Ross Program 2025: Application Problems

This document is one part of the application to the Ross Mathematics Program. The entire application can be found at

`https://apply.rossprogram.org/`

The deadline for applications is March 15, 2025. The Admissions Committee will start reading applications after that date.

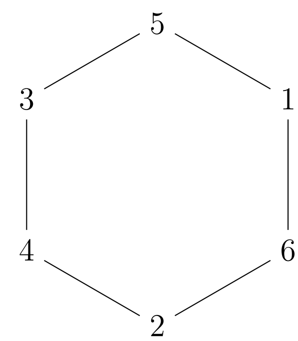
The Ross Mathematics Program values four qualities in a student above all else. These qualities are exploration, curiosity, explanation, and rigor. The following questions are meant to help you develop these aspects. You are not expected to answer every question perfectly; rather, take this application as an opportunity to explore some beautiful mathematics! We are interested in seeing how you approach unfamiliar open-ended math problems, and we encourage you to write up whatever you discover and think about, especially conjectures or variations of the questions, even if you can't prove them. The Program believes that the most valuable part of a problem is the time spent thinking about it, and your application should reflect this.

Submit your own work on these problems. If you've seen one of the problems before (e.g. in a class, online), please include a reference along with your solution. If you do use any resources, be sure to credit them.

We are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, proofs, and generalizations written in a readable format.

Problem 1: Number Wheels

An *n-wheel* is a regular $2n$-gon with a number placed at each vertex. That wheel is "balanced" if the sum of any two adjacent numbers equals the sum of the two numbers on the opposite side of the polygon. The figure provides a balanced 3-wheel.



(a) Find all of the balanced 2-wheels.

(b) Let's call an n-wheel *primitive* if each number equals the number on the opposite side of the polygon.

Verify that every primitive n-wheel is balanced.

(c) For which n does a non-primitive n-wheel exist?

Explain your reasoning.

(d) An n-wheel is *classic* if the $2n$ numbers are a rearrangement of the numbers $1, 2, 3, \ldots, 2n$.

For instance, the 3-wheel displayed above is classic.

For which n does a classic n-wheel exist?

(e) Explore n-wheels. Invent your own questions.

What progress can you make on your questions?

Problem 2: Fibonacci fill-ins

Define an f-sequence to be a list of numbers $(a_n) = (a_0, a_1, a_2, \dots)$ that satisfies the recurrence

$$a_n = a_{n-1} + a_{n-2} \quad \text{for} n \geq 2.$$

The classic Fibonacci sequence $(F_n) = (0, 1, 1, 2, 3, 5, 8, 13, \dots)$ is the f-sequence obtained by setting $F_0 = 0$ and $F_1 = 1$.

If we are given two consecutive terms in an f-sequence (a_n), then all terms a_k can be computed using that recurrence.

(a) If a_0 and a_1 are given, explain why

$$a_n = F_{n-1} \cdot a_0 + F_n \cdot a_1$$

for every $n \geq 1$.

(b) If $a_0 = a$ and $a_4 = b$, find the terms a_1, a_2, a_3 in terms of a and b.

For instance, if $a_0 = 1$ and $a_4 = 7$, then $(a_n) = \left(1, \frac{5}{3}, \frac{8}{3}, \frac{13}{3}, 7, \dots\right)$.

(c) What property must the integers a, b satisfy to ensure that when $a_0 = a$ and $a_4 = b$, all entries of the f-sequence (a_n) are integers?

(d) More generally, for a positive index k, what property must integers a and b have to ensure that when $a_0 = a$ and $a_k = b$, all entries of the f-sequence (a_n) are integers?

(e) What other questions do “Fibonacci fill-ins” inspire? Can you make progress on those questions?

Problem 3: Antiletters

Words are formed out of letters, but in the theory of antiletters, repeating letters can be deleted or introduced to yield the "same" word. So the word ABBC equals AC.

(a) Consider an alphabet of three letters A, B, and C, and form words subject to the rules that subwords matching AA, BB, CC, ABAB, BCBC, and CACA can be deleted or introduced. How many distinct words can you make?

For example, ABCA = ABCACC = ABCACAAC = ABAC = ABABBC = BC, so ABCA and BC are the same word.

(b) Now consider an alphabet of four letters A, B, C, and D, and form words where subwords matching any of

AA, BB, CC, DD, ABAB, BCBC, CDCD, DADA

can be deleted or introduced. How many distinct words can you make?

(c) Continue the pattern, now studying words built from five letters A, B, C, D, E, where subwords matching any of

AA, BB, CC, DD, EE, ABAB, BCBC, CDCD, DEDE, EAEA

can be deleted or introduced to yield the "same" word. Is there a sense in which five letters results in "more" possible words than four letters?

(d) Riff on these ideas. What happens if you invent your own patterns of subwords that can be deleted?

Problem 4: Hilbert numbers

The *Hilbert numbers* are the integers of the form $4k + 1$, for $k = 0, 1, 2, 3, \ldots$. A Hilbert number is *prime* if it is larger than 1 and not a product of two smaller Hilbert numbers. For example, check that 21, 33, 49, and 77 are Hilbert primes.

Hilbert integers do not factor uniquely into Hilbert primes, as shown by the example

$$77 \times 21 = 33 \times 49.$$

Let m be a fixed positive integer. The *m-Hilbert numbers* are the integers of the form $mk+1$, for $k = 0, 1, 2, 3, \ldots$, so the Hilbert numbers discussed above correspond to $m = 4$. Following this analogy, we similarly define m-Hilbert primes.

(a) For which values of m do you think the m-Hilbert integers factor uniquely into m-Hilbert primes?

(b) Neither the 4-Hilbert numbers (i.e., the Hilbert numbers) nor the 5-Hilbert numbers factor uniquely into their corresponding primes. Can you think of a sense in which the 4-Hilbert numbers are "closer" to having unique factorization than the 5-Hilbert numbers? Is there an m yielding even "worse" factorization than $m = 5$?

(c) What variations on these questions might you ask? Share some of your progress.