le 1 \\ x^2 - 5 & x > 1 \end{cases}$

(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 1} f(x)$

(d) $f(1)$

14. $f(x) = \begin{cases} 2x^2 + 5x - 1 & x < 0 \\ \sin x & x \ge 0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $f(0)$

15. $f(x) = \begin{cases} x^2 - 1 & x < -1 \\ x^3 + 1 & -1 \le x \le 1 \\ x^2 + 1 & x > 1 \end{cases}$

(a) $\lim_{x\to -1^-} f(x)$

(b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$

(d) $f(-1)$

(e) $\lim_{x\to 1^-} f(x)$

(f) $\lim_{x\to 1^+} f(x)$

(g) $\lim_{x\to 1} f(x)$

(h) $f(1)$

16. $f(x) = \begin{cases} \cos x & x < \pi \\ \sin x & x \ge \pi \end{cases}$

(a) $\lim_{x\to \pi^-} f(x)$

(b) $\lim_{x\to \pi^+} f(x)$

(c) $\lim_{x\to \pi} f(x)$

(d) $f(\pi)$

17. $f(x) = \begin{cases} 1 - \cos^2 x & x < a \\ \sin^2 x & x \ge a \end{cases}$,

where a is a real number.

(a) $\lim_{x\to a^-} f(x)$

(b) $\lim_{x\to a^+} f(x)$

(c) $\lim_{x\to a} f(x)$

(d) $f(a)$

18. $f(x) = \begin{cases} x+1 & x < 1 \\ 1 & x = 1 \\ x - 1 & x > 1 \end{cases}$

(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 1} f(x)$

(d) $f(1)$

19. $f(x) = \begin{cases} x^2 & x < 2 \\ x+1 & x = 2 \\ -x^2 + 2x + 4 & x > 2 \end{cases}$

(a) $\lim_{x\to 2^-} f(x)$

(b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$

(d) $f(2)$

20. $f(x) = \begin{cases} a(x-b)^2 + c & x < b \\ a(x-b) + c & x \ge b \end{cases}$,

where a, b and c are real numbers.

(a) $\lim_{x\to b^-} f(x)$

(b) $\lim_{x\to b^+} f(x)$

(c) $\lim_{x\to b} f(x)$

(d) $f(b)$

21. $f(x) = \begin{cases} \frac{|x|}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $f(0)$

Review

22. Evaluate the limit: $\lim_{x\to -1} \dfrac{x^2 + 5x + 4}{x^2 - 3x - 4}$.

23. Evaluate the limit: $\lim_{x\to -4} \dfrac{x^2 - 16}{x^2 - 4x - 32}$.

24. Evaluate the limit: $\lim_{x\to -6} \dfrac{x^2 - 15x + 54}{x^2 - 6x}$.

25. Approximate the limit numerically: $\lim_{x\to 0.4} \dfrac{x^2 - 4.4x + 1.6}{x^2 - 0.4x}$.

26. Approximate the limit numerically: $\lim_{x\to 0.2} \dfrac{x^2 + 5.8x - 1.2}{x^2 - 4.2x + 0.8}$.

1.5 Continuity

As we have studied limits, we have gained the intuition that limits measure "where a function is heading." That is, if $\lim_{x\to 1} f(x) = 3$, then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually is. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that *do not* exhibit such behavior.

Definition 3 Continuous Function

Let f be a function defined on an open interval I containing c.

1. f is **continuous at** c if $\lim_{x\to c} f(x) = f(c)$.
2. f is **continuous on** I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere**.

A useful way to establish whether or not a function f is continuous at c is to verify the following three things:

1. $\lim_{x\to c} f(x)$ exists,
2. $f(c)$ is defined, and
3. $\lim_{x\to c} f(x) = f(c)$.

Example 21 Finding intervals of continuity

Let f be defined as shown in Figure 1.25. Give the interval(s) on which f is continuous.

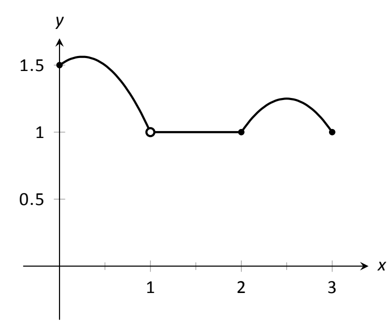


Figure 1.25: A graph of f in Example 21.

SOLUTION We proceed by examining the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ exists for all c between 0 and 3.
2. $f(c)$ is defined for all c between 0 and 3, *except for* $c = 1$. We know immediately that f cannot be continuous at $x = 1$.
3. The limit $\lim_{x\to c} f(x) = f(c)$ for all c between 0 and 3, except, of course, for $c = 1$.

We conclude that f is continuous at every point of $(0, 3)$ except at $x = 1$. Therefore f is continuous on $(0, 1) \cup (1, 3)$.

Notes:

Example 22 Finding intervals of continuity

The *floor function*, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$.) The graph of f in Figure 1.26 demonstrates why this is often called a "step function."

Give the intervals on which f is continuous.

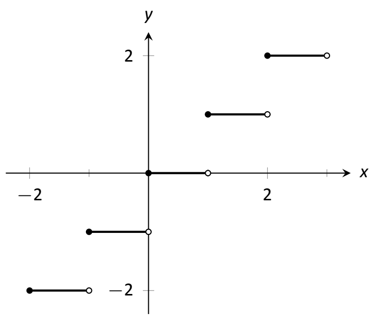


Figure 1.26: A graph of the step function in Example 22.

SOLUTION We examine the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ do not exist at the jumps from one "step" to the next, which occur at all integer values of c. Therefore the limits exist for all c except when c is an integer.
2. The function is defined for all values of c.
3. The limit $\lim\limits_{x\to c} f(x) = f(c)$ for all values of c where the limit exist, since each step consists of just a line.

We conclude that f is continuous everywhere except at integer values of c. So the intervals on which f is continuous are

$$\ldots, (-2,-1), (-1,0), (0,1), (1,2), \ldots.$$

Our definition of continuity on an interval specifies the interval is an open interval. We can extend the definition of continuity to closed intervals by considering the appropriate one-sided limits at the endpoints.

Definition 4 Continuity on Closed Intervals

Let f be defined on the closed interval $[a, b]$ for some real numbers a, b. f is **continuous on** $[a, b]$ if:

1. f is continuous on (a, b),
2. $\lim\limits_{x\to a^+} f(x) = f(a)$ and
3. $\lim\limits_{x\to b^-} f(x) = f(b)$.

We can make the appropriate adjustments to talk about continuity on half–open intervals such as $[a, b)$ or $(a, b]$ if necessary.

Notes:

Example 23 Determining intervals on which a function is continuous

For each of the following functions, give the domain of the function and the interval(s) on which it is continuous.

1. $f(x) = 1/x$
2. $f(x) = \sin x$
3. $f(x) = \sqrt{x}$
4. $f(x) = \sqrt{1-x^2}$
5. $f(x) = |x|$

Solution We examine each in turn.

1. The domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$. As it is a rational function, we apply Theorem 2 to recognize that f is continuous on all of its domain.

2. The domain of $f(x) = \sin x$ is all real numbers, or $(-\infty, \infty)$. Applying Theorem 3 shows that $\sin x$ is continuous everywhere.

3. The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. Applying Theorem 3 shows that $f(x) = \sqrt{x}$ is continuous on its domain of $[0, \infty)$.

4. The domain of $f(x) = \sqrt{1-x^2}$ is $[-1, 1]$. Applying Theorems 1 and 3 shows that f is continuous on all of its domain, $[-1, 1]$.

5. The domain of $f(x) = |x|$ is $(-\infty, \infty)$. We can define the absolute value function as $f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$. Each "piece" of this piecewise defined function is continuous on all of its domain, giving that f is continuous on $(-\infty, 0)$ and $[0, \infty)$. We cannot assume this implies that f is continuous on $(-\infty, \infty)$; we need to check that $\lim\limits_{x \to 0} f(x) = f(0)$, as $x = 0$ is the point where f transitions from one "piece" of its definition to the other. It is easy to verify that this is indeed true, hence we conclude that $f(x) = |x|$ is continuous everywhere.

Continuity is inherently tied to the properties of limits. Because of this, the properties of limits found in Theorems 1 and 2 apply to continuity as well. Further, now knowing the definition of continuity we can re–read Theorem 3 as giving a list of functions that are continuous on their domains. The following theorem states how continuous functions can be combined to form other continuous functions, followed by a theorem which formally lists functions that we know are continuous on their domains.

Notes:

Theorem 8 Properties of Continuous Functions

Let f and g be continuous functions on an interval I, let c be a real number and let n be a positive integer. The following functions are continuous on I.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as long as $g \neq 0$ on I)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on I; if n is odd, then true for all values of f on I.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on I, where the range of f on I is J, and let g be continuous on J. Then $g \circ f$, i.e., $g(f(x))$, is continuous on I.

Theorem 9 Continuous Functions

The following functions are continuous on their domains.

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
4. $f(x) = \cot x$
5. $f(x) = \sec x$
6. $f(x) = \csc x$
7. $f(x) = \ln x$
8. $f(x) = \sqrt[n]{x}$, (where n is a positive integer)
9. $f(x) = a^x$ $(a > 0)$

We apply these theorems in the following Example.

Notes:

Example 24 Determining intervals on which a function is continuous

State the interval(s) on which each of the following functions is continuous.

1. $f(x) = \sqrt{x-1} + \sqrt{5-x}$
2. $f(x) = x \sin x$
3. $f(x) = \tan x$
4. $f(x) = \sqrt{\ln x}$

Solution We examine each in turn, applying Theorems 8 and 9 as appropriate.

1. The square–root terms are continuous on the intervals $[1, \infty)$ and $(-\infty, 5]$, respectively. As f is continuous only where each term is continuous, f is continuous on $[1, 5]$, the intersection of these two intervals. A graph of f is given in Figure 1.27.

2. The functions $y = x$ and $y = \sin x$ are each continuous everywhere, hence their product is, too.

3. Theorem 9 states that $f(x) = \tan x$ is continuous "on its domain." Its domain includes all real numbers except odd multiples of $\pi/2$. Thus $f(x) = \tan x$ is continuous on

$$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots,$$

or, equivalently, on $D = \{x \in \mathbb{R} \mid x \neq n \cdot \frac{\pi}{2}$, n is an odd integer$\}$.

4. The domain of $y = \sqrt{x}$ is $[0, \infty)$. The range of $y = \ln x$ is $(-\infty, \infty)$, but if we restrict its domain to $[1, \infty)$ its range is $[0, \infty)$. So restricting $y = \ln x$ to the domain of $[1, \infty)$ restricts its output is $[0, \infty)$, on which $y = \sqrt{x}$ is defined. Thus the domain of $f(x) = \sqrt{\ln x}$ is $[1, \infty)$.

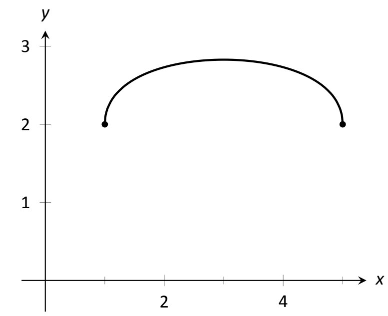


Figure 1.27: A graph of f in Example 24(1).

A common way of thinking of a continuous function is that "its graph can be sketched without lifting your pencil." That is, its graph forms a "continuous" curve, without holes, breaks or jumps. While beyond the scope of this text, this pseudo–definition glosses over some of the finer points of continuity. Very strange functions are continuous that one would be hard pressed to actually sketch by hand.

This intuitive notion of continuity does help us understand another important concept as follows. Suppose f is defined on $[1, 2]$ and $f(1) = -10$ and $f(2) = 5$. If f is continuous on $[1, 2]$ (i.e., its graph can be sketched as a continuous curve from $(1, -10)$ to $(2, 5)$) then we know intuitively that somewhere on $[1, 2]$ f must be equal to -9, and -8, and -7, -6, $\ldots$, 0, $1/2$, etc. In short, f

Notes:

takes on all *intermediate* values between -10 and 5. It may take on more values; f may actually equal 6 at some time, for instance, but we are guaranteed all values between -10 and 5.

While this notion seems intuitive, it is not trivial to prove and its importance is profound. Therefore the concept is stated in the form of a theorem.

Theorem 10 Intermediate Value Theorem

Let f be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value y, where $f(a) < y < f(b)$, there is a value c in $[a, b]$ such that $f(c) = y$.

One important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found through successive applications of this theorem. Suppose through direct computation we find that $f(a) < 0$ and $f(b) > 0$, where $a < b$. The Intermediate Value Theorem states that there is a c in $[a, b]$ such that $f(c) = 0$. The theorem does not give us any clue as to where that value is in the interval $[a, b]$, just that it exists.

There is a technique that produces a good approximation of c. Let d be the midpoint of the interval $[a, b]$ and consider $f(d)$. There are three possibilities:

1. $f(d) = 0$ – we got lucky and stumbled on the actual value. We stop as we found a root.

2. $f(d) < 0$ Then we know there is a root of f on the interval $[d, b]$ – we have halved the size of our interval, hence are closer to a good approximation of the root.

3. $f(d) > 0$ Then we know there is a root of f on the interval $[a, d]$ – again,we have halved the size of our interval, hence are closer to a good approximation of the root.

Successively applying this technique is called the **Bisection Method** of root finding. We continue until the interval is sufficiently small. We demonstrate this in the following example.

Example 25 Using the Bisection Method

Approximate the root of $f(x) = x - \cos x$, accurate to three places after the decimal.

Solution Consider the graph of $f(x) = x - \cos x$, shown in Figure 1.28. It is clear that the graph crosses the x-axis somewhere near $x = 0.8$. To start the

Notes:

Bisection Method, pick an interval that contains 0.8. We choose $[0.7, 0.9]$. Note that all we care about are signs of $f(x)$, not their actual value, so this is all we display.

Iteration 1: $f(0.7) < 0, f(0.9) > 0$, and $f(0.8) > 0$. So replace 0.9 with 0.8 and repeat.

Iteration 2: $f(0.7) < 0, f(0.8) > 0$, and at the midpoint, 0.75, we have $f(0.75) > 0$. So replace 0.8 with 0.75 and repeat. Note that we don't need to continue to check the endpoints, just the midpoint. Thus we put the rest of the iterations in Table 1.29.

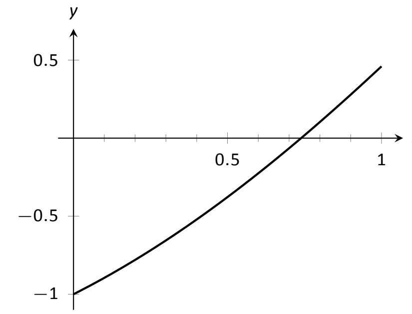


Figure 1.28: Graphing a root of $f(x) = x - \cos x$.

Notice that in the 12$^{\text{th}}$ iteration we have the endpoints of the interval each starting with 0.739. Thus we have narrowed the zero down to an accuracy of the first three places after the decimal. Using a computer, we have

$$f(0.7390) = -0.00014, \quad f(0.7391) = 0.000024.$$

Either endpoint of the interval gives a good approximation of where f is 0. The Intermediate Value Theorem states that the actual zero is still within this interval. While we do not know its exact value, we know it starts with 0.739.

This type of exercise is rarely done by hand. Rather, it is simple to program a computer to run such an algorithm and stop when the endpoints differ by a preset small amount. One of the authors did write such a program and found the zero of f, accurate to 10 places after the decimal, to be 0.7390851332. While it took a few minutes to write the program, it took less than a thousandth of a second for the program to run the necessary 35 iterations. In less than 8 hundredths of a second, the zero was calculated to 100 decimal places (with less than 200 iterations).

Iteration #	Interval	Midpoint Sign
1	$[0.7, 0.9]$	$f(0.8) > 0$
2	$[0.7, 0.8]$	$f(0.75) > 0$
3	$[0.7, 0.75]$	$f(0.725) < 0$
4	$[0.725, 0.75]$	$f(0.7375) < 0$
5	$[0.7375, 0.75]$	$f(0.7438) > 0$
6	$[0.7375, 0.7438]$	$f(0.7407) > 0$
7	$[0.7375, 0.7407]$	$f(0.7391) > 0$
8	$[0.7375, 0.7391]$	$f(0.7383) < 0$
9	$[0.7383, 0.7391]$	$f(0.7387) < 0$
10	$[0.7387, 0.7391]$	$f(0.7389) < 0$
11	$[0.7389, 0.7391]$	$f(0.7390) < 0$
12	$[0.7390, 0.7391]$	

Figure 1.29: Iterations of the Bisection Method of Root Finding

It is a simple matter to extend the Bisection Method to solve problems similar to "Find x, where $f(x) = 0$." For instance, we can find x, where $f(x) = 1$. It actually works very well to define a new function g where $g(x) = f(x) - 1$. Then use the Bisection Method to solve $g(x) = 0$.

Similarly, given two functions f and g, we can use the Bisection Method to solve $f(x) = g(x)$. Once again, create a new function h where $h(x) = f(x) - g(x)$ and solve $h(x) = 0$.

In Section **??** another equation solving method will be introduced, called Newton's Method. In many cases, Newton's Method is much faster. It relies on more advanced mathematics, though, so we will wait before introducing it.

This section formally defined what it means to be a continuous function. "Most" functions that we deal with are continuous, so often it feels odd to have to formally define this concept. Regardless, it is important, and forms the basis of the next chapter.

In the next section we examine one more aspect of limits: limits that involve infinity.

Notes:

Exercises 1.5

Terms and Concepts

1. In your own words, describe what it means for a function to be continuous.

2. In your own words, describe what the Intermediate Value Theorem states.

3. What is a "root" of a function?

4. Given functions f and g on an interval I, how can the Bisection Method be used to find a value c where $f(c) = g(c)$?

5. T/F: If f is defined on an open interval containing c, and $\lim_{x\to c} f(x)$ exists, then f is continuous at c.

6. T/F: If f is continuous at c, then $\lim_{x\to c} f(x)$ exists.

7. T/F: If f is continuous at c, then $\lim_{x\to c^+} f(x) = f(c)$.

8. T/F: If f is continuous on $[a, b]$, then $\lim_{x\to a^-} f(x) = f(a)$.

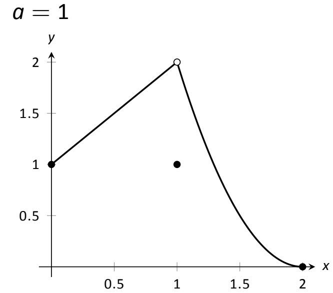
9. T/F: If f is continuous on $[0, 1)$ and $[1, 2)$, then f is continuous on $[0, 2)$.

10. T/F: The sum of continuous functions is also continuous.

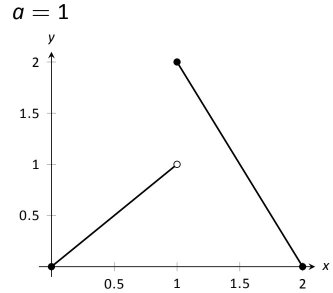
Problems

In Exercises 11 – 17, a graph of a function f is given along with a value a. Determine if f is continuous at a; if it is not, state why it is not.

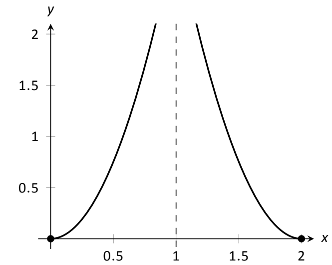
11. $a = 1$



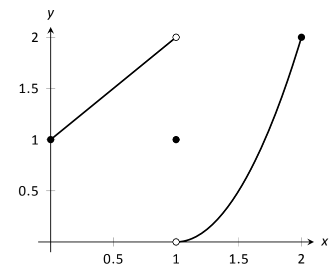
12. $a = 1$



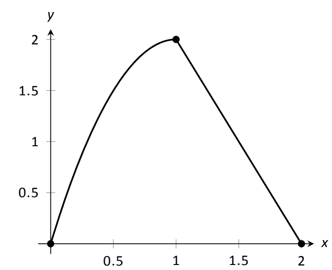
13. $a = 1$



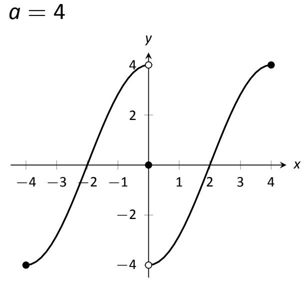
14. $a = 0$



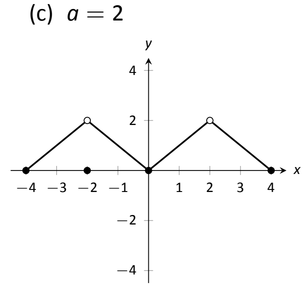
15. $a = 1$



16. $a = 4$



17. (a) $a = -2$
 (b) $a = 0$
 (c) $a = 2$



In Exercises 18 – 21, determine if f is continuous at the indicated values. If not, explain why.

18. $f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x > 0 \end{cases}$
    (a) $x = 0$
    (b) $x = \pi$

19. $f(x) = \begin{cases} x^3 - x & x < 1 \\ x - 2 & x \geq 1 \end{cases}$
    (a) $x = 0$
    (b) $x = 1$

20. $f(x) = \begin{cases} \frac{x^2+5x+4}{x^2+3x+2} & x \neq -1 \\ 3 & x = -1 \end{cases}$
    (a) $x = -1$
    (b) $x = 10$

21. $f(x) = \begin{cases} \frac{x^2-64}{x^2-11x+24} & x \neq 8 \\ 5 & x = 8 \end{cases}$
    (a) $x = 0$
    (b) $x = 8$

In Exercises 22 – 32, give the intervals on which the given function is continuous.

22. $f(x) = x^2 - 3x + 9$

23. $g(x) = \sqrt{x^2 - 4}$

24. $h(k) = \sqrt{1-k} + \sqrt{k+1}$

25. $f(t) = \sqrt{5t^2 - 30}$

26. $g(t) = \dfrac{1}{\sqrt{1-t^2}}$

27. $g(x) = \dfrac{1}{1+x^2}$

28. $f(x) = e^x$

29. $g(s) = \ln s$

30. $h(t) = \cos t$

31. $f(k) = \sqrt{1 - e^k}$

32. $f(x) = \sin(e^x + x^2)$

33. Let f be continuous on $[1, 5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$? Why/why not?

34. Let g be continuous on $[-3, 7]$ where $g(0) = 0$ and $g(2) = 25$. Does a value $-3 < c < 7$ exist such that $g(c) = 15$? Why/why not?

35. Let f be continuous on $[-1, 1]$ where $f(-1) = -10$ and $f(1) = 10$. Does a value $-1 < c < 1$ exist such that $f(c) = 11$? Why/why not?

36. Let h be a function on $[-1, 1]$ where $h(-1) = -10$ and $h(1) = 10$. Does a value $-1 < c < 1$ exist such that $h(c) = 0$? Why/why not?

In Exercises 37 – 40, use the Bisection Method to approximate, accurate to two decimal places, the value of the root of the given function in the given interval.

37. $f(x) = x^2 + 2x - 4$ on $[1, 1.5]$.

38. $f(x) = \sin x - 1/2$ on $[0.5, 0.55]$

39. $f(x) = e^x - 2$ on $[0.65, 0.7]$.

40. $f(x) = \cos x - \sin x$ on $[0.7, 0.8]$.

Review

41. Let $f(x) = \begin{cases} x^2 - 5 & x < 5 \\ 5x & x \geq 5 \end{cases}$.
    (a) $\lim\limits_{x\to 5^-} f(x)$
    (b) $\lim\limits_{x\to 5^+} f(x)$
    (c) $\lim\limits_{x\to 5} f(x)$
    (d) $f(5)$

42. Numerically approximate the following limits:
    (a) $\lim\limits_{x\to -4/5^+} \dfrac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$
    (b) $\lim\limits_{x\to -4/5^-} \dfrac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$

43. Give an example of function $f(x)$ for which $\lim\limits_{x\to 0} f(x)$ does not exist.

1.6 Limits Involving Infinity

In Definition 1 we stated that in the equation $\lim_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure 1.30. Note how, as x approaches 0, $f(x)$ grows very, very large. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

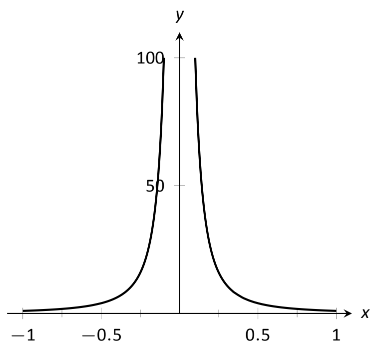


Figure 1.30: Graphing $f(x) = 1/x^2$ for values of x near 0.

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to\infty} \frac{1}{x^2} = 0.$$

We explore both types of use of ∞ in turn.

> **Definition 5 Limit of Infinity, ∞**
>
> We say $\lim_{x\to c} f(x) = \infty$ if for every $M > 0$ there exists $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$, then $f(x) \geq M$.

This is just like the ε–δ definition from Section 1.2. In that definition, given any (small) value ε, if we let x get close enough to c (within δ units of c) then $f(x)$ is guaranteed to be within ε of $f(c)$. Here, given any (large) value M, if we let x get close enough to c (within δ units of c), then $f(x)$ will be at least as large as M. In other words, if we get close enough to c, then we can make $f(x)$ as large as we want. We can define limits equal to $-\infty$ in a similar way.

It is important to note that by saying $\lim_{x\to c} f(x) = \infty$ we are implicitly stating that *the* limit of $f(x)$, as x approaches c, *does not exist.* A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive.

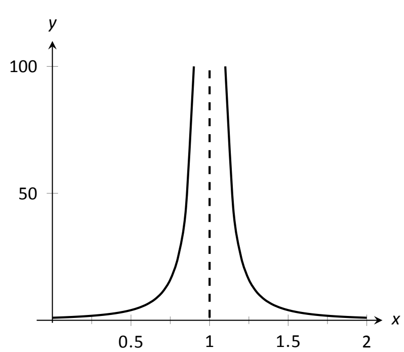


Figure 1.31: Observing infinite limit as $x \to 1$ in Example 26.

Example 26 Evaluating limits involving infinity

Find $\lim_{x\to 1} \dfrac{1}{(x-1)^2}$ as shown in Figure 1.31.

Solution In Example 4 of Section 1.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real

Notes:

number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(.99) = 10^4$, $f(.999) = 10^6$, $f(.9999) = 10^8$. A similar thing happens on the other side of 1. In general, let a "large" value M be given. Let $\delta = 1/\sqrt{M}$. If x is within δ of 1, i.e., if $|x-1| < 1/\sqrt{M}$, then:

$$\begin{aligned} |x-1| &< \frac{1}{\sqrt{M}} \\ (x-1)^2 &< \frac{1}{M} \\ \frac{1}{(x-1)^2} &> M, \end{aligned}$$

which is what we wanted to show. So we may say $\lim\limits_{x\to 1} 1/(x-1)^2 = \infty$.

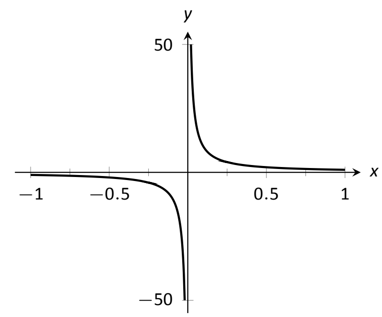


Figure 1.32: Evaluating $\lim\limits_{x\to 0} \frac{1}{x}$.

Example 27 **Evaluating limits involving infinity**

Find $\lim\limits_{x\to 0} \frac{1}{x}$, as shown in Figure 1.32.

Solution It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behavior is not consistent, we cannot say that $\lim\limits_{x\to 0} \frac{1}{x} = \infty$. However, we can make a statement about one–sided limits. We can state that $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$.

Vertical asymptotes

If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, we say the function has a **vertical asymptote** at c.

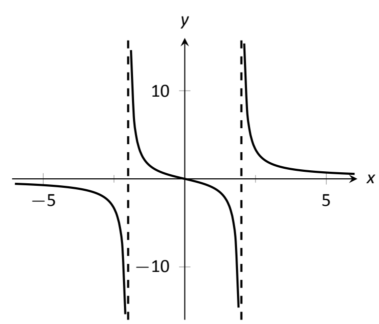


Figure 1.33: Graphing $f(x) = \frac{3x}{x^2-4}$.

Example 28 **Finding vertical asymptotes**

Find the vertical asymptotes of $f(x) = \frac{3x}{x^2-4}$.

Solution Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure 1.33. Thus the vertical asymptotes are at $x = \pm 2$.

Notes:

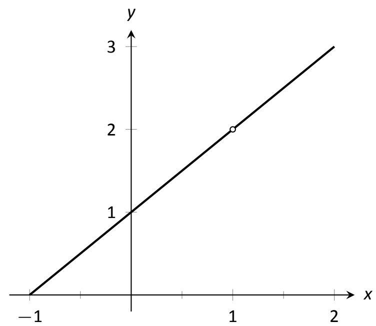


Figure 1.34: Graphically showing that $f(x) = \dfrac{x^2-1}{x-1}$ does not have an asymptote at $x = 1$.

When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 1.34. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Canceling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote, rather a hole exists in the graph at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin x)/x$. We have considered this function several times in the previous sections. We found that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the Squeeze Theorem in Section 1.3 to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

Indeterminate Forms

We have seen how the limits

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 1} \frac{x^2-1}{x-1}$$

each return the indeterminate form "0/0" when we blindly plug in $x = 0$ and $x = 1$, respectively. However, 0/0 is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up 0/0 expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called *indeterminate*.

A key concept to understand is that such limits do not really return 0/0. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

Notes:

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and canceling) or it may require a tool such as the Squeeze Theorem. In a later section we will learn a technique called l'Hospital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of evaluating a limit, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behavior of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition 6 Limits at Infinity and Horizontal Asymptote

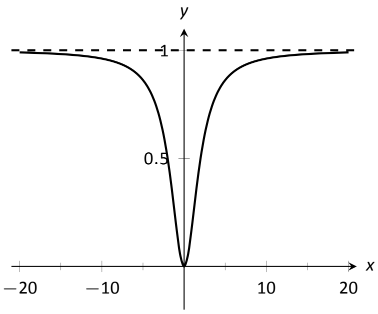
1. We say $\lim\limits_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$.

2. We say $\lim\limits_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \varepsilon$.

3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say that $y = L$ is a **horizontal asymptote** of f.

We can also define limits such as $\lim\limits_{x\to\infty} f(x) = \infty$ by combining this definition with Definition 5.

Notes:



(a)

x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
-10	0.9615
-100	0.9996
-10000	0.999996

(b)

Figure 1.35: Using a graph and a table to approximate a horizontal asymptote in Example 29.

Example 29 **Approximating horizontal asymptotes**

Approximate the horizontal asymptote(s) of $f(x) = \dfrac{x^2}{x^2+4}$.

Solution We will approximate the horizontal asymptotes by approximating the limits

$$\lim_{x\to-\infty} \frac{x^2}{x^2+4} \quad \text{and} \quad \lim_{x\to\infty} \frac{x^2}{x^2+4}.$$

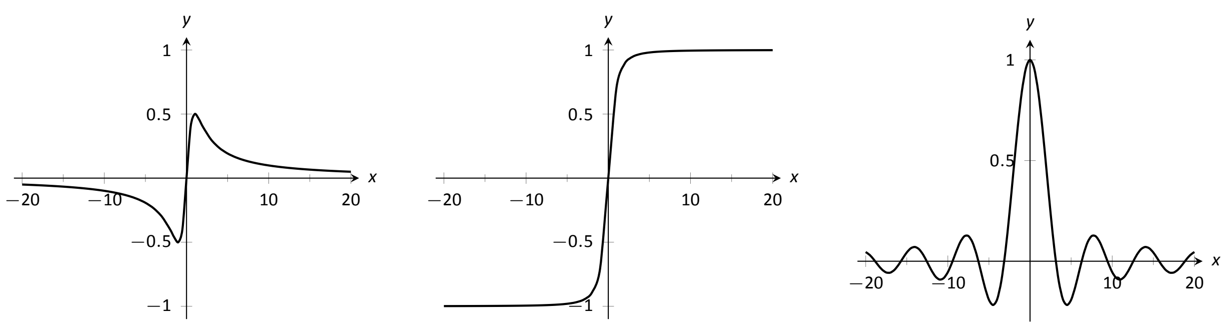
Figure 1.35(a) shows a sketch of f, and part (b) gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$.

Later, we will show how to determine this analytically.

Horizontal asymptotes can take on a variety of forms. Figure 1.36(a) shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 1.36(b) shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 1.36(c) shows that $f(x) = (\sin x)/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.



(a) (b) (c)

Figure 1.36: Considering different types of horizontal asymptotes.

We can analytically evaluate limits at infinity for rational functions once we understand $\lim\limits_{x\to\infty} 1/x$. As x gets larger and larger, the $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large

Notes:

enough value of x. Given ε, we can make $1/x < \varepsilon$ by choosing $x > 1/\varepsilon$. Thus we have $\lim_{x\to\infty} 1/x = 0$.

It is now not much of a jump to conclude the following:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence dividing by 1), which is the largest power of x to appear in the function. Doing this, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} &= \lim_{x\to\infty} \frac{1/x^3}{1/x^3} \cdot \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} \\
&= \lim_{x\to\infty} \frac{x^3/x^3 + 2x/x^3 + 1/x^3}{4x^3/x^3 - 2x^2/x^3 + 9/x^3} \\
&= \lim_{x\to\infty} \frac{1 + 2/x^2 + 1/x^3}{4 - 2/x + 9/x^3}.
\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1 + 0 + 0}{4 - 0 + 0} = \frac{1}{4}.$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Theorem 11 Limits of Rational Functions at Infinity

Let $f(x)$ be a rational function of the following form:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0},$$

where any of the coefficients may be 0 except for a_n and b_m.

1. If $n = m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \dfrac{a_n}{b_m}$.
2. If $n < m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$.
3. If $n > m$, then $\lim\limits_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are both infinite.

Notes:

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e. $n = m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for $a_n x^n / x^n$ and $b_m x^m / x^n$. Since $n = m$, this will leave us with the limit a_n / b_m. If $n < m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0. If $n > m$, and we try dividing through by x^n, we end up with all the terms in the denominator tending toward 0, while the x^n term in the numerator does not approach 0. This is indicative of some sort of infinite limit.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to $a_n x^n$, and likewise all the terms in the denominator are small compared to $b_n x^m$. If $n = m$, looking only at these two important terms, we have $(a_n x^n)/(b_n x^m)$. This reduces to a_n / b_m. If $n < m$, the function behaves like $a_n / (b_m x^{m-n})$, which tends toward 0. If $n > m$, the function behaves like $a_n x^{n-m} / b_m$, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the largest powers of x. For instance, consider again $\displaystyle \lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}$, graphed in Figure 1.36(b). When x is very large, $x^2 + 1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x|, \quad \text{and} \quad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}.$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y = 1$ and $y = -1$, respectively.

Example 30 Finding a limit of a rational function

Confirm analytically that $y = 1$ is the horizontal asymptote of $f(x) = \dfrac{x^2}{x^2+4}$, as approximated in Example 29.

Solution Before using Theorem 11, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then

Notes:

take limits.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2}{x^2+4} &= \lim_{x\to\infty}\frac{x^2/x^2}{x^2/x^2+4/x^2}\\ &= \lim_{x\to\infty}\frac{1}{1+4/x^2}\\ &= \frac{1}{1+0}\\ &= 1.\end{aligned}$$

We can also use Theorem 11 directly; in this case $n = m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., 1/1 = 1.

Example 31 Finding limits of rational functions

Use Theorem 11 to evaluate each of the following limits.

1. $\displaystyle\lim_{x\to-\infty}\frac{x^2+2x-1}{x^3+1}$
2. $\displaystyle\lim_{x\to\infty}\frac{x^2+2x-1}{1-x-3x^2}$
3. $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{3-x}$

SOLUTION

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure 1.37(a).
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure 1.37(b).
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x) = -x$ for large values of x. Therefore, the limit is $-\infty$. See Figure 1.37(c).

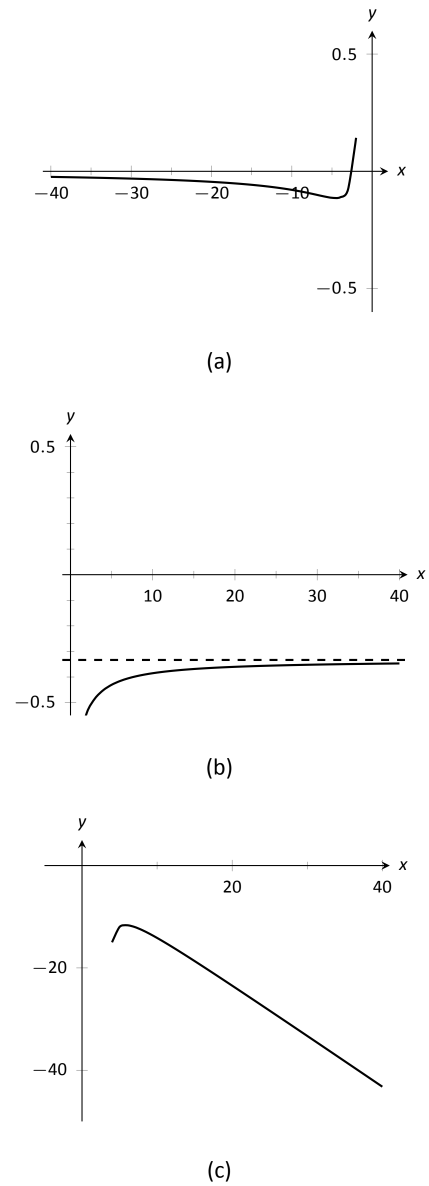


Figure 1.37: Visualizing the functions in Example 31.

Notes:

Chapter Summary

In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- developed a not–so–easy method of proving the value of a limit (ε-δ proofs),
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Later, we will want to add up an infinite list of numbers. We will do so by first adding up a finite list of numbers, then take a limit as the number of things we are adding approaches infinity. Surprisingly, this sum often is finite; that is, we can add up an infinite list of numbers and get, for instance, 42.

These are just two quick examples of why we are interested in limits. Many students dislike this topic when they are first introduced to it, but over time an appreciation is often formed based on the scope of its applicability.

Notes:

Exercises 1.6

Terms and Concepts

1. T/F: If $\lim_{x\to 5} f(x) = \infty$, then we are implicitly stating that the limit exists.

2. T/F: If $\lim_{x\to\infty} f(x) = 5$, then we are implicitly stating that the limit exists.

3. T/F: If $\lim_{x\to 1^-} f(x) = -\infty$, then $\lim_{x\to 1^+} f(x) = \infty$

4. T/F: If $\lim_{x\to 5} f(x) = \infty$, then f has a vertical asymptote at $x = 5$.

5. T/F: $\infty/0$ is not an indeterminate form.

6. List 5 indeterminate forms.

7. Construct a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

8. Let $\lim_{x\to 7} f(x) = \infty$. Explain how we know that f is/is not continuous at $x = 7$.

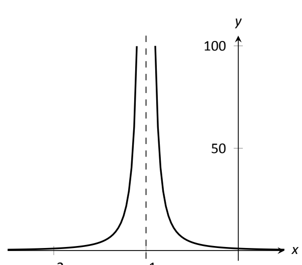
Problems

In Exercises 9 – 14, evaluate the given limits using the graph of the function.

9. $f(x) = \dfrac{1}{(x+1)^2}$

   (a) $\lim_{x\to -1^-} f(x)$

   (b) $\lim_{x\to -1^+} f(x)$



10. $f(x) = \dfrac{1}{(x-3)(x-5)^2}$.

   (a) $\lim_{x\to 3^-} f(x)$

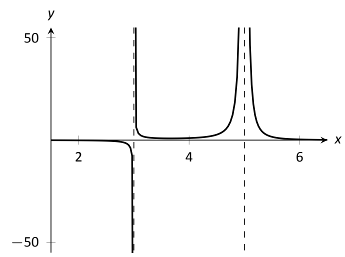
   (b) $\lim_{x\to 3^+} f(x)$

   (c) $\lim_{x\to 3} f(x)$

   (d) $\lim_{x\to 5^-} f(x)$

   (e) $\lim_{x\to 5^+} f(x)$

   (f) $\lim_{x\to 5} f(x)$



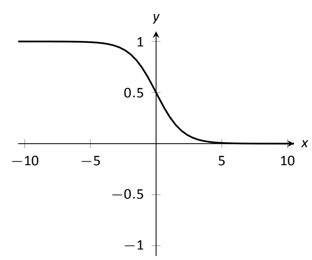
11. $f(x) = \dfrac{1}{e^x + 1}$

   (a) $\lim_{x\to -\infty} f(x)$

   (b) $\lim_{x\to \infty} f(x)$

   (c) $\lim_{x\to 0^-} f(x)$

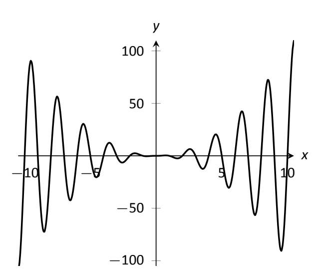
   (d) $\lim_{x\to 0^+} f(x)$



12. $f(x) = x^2 \sin(\pi x)$

   (a) $\lim_{x\to -\infty} f(x)$

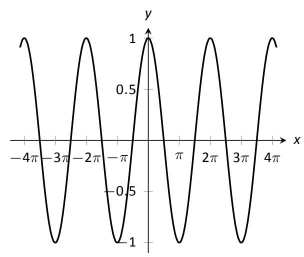
   (b) $\lim_{x\to \infty} f(x)$



13. $f(x) = \cos(x)$

(a) $\lim\limits_{x\to-\infty} f(x)$

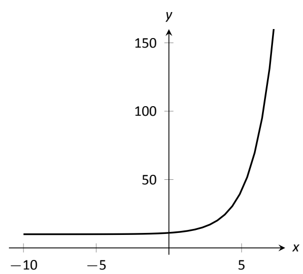
(b) $\lim\limits_{x\to\infty} f(x)$



14. $f(x) = 2^x + 10$

(a) $\lim\limits_{x\to-\infty} f(x)$

(b) $\lim\limits_{x\to\infty} f(x)$



In Exercises 15 – 18, numerically approximate the following limits:

(a) $\lim\limits_{x\to 3^-} f(x)$

(b) $\lim\limits_{x\to 3^+} f(x)$

(c) $\lim\limits_{x\to 3} f(x)$

15. $f(x) = \dfrac{x^2-1}{x^2-x-6}$

16. $f(x) = \dfrac{x^2+5x-36}{x^3-5x^2+3x+9}$

17. $f(x) = \dfrac{x^2-11x+30}{x^3-4x^2-3x+18}$

18. $f(x) = \dfrac{x^2-9x+18}{x^2-x-6}$

In Exercises 19 – 24, identify the horizontal and vertical asymptotes, if any, of the given function.

19. $f(x) = \dfrac{2x^2-2x-4}{x^2+x-20}$

20. $f(x) = \dfrac{-3x^2-9x-6}{5x^2-10x-15}$

21. $f(x) = \dfrac{x^2+x-12}{7x^3-14x^2-21x}$

22. $f(x) = \dfrac{x^2-9}{9x-9}$

23. $f(x) = \dfrac{x^2-9}{9x+27}$

24. $f(x) = \dfrac{x^2-1}{-x^2-1}$

In Exercises 25 – 28, evaluate the given limit.

25. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{x-5}$

26. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{5-x}$

27. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{x^2-5}$

28. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{5-x^2}$

Review

29. Use an $\varepsilon-\delta$ proof to show that
$\lim\limits_{x\to 1} 5x-2 = 3.$

30. Let $\lim\limits_{x\to 2} f(x) = 3$ and $\lim\limits_{x\to 2} g(x) = -1$. Evaluate the following limits.

(a) $\lim\limits_{x\to 2} (f+g)(x)$

(b) $\lim\limits_{x\to 2} (fg)(x)$

(c) $\lim\limits_{x\to 2} (f/g)(x)$

(d) $\lim\limits_{x\to 2} f(x)^{g(x)}$

31. Let $f(x) = \begin{cases} x^2-1 & x<3 \\ x+5 & x\geq 3 \end{cases}$.
Is f continuous everywhere?

32. Evaluate the limit: $\lim\limits_{x\to e} \ln x$.

2: Differential Equations

One of the strengths of calculus is its ability to describe real-world phenomena. We have seen hints of this in our discussion of the applications of derivatives and integrals in the previous chapters. The process of formulating an equation or multiple equations to describe a physical phenomenon is called *mathematical modeling.* As a simple example, populations of bacteria are often described as "growing exponentially." Looking in a biology text, we might see $P(t) = P_0e^{kt}$, where $P(t)$ is the bacteria population at time t, P_0 is the initial population at time $t = 0$, and the constant k describes how quickly the population grows. This equation for exponential growth arises from the assumption that the population of bacteria grows at a rate proportional to its size. Recalling that the derivative gives the rate of change of a function, we can describe the growth assumption precisely using the equation $P' = kP$. This equation is called a *differential equation*, and is the subject of the current chapter.

2.1 Graphical and Numerical Solutions to Differential Equations

In section **??**, we were introduced to the idea of a differential equation. Given a function $y = f(x)$, we defined a *differential equation* as an equation involving y, x, and derivatives of y. We explored the simple differential equation $y' = 2x$, and saw that a *solution* to a differential equation is simply a function that satisfies the differential equation.

Introduction and Terminology

Definition 7 Differential Equation

Given a function $y = f(x)$, a **differential equation** is an equation relating x, y, and derivatives of y.

- The variable x is called the **independent variable**.
- The variable y is called the **dependent variable**.
- The **order** of the differential equation is the order of the highest derivative of y.

Let us return to the simple differential equation

$$y' = 2x.$$

To find a solution, we must find a function whose derivative is $2x$. In other words, we seek an antiderivative of $2x$. The function

$$y = x^2$$

is an antiderivative of $2x$, and solves the differential equation. So do the functions

$$y = x^2 + 1$$

and

$$y = x^2 - 2346.$$

We call the function

$$y = x^2 + C,$$

with C an arbitrary constant of integration, the *general solution* to the differential equation.

In order to specify the value of the integration constant C, we require additional information. For example, if we know that $y(1) = 3$, it follows that $C = 2$. This additional information is called an *initial condition.*

Definition 8 Initial Value Problem

A differential equation paired with an initial condition (or initial conditions) is called an **initial value problem.**

The solution to an initial value problem is called a **particular solution**. A particular solution does not include arbitrary constants.

The family of solutions to a differential that encompasses all possible solutions is called the **general solution** to the differential equation. A general solution includes one or more arbitrary constants. The particular solution to an initial value problem is one specific member in the family of solutions.

Example 32 A simple first-order differential equation

Solve the differential equation $y' = 2y$.

Solution The solution is a function y such that differentiation yields twice the original function. Unlike our starting example, finding the solution

Notes:

here does not involve computing an antiderivative. Notice that "integrating both sides" would yield the result $y = \int 2y\,dx$, which is not useful. Without knowledge of the function y, we can't compute the indefinite integral. Later sections will explore systematic ways to find analytic solution to simple differential equations. For now, a bit of though might let us guess the solution

$$y = e^{2x}.$$

Notice that application of the chain rule yields $y' = 2e^{2x} = 2y$. Another solution is given by

$$y = -3e^{2x}.$$

In fact

$$y = Ce^{2x},$$

where C is any constant, is the *general solution* to the differential equation because $y' = 2Ce^{2x} = 2y$.

If we are provided with a single initial condition, say $y(0) = \frac{3}{2}$, we can identify $C = \frac{3}{2}$ so that

$$y = \frac{3}{2}e^{2x}$$

is the *particular solution* to the initial value problem

$$y' = 2y, \text{ with } y(0) = \frac{3}{2}.$$

Figure 2.1 shows various members of the general solution to the differential equation $y' = 2y$. Each C value yields a different member of the family, and a different function. We emphasize the particular solution corresponding to the initial condition $y(0) = \frac{3}{2}$.

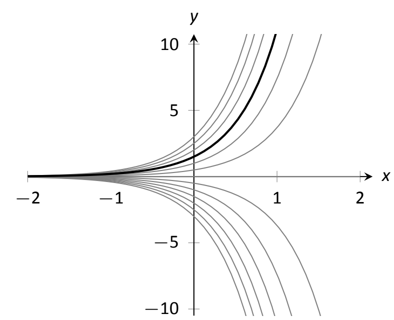


Figure 2.1: A representation of some of the members of general solution to the differential equation $y' = 2y$, including the particular solution to the initial value problem with $y(0) = \dfrac{3}{2}$, from example 32

Example 33 A second-order differential equation

Solve the differential equation $y'' + 9y = 0$.

Solution We seek a function such that two derivatives returns negative 9 multiplied by the original function. Both $\sin(3x)$ and $\cos(3x)$ have this feature. The general solution to the differential equation is given by

$$y = C_1 \sin(3x) + C_2 \cos(3x),$$

where C_1 and C_2 are arbitrary constants. To fully specify a particular solution, we require two additional conditions. For example, the initial conditions $y(0) = 1$ and $y'(0) = 3$ yield $C_1 = C_2 = 1$.

Notes:

The differential equation in example 33 is second order because the equation involves a second derivative. In general, the number of initial conditions required to specify a particular solution depends on the order of the differential equation. For the remainder of the chapter, we restrict our attention to first order differential equations and first order initial value problems.

Example 34 **Verifying a solution to the differential equation**
Which of the following is a solution to the differential equation

$$y' + \frac{y}{x} - \sqrt{y} = 0?$$

a) $y = C\left(1 + \ln x\right)^2$ b) $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ c) $y = Ce^{-3x} + \sqrt{\sin x}$

Solution Verifying a solution to a differential equation is simply an exercise in differentiation and simplification. We substitute each potential solution into the differential equation to see if it satisfies the equation.

a) Testing the potential solution $y = C\left(1 + \ln x\right)^2$:

Differentiating, we have $y' = \dfrac{2C(1+\ln x)}{x}$. Substituting into the differential equation,

$$\begin{aligned}
&\frac{2C(1+\ln x)}{x} + \frac{C(1+\ln x)^2}{x} - \sqrt{C}(1+\ln x)\\
&= (1+\ln x)\left(\frac{2C}{x} + \frac{C(1+\ln x)}{x} - \sqrt{C}\right)\\
&\neq 0.
\end{aligned}$$

Since it doesn't satisfy the differential equation, $y = C(1+\ln x)^2$ is *not* a solution.

b) Testing the potential solution $y = \left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)^2$:

Differentiating, we have $y' = 2\left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)\left(\dfrac{1}{3} - \dfrac{C}{2x^{3/2}}\right)$. Substituting into the differential equation,

$$\begin{aligned}
&2\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{1}{3} - \frac{C}{2x^{3/2}}\right) + \frac{1}{x}\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2 - \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\\
&= \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{2}{3} - \frac{C}{x^{3/2}} + \frac{1}{3} + \frac{C}{x^{3/2}} - 1\right)\\
&= 0.
\end{aligned}$$

Notes:

Thus $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ *is* a solution to the differential equation.

c) Testing the potential solution $y = Ce^{-3x} + \sqrt{\sin x}$:

Differentiating, $y' = -3Ce^{-3x} + \dfrac{\cos x}{2\sqrt{\sin x}}$. Substituting into the differential equation,

$$-3Ce^{-3x} + \frac{\cos x}{2\sqrt{\sin x}} + \frac{Ce^{-3x} + \sqrt{\sin x}}{x} - \sqrt{Ce^{-3x} + \sqrt{\sin x}} \neq 0.$$

The function $y = Ce^{-3x} + \sqrt{\sin x}$ is *not* a solution to the differential equation.

Example 35 Verifying a Solution to a Differential Equation

Verify that $x^2 + y^2 = Cy$ is a solution to $y' = \dfrac{2xy}{x^2 - y^2}$.

Solution The solution in this example is called an *implicit solution.* That means the dependent variable y is a function of x, but has not been explicitly solved for. Verifying the solution still involves differentiation, but we must take the derivatives implicitly. Differentiating, we have

$$2x + 2yy' = Cy'.$$

Solving for y', we have

$$y' = \frac{2x}{C - 2y}.$$

From the solution, we know that $C = \dfrac{x^2 + y^2}{y}$. Then

$$\begin{aligned} y &= \frac{2x}{\frac{x^2 + y^2}{y} - 2y} \\ &= \frac{2xy}{x^2 + y^2 - 2y^2} \\ &= \frac{2xy}{x^2 - y^2}. \end{aligned}$$

We have verified that $x^2 + y^2 = Cy$ is a solution to $y' = \dfrac{2xy}{x^2 - y^2}$.

Graphical Solutions to Differential Equations

The solutions to the differential equations we have explored so far are called *analytic solutions.* We have found exact forms for the functions that solve the

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Notes:

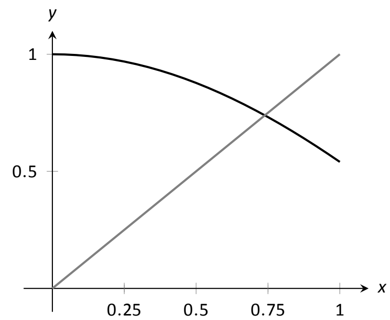


Figure 2.2: Graphically finding an approximate solution to $\cos x = x$.

differential equations. Many times a differential equation will have a solution, but it is difficult or impossible to find the solution analytically. This is analogous to algebraic equations. The algebraic equation $x^2 + 3x - 1 = 0$ has two real solutions that can be found analytically by using the quadratic formula. The equation $\cos x = x$ has one real solution, but we can't find it analytically. As shown in figure 2.2, we can find an approximate solution graphically by plotting $\cos x$ and x and observing the x-value of the intersection. We can similarly use graphical tools to understand the qualitative behavior of solutions to a first order-differential equation.

Consider the first-order differential equation

$$y' = f(x, y).$$

The function f could be any function of the two variables x and y. Written in this way, we can think of the function f as providing a formula to find the slope of a solution at a given point in the xy-plane. In other words, suppose a solution to the differential equation passes through the point (x_0, y_0). At the point (x_0, y_0), the slope of the solution curve will be $f(x_0, y_0)$. Since this calculation of the slope is possible at any point (x, y) where the function $f(x, y)$ is defined, we can produce a plot called a *slope field* (or direction field) that shows the slope of a solution at any point in the xy-plane where the solution is defined. Further, this process can be done purely by working with the differential equation itself. In other words, we can draw a slope field and use it to determine the qualitative behavior of solutions to a differential equation without having to solve the differential equation.

Definition 9 Slope Field

A **slope field** for a first-order differential equation $y' = f(x, y)$ is a plot in the xy-plane made up of short line segments or arrows. For each point (x_0, y_0) where $f(x, y)$ is defined, the slope of the line segment is given by $f(x_0, y_0)$. Plots of solutions to a differential equation are tangent to the line segments in the slope field.

Example 36 Finding a slope field

Find a slope field for the differential equation $y' = x + y$.

Solution Because the function $f(x, y) = x + y$ is defined for all points (x, y), every point in the xy-plane has an associated line segment. It is not practical to draw an entire slope field by hand, but many tools exist for drawing slope fields on a computer. Here, we explicitly calculate and plot a few of the line seg-

Notes:

ments in the slope field.

- The slope of the line segment at $(0, 0)$ is $f(0, 0) = 0 + 0 = 0$.
- The slope of the line segment at $(1, 1)$ is $f(1, 1) = 1 + 1 = 2$.
- The slope of the line segment at $(1, -1)$ is $f(1, -1) = 1 - 1 = 0$.
- The slope of the line segment at $(-2, 3)$ is $f(-2, -1) = -2 - 1 = -3$.

Continuing the above process and plotting the line segments with appropriate slopes results in the slope field shown in figure 2.3.

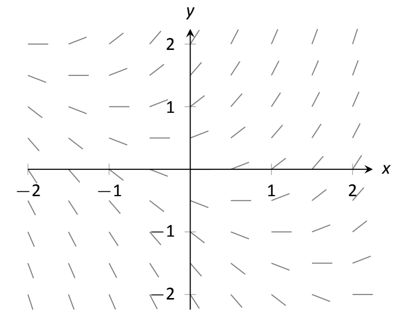


Figure 2.3: Slope field for $y' = x + y$ from example 36.

Example 37 **Finding a graphical solution to an initial value problem**
Find a graphical solution to the initial value problem $y' = x + y$, with $y(1) = -1$.

Solution The solution to the initial value problem should be a continuous smooth curve. Using the slope field, we can draw of a sketch of the solution using the following two criteria:

1. The solution must pass through the point $(1, -1)$.
2. When the solution passes through a point (x_0, y_0) it must be tangent to the line segment at (x_0, y_0).

Essentially, we sketch a solution to the initial value problem by starting at the point $(1, -1)$ and "following the lines" in either direction. A sketch of the solution is shown in figure 2.4.

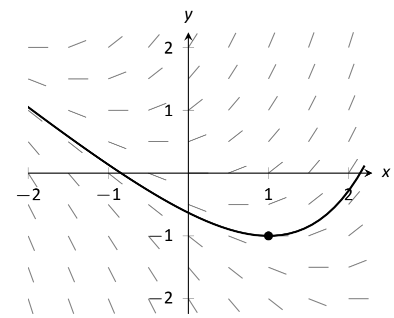


Figure 2.4: Solution to the initial value problem $y' = x + y$, with $y(1) = -1$ from example 37

Example 38 **Using a slope field to predict long term behavior**
Use the slope field for the differential equation $y' = y(1 - y)$, shown in figure 2.5, to predict long term behavior of solutions to the equation.

Solution This differential equation, called the *logistic differential equation*, often appears in population biology to describe the size of a population. For that reason, we use t (time) as the independent variable instead of x. We also often restrict attention to non-negative y-values because negative values correspond to a negative population.

Looking at the slope field in figure 2.5, we can predict long term behavior for a given initial condition.

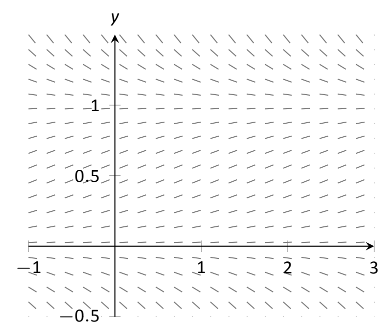


Figure 2.5: Slope field for the logistic differential equation $y' = y(1 - y)$ from example 38.

Notes:

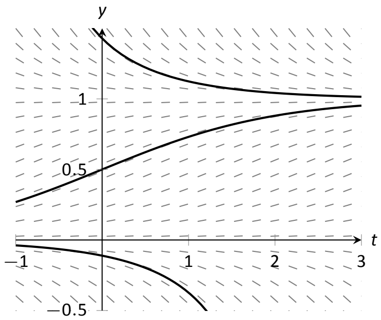


Figure 2.6: Slope field for the logistic differential equation $y' = y(1-y)$ from example 38 with a few representative solution curves.

- If the initial y-value is negative ($y(0) < 0$), the solution curve must pass though the point $(0, y(0))$ and follow the slope field. We expect the solution y to become more and more negative as time increases. Note that this result is not physically relevant when considering a population.
- If the initial y-value is greater than 0 but less than 1, we expect the solution y to increase and level off at $y = 1$.
- If the initial y-value is greater than 1, we expect the solution y to decrease and level off at $y = 1$.

The slope field for the logistic differential equation, along with representative solution curves, is shown in figure 2.6. Notice that any solution curve with positive initial value will tend towards the value $y = 1$. We call this the *carrying capacity.*

Numerical Solutions to Differential Equations: Euler's Method

While the slope field is an effective way to understand the qualitative behavior of solutions to a differential equation, it is difficult to use a slope field to make quantitative predictions. For example, if we have the slope field for the differential equation $y' = x + y$ from example 36 along with the initial condition $y(0) = 1$, we can understand the qualitative behavior of the solution to the initial value problem, but will struggle to predict a specific value, $y(2)$ for example, with any degree of confidence. The most straight forward way to predict $y(2)$ is to find the analytic solution to the the initial value problem and evaluate it at $x = 2$. Unfortunately, we have already mentioned that it is impossible to find analytic solutions to many differential equations. In the absence of an analytic solution, a *numerical solution* can serve as an effective tool to make quantitative predictions about the solution to an initial value problem.

There are many techniques for computing numerical solutions to initial value problems. A course in numerical analysis will discuss various techniques along with their strengths and weaknesses. The simplest technique is called *Euler's Method* (pronounced "oil-er," not "you-ler"). Consider the first-order initial value problem

$$y' = f(x, y), \text{ with } y(x_0) = y_0.$$

Using the definition of the derivative,

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}.$$

If we remove the limit but restrict h to be "small," we have

$$y'(x) \approx \frac{y(x+h) - y(x)}{h},$$

Notes:

so that

$$f(x, y) \approx \frac{y(x+h) - y(x)}{h},$$

because $y' = f(x, y)$ according to the differential equation. Rearranging terms,

$$y(x+h) \approx y(x) + hf(x, y).$$

This statement says that if we know the solution (y-value) to the initial value problem for some given x-value, we can find an approximation for the solution at the value $x+h$ by taking our y-value and adding h times the function f evaluated at the x and y values. Euler's method uses the initial condition of an initial value problem as the starting point, and then uses the above idea to find approximate values for the solution y at later x-values. The algorithm is summarized in key idea 1.

Key Idea 1 Euler's Method

Consider the initial value problem

$$y' = f(x, y) \text{ with } y(x_0) = y_0.$$

Let h be a small positive number and N be an integer.

1. For $i = 0, 1, 2, \ldots, N$, define

$$x_i = x_0 + ih.$$

2. The value y_0 is given by the initial condition.
   For $i = 0, 1, 2, \ldots, N-1$, define

$$y_{i+1} = y_i + hf(x_i, y_i).$$

This process yields a sequence of $N+1$ points (x_i, y_i) for $i = 0, 1, 2, \ldots, N$, where (x_i, y_i) is an approximation for $(x_i, y(x_i))$.

Let's practice Euler's Method using a few concrete examples.

Example 39 Using Euler's Method 1

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.5$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.5$, it takes $N = 2$ steps to get to $x = 2$. Using steps 1 and

Notes:

2 from the Euler's Method algorithm,

$x_0 = 1$	$y_0 = -1$
$x_1 = x_0 + h$ $= 1 + 0.5$ $= 1.5$	$y_1 = y_0 + hf(x_0, y_0)$ $= -1 + 0.5(1 - 1)$ $= -1$
$x_2 = x_0 + 2h$ $= 1 + 2(0.5)$ $= 2$	$y_2 = y_1 + hf(x_1, y_1)$ $= -1 + 0.5(1.5 - 1)$ $= -0.75$

Using Euler's method, we find the approximate $y(2) \approx -0.75$.

To help visualize the Euler's method approximation, these three points (connected by line segments) are plotted along with the analytical solution to the initial value problem in figure 2.7.

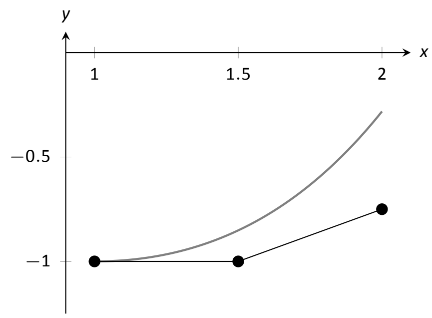


Figure 2.7: Euler's Method approximation to $y' = x + y$ with $y(1) = -1$ from example 39, along with the analytical solution to the initial value problem.

Let's repeat the previous example using a smaller h-value.

Example 40 Using Euler's Method 2

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.25$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.25$, it takes $N = 4$ steps to get to $x = 2$. Using steps 1 and 2 from the Euler's Method algorithm (and rounding to 4 decimal points), we have

x_0	$= 1$	y_0	$= -1$
x_1	$= 1.25$	y_1	$= -1 + 0.25(1 - 1)$ $= -1$
x_2	$= 1.5$	y_2	$= -1 + 0.25(1.25 - 1)$ $= -0.9375$
x_3	$= 1.75$	y_3	$= -0.9375 + 0.25(1.5 - 0.9375)$ $= -0.7969$
x_4	$= 2$	y_4	$= -0.7969 + 0.25(1.75 - 0.7969)$ $= -0.5586$

Using Euler's method, we find the approximate $y(2) \approx -0.5584$.

These five points, along with the points from example 39 and the analytic solution, are plotted in figure 2.8.

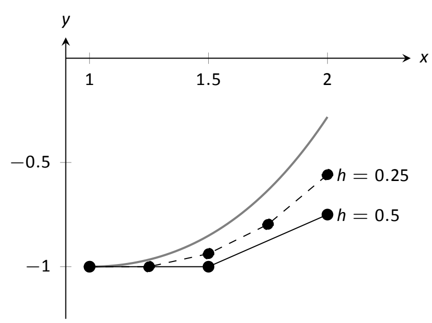


Figure 2.8: Euler's Method approximations to $y' = x + y$ with $y(1) = -1$ from examples 39 and 40, along with the analytical solution.

Using the results from examples 39 and 40, we can make a few observations about Euler's method. First, the Euler approximation gets successively worse as we get farther from the initial condition. This is because Euler's method involves two sources of error. The first comes from the fact that we're using a positive

Notes:

h-value in the derivative approximation instead of using a limit as h approaches zero. Essentially, we're using a linear approximation to the solution y (similar to the process described in section **??** on differentials.) This error is often called the *local truncation error.* The second source of error comes from the fact that every step in Euler's method uses the result of the previous step. That means we're using an approximate y-value to approximate the next y-value. Doing this repeatedly causes the errors to build on each other. This second type of error is often called the *propagated* or *accumulated error.* A second observation is that the Euler approximation is more accurate for smaller h-values. This accuracy comes at a cost, though. Example 40 is more accurate than example 39, but takes twice as many computations. In general, numerical algorithms (even when performed by a computer program) require striking a balance between a desired level of accuracy and the amount of computational effort we are willing to undertake.

Let's do one final example of Euler's Method.

Example 41 Using Euler's Method 3

Find an approximation for the solution to the logistic differential equation $y' = y(1-y)$ with $y(0) = 0.25$, for $0 \le y \le 4$. Use $N = 10$ steps.

Solution The logistic differential equation is what is called an *autonomous equation*. An autonomous differential equation has no explicit dependence on the independent variable (t in this case). This has no real effect on the application of Euler's method other than the fact that the function $f(t, y)$ is really just a function of y. To take steps in the y variable, we use

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + hy_i(1 - y_i).$$

Using $N = 10$ steps requires $h = \dfrac{4-0}{10} = 0.4$. Implementing Euler's Method, we have

x_0	$= 0$	y_0	$= 0.25$
x_1	$= 0.4$	y_1	$= 0.25 + 0.4(0.25)(1 - 0.25)$ $= 0.325$
x_2	$= 0.8$	y_2	$= 0.325 + 0.4(0.325)(1 - 0.325)$ $= 0.41275$
x_3	$= 1.2$	y_3	$= 0.41275 + 0.4(0.41275)(1 - 0.41275)$ $= 0.50970$
x_4	$= 1.6$	y_4	$= 0.50970 + 0.4(0.50970)(1 - 0.50970)$ $= 0.60966$
x_5	$= 2.0$ $=$	y_5	$= 0.60966 + 0.4(0.60966)(1 - 0.60966)$ $= 0.70485$

Notes:

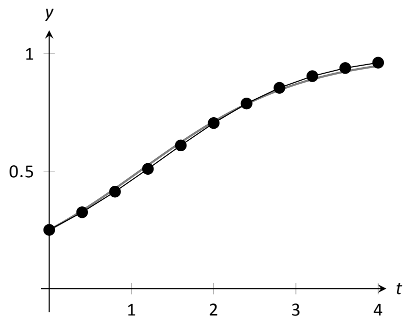


Figure 2.9: Euler's Method approximation to $y' = y(1-y)$ with $y(0) = 0.25$ from example 41, along with the analytical solution.

x_6	$= 2.4$	y_6	$= 0.70485 + 0.4(0.70485)(1 - 0.70485)$ $= 0.78806$
x_7	$= 2.8$	y_7	$= 0.78806 + 0.4(0.78806)(1 - 0.78806)$ $= 0.85487$
x_8	$= 3.2$	y_8	$= 0.85487 + 0.4(0.85487)(1 - 0.85487)$ $= 0.90450$
x_9	$= 3.6$	y_9	$= 0.90450 + 0.4(0.90450)(1 - 0.90450)$ $= 0.93905$
x_{10}	$= 4.0$	y_{10}	$= 0.93905 + 0.4(0.93905)(1 - 0.93905)$ $= 0.96194$

These 11 points, along with the the analytic solution, are plotted in figure 2.9.

The study of differential equations is a natural extension of the study of derivatives and integrals. The equations themselves involve derivatives, and methods to find analytic solutions often involve finding antiderivatives. In this section, we focus on graphical and numerical techniques to understand solutions to differential equations. We restrict our examples to relatively simple initial value problems that permit analytic solution to the equations, but should remember that this is only for comparison purposes. In reality, many differential equations, even some that appear straight forward, do not have solutions we can find analytically. Even so, we can use the techniques presented in this section to understand the behavior of solutions. In the next two sections, we explore two techniques to find analytic solutions to two different classes of differential equations.

Notes:

Exercises 2.1

Terms and Concepts

1. In your own words, what is an initial value problem, and how is it different than a differential equation?

2. In your own words, describe what it means for a function to be a solution to a differential equation.

3. How can we verify that a function is a solution to a differential equation?

4. Describe the difference between a particular solution and a general solution.

5. Why might we use a graphical or numerical technique to study solutions to a differential equation instead of simply solving the differential equation to find an analytic solution?

6. Describe the considerations that should be made when choosing an h value to use in a numerical method like Euler's Method.

Problems

In Exercises 7 – 10, verify that the given function is a solution to the differential equation or initial value problem.

7. $y = Ce^{-6x^2}$; $y' = -12xy$.

8. $y = x\sin x$; $y' - x\cos x = (x^2+1)\sin x - xy$, with $y(\pi) = 0$.

9. $2x^2 - y^2 = C$; $yy' - 2x = 0$

10. $y = xe^x$; $y'' - 2y' + y = 0$

In Exercises 11 – 12, verify that the given function is a solution to the differential equation and find the C value required to make the function satisfy the initial condition.

11. $y = 4e^{3x}\sin x + Ce^{3x}$; $y' - 3y = 4e^{3x}\cos x$, with $y(0) = 2$

12. $y(x^2+y) = C$; $2xy + (x^2+2y)y' = 0$, with $y(1) = 2$

In Exercises 13 – 16, sketch a slope field for the given differential equation.

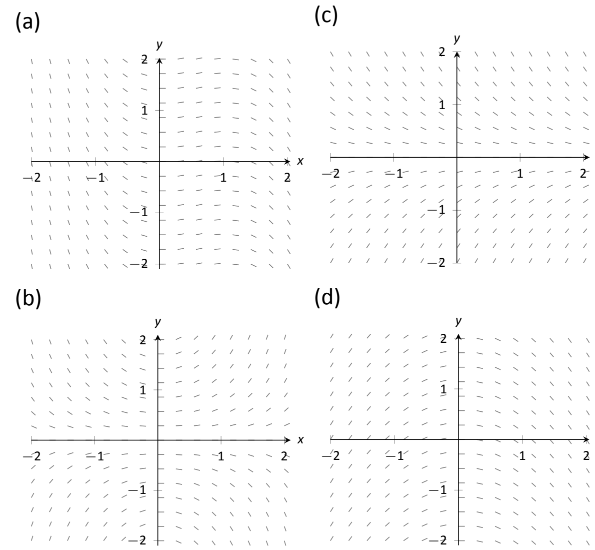
13. $y' = y - x$

14. $y' = \dfrac{x}{2y}$

15. $y' = \sin(\pi y)$

16. $y' = \frac{y}{4}$

In Exercises 17 – 20, match the slope field with the appropriate differential equation.



17. $y' = xy$

18. $y' = -y$

19. $y' = -x$

20. $y' = x(1-x)$

In Exercises 21 – 24, sketch the slope field for the differential equation, and use it to draw a sketch of the solution to the initial value problem.

21. $y' = \dfrac{y}{x} - y$, with $y(0.5) = 1$.

22. $y' = y\sin x$, with $y(0) = 1$.

23. $y' = y^2 - 3y + 2$, with $y(0) = 2$.

24. $y' = -\dfrac{xy}{1+x^2}$, with $y(0) = 1$.

In Exercises 25 – 28, use Euler's Method to make a table of values that approximates the solution to the initial value problem on the given interval. Use the specified h or N value.

25. $y' = x + 2y$
    $y(0) = 1$
    interval: $[0, 1]$
    $h = 0.25$

26. $y' = xe^{-y}$
    $y(0) = 1$
    interval: $[0, 0.5]$
    $N = 5$

27. $y' = y + \sin x$
    $y(0) = 2$
    interval: $[0, 1]$
    $h = 0.2$

28. $y' = e^{x-y}$
    $y(0) = 0$
    interval: $[0, 2]$
    $h = 0.5$

In Exercises 29 – 30, use the provided solution $y(x)$ and Euler's Method with the $h = 0.2$ and $h = 0.1$ to complete the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
$y(x)$						
$h = 0.2$						
$h = 0.1$						

29. $y' = xy^2$
    $y(0) = 1$
    solution: $y(x) = \dfrac{2}{1 - x^2}$

30. $y' = xe^{x^2} + \dfrac{1}{2}xy$
    $y(0) = \dfrac{1}{2}$
    solution: $y(x) = \dfrac{1}{2}(x^2 + 1)e^{x^2}$

2.2 Separable Differential Equations

Similar to algebraic equations, there are specific techniques that can be used to solve specific types of differential equations. In algebra, we can use the quadratic formula to solve a quadratic equation, but not a linear or cubic equation. In the same way, techniques that can be used for a specific type of differential equation are ineffective for a differential equation of a different type. In this section, we describe and practice a technique to solve a class of differential equations called *separable equations.*

Definition 10 Separable Differential Equation

A **separable differential equation** is one that can be written in the form

$$n(y)\frac{dy}{dx} = m(x),$$

where n is a function that depends only on the dependent variable y, and m is a function that depends only on the independent variable x.

Below, we show a few examples of separable differential equations, along with similar looking equations that are not separable.

Separable	Not Separable
1. $\frac{dy}{dx} = x^2y$	1. $\frac{dy}{dx} = x^2 + y$
2. $y\sqrt{y^2-5}\frac{dy}{dx} - \sin x \cos x = 0$	2. $y\sqrt{y^2-1}\frac{dy}{dx} - \sin x \cos y = 0$
3. $\frac{dy}{dx} = \frac{(x^2+1)e^y}{y}$	3. $\frac{dy}{dx} = \frac{(xy+1)e^y}{y}$

Notice that a separable equation requires that the functions of the dependent and independent variables be multiplied, not added (like example 1 of the not separable column). An alternate definition of a separable differential equation states that an equation is separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

for some functions f and g.

Notes:

Separation of Variables

Let's find a formal solution to the separable equation

$$n(y)\frac{dy}{dx} = m(x).$$

Since the functions on the left and right hand sides of the equation are equal, their antiderivatives should be equal up to an arbitrary constant of integration. That is

$$\int n(y)\frac{dy}{dx}\,dx = \int m(x)\,dx + C.$$

Though the integral on the left may look a bit strange, recall that y itself is a function of x. Consider the substitution $u = y(x)$. The differential is $du = \frac{dy}{dx}\,dx$. Using this substitution, the above equation becomes

$$\int n(u)\,du = \int m(x)\,dx + C.$$

Let $N(u)$ and $M(x)$ be antiderivatives of $n(u)$ and $m(x)$, respectively. Then

$$N(u) = M(x) + C.$$

Since $u = y(x)$, this is

$$N(y) = M(x) + C.$$

This relationship between y and x is an implicit form of the solution to the differential equation. Sometimes (but not always) it is possible to solve for y to find an explicit version of the solution.

Though the technique outlined above is formally correct, what we did essentially amounts to integrating the function n with respect to its variable and integrating the function m with respect to its variable. The informal way to solve a separable equation is to treat the derivative $\frac{dy}{dx}$ as if it were a fraction. The separated form of the equation is

$$n(y)\,dy = m(x)\,dx.$$

To solve, we integrate the left hand side with respect to y and the right hand side with respect to x and add a constant of integration. As long as we are able to find the antiderivatives, we can find an implicit form for the solution. Sometimes we are able to solve for y in the implicit solution to find an explicit form of the solution to the differential equation. We practice the technique by solving the three differential equations listed in the separable column above, and conclude by revisiting and finding the general solution to the logistic differential equation from section 2.1

Notes:

Example 42 **Solving a Separable Differential Equation**

Find the general solution to the differential equation $\frac{dy}{dx} = x^2y$.

Solution Using the informal solution method outlined above, we treat $\frac{dy}{dx}$ as a fraction, and write the separated form of the differential equation as

$$\frac{dy}{y} = x^2dx.$$

Integrating the left hand side of the equation with respect to y and the right hand side of the equation with respect to x yields

$$\ln|y| = \frac{1}{3}x^3 + C.$$

This is an implicit form of the solution to the differential equation. Solving for y yields an explicit form for the solution. Exponentiating both sides, we have

$$|y| = e^{\frac{x^3}{3}+C} = e^{\frac{x^3}{3}}e^C.$$

This solution is a bit problematic. First, the absolute value makes the solution difficult to understand. The second issue comes from our desire to find the *general solution.* Recall that a general solution includes all possible solutions to the differential equation. In other words, for any given initial condition, the general solution must include the solution to that specific initial value problem. We can often satisfy any given initial condition by choosing an appropriate C value. When solving separable equations, though, it is possible to lose solutions that have the form $y =$ constant. Notice that $y = 0$ solves the differential equation, but it is not possible to choose a finite C to make our solution look like $y = 0$. Our solution cannot solve the initial value problem $\frac{dy}{dx} = x^2y$, with $y(a) = 0$ (where a is any value). Thus, we haven't actually found a general solution to the problem. We can clean up the solution and recover the missing solution with a bit of clever thought.

Recall the formal definition of the absolute value: $|y| = y$ if $y \geq 0$ and $|y| = -y$ if $y < 0$. Our solution is either $y = e^Ce^{\frac{x^3}{3}}$ or $y = -e^Ce^{\frac{x^3}{3}}$. Further, note that C is constant, so e^C is also constant. If we write our solution as $y = Ce^{\frac{x^3}{3}}$, and allow C to take on both positive and negative values, we incorporate both cases of the absolute value. Finally, if we allow C to be zero, we recover the missing solution discussed above. The best way to express the general solution to our differential equation is

$$y = Ce^{\frac{x^3}{3}}.$$

Note: The indefinite integrals $\int \frac{dy}{y}\,dy$ and $\int x^2\,dx$ both produce arbitrary constants. Since both constants are arbitrary, we combine them into a single constant of integration.

Note: Missing constant solutions can't always be recovered by cleverly redefining the arbitrary constant. The differential equation $\frac{dy}{dx} = y^2 - 1$ is an example of this fact. Both $y = 1$ and $y = -1$ are constant solutions to this differential equation. Separation of variables yields a solution where $y = 1$ can be attained by choosing an appropriate C value, but $y = -1$ can't. The general solution is the set containing the solution produced by separation of variables *and* the missing solution $y = -1$. We should always be careful to look for missing constant solutions when seeking the general solution to a separable differential equation.

Notes:

Example 43 **Solving a Separable Initial Value Problem**

Solve the initial value problem $y\sqrt{y^2-5}\,\frac{dy}{dx} - \sin x\cos x = 0$, with $y(0) = -3$.

Solution We first put the differential equation in separated form

$$y\sqrt{y^2-5}\,dy = \sin x\cos x\,dx.$$

The indefinite integral $\int y\sqrt{y^2-5}\,dy$ requires the substitution $u = y^2 - 5$. Using this substitute yields the antiderivative $\frac{1}{3}(y^2-5)^{3/2}$. The indefinite integral $\int \sin x\cos x\,dx$ requires the substitution $u = \sin x$. Using this substitution yields the antiderivative $\frac{1}{2}\sin^2 x$. Thus, we have an implicit form of the solution to the differential equation given by

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + C.$$

The initial condition says that y should be -3 when x is 0, or

$$\frac{1}{3}((-3)^2-5)^{3/2} = \frac{1}{2}\sin^2 0 + C.$$

This is $C = 8/3$, yielding the particular solution to the initial value problem

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + \frac{8}{3}.$$

Example 44 **Solving a Separable Differential Equation**

Find the general solution to the differential equation $\frac{dy}{dx} = \frac{(x^2+1)e^y}{y}$.

Solution We start by observing that there are no constant solutions to this differential equation because there are no constant y values that make the right hand side of the equation identically zero. Thus, we need not worry about losing solutions during the separation of variables process. The separated form of the equation is given by

$$ye^{-y}\,dy = (x^2+1)\,dx.$$

The antiderivative of the left hand side requires integration by parts. Evaluating both indefinite integrals yields the implicit solution

$$-(y+1)e^{-y} = \frac{1}{3}x^3 + x + C.$$

Since we cannot solve for y, we cannot find an explicit form of the solution.

Notes:

Example 45 **Solving the Logistic Differential Equation**

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right)$

Solution We looked at a slope field for this equation in section 2.1 in the specific case of $k = M = 1$. Here, we use separation of variables to find an analytic solution to the more general equation. Notice that the independent variable t does not explicitly appear in the differential equation. We mentioned that an equation of this type is called *autonomous*. All autonomous differential equation are separable.

We start by making the observation that both $y = 0$ and $y = M$ are constant solutions to the differential equation. We must check that these solutions are not lost during the separation of variables process. The separated form of the equation is

$$\frac{1}{y\left(1 - \dfrac{y}{M}\right)}\,dy = k\,dt.$$

The antiderivative of the left hand side of the equation can be found by making use of partial fractions. Using the techniques discussed in section **??**, we write

$$y\left(1 - \frac{y}{M}\right) = \frac{1}{y} + \frac{1}{M - y}.$$

Then an implicit form of the solution is given by

$$\ln|y| - \ln|M - y| = kt + C.$$

Combining the logarithms,

$$\ln\left|\frac{y}{M - y}\right| = kt + C.$$

Similarly to example 42, we can write

$$\frac{y}{M - y} = Ce^{kt}.$$

Letting C take on positive values or negative values incorporates both cases of the absolute value. This is another implicit form of the solution. Solving for y gives the explicit form

$$y = \frac{M}{1 + be^{-kt}},$$

where b is an arbitrary constant. Notice that $b = 0$ recovers the constant solution $y = M$. The constant solution $y = 0$ cannot be produced with a finite b value, and has been lost. The general solution the logistic differential equation is the set containing $y = \dfrac{M}{1 + be^{-kt}}$ and $y = 0$.

Note: Solving for y initially yields the explicit solution $y = \dfrac{CMe^{kt}}{1 + Cekt}$. Dividing numerator and denominator by Ce^{kt} and defining $b = 1/C$ yields the commonly presented form of the solution given in example 45.

Notes:

Exercises 2.2

Problems

In Exercises 1 – 4, decide whether the differential equation is separable or not separable. If the equation is separable. write it in separated form.

1. $\dfrac{dy}{dx} = y^2 - y$

2. $x\dfrac{dy}{dx} + x^2 y = \dfrac{\sin x}{x - y}$

3. $(y+3)\dfrac{dy}{dx} + \ln x\dfrac{dy}{dx} - x\sin y = (y+3)\ln x$

4. $\dfrac{dy}{dx} - x^2\cos y + y = \cos y - x^2 y$

In Exercises 5 – 12, find the general solution to the separable differential equation. Be sure to check for missing constant solutions.

5. $\dfrac{dy}{dx} + 1 - y^2 = 0$

6. $\dfrac{dy}{dx} = y - 2$

7. $x\dfrac{dy}{dx} = 4y$

8. $y\dfrac{dy}{dx} = 4x$

9. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

10. $(x^2+1)\dfrac{dy}{dx} = \dfrac{x}{y-1}$

11. $\dfrac{dy}{dx} = \dfrac{x\sqrt{1-4y^2}}{x^4 + 2x^2 + 2}$

12. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

In Exercises 13 – 20, find the particular solution to the separable initial value problem.

13. $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos y}$, with $y(0) = \dfrac{\pi}{2}$

14. $\dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$, with $y(0) = 1$

15. $\dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y}$, with $y(0) = -4$

16. $x + ye^{-x}\dfrac{dy}{dx} = 0$, with $y(0) = -2$

17. $\dfrac{dy}{dx} = \dfrac{x\ln(x^2+1)}{y-1}$, with $y(0) = 2$

18. $\sqrt{1-x^2}\,\dfrac{dy}{dx} - \dfrac{\arcsin x}{y\cos(y^2)} = 0$, with $y(0) = \sqrt{\dfrac{7\pi}{6}}$

19. $\dfrac{dy}{dx} = (\cos^2 x)(\cos^2 2y)$, with $y(0) = 0$

20. $\dfrac{dy}{dx} = \dfrac{y^2\sqrt{1-y^2}}{x}$, with $y(0) = 1$

2.3 First Order Linear Differential Equations

In the previous section, we explored a specific techique to solve a specific type of differential equation; a separable differential equation. In this section, we develop and practice a technique to solve a type of differential equation called a *first order linear* differential equation.

Recall than a linear algebraic equation in one variable is one that can be written $ax + b = 0$, where a and b are real numbers. Notice that the variable x appears to the first power. The equations $\sqrt{x} + 1 = 0$ and $\sin(x) - 3x = 0$ are both nonlinear. A linear differential equation is one in which the dependent variable and its derivatives appear only to the first power. We focus on first order equations, which involve first (but not higher order) derivatives of the dependent variable.

Definition 11 First Order Linear Differential Equation

A **first order linear differential equation** is a differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

where p and q are arbitrary functions of the independent variable x.

Example 46 Classifying Differential Equations

Classify each differential equation as first order linear, separable, both, or neither.

(a) $\dfrac{dy}{dx} = xy$

(b) $\dfrac{dy}{dx} = e^y + 3x$

(c) $\dfrac{dy}{dx} - (\cos x)y = \cos x$

(d) $y\dfrac{dy}{dx} - 3xy = 4\ln x$

Solution (a) Both. We identify $p(x) = -x$ and $q(x) = 0$. The separated form of the equation is $\dfrac{dy}{y} = x\,dx$.

(b) Neither. The e^y term makes the equation nonlinear. Because of the addition, it is not possible to write the equation in separated form.

(c) First order linear. We identify $p(x) = -\cos x$ and $q(x) = \cos x$. The equation cannot be written in separated form.

(d) Neither. Notice that dividing by y results in the nonlinear term $\dfrac{4\ln x}{y}$. It is not possible to write the equation in separated form.

Notes:

Notice that linearity depends on the dependent variable y, not the independent variable x. The functions $p(x)$ and $q(x)$ need not be linear, as demonstrated in part (c) of example 46. Neither $\cos x$ nor $\sin x$ are linear functions of x, but the differential equation is still linear.

Solving First Order Linear Equations

We motivate the solution technique by way of an observation and an example. We first observe that the expression $\frac{d}{dx}(xy)$ can be expanded via the product rule and implicit differentiation to the expression $x\frac{dy}{dx}+y$. Now we look at an example. Consider the first order linear differential equation

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{\sin x\cos x}{x}.$$

If we multiply both sides of the differential equation by x and use our observation, we see that the differential equation can be written

$$\frac{d}{dx}(xy)=\sin x\cos x.$$

We can now integrate both sides of the differential equation with respect to x. On th left, the antiderivative of the derivative is simply the function xy. Using the substitution $u=\sin x$ on the right results in the implicit solution

$$xy=\frac{1}{2}\sin^2 x+C.$$

Solving for y yields the explicit solution

$$y=\frac{\sin^2 x}{2x}+\frac{C}{x}.$$

Note: In the examples in the previous section, we performed operations on the arbitrary constant C, but still called the result C. The justification is that the result after the operation is *still* an arbitrary contant. Here, we divide C by x, so the result depends explicitly on the independent variable x. Since C/x is *not* contant, we can't just call it C.

As motivated by the problem we just solved, the basic idea behind solving first order linear differential equations is to multiply both sides of the differential equation by a function, called an *integrating factor*, that makes the left hand side of the equation look like an expanded product rule. We then condense the left hand side into the derivative of a product and integrate both sides. An obvious question is how to find the integrating factor.

Consider the first order linear equation

$$\frac{dy}{dx}+p(x)y=q(x).$$

Notes:

Let's call the integrating factor $\mu(x)$. We multiply both sides of the differential equation by $\mu(x)$ to get

$$\mu(x)\left(\frac{dy}{dx}+p(x)y\right)=\mu(x)q(x).$$

Our goal is to choose $\mu(x)$ so that the left hand side of the differential equation looks like the result of a product rule. The left hand side of the equation is

$$\mu(x)\frac{dy}{dx}+\mu(x)p(x)y.$$

Using the product rule and implicit differentiation,

$$\frac{d}{dx}\big(\mu(x)y\big)=\frac{d\mu}{dx}y+\mu(x)\frac{dy}{dx}.$$

Equating these two gives

$$\frac{d\mu}{dx}y+\mu(x)\frac{dy}{dx}=\mu(x)\frac{dy}{dx}+\mu(x)p(x)y,$$

or

$$\frac{d\mu}{dx}=\mu(x)p(x).$$

In order for the integrating factor $\mu(x)$ to perform its job, it must solve the differential equation above. But that differential equation is separable, so we can solve it. The separated form is

$$\frac{d\mu}{\mu}=p(x)\,dx.$$

Integrating,

$$\ln\mu=\int p(x)\,dx,$$

or

$$\mu(x)=e^{\int p(x)\,dx}.$$

If $\mu(x)$ is chosen this way, after multiplying by $\mu(x)$, we can always write the differential equation in the form

$$\frac{d}{dx}\big(\mu(x)y\big)=\mu(x)q(x).$$

Integrating and solving for y, the explicit solution is

$$y=\frac{1}{\mu(x)}\int\big(\mu(x)q(x)\big)\,dx.$$

Note: Following the steps outlined in the previous section, we should technically end up with $\mu(x)=Ce^{\int p(x)\,dx}$, where C is an arbitrary constant. Because we multiply both sides of the differential equation by $\mu(x)$, the arbitrary constant cancels, and we omit it when finding the integrating factor.

Notes:

Though this formula can be used to write down the solution to a first order linear equation, we shy away from simply memorizing a formula. The process is lost, and it's easy to forget the formula. Rather, we always always follow the steps outlined in key idea 2 when solving equations of this type.

Key Idea 2 Solving First Order Linear Equations

1. Write the differential equation in the form
$$\frac{dy}{dx} + p(x)y = q(x).$$
2. Compute the integrating factor
$$\mu(x) = e^{\int p(x)\,dx}.$$
3. Multiply both sides of the differential equation by $\mu(x)$, and condense the left hand side to get
$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$
4. Integrate both sides of the differential equation with respect to x, taking care to remember the arbitrary constant.
5. Solve for y to find the explicit solution to the differential equation.

Let's practice the process by solving the two first order linear differential equations from example 46.

Example 47 Solving a First Order Linear Equation

Find the general solution to $\frac{dy}{dx} = xy$.

Solution We solve by following the steps in key idea 2. Unlike the process for solving separable equations, we need not worry about losing constant solutions. The answer we find *will* be the general solution to the differential equation. We first write the equation in the form

$$\frac{dy}{dx} - xy = 0.$$

Notes:

By identifying $p(x) = -x$, we can compute the integrating factor

$$\mu(x) = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}.$$

Multiplying both side of the differential equation by $\mu(x)$, we have

$$e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right) = 0.$$

The left hand side of the differential equation condenses to yield

$$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right) = 0.$$

We integrate both sides with respect to x to find the implicit solution

$$e^{-\frac{1}{2}x^2}y = C,$$

or the explicit solution

$$y = Ce^{\frac{1}{2}x^2}.$$

Note: The step where the left hand side of the differential equation condenses to the derivative of a product can feel a bit magical. The reality is that we choose $\mu(x)$ so that we can get exactly this condensing behavior. It's not magic, it's math! If you're still skeptical, try using the product rule and implicit differentiation to evaluate $\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right)$, and verify that it becomes $e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right)$.

Example 48 **Solving a First Order Linear Equation**

Find the general solution to $\frac{dy}{dx} - (\cos x)y = \cos x$.

Solution The differential equation is already in the correct form. The integrating factor is given by

$$\mu(x) = e^{-\int \cos x\,dx} = e^{-\sin x}.$$

Multiplying both sides of the equation by the integrating factor and condensing,

$$\frac{d}{dx}\left(e^{-\sin x}y\right) = (\cos x)e^{-\sin x}$$

Using the substitution $u = -\sin x$, we can integrate to find the implicit solution

$$e^{-\sin x}y = -e^{-\sin x} + C.$$

The explicit form of the general solution is

$$y = -1 + Ce^{\sin x}.$$

We continue our practice by finding the particular solution to an initial value problem.

Notes:

Example 49 **Solving a First Order Linear Initial Value Problem**

Solve the initial value problem $x\dfrac{dy}{dx} - y = x^3 \ln x$, with $y(1) = 0$.

SOLUTION We first divide by x to get

$$\frac{dy}{dx} - \frac{1}{x}y = x^2 \ln x.$$

The integrating factor is given by

$$\begin{aligned}\mu(x) &= e^{-\int \frac{1}{x}\,dx} \\ &= e^{-\ln x} \\ &= e^{\ln x^{-1}} \\ &= x^{-1}.\end{aligned}$$

Multiplying both sides of the differential equation by the integrating factor and condensing the left hand side, we have

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x \ln x.$$

Using integrating by parts to find the antiderivative of $x \ln x$, we find the implicit solution

$$\frac{y}{x} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.$$

Solving for y, the explicit solution is

$$y = \frac{1}{2}x^3 \ln x - \frac{1}{4}x^3 + Cx.$$

The initial condition $y(1) = 0$ yields $C = 1/4$. The solution to the initial value problem is

$$y = \frac{1}{2}x^3 \ln x - \frac{1}{4}x^3 + \frac{1}{4}x.$$

Differential equations are a valuable tool for exploring various physical problems. This process of using equations to describe real world situations is called mathematical modeling, and is the topic of the next section. The last two examples in this section begin our discussion of mathematical modeling.

Example 50 **A Falling Object Without Air Resistance**

Suppose an object with mass m is dropped from an airplane. Find and solve a

Notes:

differential equation describing the vertical velocity of the object assuming no air resistance.

Solution The basic physical law at play is Newton's second law,

$$\text{mass} \times \text{acceleration} = \text{the sum of the forces.}$$

Using the fact that acceleration is the derivative of velocity, mass $\times$ acceleration can be writting mv'. In the absence of air resistance, the only force of interest is the force due to gravity. This force is approximately constant, and is given by mg, where g is the gravitational constant. The word equation above can be written as the differential equation

$$m\frac{dv}{dt} = mg.$$

Because g is constant, this differential equation is simply an integration problem, and we find

$$v = gt + C.$$

Since $v = C$ with $t = 0$, we see that the arbitrary constant here corresponds to the initial vertical velocity of the object.

The process of mathematical modeling does not stop simply because we have found an answer. We must examine the answer to see how well it can describe real world observations. In the previous example, the answer may be somewhat useful for short times, but intuition tells us that something is missing. Our answer says that a falling object's velocity will increase linearly as a function of time, but we know that a falling object does not speed up indefinitely. In order to more fully describe real world behavior, our mathematical model must be revised.

Example 51 **A Falling Object with Air Resistance**

Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object, taking air resistance into account.

Solution We still begin with Newon's second law, but now we assume that the forces in the object come both from gravity and from air resistance. The gravitational force is still given by mg. For air resistance, we assume the force is related to the velocity of the object. A simple way to describe this assumption might be kv^p, where k is a proportionality constant and p is a positive real number. The value k depends on various factors such as the density of the object, surface area of the object, and density of the air. The value p affects how changes in the velocity affect the force. Taken together, a function of the form

Notes:

kv^p is often called a *power law*. The differential equation for the velocity is given by

$$m\frac{dv}{dt} = mg - kv^p.$$

(Notice that the force from air resistance opposes motion, and points in the opposite direction as the force from gravity.) This differential equation is separable, and can be written in the separated form

$$\frac{m}{mg - kv^p}\, dv = dt.$$

For arbitrary positive p, the integration is difficult, making this problem hard to solve analytically. In the case that $p = 1$, the differential equation becomes linear, and is easy to solve either using either separation of variables or integrating factor techniques. We assume $p = 1$, and proceed with an integrating factor so we can continue practicing the process. Writing

$$\frac{dv}{dt} + \frac{k}{m}v = g,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{k}{m}\, dt} = e^{\frac{k}{m}t}.$$

Then

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = ge^{\frac{k}{m}t},$$

so

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C,$$

or

$$v = \frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

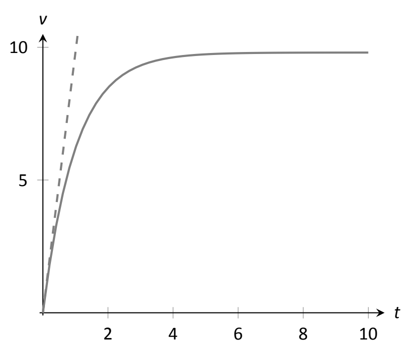


Figure 2.10: The velocity function from examples 50 (dashed) and 51 (solid) under the assumption that $v(0) = 0$, with $g = 9.8, m = 1$ and $k = 1$.

In the solution above, the exponential term decays as time increases, causing the velocity to approach the constant value mg/k in the limit as t approaches infinity. This value is called the *terminal velocity*. If we assume a zero initial velocity (the object is dropped, not thrown from the plane), the velocities from examples 50 and 51 are given by $v = gt$ and $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$, respectively. These two functions are shown in figure **??**, with $g = 9.8, m = 1$, and $k = 1$. Notice that the two curves agree well for short times, but have dramatically different behaviors as t increases. Part of the art in mathematical modeling is deciding on the level of detail required to answer the question of interest. If we are only interested in the initial behavior of the falling object, the simple model in example 50 may be sufficient. If we are interested in the longer term behavior of the object, the simple model is not sufficient, and we should consider a more complicated model.

Notes:

2.4 Modeling with Differential Equations

In the first three sections of this chapter, we focused on the basic ideas behind differential equations and the mechanics of solving certain types of differential equations. We have only hinted at their practical use. In this section, we use differential equations for mathematical modeling, the process of using equations to describe real world processes. We explore a few different mathematical models with the goal of gaining an introduction to this large field of applied mathematics.

Models Involving Proportional Change

Some of the simplest differential equation models involve one quantity that changes at a rate proportional to another quantity. In the introduction to this chapter, we consider a population that grows at a rate proportional to the current population. The words in this assumption can be directly translated into a differential equation as shown below.

is proportional to

$$\frac{dp}{dt} = kp$$

The rate of change of the population the population.

Figure 2.11: Translating words into a differential equation.

There are some key ideas that can be helpful when translating words into a differential equation. Any time we see something about rates or changes, we should think about derivatives. The word "is" usually corresponds to an equal sign in the equation. The words "proportional to" mean we have a constant multiplied by something.

The differential equation in figure 2.11 is easily solved using separation of variables. We find

$$p = Ce^{kt}.$$

Notice that we need values for both C and k before we can use this formula to predict population size. We require information about the population at two different times in order to fully determine the population model.

Example 52 Bacterial Growth

Suppose a population of *e-coli* bacteria grows at a rate proportional to the current population. If an initial popluation of 200 bacteria has grown to 1600 three hours later, find a function for the size of the population at time t, and use it to predict when the population size will reach 10,000.

Notes:

Solution We already know that the population at time t is given by $p = Ce^{kt}$ for some C and k. The information about the initial size of the population means that $p(0) = 200$. Thus $C = 200$. Our knowledge of the population size after three hours allows us to solve for k via the equation

$$1600 = 200e^{3k}.$$

Solving this exponential equation yields $k = \ln(8)/3 \approx 0.6931$. The popluation at time t is given by

$$p = 200e^{(\ln(8)/3)t}.$$

Solving

$$10000 = 200e^{(\ln(8)/3)t}$$

yields $t = (3\ln 50)/\ln 8 \approx 5.644$. The population is predicted to reach 10,000 bacteria in slightly more than five and a half hours.

Another example of porportional change is **Newton's Law of Cooling.** The laws of thermodynamics state that heat flows from areas of high temperature to areas of lower temperature. A simple example is a hot object that cools down when placed in a cool room. Newton's Law of Cooling is the simple assumption that the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the ambient temperature of the room. If T is the temperature of the object, and A is the constant ambient temperature, Newton's Law of Cooling can be expressed as the differential equation

$$\frac{dT}{dt} = k(A - T.)$$

Note: The equation $\dfrac{dT}{dt} = k(T-A)$ is also a valid representation of Newton's Law of Cooling. Intuition tells us that T will increase if T is less than A and decrease if T is greater then A. The form we use in the text follows this intuition with a positive k value. The form above will require that k take on a negative value. In the end, both forms result in the same general solution.

This differential equation is both linear and separable. The separated form is

$$\frac{1}{A - T}\,dT = k\,dt.$$

Then an implicit definition of the temperature is given by

$$-\ln|A - T| = kt + C.$$

If we solve for T, we find the explicit temperature

$$T = A - Ce^{-kt}.$$

Though we didn't show the steps, the explicit solution involves the typical process of renaming the constant $\pm e^{-C}$ as C, and allowing C to be positive, negative, or zero to account for both cases of the absolution value and to catch the constant solution $T = A$. Notice that the temperature of the object approaches the ambient temperature in the limit as $t \to \infty$.

Notes:

Example 53 **Hot Coffee**

A freshly brewed cup of coffee is set on the counter and has a temperature of 200° fahrenheit. After 3 minutes, it has cooled to 190°, but is still too hot to drink. If the room is 72° and the coffee cools according to Newton's Law of Cooling, how long will the impatient coffee drinker have to wait until the coffee has cooled to 165°?

SOLUTION Since we have already solved the differential equation for Newton's Law of Cooling, we can immediately use the function

$$T = A - Ce^{-kt}.$$

Since the room is 72°, we know $A = 72$. The initial temperature is 200°, which means $C = -128$. At this point, we have

$$T = 72 + 128e^{-kt}$$

The information about the coffee cooling to 190° in 3 minutes leads to the equation

$$190 = 72 + 128e^{-3k}.$$

solving the exponential equation for k, we have

$$k = -\frac{1}{3}\ln\left(\frac{59}{64}\right) \approx 0.0271.$$

Finally, we finish the problem by solving the exponential equation

$$165 = 72 + 128e^{\frac{1}{3}\ln\left(\frac{59}{64}\right)t}.$$

The coffee drinker must wait $t = \dfrac{3\ln\left(\frac{93}{128}\right)}{\ln\left(\frac{59}{64}\right)} \approx 11.78$ minutes.

We finish our discussion of models of proportional change by exploring three different models of disease spread through a population. In all of the models, we let y denote the proportion of the population that is sick ($0 \leq y \leq 1$). We assume a proportion of 0.05 is initially sick and that a proportion of 0.1 is sick 1 week later.

Example 54 **Disease Spread 1**

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the popoulation that is sick at time t.

Notes:

Solution The assumption here seems to have some merit because it matches our intuition that a disease should spread more rapidly when more individuals are sick. The differential equation is simply

$$\frac{dy}{dt} = ky,$$

with solution

$$y = Ce^{kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ lead to $C = 0.05$ a and $k = \ln 2$, so the function is

$$y = 0.05e^{(\ln 2)t}.$$

We should point out a glaring problem with this model. The variable y is a proportion and should take on values between 0 and 1, but the function $y = 0.05e^{2t}$ grows without bound. After $t \approx 4.32$ weeks, y exceeds 1, and the model ceases to make physical sense.

Example 55 **Disease Spread 2**

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are not sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the popoulation that is sick at time t

Solution The intuition behind the assumption here is that a disease can only spread if there are individuals who are susceptible to the infection. As fewer and fewer people are able to be infected, the disease spread should slow down. Since y is proportion of the population that is sick, $1 - y$ is the proportion who are not sick, and the differential equation is

$$\frac{dy}{dt} = k(1 - y).$$

Though the context is quite different, the differential equation is identical to the differential equation for Newton's Law of Cooling, with $A = 1$. The solution is

$$y = 1 - Ce^{-kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $C = 0.95$ and $k = -\ln\left(\frac{18}{19}\right) \approx 0.0541$, so the final function is

$$y = 1 - .95e^{\ln\left(\frac{18}{19}\right)t}.$$

Notice that this function approches $y = 1$ in the limit as $t \to \infty$, and does not suffer from the non-physical behavior described in example 54.

Notes:

In example 54, we assumed disease spread depends on the number of infected individuals. In example 55, we assumed disease spread depends on the number of susceptible individuals who are able to become infected. In reality, we would expect many diseases to require the interaction of both infected on susceptible individuals in order to spread. One of the simplest ways to model this required interaction is to assume disease spread depends on the product of the proportions of infected and uninfected individuals. This assumption is often called the *law of mass action.*

Example 56 Disease Spread 3

Suppose a disease spreads through a population at a rate proportional to the product of the number of infected and uninfected individuals. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the population that is sick at time t

Solution The differential equation is

$$\frac{dy}{dt} = ky(1-y).$$

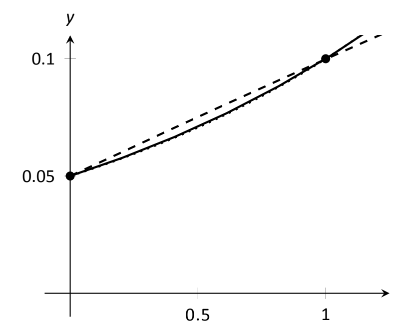
This is exactly the logistic equation with $M = 1$. We solve this differential equation in example 45, and find

$$y = \frac{1}{1 + be^{-kt}}.$$

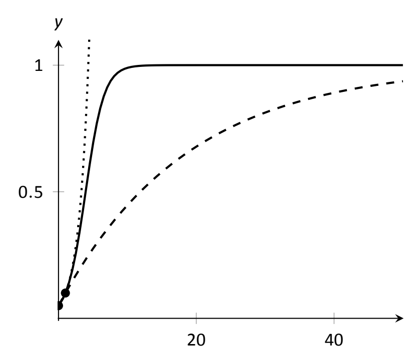
The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $b = 19$ and $k = -\ln\left(\frac{9}{19}\right) \approx 0.7472$. The final function is

$$y = \frac{1}{1 + 19e^{\ln\left(\frac{9}{19}\right)t}}.$$

Based on the three different assumptions about the rate of disease spread explored in the last three examples, we now have three different functions giving the proportion of a population that is sick at time t. Each of the three functions meets the conditions $y(0) = 0.05$ and $y(1) = 0.1$. The three functions are shown in figure 2.12. Notice that the logistic function mimics specific parts of the functions from examples 54 and 55. We see in figure 2.12 (a) that the logistic and exponential functions are virtually indistinguishable for small t values. When there are few infected individuals and lots of susceptible individuals, the spread of a disease is largely determined by the number of sick people. The logistic curve captures this feature, and is "almost exponential" early on. In figure 2.12, we see that the logistic curve leaves the exponential curve from example 54 and approaches the curve from example 55. This result implies that when most



(a)



(b)

Figure 2.12: Plots of the functions from example 54 (dotted), example 55 (dashed), and example 56 (solid).

Notes:

of the population is sick, the spread of the disease is largely dependent on the number of susceptible individuals. Though there are much more sophisticated mathematical models describing the spread of infections, we could argue that the logistic model presented in this example is the "best" of the three.

Rate-in Rate-out Problems

One of the classic ways to build a mathematical model involves tracking the way the amount of something can change. Consider a box with a specific type of stuff inside. The amount of stuff of the specific type in the box can only change in three ways; we can add more stuff to the box, we can remove some of the stuff from the box, or some of the stuff can change into stuff of a different type. In the examples that follow, we assume stuff doesn't change type, so we only need to keep track of stuff coming into the box and stuff leaving the box. To derive a differential equation, we track rates:

$$\text{rate of change of some quantity} = \text{rate in} - \text{rate out}.$$

Though we stick to relatively simple examples, this basic idea can be used to derive some very important differential equations in mathematics and physics.

The examples to follow involve tracking the amount of a chemical in solution. We assume liquid containing some chemical flows into a container at some rate. That liquid mixes instantaneously with the liquid already in the container. Then the liquid from the container flows out at some (potentially different) rate.

Note: The assumption about instantaneous mixing, though not physically accurate, leads to a differential equation we have hope of solving. In reality, the amount of chemical at a specific location in the container depends both on the location and how long we have been waiting. This dependence on both space and time leads to a type of differential equation called a *partial differential equation.* Differential equations of this type are more interesting, but significantly harder to study. Instantaneous mixing removes any spatial dependence from the problem, and leaves us with an *ordinary differential equation*.

Example 57 Equal Flow Rates

Suppose a 10 liter bucket has 5 liters of salt solution in it. The initial concentration of the salt solution is 1 g/L. A salt solution with concentration 3 g/L flows into the bucket at a rate of 2 L/min. Suppose the salt solution mixes instantaneously with the solution already in the bucket and that the mixed solution from the bucket flows out at a rate of 2 L/min. Find a function that gives the amount of salt in the bucket at time t.

Solution We use the rate in - rate out setup described above. The quantity here is the amount (in grams) of salt in the bucket at time t. Let y denote the amount of salt. In words, the differential equation is given by

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}.$$

Thinking in terms of units can help fill in the details of the differential equation. Since y has units of grams, the left hand side of the equation has units g/min. Both terms on the right hand side must have these same units. Notice that the product of a concentration (with units g/L) and a flow rate (with units L/min)

Notes:

results in a quantity with units g/min. Both terms on the right hand side of the equation will include a concentration multiplied by a flow rate.

For the rate in, we multiply the inflow concentration by the rate that fluid is flowing into the bucket. This is $(3)(2) = 6$ g/min.

The rate out is more complicated. The flow rate is still 2 L/min, meaning that the overall volume of the fluid in the bucket is the constant 5 L. The salt concentration in the bucket is not constant though, meaning that the outflow concentration is not constant. In particular, the outflow concentration is *not* the constant 1 g/L. This is simply the initial concentration. To find the concentration at any time, we need the amount of salt in the bucket at that time and the volume of liquid in the bucket at that time. The volume of liquid is the constant 5 L, and the amount of salt is given by the dependent variable y. Thus, the outflow concentration is $y/5$, yielding a rate out given by $\left(\frac{y}{5}\right)(2) = \frac{2y}{5}$.

The differential equation we wish to solve is given by

$$\frac{dy}{dt} = 6 - \frac{2y}{5}.$$

To furnish an initial condition, we must convert the initial salt concentration into an initial amount of salt. This is $(1)(5) = 5$ g, so $y(0) = 5$ is our initial condition.

Our differential equation is both separable and linear. We solve using separation of variables. The separated form of the differential equation is

$$\frac{5}{30 - 2y}\,dy = dt.$$

Integrating, yields the implicit solution

$$-\frac{5}{2}\ln|30 - 2y| = t + C.$$

Solving for y (end redefining the arbitrary constant C as necessary) yields the explicit solution

$$y = 15 + Ce^{-\frac{2}{5}t}.$$

The initial condition $y(0) = 5$ means that $C = -10$ so that

$$y = 15 - 10e^{-\frac{2}{5}t}$$

Is the particular solution to our initial value problem.

This function is plotted in figure 2.13. Notice that in the limit as $t \to \infty$, y approaches 15. This corresponds to a bucket concentration of $15/5 = 3$ g/L. It should not be surprising that salt concentration inside the tank will move to match the inflow salt concentration.

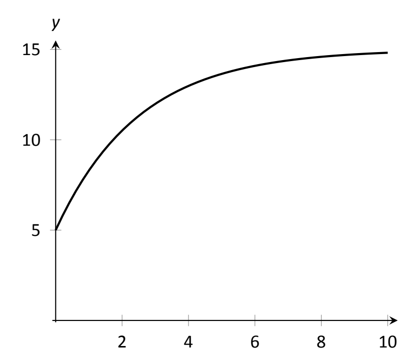


Figure 2.13: Salt concentration at time t, from example 57.

Example 58 **Unequal Flow Rates**

Suppose the setup is identical to the setup in example 57 except that now liquid

Notes:

flows out of the bucket at a rate of 1 L/min. Find a function that gives the amount of salt in the bucket at time t. What is the salt concentration when the solution ceases to be valid?

Solution Because the inflow and outflow rates no longer match, the volume of liquid in the bucket is not the constant 5 L. In general, we can find the volume of liquid via the equation

$$\text{volume} = \text{initial volume} + (\text{inflow rate - outflow rate})t.$$

In this example, the volume at time t is $5 + t$. Because the total volume of the bucket is only 10 L, it follows that our solution will only be valid for $0 \leq t \leq 5$. At that point it is no longer possible to have liquid flow into a the bucket at a rate of 2 L/min and out of the bucket at a rate of 1 L/min.

To update the differential equation, we must modify the rate out. Since the volume is $5+t$, the concentration at time t is given by $\frac{y}{5+t}$. Thus for rate out, we must use $\left(\frac{y}{5+t}\right)(1)$. The initial value problem is

$$\frac{dy}{dt} = 6 - \frac{y}{5+t}, \text{ with } y(0) = 5.$$

Unlike example 57, where we had equal flow rates, this differential equation is no longer separable. We must proceed with an integrating factor. Writing the differential equation in the form

$$\frac{dy}{dt} + \frac{1}{5+t}y = 6,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{1}{5+t}\,dt} = e^{\ln(5+t)} = 5 + t.$$

Then

$$\frac{d}{dt}\big((5+t)y\big) = 6(5+t),$$

yielding the implicit solution

$$(5+t)y = 30t + 3t^2 + C.$$

The initial condition $y(0) = 5$ implies $C = 25$, so the explicit solution to our initial value problem is given by

$$y = \frac{3t^2 + 30t + 25}{5+t}.$$

Notes:

This solution ceases to be valid at $t = 5$. At that time, there are 25 g of salt in the tank. The volume of liquid is 10 L, resulting in a salt concentration of 2.5 g/L.

Differential equations are powerful tools that can be used to help describe the world around us. Though relatively simple in concept, the ideas of proportional change and matching rates can serve as building blocks in the development of more sophisticated mathematical models. As we saw in this section, some simple mathematical models can be solved analytically using the techniques developed in this chapter. Most more sophicticated mathematical models don't allow for analytic solutions. Even so, there are an array of graphical and numerical techniques that can be used to analyze the model to make predictions and infer information about real world phenomenon.

Notes:

A: Solutions To Selected Problems

Chapter 1

Section 1.1

1. Answers will vary.
2. An indeterminate form.
3. F
4. The function may approach different values from the left and right, the function may grow without bound, or the function might oscillate.
5. Answers will vary.
6. -1
7. -5
8. Limit does not exist
9. 2
10. 1.5
11. Limit does not exist.
12. Limit does not exist.
13. 7
14. 1
15. Limit does not exist.
16.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-7
-0.01	-7
0.01	-7
0.1	-7

The limit seems to be exactly 7.

17.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	9
-0.01	9
0.01	9
0.1	9

The limit seems to be exactly 9.

18.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	4.9
-0.01	4.99
0.01	5.01
0.1	5.1

The limit is approx. 5.

19.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.114943
-0.01	-0.111483
0.01	-0.110742
0.1	-0.107527

The limit is approx. -0.11.

20.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	29.4
-0.01	29.04
0.01	28.96
0.1	28.6

The limit is approx. 29.

21.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	0.202027
-0.01	0.2002
0.01	0.1998
0.1	0.198026

The limit is approx. 0.2.

22.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.998334
-0.01	-0.999983
0.01	-0.999983
0.1	-0.998334

The limit is approx. -1.

23.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.0499583
-0.01	-0.00499996
0.01	0.00499996
0.1	0.0499583

The limit is approx. 0.005.

Section 1.2

1. ε should be given first, and the restriction $|x-a|<\delta$ implies $|f(x)-K|<\varepsilon$, not the other way around.
2. The y–tolerance.
3. T
4. T
5. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-5|<\delta$, $|f(x)-(-2)|<\varepsilon$.
   Consider $|f(x)-(-2)|<\varepsilon$:
   $$\begin{aligned}|f(x)+2|&<\varepsilon\\ |(3-x)+2|&<\varepsilon\\ |5-x|&<\varepsilon\\ -\varepsilon<5-x&<\varepsilon\\ -\varepsilon<x-5&<\varepsilon.\end{aligned}$$
   This implies we can let $\delta=\varepsilon$. Then:
   $$\begin{aligned}|x-5|&<\delta\\ -\delta<x-5&<\delta\\ -\varepsilon<x-5&<\varepsilon\\ -\varepsilon<(x-3)-2&<\varepsilon\\ -\varepsilon<(-x+3)-(-2)&<\varepsilon\\ |3-x-(-2)|&<\varepsilon,\end{aligned}$$
   which is what we wanted to prove.
6. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-3|<\delta$, $|f(x)-6|<\varepsilon$.
   Consider $|f(x)-6|<\varepsilon$, keeping in mind we want to make a statement about $|x-3|$:
   $$\begin{aligned}|f(x)-6|&<\varepsilon\\ |x^2-3-6|&<\varepsilon\\ |x^2-9|&<\varepsilon\\ |x-3|\cdot|x+3|&<\varepsilon\\ |x-3|&<\varepsilon/|x+3|\end{aligned}$$
   Since x is near 3, we can safely assume that, for instance, $2<x<4$. Thus
   $$\begin{aligned}2+3<x+3&<4+3\\ 5<x+3&<7\\ \frac{1}{7}<\frac{1}{x+3}&<\frac{1}{5}\\ \frac{\varepsilon}{7}<\frac{\varepsilon}{x+3}&<\frac{\varepsilon}{5}\end{aligned}$$

Let $\delta = \frac{\varepsilon}{7}$. Then:

$$\begin{aligned} |x-3| &< \delta \\ |x-3| &< \frac{\varepsilon}{7} \\ |x-3| &< \frac{\varepsilon}{x+3} \\ |x-3|\cdot|x+3| &< \frac{\varepsilon}{x+3}\cdot|x+3| \end{aligned}$$

Assuming x is near 3, $x+3$ is positive and we can drop the absolute value signs on the right.

$$\begin{aligned} |x-3|\cdot|x+3| &< \frac{\varepsilon}{x+3}\cdot(x+3) \\ |x^2-9| &< \varepsilon \\ |(x^2-3)-6| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

7. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-4| < \delta$, $|f(x)-15| < \varepsilon$.
   Consider $|f(x)-15| < \varepsilon$, keeping in mind we want to make a statement about $|x-4|$:

$$\begin{aligned} |f(x)-15| &< \varepsilon \\ |x^2+x-5-15| &< \varepsilon \\ |x^2+x-20| &< \varepsilon \\ |x-4|\cdot|x+5| &< \varepsilon \\ |x-4| &< \varepsilon/|x+5| \end{aligned}$$

Since x is near 4, we can safely assume that, for instance, $3 < x < 5$. Thus

$$\begin{aligned} 3+5 < x&+5 < 5+5 \\ 8 < x&+5 < 10 \\ \frac{1}{10} < &\frac{1}{x+5} < \frac{1}{8} \\ \frac{\varepsilon}{10} < &\frac{\varepsilon}{x+5} < \frac{\varepsilon}{8} \end{aligned}$$

Let $\delta = \frac{\varepsilon}{10}$. Then:

$$\begin{aligned} |x-4| &< \delta \\ |x-4| &< \frac{\varepsilon}{10} \\ |x-4| &< \frac{\varepsilon}{x+5} \\ |x-4|\cdot|x+5| &< \frac{\varepsilon}{x+5}\cdot|x+5| \end{aligned}$$

Assuming x is near 4, $x+5$ is positive and we can drop the absolute value signs on the right.

$$\begin{aligned} |x-4|\cdot|x+5| &< \frac{\varepsilon}{x+5}\cdot(x+5) \\ |x^2+x-20| &< \varepsilon \\ |(x^2+x-5)-15| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

8. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-2| < \delta$, $|f(x)-7| < \varepsilon$.
   Consider $|f(x)-7| < \varepsilon$, keeping in mind we want to make a statement about $|x-2|$:

$$\begin{aligned} |f(x)-7| &< \varepsilon \\ |x^3-1-7| &< \varepsilon \\ |x^3-8| &< \varepsilon \\ |x-2|\cdot|x^2+2x+4| &< \varepsilon \\ |x-3| &< \varepsilon/|x^2+2x+4| \end{aligned}$$

Since x is near 2, we can safely assume that, for instance, $1 < x < 3$. Thus

$$\begin{aligned} 1^2+2\cdot1+4 < x^2&+2x+4 < 3^2+2\cdot3+4 \\ 7 < x^2&+2x+4 < 19 \\ \frac{1}{19} < &\frac{1}{x^2+2x+4} < \frac{1}{7} \\ \frac{\varepsilon}{19} < &\frac{\varepsilon}{x^2+2x+4} < \frac{\varepsilon}{7} \end{aligned}$$

Let $\delta = \frac{\varepsilon}{19}$. Then:

$$\begin{aligned} |x-2| &< \delta \\ |x-2| &< \frac{\varepsilon}{19} \\ |x-2| &< \frac{\varepsilon}{x^2+2x+4} \\ |x-2|\cdot|x^2+2x+4| &< \frac{\varepsilon}{x^2+2x+4}\cdot|x^2+2x+4| \end{aligned}$$

Assuming x is near 2, x^2+2x+4 is positive and we can drop the absolute value signs on the right.

$$\begin{aligned} |x-2|\cdot|x^2+2x+4| &< \frac{\varepsilon}{x^2+2x+4}\cdot(x^2+2x+4) \\ |x^3-8| &< \varepsilon \\ |(x^3-1)-7| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

9. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-2| < \delta$, $|f(x)-5| < \varepsilon$. However, since $f(x)=5$, a constant function, the latter inequality is simply $|5-5| < \varepsilon$, which is always true. Thus we can choose any δ we like; we arbitrarily choose $\delta = \varepsilon$.

10. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-0| < \delta$, $|f(x)-0| < \varepsilon$.
    Consider $|f(x)-0| < \varepsilon$, keeping in mind we want to make a statement about $|x-0|$ (i.e., $|x|$):

$$\begin{aligned} |f(x)-0| &< \varepsilon \\ |e^{2x}-1| &< \varepsilon \\ -\varepsilon < e^{2x}&-1 < \varepsilon \\ 1-\varepsilon < e&^{2x} < 1+\varepsilon \\ \ln(1-\varepsilon) < &2x < \ln(1+\varepsilon) \\ \frac{\ln(1-\varepsilon)}{2} < &x < \frac{\ln(1+\varepsilon)}{2} \end{aligned}$$

Let $\delta = \min\left\{\left|\frac{\ln(1-\varepsilon)}{2}\right|, \frac{\ln(1+\varepsilon)}{2}\right\} = \frac{\ln(1+\varepsilon)}{2}$.

Thus:

$$
\begin{gathered}
|x| < \delta \\
|x| < \frac{\ln(1+\varepsilon)}{2} < \left|\frac{\ln(1-\varepsilon)}{2}\right| \\
\frac{\ln(1-\varepsilon)}{2} < x < \frac{\ln(1+\varepsilon)}{2} \\
\ln(1-\varepsilon) < 2x < \ln(1+\varepsilon) \\
1-\varepsilon < e^{2x} < 1+\varepsilon \\
-\varepsilon < e^{2x} - 1 < \varepsilon \\
|e^{2x} - 1 - (0)| < \varepsilon,
\end{gathered}
$$

which is what we wanted to prove.

11. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 0| < \delta$, $|f(x) - 0| < \varepsilon$. In simpler terms, we want to show that when $|x| < \delta$, $|\sin x| < \varepsilon$.

    Set $\delta = \varepsilon$. We start with assuming that $|x| < \delta$. Using the hint, we have that $|\sin x| < |x| < \delta = \varepsilon$. Hence if $|x| < \delta$, we know immediately that $|\sin x| < \varepsilon$.

Section 1.3

1. Answers will vary.
2. Answers will vary.
3. Answers will vary.
4. Answers will vary.
5. As x is near 1, both f and g are near 0, but f is approximately twice the size of g. (I.e., $f(x) \approx 2g(x)$.)
6. 9
7. 6
8. 0
9. Limit does not exist.
10. 3
11. Not possible to know; as x approaches 6, $g(x)$ approaches 3, but we know nothing of the behavior of $f(x)$ when x is near 3.
12. 3
13. -45
14. 1
15. -1
16. 0
17. π
18. 7
19. $-0.000000015 \approx 0$
20. $1/2$
21. Limit does not exist
22. 64
23. 2
24. 0
25. $\frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3} \approx 0.6064$
26. $\frac{3\pi+1}{1-\pi}$
27. -8
28. -1
29. 10
30. -2
31. $-3/2$
32. $-7/8$
33. 0
34. 0
35. 1
36. 9
37. 3
38. $5/8$
39. 1
40. $\pi/180$
41. (a) Apply Part 1 of Theorem 1.
    (b) Apply Theorem 6; $g(x) = \frac{x}{x}$ is the same as $g(x) = 1$ everywhere except at $x = 0$. Thus $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} 1 = 1$.
    (c) The function $f(x)$ is always 0, so $g\big(f(x)\big)$ is never defined as $g(x)$ is not defined at $x = 0$. Therefore the limit does not exist.
    (d) The Composition Rule requires that $\lim\limits_{x\to 0} g(x)$ be equal to $g(0)$. They are not equal, so the conditions of the Composition Rule are not satisfied, and hence the rule is not violated.

Section 1.4

1. The function approaches different values from the left and right; the function grows without bound; the function oscillates.
2. F
3. F
4. T
5. (a) 2
   (b) 2
   (c) 2
   (d) 1
   (e) As f is not defined for $x < 0$, this limit is not defined.
   (f) 1
6. (a) 1
   (b) 2
   (c) Does not exist.
   (d) 2
   (e) 0
   (f) As f is not defined for $x > 2$, this limit is not defined.
7. (a) Does not exist.
   (b) Does not exist.
   (c) Does not exist.
   (d) Not defined.
   (e) 0
   (f) 0
8. (a) 2
   (b) 0
   (c) Does not exist.
   (d) 1
9. (a) 2
   (b) 2
   (c) 2

(d) 2

10. (a) 4
    (b) -4
    (c) Does not exist.
    (d) 0
11. (a) 2
    (b) 2
    (c) 2
    (d) 0
    (e) 2
    (f) 2
    (g) 2
    (h) Not defined
12. (a) $a-1$
    (b) a
    (c) Does not exist.
    (d) a
13. (a) 2
    (b) -4
    (c) Does not exist.
    (d) 2
14. (a) -1
    (b) 0
    (c) Does not exist.
    (d) 0
15. (a) 0
    (b) 0
    (c) 0
    (d) 0
    (e) 2
    (f) 2
    (g) 2
    (h) 2
16. (a) -1
    (b) 0
    (c) Does not exist.
    (d) 0
17. (a) $1-\cos^2 a = \sin^2 a$
    (b) $\sin^2 a$
    (c) $\sin^2 a$
    (d) $\sin^2 a$
18. (a) 2
    (b) 0
    (c) Does not exist
    (d) 1
19. (a) 4
    (b) 4
    (c) 4
    (d) 3
20. (a) c
    (b) c
    (c) c
    (d) c
21. (a) -1
    (b) 1
    (c) Does not exist
    (d) 0
22. $-3/5$
23. $2/3$
24. 2.5
25. -9
26. -1.63

Section 1.5

1. Answers will vary.
2. Answers will vary.
3. A root of a function f is a value c such that $f(c)=0$.
4. Consider the function $h(x)=g(x)-f(x)$, and use the Bisection Method to find a root of h.
5. F
6. T
7. T
8. F
9. F
10. T
11. No; $\lim_{x\to 1} f(x)=2$, while $f(1)=1$.
12. No; $\lim_{x\to 1} f(x)$ does not exist.
13. No; $f(1)$ does not exist.
14. Yes
15. Yes
16. Yes
17. (a) No; $\lim_{x\to -2} f(x) \neq f(-2)$
    (b) Yes
    (c) No; $f(2)$ is not defined.
18. (a) Yes
    (b) Yes
19. (a) Yes
    (b) No; the left and right hand limits at 1 are not equal.
20. (a) Yes
    (b) Yes
21. (a) Yes
    (b) No. $\lim_{x\to 8} f(x)=16/5\neq f(8)=5$.
22. $(-\infty,\infty)$
23. $(-\infty,-2]\cup[2,\infty)$
24. $[-1,1]$
25. $(-\infty,-\sqrt{6}]\cup[\sqrt{6},\infty)$
26. $(-1,1)$
27. $(-\infty,\infty)$
28. $(-\infty,\infty)$

29. $(0, \infty)$

30. $(-\infty, \infty)$

31. $(-\infty, 0]$

32. $(-\infty, \infty)$

33. Yes, by the Intermediate Value Theorem.

34. Yes, by the Intermediate Value Theorem. In fact, we can be more specific and state such a value c exists in $(0, 2)$, not just in $(-3, 7)$.

35. We cannot say; the Intermediate Value Theorem only applies to function values between -10 and 10; as 11 is outside this range, we do not know.

36. We cannot say; the Intermediate Value Theorem only applies to continuous functions. As we do know know if h is continuous, we cannot say.

37. Approximate root is $x = 1.23$. The intervals used are:
$[1, 1.5]$ $[1, 1.25]$ $[1.125, 1.25]$
$[1.1875, 1.25]$ $[1.21875, 1.25]$ $[1.234375, 1.25]$
$[1.234375, 1.2421875]$ $[1.234375, 1.2382813]$

38. Approximate root is $x = 0.52$. The intervals used are:
$[0.5, 0.55]$ $[0.5, 0.525]$ $[0.5125, 0.525]$
$[0.51875, 0.525]$ $[0.521875, 0.525]$

39. Approximate root is $x = 0.69$. The intervals used are:
$[0.65, 0.7]$ $[0.675, 0.7]$ $[0.6875, 0.7]$
$[0.6875, 0.69375]$ $[0.690625, 0.69375]$

40. Approximate root is $x = 0.78$. The intervals used are:
$[0.7, 0.8]$ $[0.75, 0.8]$ $[0.775, 0.8]$
$[0.775, 0.7875]$ $[0.78125, 0.7875]$
(A few more steps show that 0.79 is better as the root is $\pi/4 \approx 0.78539$.)

41. (a) 20
    (b) 25
    (c) Limit does not exist
    (d) 25

42.

x	$f(x)$
-0.81	-2.34129
-0.801	-2.33413
-0.79	-2.32542
-0.799	-2.33254

The top two lines give an approximation of the limit from the left: -2.33. The bottom two lines give an approximation from the right: -2.33 as well.

43. Answers will vary.

Section 1.6

1. F
2. T
3. F
4. T
5. T
6. Answers will vary.
7. Answers will vary.
8. The limit of f as x approaches 7 does not exist, hence f is not continuous. (Note: f could be defined at 7!)

9. (a) ∞
   (b) ∞

10. (a) $-\infty$
    (b) ∞
    (c) Limit does not exist
    (d) ∞
    (e) ∞
    (f) ∞

11. (a) 1
    (b) 0
    (c) $1/2$
    (d) $1/2$

12. (a) Limit does not exist
    (b) Limit does not exist

13. (a) Limit does not exist
    (b) Limit does not exist

14. (a) 10
    (b) ∞

15. Tables will vary.

(a)

x	$f(x)$
2.9	-15.1224
2.99	-159.12
2.999	-1599.12

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	16.8824
3.01	160.88
3.001	1600.88

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x)$ does not exist.

16. Tables will vary.

(a)

x	$f(x)$
2.9	-335.64
2.99	-30350.6

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	-265.61
3.01	-29650.6

It seems $\lim_{x\to 3^+} f(x) = -\infty$.

(c) It seems $\lim_{x\to 3} f(x) = -\infty$.

17. Tables will vary.

(a)

x	$f(x)$
2.9	132.857
2.99	12124.4

It seems $\lim_{x\to 3^-} f(x) = \infty$.

(b)

x	$f(x)$
3.1	108.039
3.01	11876.4

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x) = \infty$.

18. Tables will vary.

(a)

x	$f(x)$
2.9	-0.632
2.99	-0.6032
2.999	-0.60032

It seems $\lim_{x\to 3^-} f(x) = -0.6$.

(b)

x	$f(x)$
3.1	-0.5686
3.01	-0.5968
3.001	-0.59968

It seems $\lim_{x\to 3^+} f(x) = -0.6$.

(c) It seems $\lim_{x\to 3} f(x) = -0.6$.

19. Horizontal asymptote at $y = 2$; vertical asymptotes at $x = -5, 4$.

20. Horizontal asymptote at $y = -3/5$; vertical asymptote at $x = 3$.

21. Horizontal asymptote at $y = 0$; vertical asymptotes at $x = -1, 0$.

22. No horizontal asymptote; vertical asymptote at $x = 1$.

23. No horizontal or vertical asymptotes.

24. Horizontal asymptote at $y=-1$; no vertical asymptotes

25. ∞

26. $-\infty$

27. $-\infty$

28. ∞

29. Solution omitted.

30. (a) 2
    (b) -3
    (c) -3
    (d) $1/3$

31. Yes. The only "questionable" place is at $x=3$, but the left and right limits agree.

32. 1

Chapter 2

Section 2.1

1. An initial value problems is a differential equation that is paired with one or more initial conditions. A differential equation is simply the equation without the initial conditions.

2. Answers will vary.

3. Substitute the proposed function into the differential equation, and show the the statement is satisfied.

4. A particular solution is one specifica member of a family of solutions, and has no arbitrary constants. A general solution is a family of solutions, includes all possible solutions to the differential equation, and typically includes one or more arbitrary constants.

5. Many differential equations are impossible to solve analytically.

6. A smaller h value leads to a numerical solution that is closer to the true solution, but decreasing the h value leads to more computational effort.

7. Answers will vary.

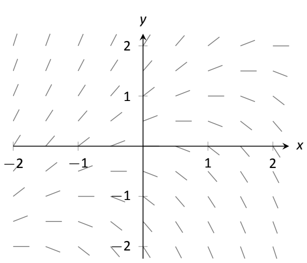
8. Answers will vary.

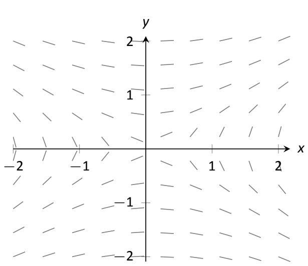
9. Answers will vary.

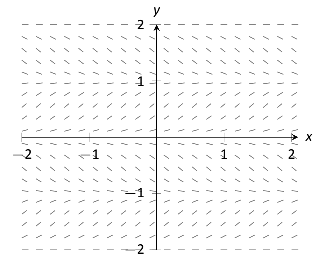
10. Answers will vary.

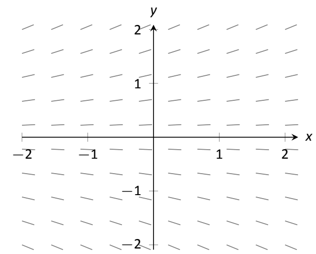
11. $C=2$

12. $C=6$

13. 

14. 

15. 

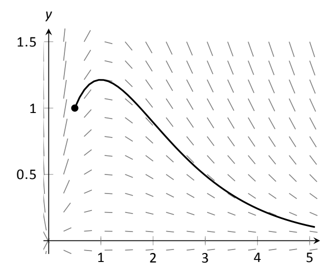
16. 

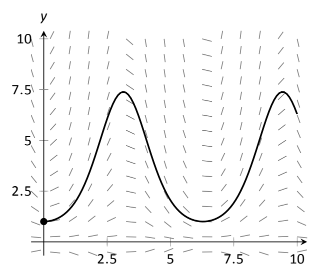
17. b

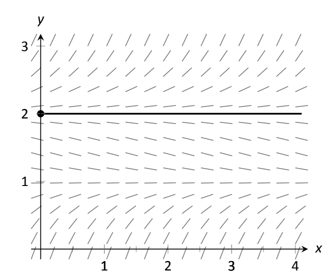
18. c

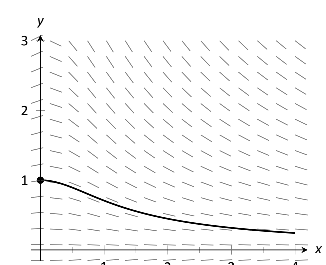
19. d

20. a

21. 

22. 

23. 

24. 

25.

x_i	y_i
0.00	1.0000
0.25	1.5000
0.50	2.3125
0.75	3.5938
1.00	5.5781

26.

x_i	y_i
0.0	1.0000
0.1	1.0000
0.2	1.0037
0.3	1.0110
0.4	1.0219
0.5	1.0363

27.

x_i	y_i
0.0	2.0000
0.2	2.4000
0.4	2.9197
0.6	3.5816
0.8	4.4108
1.0	5.4364

28.

x_i	y_i
0.0	0.0000
0.5	0.5000
1.0	1.8591
1.5	10.5824
2.0	88378.1190

29.

x_i	$y(x)$	h=0.2	$h = 0.1$
0.0	1.0000	1.0000	1.0000
0.2	1.0204	1.0000	1.0100
0.4	1.0870	1.0400	1.0623
0.6	1.2195	1.1265	1.1687
0.8	1.4706	1.2788	1.3601
1.0	2.0000	1.5405	1.7129

30.

x_i	$y(x)$	$h = 0.2$	$h = 0.1$
0.0	0.5000	0.5000	0.5000
0.2	0.5412	0.5000	0.5201
0.4	0.6806	0.5816	0.6282
0.6	0.9747	0.7686	0.8622
0.8	1.5551	1.1250	1.3132
1.0	2.7183	1.7885	2.1788

Section 2.2

1. Separable. $\dfrac{1}{y^2 - y}\,dy = dx$
2. Not separable.
3. Not separable.
4. Separable. $\dfrac{1}{\cos y - y}\,dy = (x^2+1)\,dx$
5. $\left\{y = \dfrac{1+Ce^{2x}}{1-Ce^{2x}}, y = -1\right\}$
6. $y = 2 + Ce^x$
7. $y = Cx^4$
8. $y^2 - 4x^2 = C$
9. $(y-1)e^y = -e^{-x} - \dfrac{1}{3}e^{-3x} + C$
10. $(y-1)^2 = \ln(x^2+1) + C$
11. $\left\{\arcsin 2y - \arctan(x^2+1) = C, y = \pm\dfrac{1}{2}\right\}$
12. $\left\{y = \dfrac{1}{C - \arctan x}, y = 0\right\}$
13. $\sin y + \cos x = 2$
14. $-x^3 + 3y - y^3 = 2$
15. $\frac{1}{2}y^2 - \ln(1+x^2) = 8$
16. $y^2 + 2xe^x - 2e^x = 2$
17. $\dfrac{1}{2}y^2 - y = \dfrac{1}{2}\big((x^2+1)\ln(x^2+1) - (x^2+1)\big) + \dfrac{1}{2}$
18. $\sin(y^2) - (\arcsin x)^2 = -\frac{1}{2}$
19. $2\tan 2y = 2x + \sin 2x$
20. $x = exp\left(-\dfrac{\sqrt{1-y^2}}{y}\right)$

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