Ross Program 2024 Application Problems

This document is one part of the application to the *Ross Mathematics Program*, and will remain posted at https://rossprogram.org/students/to-apply from January through March. The deadline for applications is March 15, 2024. The Admissions Committee will start reading applications on March 16.

You are not expected to answer every question perfectly; rather, take this application as an opportunity to explore some beautiful mathematics! We are interested in seeing how you approach unfamiliar, open-ended math problems, and we encourage you to write up your discoveries and conjectures, even if you can't prove them.

We believe that the most valuable part of a problem is the time spent thinking on it, and your application should reflect this: we are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, proofs, and generalizations written in a readable format. If you make progress on these five problems (even if you don't solve them completely), we encourage you to submit your Ross application.

Submit your own work on these problems. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution. Admission factors include the quality of mathematical exposition and the questions you pose, as well as the completeness and correctness of your solutions to those questions.

Problem 1

A polynomial is *integral* when it has integer coefficients. The square root of 2 is a solution to the integral polynomial equation $x^2 - 2 = 0$.

A number is *rational* when it can be expressed as $\frac{a}{b}$ for integers a and b (with $b \neq 0$).

A number is *irrational* when it is not rational.

(a) Suppose c is a non-square integer. (That is, $c \neq n^2$ for any n.) Explain why $\sqrt{c}$ is not rational. Similarly, if c is a non-cube integer, does it follow that $\sqrt[3]{c}$ is irrational?

(b) Find an integral polynomial equation that has $\alpha = \sqrt{3} + \sqrt{5}$ as a solution. Show that α is irrational.

(c) Let a and b be integers. Find an integral polynomial equation which has $\sqrt{a} + \sqrt{b}$ as a solution. Must $\sqrt{a} + \sqrt{b}$ be irrational? If $a \neq b$, must $\sqrt{a} - \sqrt{b}$ be irrational?

(d) Formulate some generalizations. As a starting point, is $\beta = \sqrt{3} + \sqrt{5} + \sqrt{7}$ irrational? What about numbers like $\gamma = 3\sqrt{2} - 2\sqrt{3} - 3\sqrt{5} + \sqrt{6}$ and $\delta = \sqrt[3]{5} - \sqrt{2}$?

Problem 2

Here's a (paraphrased) conversation that took place between two Ross students.

A: Hey, want to see a magic trick?
B: Sure, how does it go?
A: Think of any number. Any nonnegative integer, I should say.
B: Okay.
A: Multiply it by 3.
B: Okay....
A: Now divide it by 2, and if you get a decimal, then round down. Tell me if you rounded down or not.
B: I did have to round down.
A: Multiply it by 3 then divide it by 2 again. Again, tell me if you rounded down.
B: Okay, hang on... I did not have to round down this time.
A: Great, now just tell me: how many times does 9 go into this last number?
B: You want me to divide by 9?
A: Yes, just the quotient.
B: Um, the quotient is 4.
A: So your original number was 19.
B: That's right! How did you do that? Let me think....

(a) Figure out how A's magic trick works, and write up a clear, mathematical explanation of how to perform it.

(b) What variants of this trick can you come up with using the same principles? Can you change the numbers 2, 3, and 9 in the trick, or maybe the operations involved? What about the information you ask for?

Problem 3

Franziska and Safua play the game Halvsies. The game is set up with one pile of 5 stones, one pile of 8 stones, and one pile of 11 stones. Franziska, who's younger than Safua, always goes first. On their turn, a player does one of the following:

- They may remove stones from a single pile. They remove at least one stone and no more than half of the pile (rounded up).

- They may split one pile, consisting of more than one stone, into two new piles as evenly as possible.

Whichever player removes the last stone is the winner.

Of course, Halvsies could be played starting with any number of piles and any numbers of stones. Answer the following questions to generate some avenues of exploration and perhaps solve the original game.

(a) Show that if the game were set up with two piles of equal size, Safua can guarantee a win. Conclude that if they played Halvsies with a single pile with an even number of stones, Franziska can guarantee a win.

(b) Suppose Safua wins in a single-pile game with A stones and also in a single-pile game with B stones. Does she win in the game which starts with one pile of A and one pile of B?

(c) Suppose Franziska wins in a single-pile game with A stones and also in a single-pile game with B stones, and $A \neq B$. Does she win in the game which starts with one pile of A and one pile of B?

(d) Hopefully there are some questions that naturally arose that you can pursue now. Can you determine who wins in other games of Halvsies? Who can guarantee a win in the original game set-up?

Problem 4

It's hard to keep up with the fashionable trends of the day, and mathematical fashion is no exception. The fashion season of 2024 has the following unspoken fashion rules:

- 1 is fashionable.

- If a and b are fashionable numbers, then so is $\frac{1}{a+b}$.

- All other numbers are "so last year" and thus not fashionable.

(a) Find the smallest and largest fashionable numbers, and prove that they are the smallest and largest.

(b) Which numbers are fashionable? Can you characterize the set of all fashionable numbers?

(c) Word on the street is that in 2025, the fashion rules will be the same except for "1 is fashionable". Explore how the set of fashionable numbers changes if we begin with a different starting set of numbers. For example, what if 2 and $\sqrt{3}$ are the given fashionable numbers?

Problem 5

We'd like for you to define the *squarilarity* of two-dimensional shapes. The squarilarity should be a number between 0 and 1 quantifying how much a shape looks like a square. All squares should have squarilarity equal to 1, but aside from that restriction you can decide how to assign numbers to various shapes.

We welcome and are interested in seeing any ideas you have. It may be a good idea to start thinking of squarilarity definitions for simple shapes, and then expanding to more complicated shapes. Ultimately, for whatever definition(s) you come up with, we'd like for you to do the following.

(a) Specify the class of shapes for which your definition applies.

(b) State your definition of squarilarity for the shapes you're considering.

(c) Verify that the squarilarity is always between 0 and 1, and verify that the squarilarity of any square is 1.

(d) Explore the implications of your definition! Compute the squarilarity of a few shapes, and make observations of patterns you notice. Is your definition of squarilarity a good one?