Ross Program 2022 Application Problems

This document is one part of the application to the *Ross Mathematics Program*, and will remain posted at `https://rossprogram.org/students/to-apply` from January through March.

The deadline for applications is March 31, 2022. The Admissions Committee will start reading applications in April.

Work independently on the problems below. We are interested in seeing how you approach and explore unfamiliar open-ended math problems, not whether you can find answers by searching through web sites or books, or by asking other people.

Submit your own work on these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those guesses, and write logical proofs when possible. Where were you led by your experimenting?

Include your thoughts (but not your scratch-paper) even if you might not have found a complete solution. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution.

We are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, proofs, and generalizations written in a readable format.

The quality of mathematical exposition, the questions you pose, as well as the correctness and completeness of your solutions to those questions, are factors in admission decisions.

PDF format is required.

You may type your solutions using LaTeX or with a word processor, and then convert the output to PDF format.
Alternatively, you may scan your solutions from a handwritten paper copy, and convert that file to PDF. (Use dark pencil or pen and write on only one side of the paper.) Submitting photos of your work is not recommended since file sizes of photos are often too large. (The Ross system cannot accept files much larger than 5 megabytes.) Rather than photographs, you might use a "scan" feature on your camera.

Problem 1

Let $\mathbb{Z}$ denote the set of integers. If m is a positive integer, we write $\mathbb{Z}_m$ for the system of "integers modulo m." Some authors write $\mathbb{Z}/m\mathbb{Z}$ for that system.

> For completeness, we include some definitions here. The system $\mathbb{Z}_m$ can be represented as the set $\{0, 1, \ldots, m-1\}$ with operations $\oplus$ (addition) and $\odot$ (multiplication) defined as follows. If a, b are elements of $\{0, 1, \ldots, m-1\}$, define:
>
> $a \oplus b =$ the element c of $\{0, 1, \ldots, m-1\}$ such that $a + b - c$ is an integer multiple of m.
>
> $a \odot b =$ the element d of $\{0, 1, \ldots, m-1\}$ such that $ab - d$ is an integer multiple of m.
>
> For example, $3 \oplus 4 = 2$ in $\mathbb{Z}_5$, and $3 \odot 3 = 1$ in $\mathbb{Z}_4$.
>
> To simplify notations (at the expense of possible confusion), we abandon that new notation and write $a + b$ and ab for the operations in $\mathbb{Z}_m$, rather than writing $a \oplus b$ and $a \odot b$.

The sequence of 2-powers is $(2^n) = (2^1, 2^2, 2^3, 2^4, \ldots)$.
Evaluating that sequence in $\mathbb{Z}_{10}$, $\mathbb{Z}_{28}$, and $\mathbb{Z}_{48}$, we find

$$(2^n) = (2, 4, 8, 6, 2, 4, 8, 6, \ldots) \qquad \text{(in } \mathbb{Z}_{10}\text{)}.$$
$$(2^n) = (2, 4, 8, 16, 4, 8, 16, \ldots) \qquad \text{(in } \mathbb{Z}_{28}\text{)}.$$
$$(2^n) = (2, 4, 8, 16, 32, 16, 32, \ldots) \qquad \text{(in } \mathbb{Z}_{48}\text{)}.$$

Compute (2^n) in $\mathbb{Z}_m$ for several other numbers m. What patterns do you observe?

Here are some questions to guide your work.

(a) That sequence (2^n) appears to repeat after a few initial terms.
Why must such repetition occur for every m?

(b) Let $\rho(m)$ be length of the "tail" of terms that occur before the repeating part begins. For instance, $\rho(10) = 0$, $\rho(28) = 1$ and $\rho(48) = 3$.

For which m does $\rho(m) = 0$? When does $\rho(m) = 1$? When does $\rho(m) = 2$?

(c) Let $o(m)$ be the period (length of the periodic cycle) of the sequence (2^n) in $\mathbb{Z}_m$. For instance, $o(10) = 4$ and $o(28) = 3$ and $o(48) = 2$.

How is $o(m)$ related to $o(2m)$? What about $o(m)$ and $o(3m)$?

How is $o(35)$ related to $o(5)$ and $o(7)$?

(d) Make some conjectures about the patterns mentioned in (b) and (c) above. Investigate other patterns that such sequences in $\mathbb{Z}_m$ seem to satisfy, and make some conjectures.

Can you prove some parts of your conjectures?

Problem 2

The number systems $\mathbb{Z}$ and $\mathbb{Z}_m$ (for a positive integer m) were defined in the previous problem. We also use the notations:

$\mathbb{Q}$ is the system of rational numbers.

$4\mathbb{Z}$ is the set of multiples of 4 in $\mathbb{Z}$. Similarly for $4\mathbb{Z}_{12}$.

Consider the following number systems:

$$\mathbb{Z}, \quad \mathbb{Q}, \quad 4\mathbb{Z}, \quad \mathbb{Z}_3, \quad \mathbb{Z}_8, \quad \mathbb{Z}_9, \quad 4\mathbb{Z}_{12}, \quad \mathbb{Z}_{13}.$$

One system may be viewed as *similar* to another in several different ways.

(a) Let's measure similarity using algebraic properties. First consider the following sample properties:

(i) If $a^2 = 1$, then $a = \pm 1$.

(ii) If $2x = 0$, then $x = 0$.

(iii) If $c^2 = 0$, then $c = 0$.

Which of the systems above have properties (i), (ii), and/or (iii)?

(b) Formulate another algebraic property and determine which of those systems have that property.

Write down some additional algebraic properties and investigate them.

(c) In your opinion, which of the listed systems are "most similar" to each another?

Problem 3

Rossie is a simple robot in the plane, with Start position at the origin O and facing the positive x-axis.
An angle θ is entered into Rossie's memory. Rossie can take only two actions:

> S: Rossie steps one meter in the direction Rossie is facing.
> R: Rossie stays in place and rotates counterclockwise through angle θ.

Notation: A string of symbols S and R (read from left to right) represents a sequence of Rossie's moves. For instance, $SRRSS$ indicates that Rossie steps one meter along the x-axis, rotates through angle 2θ, and then steps two meters in that new direction. In the questions below, consider only those sequences that begin with an S.

(a) When $\theta = 2\pi/3 = 120°$, the actions $SRSRSR$ cause Rossie to trace an equilateral triangle and return to Start.

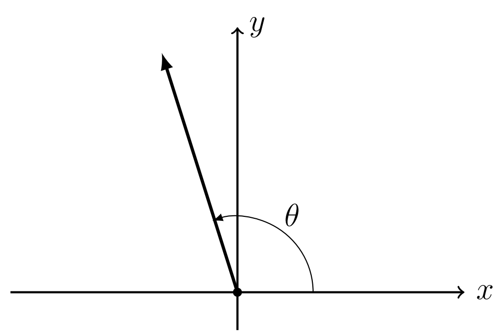
What path does Rossie trace when $\theta = 4\pi/5 = 144°$?

Question: For which θ can some sequence of S and R actions return Rossie to Start?

(Then by repeating that sequence of actions, Rossie will retrace the same path.)

(b) Suppose θ is the angle pictured below, with $\cos(\theta) = -1/3$. Note that θ is approximately $109.47°$.

With this angle θ, explain why the actions
$SSSRSSRSSS$
cause Rossie to return to O.



Note: Rossie returns to O but is not facing the positive x-axis.
That is: Rossie ends up at O but has not returned to Start.

Question: With that θ, is there some sequence of the actions S and R that returns Rossie to Start? Justify your answer.

(c) Provide more examples of θ that allow Rossie to return to O but not to Start. Is there some way to describe <u>all</u> such angles θ ?

(d) Are there some angles θ for which Rossie can never return to O?
Explain your reasoning.