APEX CALCULUS
Version 3.0

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Preface

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three–volume series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.apexcalculus.com`.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactive the interactivity, one can right-click and choose the “Disable Content” option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

1: Differential Equations

One of the strengths of calculus is its ability to describe real-world phenomena. We have seen hints of this in our discussion of the applications of derivatives and integrals in the previous chapters. The process of formulating an equation or multiple equations to describe a physical phenomenon is called *mathematical modeling.* As a simple example, populations of bacteria are often described as "growing exponentially." Looking in a biology text, we might see $P(t) = P_0e^{kt}$, where $P(t)$ is the bacteria population at time t, P_0 is the initial population at time $t = 0$, and the constant k describes how quickly the population grows. This equation for exponential growth arises from the assumption that the population of bacteria grows at a rate proportional to its size. Recalling that the derivative gives the rate of change of a function, we can describe the growth assumption precisely using the equation $P' = kP$. This equation is called a *differential equation*, and is the subject of the current chapter.

1.1 Graphical and Numerical Solutions to Differential Equations

In section **??**, we were introduced to the idea of a differential equation. Given a function $y = f(x)$, we defined a *differential equation* as an equation involving y, x, and derivatives of y. We explored the simple differential equation $y' = 2x$, and saw that a *solution* to a differential equation is simply a function that satisfies the differential equation.

Introduction and Terminology

Definition 1 Differential Equation

Given a function $y = f(x)$, a **differential equation** is an equation relating x, y, and derivatives of y.

- The variable x is called the **independent variable**.
- The variable y is called the **dependent variable**.
- The **order** of the differential equation is the order of the highest derivative of y.

Let us return to the simple differential equation

$$y' = 2x.$$

To find a solution, we must find a function whose derivative is $2x$. In other words, we seek an antiderivative of $2x$. The function

$$y = x^2$$

is an antiderivative of $2x$, and solves the differential equation. So do the functions

$$y = x^2 + 1$$

and

$$y = x^2 - 2346.$$

We call the function

$$y = x^2 + C,$$

with C an arbitrary constant of integration, the *general solution* to the differential equation.

In order to specify the value of the integration constant C, we require additional information. For example, if we know that $y(1) = 3$, it follows that $C = 2$. This additional information is called an *initial condition.*

Definition 2 Initial Value Problem

A differential equation paired with an initial condition (or initial conditions) is called an **initial value problem.**

The solution to an initial value problem is called a **particular solution** to the initial value problem.
The solution to a differential that encompasses all possible solutions is called the **general solution** to the differential equation.

Example 1 A simple first-order differential equation
Solve the differential equation $y' = 2y$.

Solution The solution is a function y such that differentiation yields twice the original function. Unlike our starting example, finding the solution here does not involve computing an antiderivative. Notice that "integrating both sides" would yield the result $y = \int 2y\,dx$, which is not useful. Without knowledge of the function y, we can't compute the indefinite integral. Later sections

Notes:

will explore systematic ways to find analytic solution to simple differential equations. For now, a bit of though might let us guess that solution

$$y = e^{2x}.$$

Notice that application of the chain rule yields $y' = 2e^{2x} = 2y$. Another solution is given by

$$y = -3e^{2x}.$$

In fact

$$y = Ce^{2x},$$

where C is any constant, is the *general solution* to the differential equation because $y' = 2Ce^{2x} = 2y$.

If we are provided with a single initial condition, say $y(0) = 3$, we can identify $C = 3$ so that

$$y = 3e^{2x}$$

is the *particular solution* to the initial value problem

$$y' = 2y, \text{ with } y(0) = 3.$$

Example 2 A second-order differential equation

Solve the differential equation $y'' + 9y = 0$.

Solution We seek a function such that two derivatives returns negative 9 multiplied by the original function. Both $\sin(3x)$ and $\cos(3x)$ have this feature. The general solution to the differential equation is given by

$$y = C_1 \sin(3x) + C_2 \cos(3x),$$

where C_1 and C_2 are arbitrary constants. To fully specify a particular solution, we require two additional conditions. For example, the initial conditions $y(0) = 1$ and $y'(0) = 3$ yield $C_1 = C_2 = 1$.

The differential equation in example 2 is second order because the equation involves a second derivative. In general, the number of initial conditions required to specify a particular solution depends on the order of the differential equation. For the remainder of the chapter, we restrict our attention to first order differential equations and first order initial value problems.

Example 3 Verifying a solution to the differential equation

Which of the following is a solution to the differential equation

$$y' + \frac{y}{x} - \sqrt{y} = 0?$$

Notes:

a) $y = C\,(1+\ln x)^2$ b) $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ c) $y = Ce^{-3x} + \sqrt{\sin x}$

Solution Verifying a solution to a differential equation is simply an exercise in differentiation and simplification. We substitute each potential solution into the differential equation to see if it satisfies the equation.

a) Testing the potential solution $y = C\,(1+\ln x)^2$:

Differentiating, we have $y' = \dfrac{2C(1+\ln x)}{x}$. Substituting into the differential equation,

$$\begin{aligned}&\frac{2C(1+\ln x)}{x} + \frac{C(1+\ln x)^2}{x} - \sqrt{C}(1+\ln x)\\ &= (1+\ln x)\left(\frac{2C}{x} + \frac{C(1+\ln x)}{x} - \sqrt{C}\right)\\ &\neq 0.\end{aligned}$$

Since it doesn't satisfy the differential equation, $y = C(1+\ln x)^2$ is *not* a solution.

b) Testing the potential solution $y = \left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)^2$:

Differentiating, we have $y' = 2\left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)\left(\dfrac{1}{3} - \dfrac{C}{2x^{3/2}}\right)$. Substituting into the differential equation,

$$\begin{aligned}&2\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{1}{3} - \frac{C}{2x^{3/2}}\right) + \frac{1}{x}\left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2 - \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\\ &= \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)\left(\frac{2}{3} - \frac{C}{x^{3/2}} + \frac{1}{3} + \frac{C}{x^{3/2}} - 1\right)\\ &= 0.\end{aligned}$$

Thus $y = \left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)^2$ *is* a solution to the differential equation.

c) Testing the potential solution $y = Ce^{-3x} + \sqrt{\sin x}$:

Differentiating, $y' = -3Ce^{-3x} + \dfrac{\cos x}{2\sqrt{\sin x}}$. Substituting into the differential equation,

$$-3Ce^{-3x} + \frac{\cos x}{2\sqrt{\sin x}} + \frac{Ce^{-3x} + \sqrt{\sin x}}{x} - \sqrt{Ce^{-3x} + \sqrt{\sin x}} \neq 0.$$

Notes:

The function $y = Ce^{-3x} + \sqrt{\sin x}$ is *not* a solution to the differential equation.

Graphical Solutions to Differential Equations

The solutions to the differential equations we have found so far are called *analytic solutions.* We have found exact forms for the functions that solve the differential equations. Many times a differential equation will have a solution, but it is difficult or impossible to find the solution analytically. This is analogous to algebraic equations. The algebraic equation $x^2+3x-1=0$ has two real solutions that can be found analytically by using the quadratic formula. The equation $\cos x = x$ has one real solution, but we can find it analytically. As shown in figure 1.1, we can find an approximate solution graphically by plotting $\cos x$ and x and observing the x-value of the intersection. We can similarly use graphical tools to understand the qualitative behavior of solutions to a first order-differential equation.

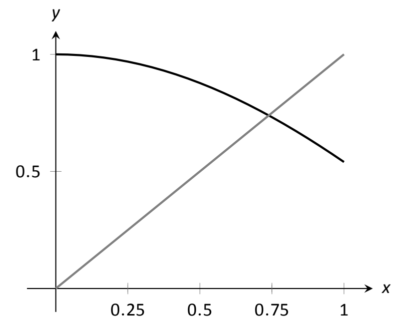

Figure 1.1: Graphically finding an approximate solution to $\cos x = x$.

Consider the first-order differential equation

$$y' = f(x, y).$$

The function f could be any function of the two variables x and y. Written in this way, we can think of the function f as providing a formula to find the slope of a solution at a given point in the xy-plane. In other words, suppose a solution to the differential equation passes through the point (x_0, y_0). Then, at the point (x_0, y_0) the slope of the solution curve will be $f(x_0, y_0)$. Since this calculation of the slope is possible at any point (x, y) where the function $f(x, y)$ is defined, we can produce a plot called a *slope field* that shows the slope of a solution at any point in the xy-plane where the solution is defined. Further, this process can be done purely by working with the differential equation itself. In other words, we can draw a slope field and use it to determine the qualitative behavior of solutions to a differential equation without having to solve the differential equation.

Definition 3 Slope Field

A **slope field** for a first-order differential equation $y' = f(x, y)$ is a plot in the xy-plane made up of short line segments or arrows. For each point (x_0, y_0) where $f(x, y)$ is defined, the slope of the line segment is given by $f(x_0, y_0)$. Plots of solutions to a differential equation are tangent to the line segments in the slope field.

Example 4 Finding a slope field

Find a slope field for the differential equation $y' = x + y$.

Notes:

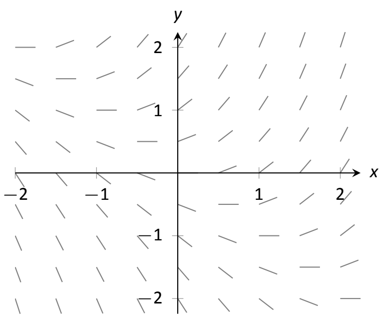

Figure 1.2: Slope field for $y' = x + y$ from example 4.

Solution Because the function $f(x, y) = x + y$ is defined for all points (x, y), every point in the xy-plane has an associated line segment. It is not practical to draw an entire slope field by hand, but many tools exist for drawing slope fields on a computer. We will explicitly calculate and plot a few of the line segments in the slope field.

The slope of the line segment at $(0, 0)$ is given by $f(0, 0) = 0 + 0 = 0$.

The slope of the line segment at $(1, 1)$ is given by $f(1, 1) = 1 + 1 = 2$.

The slope of the line segment at $(1, -1)$ is given by $f(1, -1) = 1 - 1 = 0$.

The slope of the line segment at $(-2, 3)$ is given by $f(-2, -1) = -2 - 1 = -3$.

Continuing the above process and plotting the line segments with appropriate slopes results in the slope field shown in figure 1.2.

Example 5 **Finding a graphical solution to an initial value problem**

Find a graphical solution to the initial value problem $y' = x + y$ with $y(1) = -1$.

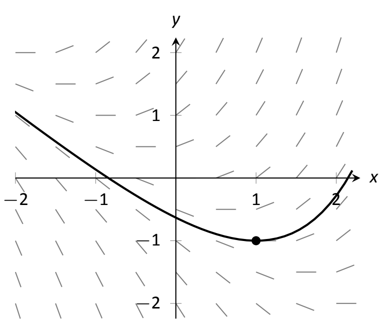

Figure 1.3: Solution to the initial value problem $y' = x + y$ with $y(1) = -1$ from example 5

Solution The solution to the initial value problem should be a continuous smooth curve. Using the slope field, we can draw of a sketch of the solution using the following two criteria:

1. The solution must pass through the point $(1, -1)$.
2. When the solution passes through a point (x_0, y_0) it must be tangent to the line segment at (x_0, y_0).

Essentially, we sketch a solution to the initial value problem by starting at the point $(1, -1)$ and "following the lines" in either direction. A sketch of the solution is shown in figure 1.3.

Example 6 **Using a slope field to predict long term behavior**

Use the slope field for the differential equation $y' = y(1 - y)$, shown in figure 1.4, to predict long term behavior of solutions to the equation.

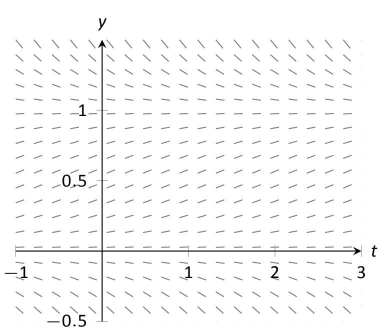

Figure 1.4: Slope field for the logistic differential equation $y' = y(1 - y)$ from example 6.

Solution This differential equation, called the *logistic differential equation*, often appears in population biology to describe the size of a population. For that reason, we use t (time) as the independent variable instead of x. We also often restrict attention to non-negative y-values because negative values correspond to a negative population.

Notes:

Looking at the slope field in figure 1.4, we can predict long term behavior for a given initial condition.

- If the initial y-value is negative ($y(0) < 0$), the solution curve must pass though the point $(0, y(0))$ and follow the slope field. We expect the solution y to become more and more negative as time increases. Note that this result is not physically relevant when considering a population.
- If the initial y-value is greater than 0 but less than 1, we expect the solution y to increase and level off it $y = 1$.
- If the initial y-value is greater than 1, we expect the solution y to decrease and level off at $y = 1$.

The slope field for the logistic differential equation, along with representative solution curves, is shown in figure 1.5. Notice that any solution curve with positive initial value will tend towards the value $y = 1$. We call this the *carrying capacity.*

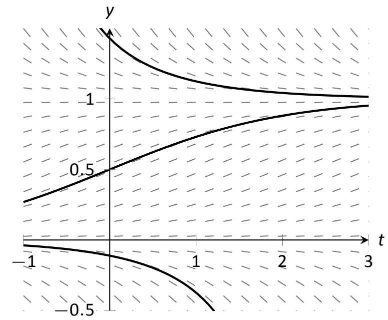

Figure 1.5: Slope field for the logistic differential equation $y' = y(1 - y)$ from example 6 with a few representative solution curves.

Numerical Solutions to Differential Equations: Euler's Method

While the slope field is an effective way to understand the qualitative behavior of solutions to a differential equation, it is difficult to use a slope field to make quantitative predictions. For example, if we have the slope field for the differential equation $y' = x + y$ from example 4 along with the initial condition $y(0) = 1$, we can understand the qualitative behavior of the solution to the initial value problem, but will struggle to predict a specific value, $y(2)$ for example, with any degree of confidence. The most straight forward way to predict $y(2)$ is to find the analytic solution to the the initial value problem and evaluate it at $x = 2$. Unfortunately, we have already mentioned that it is impossible to find analytic solutions to many differential equations. In the absence of an analytic solution, a *numerical solution* can serve as an effective tool to make quantitative predictions about the solution to an initial value problem.

There are many techniques for computing numerical solutions to initial value problems. A course in numerical analysis will discuss various techniques along with their strengths and weaknesses. The simplest technique is called *Euler's Method* (pronounced "oil-er," not "you-ler"). Consider the first-order initial value problem

$$y' = f(x, y), \text{ with } y(x_0) = y_0.$$

Using the definition of the derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

Notes:

If we remove the limit but restrict h to be "small," we have

$$y'(x) \approx \frac{y(x+h) - y(x)}{h},$$

so that

$$f(x,y) \approx \frac{y(x+h) - y(x)}{h},$$

because $y' = f(x,y)$ according to the differential equation. Rearranging terms,

$$y(x+h) \approx y(x) + hf(x,y).$$

This statement says that if we know the solution (y-value) to the initial value problem for some given x-value, we can find an approximation for the solution at the value $x+h$ by taking our y-value and adding h times the function f evaluated at the x and y values. Euler's method uses the initial condition of an initial value problem as the starting point, and then uses the above idea to find approximate values for the solution y at later x-values. The algorithm is summarized in key idea 1.

Key Idea 1 Euler's Method

Consider the initial value problem

$$y' = f(x,y) \text{ with } y(x_0) = y_0.$$

Let h be a small positive number and N be an integer.

1. The value x_0 is given by the initial condition.
   For $i = 0, 1, 2, \ldots, N-1$, define
   $$x_{i+1} = x_0 + ih.$$
2. The value y_0 is given by the initial condition.
   For $i = 0, 1, 2, \ldots, N-1$, define
   $$y_{i+1} = y_i + hf(x_i, y_i).$$

This process yields a sequence of $N+1$ points (x_i, y_i) for $i = 0, 1, 2, \ldots, N$, where (x_i, y_i) is an approximation for $(x_i, y(x_i))$ for each i.

Let's practice Euler's Method using a few concrete examples.

Notes:

Example 7 **Using Euler's Method 1**

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.5$.

SOLUTION Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.5$, it takes $N = 2$ steps to get to $x = 2$. Using steps 1 and 2 from the Euler's Method algorithm,

x_0	$= 1$	y_0	$= -1$
x_1	$= x_0 + h$	y_1	$= y_0 + hf(x_0, y_0)$
	$= 1 + 0.5$		$= -1 + 0.5(1 - 1)$
	$= 1.5$		$= -1$
x_2	$= x_0 + 2h$	y_2	$= y_1 + hf(x_1, y_1)$
	$= 1 + 2(0.5)$		$= -1 + 0.5(1.5 - 1)$
	$= 2$		$= -0.75$

Using Euler's method, we find the approximate $y(2) \approx -0.75$.

To help visualize the Euler's method approximation, these three points (connected by line segments) are plotted along with the analytical solution to the initial value problem in figure 1.6.

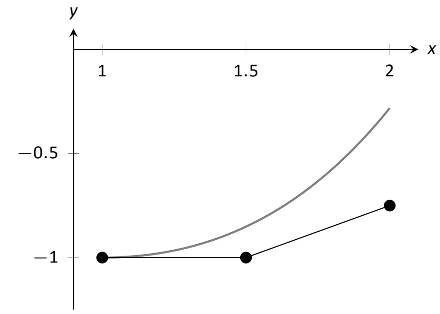

Figure 1.6: Euler's Method approximation to $y' = x + y$ with $y(1) = -1$ from example 7, along with the analytical solution to the initial value problem.

Let's repeat the previous example using a smaller h-value.

Example 8 **Using Euler's Method 2**

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.25$.

SOLUTION Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.25$, it takes $N = 4$ steps to get to $x = 2$. Using steps 1 and 2 from the Euler's Method algorithm (and rounding to 4 decimal points), we have

x_0	$= 1$	y_0	$= -1$
x_1	$= 1.25$	y_1	$= -1 + 0.25(1 - 1)$
			$= -1$
x_2	$= 1.5$	y_2	$= -1 + 0.25(1.25 - 1)$
			$= -0.9375$
x_3	$= 1.75$	y_3	$= -0.9375 + 0.25(1.5 - 0.9375)$
			$= -0.7969$
x_4	$= 2$	y_4	$= -0.7969 + 0.25(1.75 - 0.7969)$
			$= -0.5586$

Using Euler's method, we find the approximate $y(2) \approx -0.5584$.

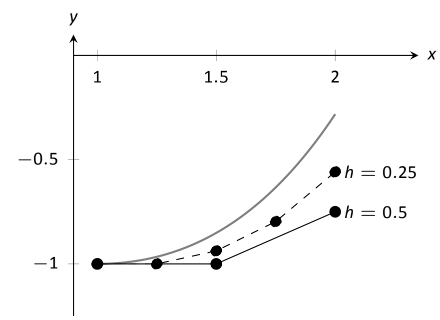

Figure 1.7: Euler's Method approximations to $y' = x + y$ with $y(1) = -1$ from examples 7 and 8, along with the analytical solution.

Notes:

These five points, along with the points from example 7 and the analytic solution, are plotted in figure 1.7.

Using the results from examples 7 and 8, we can make a few observations about Euler's method. First, the Euler approximation gets successively worse as we get farther from the initial condition. This is because Euler's method involves two sources of error. The first comes from the fact that we're using a positive h-value in the derivative approximation instead of using a limit as h approaches zero. Essentially, we're using a linear approximation to the solution y (similar to the process described in section **??** on differentials.) This error is often called the *local truncation error.* The second source of error comes from the fact that every step in Euler's method uses the result of the previous step. That means we're using an approximate y-value to approximate the next y-value. Doing this repeatedly cases the errors to build on each other. This second type of error is often called the *propagated* or *accumulated error.* A second observation is that the Euler approximation is more accurate for smaller h-values. This accuracy comes at a cost, though. Example 8 is more accurate than example 7, but takes twice as many computations. In general, numerical algorithms (even when performed by a computer program) require striking a balance between a desired level of accuracy and the amount of computational effort we are willing to undertake.

Let's do one final example of Euler's Method.

Example 9 Using Euler's Method 3

Find an approximation for the solution to the logistic differential equation $y' = y(1-y)$ with $y(0) = 0.25$, for $0 \le y \le 4$. Use $N = 10$ steps.

Solution The logistic differential equation is what is called an *autonomous equation*. An autonomous differential equation has no explicit dependence on the independent variable (t in this case). This has no real effect on the application of Euler's method other than the fact that the function $f(t, y)$ is really just a function of y. To take steps in the y variable, we use

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + hy_i(1-y_i).$$

Using $N = 10$ steps requires $h = \dfrac{4-0}{10} = 0.4$. Implementing Euler's Method, we have

Notes:

x_0	$= 0$	y_0	$= 0.25$
x_1	$= 0.4$	y_1	$= 0.25 + 0.4(0.25)(1 - 0.25)$
			$= 0.325$
x_2	$= 0.8$	y_2	$= 0.325 + 0.4(0.325)(1 - 0.325)$
			$= 0.41275$
x_3	$= 1.2$	y_3	$= 0.41275 + 0.4(0.41275)(1 - 0.41275)$
			$= 0.50970$
x_4	$= 1.6$	y_4	$= 0.50970 + 0.4(0.50970)(1 - 0.50970)$
			$= 0.60966$
x_5	$= 2.0$	y_5	$= 0.60966 + 0.4(0.60966)(1 - 0.60966)$
	$=$		$= 0.70485$
x_6	$= 2.4$	y_6	$= 0.70485 + 0.4(0.70485)(1 - 0.70485)$
			$= 0.78806$
x_7	$= 2.8$	y_7	$= 0.78806 + 0.4(0.78806)(1 - 0.78806)$
			$= 0.85487$
x_8	$= 3.2$	y_8	$= 0.85487 + 0.4(0.85487)(1 - 0.85487)$
			$= 0.90450$
x_9	$= 3.6$	y_9	$= 0.90450 + 0.4(0.90450)(1 - 0.90450)$
			$= 0.93905$
x_{10}	$= 4.0$	y_{10}	$= 0.93905 + 0.4(0.93905)(1 - 0.93905)$
			$= 0.96194$

These 11 points, along with the the analytic solution, are plotted in figure 1.8.

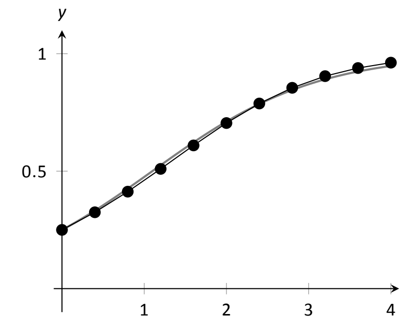

Figure 1.8: Euler's Method approximation to $y' = y(1 - y)$ with $y(0) = 0.25$ from example 9, along with the analytical solution.

The study of differential equations is a natural extension of the study of derivatives and integrals. The equations themselves involve derivatives, and methods to find analytic solutions often involve finding antiderivatives. In this section, we focus on graphical and numerical techniques to understand solutions to differential equations. We restrict our examples to relatively simple initial value problems where we can find analytic solution to the equations, but should remember that this is only for comparison purposes. In reality, many differential equations, even some that appear straight forward, do not have solutions we can find analytically. Even so, we can use the techniques presented in this section to understand the behavior of solutions. In the next two sections, we explore two techniques to find analytic solutions to two different classes of differential equations.