A STUDY OF NURTURE OF MATHEMATICALLY

TALENTED HIGH SCHOOL CHILDREN

DISSERTATION

Presented in Partial Fulfillment of the Requirements for

the Degree Doctor of Philosophy in the Graduate

School of the Ohio State University

By

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* * * * *

The Ohio State University

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The intent of the research was to gain further insight into suitable methods of nurture of the mathematical talent of high ability children.

In particular, this dissertation followed a group of high ability children through an eight week summer program, at The Ohio State University, whose purpose was to nurture mathematical talent and develop "mathematical maturity." Methods of nurture were observed over two summers. Levels of "maturity" were recognized and defined operatively on three general levels and were represented by types of problems (numericals, well formulated straight-forward proof problems, and ingenuity problems or abstract generalizations) on the daily problem sets worked by the children.

The components (problem sets, counsellors, lectures, seminars, group interaction, living conditions, books) of the nurture process were observed to be combined into a dynamic whole, but were observed flexible enough to be adaptable to different settings. Each component was observed to have its individual role in the whole nurture process

and this role was studied, with recorded observations of counsellors, seminar instructors, students; and with written case histories of students who attended the mathematics summer institutes of 1979 and 1980.

Some conclusions:

Problem Sets had to be designed to teach with levels of mathematical maturity written into the order of questions and time of appearance.

Counsellors were an effective group of peer teachers and the pyramidal nurture system successfully trained the counsellor through the medium of problem sets.

Expository lectures were hard for the inexperienced student to follow.

Seminars followed progress closely enough to suggest necessary changes in the nurture process.

Group interactions were an effective stimulant for mathematical discoveries.

The dormitory setup for participants effectively arranged for a place and method of disciplined study. Mathematical nurture of commuters was much more difficult.

The ability and desire to read mathematics reference books seemed to be a necessity for second year participants.

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FIELDS OF STUDY

Major Field: Mathematics Education

Studies in Mathematics Education. Professors Jon L. Higgins and Harold C. Trimble

Studies in Mathematics. Professor Arnold E. Ross

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CHAPTER I

This study monitored both the process and the product of mathematical nurturing within the summer program of mathematics for high ability secondary school children at The Ohio State University in the summer of 1979 and the summer of 1980.

Chapter I is organized in the following ways:

A. Introduction of the problem

B. Problem statement

C. Methods and procedures of the analyst

   (a) introduction
   (b) methods of the analyst
   (c) procedures of the analyst

D. Setting for the study

Introduction

The accomplished representatives of every science in which discovery or invention play a role have traditionally been concerned with identifying the talented youngster in order to promote, within the youngster, the disciplined kind of creative intuition needed for scientific research. Mass education has pushed the contact between professional and young talent further and further back into the educational system, so that contact may not come until graduate school. No one in his generation has been more concerned with this dilemma than Arnold E. Ross, director of the summer institute of mathematics for high ability children at The Ohio State University in 1979 and 1980.

(In 1979 and 1980 the summer program was officially called, THE OHIO STATE UNIVERSITY DEPARTMENT OF MATHEMATICS STUDENT TRAINING PROGRAM FOR HIGH-ABILITY SECONDARY SCHOOL STUDENTS, but this study will refer to it as the summer institute or summer program.) Professor Ross has been director of similar summer programs for gifted children dating back twenty-three years to his first program at the University of Notre Dame in 1957. The following statement from his article, "Nature or Nurture," points to his deep belief in the existence of mathematical talent in some individuals at an early age, and his belief in the need for opportunities to develop that talent early.

> It is quite usual for scientific or mathematical talent to manifest itself at an early age... often in early teens. In the instances of the successful maturing of such young talent and of the development of high competence, one finds more often than not the presence of continued opportunity for contact with good mathematical and scientific ideas, and with people who are capable of providing encouragement and guidance toward significant challenges. It appears that very vivid early impressions leave their mark upon the nature of the ultimate achievement. ...This need today cannot be met through the traditional practices of an educational system overwhelmed not only by the magnitude of its student population but also by the growing diversity in the cultural and economic backgrounds in the new body of students. ...Our summer program is directed to those young people who have been able to develop a strong interest in mathematics and whose mathematical experience has been richer than usual as a result of some happy combination of circumstances such as an accidental encounter with a challenging idea, or encouragement by a dedicated teacher, or a stimulus of regional competition, or a special involvement in a stimulating group effort, or some other irresistable temptation to explore. Their concern with mathematics may constitute their principal academic interest or it may be concomitant with another strong academic involvement which may, but need not necessarily be, connected to their interest in mathematics.
> (Ross, 1973, p.1)

Over the twenty-three years of its existence, many mathematics departments have had contact either with a Ross summer program directly or with its participants of previous years. Felix Browder, as Chairman of the University of Chicago Mathematics Department in 1975, wrote an article entitled "Youthful Enjoyment of Rigorous Thinking." He commented:

> In our department, as well as as in other leading mathematics departments in this country, the results can be evaluated in terms of the number and quality of the students we have had who had earlier been in Ross's program. As an example, over the past six years the University of Chicago has entered a team in the Lowell Putman national undergraduate mathematics competition, which embraces all the universities and is taken seriously by all of them. During five of those six years, our team has taken one of the first three places, (first place for one year, second place for two years, third place for two years), a record ahead of Harvard, MIT, Princeton, and sometimes even Cal. Tech. Each one of those teams has had at least one veteran of the SSTP program at Ohio State on it, and in most cases, two. ....More than 1000 high school students have, over the years, participated in the program, and its graduates have established outstanding records in their subsequent careers in higher education. Each year a significant proportion of the country's strongest new Ph.D.s in mathematical fields can be dated to their recipients' participation in this summer science experience... (Browder, 1975)

Is creativity <u>born</u> to the human being or is it a <u>culturally developed</u> trait? Many opinions, including the following from Poincare, the well known nineteenth century French mathematician, do not answer in the affirmative to either of the extremes of this question. Many believe creativity lies somewhere between inborn talent and cultural development;

and that indeed, an inborn talent must first exist and then must be nurtured and developed to be successful, and development of mathematical intuition (sometimes referred to as mathematical maturity) in the talented individual is a prerequisite to scientific creativity and achievement. From Poincare comes the following insight into the existence and development of mathematical talent:

> We can understand that this feeling, this intuition of mathematical order, which enables us to guess hidden harmonies and relations, cannot belong to everyone. Some have neither this delicate feeling that is difficult to define, nor a power of memory and attention above the common, and so they are absolutely incapable of understanding even the first steps of higher mathematics. This applies to the majority of people. Others have the feeling only in a slight degree, and they are gifted with an uncommon memory and a great capacity for attention. They learn the details one after the other by heart, they can understand mathematics and sometimes apply them, but they are not in a condition to create. Lastly, others possess the special intuition I have spoken of more or less highly developed, and they can not only understand mathematics, even though their memory is in no way extraordinary, <u>but they can become creators, and seek to make discovery with more or less chance of success, according as their intuition is more or less developed.</u> (emphasis added) (Poincare, 1952, p. 50)

The Problem

This dissertation is concerned with two investigations:

1. The identification of student behaviors associated with good progress* or lack of good progress in the development of mathematical intuition. (mathematical maturation)

2. The identification of methods successful for the nurturing of mathematical talent.

To this end the study presents information from:

(a) direct observation of those students who came to the summer institute at The Ohio State University as first year participants in the summer of 1979 and especially those students who returned to the summer institute of 1980 as second year participants and junior counsellors.

(b) the observation and hindsight/insight of junior counsellors concerning the mathematical nurturing and student behaviors of their peers in 1979:

   (i) from their point of view as the counsellor in 1980

   (ii) from their point of view as the student in 1979

(c) the history of the program during its twenty-three years of existence.

* good progress as judged by the group of professors and/or counsellors involved in teaching during the summer program of 1979 or 1980, and defined on the basis of written solutions (from participants) to problem sets, written examinations, oral discussions, and seminar participation.

Methods and Procedures of the Analyst

Introduction

A certain high level of mathematical maturity or development of mathematical intuition is said by mathematicians to be necessary for the understanding of higher mathematics. Is it possible to outline tasks that represent a level of maturity or does one just point a finger at the student representative? What is mathematical maturity and can it be developed?

In this study the investigator will resist the temptation to collect definitions. Instead, she will follow the movement of youngsters through a program of mathematics for high ability secondary school children where something called mathematical maturity may happen. In past years, this program has had successful graduates who finally could be called mathematically mature.

The investigator will follow the nurture of youngsters fourteen years of age through eighteen years of age in an eight week summer program for the nurture and development of mathematical talent, first designed by Professor Ross in 1957 and modified over the many summers since then.

Levels of mathematical maturity will be defined within the limits of the study, as represented by the levels of difficulty (as perceived by the staff of the summer program) of the mathematical questions asked on the problem sets and in lectures.

Methods

In this study the investigator chose to examine test results and sets of solutions to assigned problems only as the end result of methods of nurture, and to identify student behaviors associated with good or bad test results and solution sets.

The year 1979 was a particularly good one to observe first year participants because they were not a homogeneous group. They were invited to participate, but not because they had shown they could solve diffitcult problems. They represented a wide range of potential abilities. To begin with, their individual problem solving abilities were unknown. They were younger than usual (fifteen years of age average); but they had convinced someone that they had good potential ability. The program was confronted with a nonhomogeneous group with very little previous mathematics experience. Also included in this group was a large percentage of commuters so one could examine the importance of the dormitory environment on the first year participant.

When the study began, the investigator sought to observe the students in their daily routine and in such a way that she would be an unobtrusive and natural part of that daily routine. She knew that the young first year participant would be too busy with the daily mathematics assignments to want to discuss any topics, either personal or mathematical, unrelated to possible solutions of daily problems. Since she had been teaching mathematics courses for fifteen years at the university undergraduate level, and one year at the high school level, the role of student-teacher within the program was a most convenient position from which to observe. The analyst started in the summer of 1979 by sitting in all classes, both lectures and seminars, and taking notes of everything that happened in class both mathematical and non-mathematical. She kept a notebook with mathematical notes down the middle, and comments in the margins and in-between mathematical statements. She listened to interesting problem solutions from participants and some even more interesting side comments about everything from methods of teaching to personal frustrations. She often discussed the day's lecture or mathematical problems of current interest with participants/counsellors. The students perceived her as a mother-counsellor and assistant to Ross and within a week after the start of the program in 1979, the perception became the reality. She spent much of the day looking after students' problems and often visited the student dormitory to feel the students' moods concerning their work. Some problems were easy to solve like having "quadratic reciprocity" T-shirts made up for interested participants and sitting with non-seriously injured students in emergency rooms. Some problems in the dormitory were harder to solve, and mathematical problems ranged from easy to impossible. She became the link between Ross and the students. Serious problems were conveyed immediately to Ross. Early in the day, she sat next to a student with problems; and later in the day, she sat with Ross or counsellors or both and talked about the students with problems. They discussed their communication skills in seminar or on solution sets, the anatomy of good or bad attempts at proof, or a student's individual weaknesses and strengths, and the nature of his/her progress.

Procedures

Observations were conducted by the analyst from her position as a:

(i) student-teacher in class learning with the participants

(ii) mother-counsellor and the grapevine link between the participant in the summer institute and Ross

(iii) an assistant to Ross

The learning process of the first year participants of 1979 was observed from more than one point of view; direct observation by the analyst of the first year participants in 1979 was only one point of view.

Eight first year participants of 1979 were returned in the summer of 1980 as junior counsellors and the analyst had the opportunity to observe the 1979 nurture process from (a second point of view) the view of the counsellor recalling his/her first year of nurture as a first year participant and able to relate his/her experience as the counsellor responsible for assisting in the nurture of the first year participants in 1980. The second year student-counsellors' views naturally emphasized the importance of nurture methods that were most important to themselves. They were also candid in pointing out behaviors thought to be negative or positive with respect to either their own progress or the progress of another participant.

Observations of the methods of nurture and development of mathematical talent that were used in the summer institute in mathematics for gifted high school children were started in the summer of 1979, and continued through the summer of 1980 at The Ohio State University. In particular, observations were made of the nurture and development of mathematical talent of the high school participants who started in the summer institute in the summer of 1979 and returned in the summer of 1980 as junior counsellors. Direct observations were also made of those participants who came to the summer institute program for one summer only in 1979.

Summer Institute of 1979

> Direct Observations (by the analyst) were recorded from classes, seminars, counsellor meetings, dormitory discussions, written solutions to problem sets, discussions with teachers of participants in the summer institute, and written examinations.

Summer Institute of 1980

(i) Direct observations (by the analyst) were recorded of the participant who had returned to the institute of 1980 as a second year participant. Observations were made of him/her as both a counsellor and an advanced student taking mathematics courses.

(ii) Observations of "the 1979 experience" (by the second year participants) were recalled and described to the analyst.

(iii) Observations (by the second year participants) of the first year participants from the view of the "counsellor" were described to the analyst.

From the observations made in 1979 and 1980, case studies of each of the first year participants of 1979 were written, and the important components of the summer institute and its nurture process were reviewed.

Information for the study, then, came from four accessible areas:

1. direct observations
2. case studies
3. analysis (counsellors' opinions included in here)
4. history and experience of twenty-three summer programs.

Finally, instead of an analysis of test results, the study uses as data:

(i) important behaviors exhibited

(ii) important methods of nurture

The Setting

During the summer of 1979 The Ohio State University sponsored a mathematics program, directed by Professor Arnold E. Ross, for high ability secondary school children. The program operated for eight weeks from June 17 to August 10. Lectures and seminars were conducted in classrooms on the Main Campus of the University. Most of the high school participants lived in dormitories on the campus for the eight weeks of the program. There were twenty-six participants in all. Twenty of these were first year participants; thirteen lived on campus and seven were commuters; the remaining six participants were junior counsellors (second or third year participants).

The program had the advantage of twenty-two years of experience in the recognition and early development of mathematical talent. It all started with the insertion of seven bright high school students within a summer program for secondary mathematics teachers at Notre

Dame University in 1957. These high school visitors grasped ideas which were traditionally considered inaccessible to the very young. The program later shifted its emphasis to a summer program specifically designed for high ability secondary school students. The program continued to operate every summer with Professor Arnold E. Ross as its major organizer and lecturer from 1957 - 1963 at Notre Dame; 1964 - 1974 at The Ohio State University, 1975 - 1978 at the University of Chicago, and again at The Ohio State University in 1979 - 1980.

The tradition of the problem-oriented method of teaching by discovery (sometimes called the inductive method of teaching -- see Hendrix, 1961) a method used by E. H. Moore (1862 - 1932), a mathematician and educator at the University of Chicago, had been refined and applied by A.E. Ross to an eight week program in number theory for the nurture of young talent. The role of the number theory in the nurture process is described by A. E. Ross in his article, Talent Search and Development.

> Since 1957, we have used number theory as the basic vehicle for the development of the student's capacity for observation, invention, the use of language, and all those traits of character which constitute intellectual discipline. We have chosen number theory because of its wealth of accessible, yet fundamental and deep, mathematical ideas and for its wealth of challenging but tractable problems. (Ross, 1978, p3)

All summer program participants attended daily lectures taught by A. E. Ross at 9 a.m. First year participants attended problem seminarts three times a week in groups of less than ten. The more advanced student-counsellor participants who had returned to the program, for at least a second summer, attended lectures and seminars in advanced mathematics on the graduate level and taught by research mathematicians on the faculty at The Ohio State University (or by visiting faculty). Most first year participants spent their time between classes working in the dormitory on the problem sets received daily in lecture. Each first year participant was expected to give written solutions to the student-counsellors who lived in the dormitory with the first year participants. The student-counsellors served as teachers, coaches, graders, and friends to the first year participants.

The year 1979 was an unusual one for the program. In previous years the participants were chosen by a combination of recommendations from teachers and responses from candidates to questions and problems sent through the mail. In response to requests by schools or teachers or interested individuals, packets of materials were mailed out. The program did very little advertising to seek out potential candidates. This was a grapevine recruitment system supported by interested individuals. Over the twenty-three years, the grapevine grew and the program received applications from many good candidates. Normally, the packets contained a description of the program, an extensive questionnaire, and a set of problems that were to be returned by the applicant. The problems, on elementary mathematical topics, tested the respondent's ingenuity and interest rather than breadth of experience. The problem solutions were then carefully reviewed, together with essays (included in the questionnaire), recommendations from teachers, and school records. However, in the summer of 1979 participants were chosen differently. Professor Ross realized only very late in the spring of 1979 that the program would not operate at the University of Chicago in the summer of 1979. The summer of 1979 was scheduled to be summer of retirement for Ross from the gifted children's mathematics program at the University of Chicago. He decided the children's summer program should be continued and could be reinstituted at The Ohio State University provided no time was allowed to elapse and the existing grapevine was immediately utilized. There was little time left to send out the usual lengthy set of problems. Although two problems were included in most applications, some students did not have enough time to work on these problems. As a result, the acceptance of participants depended heavily on the recommendations of teachers, colleagues, or other interested persons. The summer program of 1979 had many younger than usual first year participants and seven commuters (Previously, commuters were unusual).

The unusual makeup of program participants in 1979 provided observation of more than the usual variety of students and gave the investigator a broader base for study. She could more easily observe what elements in the program would provide effective nurture for which kind of student.

CHAPTER II

REVIEW OF LITERATURE

Introduction

This chapter is a review of the literature related to the mathematically gifted. Several aspects of concern for the mathematically gifted are addressed.

Recent history of concern for the mathematically gifted in the United States is summarized, including the influence of current political issues, and the effects of public support on research related to the mathematically gifted individual.

Distinctions are made between the nature and the nurture of mathematically gifted individuals. Definitions of nature and nurture used in this study are discussed.

The nature of the mathematically gifted individual is reviewed from two distinct points of view: that of the psychologist, and that of the mathematician.

Methods for nurturing the mathematically gifted individual are reviewed from the perspectives of mathematicians, mathematics educators, public school teachers, educational administrators, and educational psychologists.

Methods of nurture are sorted into two categories: enrichment, and acceleration. The controversy over relative merits of enrichments versus acceleration is reviewed.

The main sections below serve to organize chapter II.

A. Recent concern for the mathematically gifted.

B. Definitions of nature and nurture of the mathematically gifted.

C. Nature of the mathematically gifted.

   1. Psychologists' concern for the nature of the mathematically gifted.

   2. Mathematicians' concern for the nature of the mathematically gifted.

D. Nurture of the mathematically gifted.

   1. Educators' concern for the nurture of the mathematically gifted.

   2. Methods used by educators to nurture the mathematically gifted.

   3. Enrichment versus acceleration, as methods of nurture for gifted children.

Recent Concern for the Mathematically Gifted

Public support has been a strong force in progress toward the discovery of successful methods for the recognition and nurture of the mathematically tifted. Public concern has depended upon the economic, social, and political realities of the time, and these economic, social, and political realities have been more or less noticeably affected by scientific progress.

In recent years, public support for scientific research and the development of the research scientist has fluctuated from extreme concern to indifference. Perhaps because Americans felt no political or economic threat to what they perceived to be their country's scientific leadership in the pre-sputnik 1950's they showed little interest in scientific research and little interest in science or mathematics education. In 1957, Sputnik shocked the nation into believing that the United States was falling behind Russia (the perceived enemy) in scientific progress. This produced, in the late 1950's and 1960's, a wave of interest in the scientifically gifted individual and prompted the nation to look for mathematically gifted school children. Kalin (1967) noted:

> In 1967, there were 135 programs (Summer Science Training Programs, Usually at Universities and supported by The National Science Foundation) of which 30 were devoted solely to mathematics while an additional 24 were part mathematics, part science. There were 1,542 participants in the mathematics-only programs, mostly rising seniors or juniors. Although financed in a large measure by NSF, each department of mathematics had the freedom to devise its own program. (Kalin, 1967, p.63)

Tannenbaum described the renewed interest by schools in the gifted student brought about by Sputnik:

> Indeed, the reaction to Sputnik might not have been so swift and strong if the critics' cries for change in our schools had not had a cumulative effect. When the educational community finally took action on behalf of the gifted, it did so with alacrity. Public and private funds became available to assist in the pursuit of excellence, primarily in the fields of science and technology. Academic coursework was telescoped and stiffened to test the brainpower of the gifted. ...There were even special efforts made to locate and nurture giftedness among the socially disadvantaged, most notably through the P.S. 43 Project in New York City, which later became

> the widely heralded but eventually ill-fated Higher Horizons Program. Interest spread also to school systems in rural areas and to colleges and universities where the gifted were provided with enrichment experiences never before extended to them... Much of what is taught today in the mathematics and sciences, for example, is a legacy of post-Sputnik designs in gifted education. (Tannenbaum, 1979, p.10)

In the middle 1950's, school desegregation started in the United States and set off a movement to equalize education. Disadvantaged children were suddenly very visible and attention was focused on the disadvantaged learner. It appeared that public support for research concerning the gifted learner was incompatible with research concerning the disadvantaged learner. Intelligence tests and achievement tests used to find the gifted child were claimed by some to be biased against racial minorities and the socio-economically depressed. As a result, some urban centers discontinued their use.

At a conference of school principals, Herrnstein Chairman of the Department of Psychology at Harvard University from 1967 - 1971, commented at length about the effects of banning the IQ and achievement tests for the purpose of equalizing the educational treatment of children from different socio-economic classes. Some of his comments are contained in the following:

> Where the IQ gets banned, as in New York City, other kinds of tests tend to replace it: reading scores, arithmetical problem solving, a raft of other kinds of achievement, as distinguished from aptitude tests. Achievement and aptitude tests do differ, but the difference is not what people seem to think. Achievement tests individually correlate with IQ, and a whole group of achievement tests correlates almost as much with any IQ test as different IQ tests correlate with each other, in the range of 80 to 90%. Deprived of IQ tests, teachers and parents are likely to substitute something almost equivalent... Unfortunately achivement tests are

> individually probably less heritable than the IQ. It is unfortunate because if they are less heritable, they will be affected more by the family's cultural or socio-economic level. The class bias that some people think they find in the IQ probably taints achievement tests substantially more. The main practical effect of banning the IQ while retaining achievement tests is to miss spotting some children whose high aptitude has not produced outstanding academic achievement, the very children for whom remedial education or counselling might be expected to do the most. They are also likely to be the emotionally disturbed or the culturally disadvantaged.
>
> If achievement tests are culturally biased even more than the IQ, there will be attempts to ban them too. The impulse would be egalitarian since different socio-economic backgrounds differ on the average. But by not giving any standardized tests at all, schools would be thrown back on grades and the subjective impressions of teachers. Of all the ways to assess and dignose children, grades and personal impressions are probably the most contaminated by cultural factors. They reflect the social class veneer the most, mainly to the disadvantage of lower-class children who have not had the benefits of educated parents with lots of leisure time. This, indeed, was the situation that impelled the early testers, such as Cyril Burt in England and Lewis Terman in the United States, to create tests in the first place. They wanted to cut through the differences between children arising in the style of home life, to get at the essential capacities underneath by measuring native intelligence. To some extent, they succeeded, as the IQ's heritability and predictiveness show. Yet, the test has become unpopular with the very people who, in their liberal sentiments, are descendants of Burt, Terman, and others in the testing movement. The disillusionment is easily explained. IQ tests, while compensating for some of the cultural differences that push grades up or down, have not equalized the children from all social classes. Proportionately, upper classes are favored by measures of scholastic aptitude and achievement on the average. The heritability of the test scores seems to sound an unwelcome fatalistic note about the limits of educational equalization. (Herrnstein, 1976, p.118)

Yet there were many educators who strongly disagreed with the heritability of intelligence or the IQ score as a measure of "true ability." In the seventy-eighth yearbook of the National Society for the Study of Education, Gallegher discussed the problems in recognizing the talented when the educators operated under the "incorrect assumption" of the Genetic Theory of Intelligence:

> The acceptance of the purely genetic nature of intelligence led to some embarrassing and troubling results, such as the consistent racial and ethnic differences found in the proportions of children testing as gifted. Such results were not widely quoted or displayed, although there have been clear and consistent findings on the question. The accumulation of evidence from studies in child development suggests that there is a subtle and complex interaction between environment and native ability, the result of which is what is measured by a score on an intelligence test. The full range of implication of this fact has not been fully grasped or acted upon by educators of the gifted, although some initial steps have been taken. These steps include the use of many different techniques to identify gifted children in minority groups. But we still have not fully accepted the interaction concept - the concept that we can create giftedness through designing enriched environments and opportunities, or that we can destroy it by failing to create those environments and opportunities. (Gallegher, 1979, p.29)

In recent years, it seemed impossible for educators to design a methodology for use in the schools that could suitably enrich the environment and devlop both the talents of the disadvantaged and the gifted student at the same time. The operation of separate programs for the disadvantaged and the gifted child either in the school or outside of the school was considered by many influential groups as biased in favor of the advantaged learner and a form of elitism.

In 1967, as previously noted, there were 30 Summer Institutes devoted solely to mathematics. But in 1980 there were fewer than ten Summer Institutes (open to students that did not have limited mathematics opportunity) devoted solely to mathematics and supported by the National Science Foundation (National Science Foundation, 1980).

Even well established projects such as the Study of Mathematically Precocious Youth (SMPY) at the John Hopkins University had enough problems with respect to support of it's research to prompt Bereiter, in a discussion and critique of SMPY to make the following statement:

> As I suggested earlier, the preponderance of educational concern today is with the underdog, and so a project like SMPY faces problems in establishing a basis for support if it is to grow beyond demonstration and have a substantial, continuing impact on education. The kinds of things that SMPY is doing are amply justified both in terms of enriching society and of enriching the lives of certain of its deserving members, but in the practical world it is not enough to be on the side of the angels. ...Thus I hope I will not be thought devoid of idealism when I suggest that SMPY ought to put some effort into showing that what it is doing pays monetary and not just spiritual dividends. As a parallel case, there is a compensatory preschool program for low - IQ children that has won continuing support for years from a financially troubled legislature on the strength of evidence that it saved the tax-payers money... Saving tax dollars was no more the purpose of that program than it is the purpose of SMPY, but saving tax dollars proved to be its key to survival... I think this kind of analysis is needed, because otherwise SMPY may tend to be regarded as just another kind of welfare service offered to a not very needy clientele, and it will always be vulnerable to the objection that the service is not offered equally to all who deserve or could profit from it. Its political justification has to rest, not on what it does for individual students but on what it does for society. (Bereiter, 1979, p.314)

Tennenbaum noted the indirect negative impact that the recent movement toward equalizing education has had on educational programs for gifted children:

> Quality integrated education is as much a dream today as it has ever been. A prodigious about of work yet remains to be done before underpriveleged children can begin to derive their rightful benefits at school. That kind of investment of effort in compensatory programs usually draws attention away from curricular enrichment for the gifted... The 1970's have experienced hard times and drastic cutbacks in expenditures for education. Programs for the gifted are usually the

> most expendable ones when budgetary considerations force cutbacks in services for children. (Tannenbaum, 1979, p.25)

Defining Giftedness

What makes giftedness? This was the question on which a recent discussion of the teaching of mathematics to gifted children and youth (The National Council of Teachers of Mathematics Professional Series (NCTM), 1981) was focused.

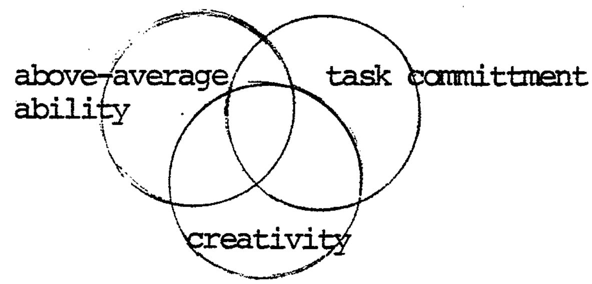
> This general concern, coupled with the more practical responsibilities of formal education systems, has caused psychologists and educators to raise two related questions:
>
> 1. How can we identify persons who have the highest potential for superior performance?
>
> 2. What types of learning experiences can we provide to develop this potential?
>
> (Ridge and Renzulli, NCTM, 1981, p.192)

These two questions can be characterized as:

1. The question of the NATURE of the gifted individual.

2. The question of the NURTURE of the gifted individual.

The NCTM discussion scrutinized the many definitions of giftedness, from conservative (gifted are the scorers in the top 1% on the Stanford-Binet Intelligence Scale or its equivalent) to liberal (any child is gifted whose performance, in a potentially valuable line of human activity is consistently remarkable). Renzulli operationally defines giftedness as the intersection of the triple cluster (shaded area below) of traits that must be present in the gifted individual. (Ridge & Renzulli, NCTM, 1981, p.197) The intersection represents the interaction of the three traits upon "creative-productive accomplishment."



The Separation of Questions of Nature and Questions of Nurture

The foci of the work of concerned persons with gifted individuals in mathematics followed two paths: one in the direction of defining the nature of the gifted individual and the other in the direction of developing methods of nurture of the gifted individual. It is not possible to give a definition of nature or nurture that has been consistent from person to person or publication to publication. However, within the confines of this study it becomes necessary to define what will be meant by the "nature" or "nurture" of the mathematically gifted individual.

Many programs for the gifted have concerned themselves with:

1) recognition of the gifted (nature) and/or

2) development of talent (nurture)

Under the heading of nature we place the methods for recognition of the mathematically gifted individual. Methods of recognition range historically from the definition of suitable "gifted group traits" by educators, psychologists, and mathematicians, to the "measurement of talent" by giving stage defining tasks or tests defining mental age, or tests indicating achievement/aptitude, or tests suggesting creative ability.

Under the heading of nurture we place the methods utilized for the creation of an environment conducive to the development of mathematical talent by enrichment or acceleration. Although psychologists and educators interested in the talented or gifted generally agree that some kind of nurture of talent is necessary, there has been much controversy about the relative merits of acceleration or enrichment as methods of nurture. These differences of opinion between some groups of psychologists have spilled over to some educators who justify development of one method of nurture over another on the basis of a current perception of the argument.

Although nurture of the mathematically talented is the main concern of this study, the question of nature is not ignored. It is conjectured that the nurture and successful development of talent depend upon certain personality traits; and these traits must be defined, recognized, and then strengthened.

It would appear that early recognition and development of talent is necessary to give the individual more choices for lifetime work before time and circumstance obliterate the practicality of some choices. This would seem to maximize his/her possible contribution to society. But what clues toward a young person's unrealized potential capabilities does an educator have? The personalities of proven talented individuals are interesting in their own right but are not of practical use to an educator unless the factors of success in a special area can be extracted and applied toward the recognition and nurture of the individual of unproven talent in that same area.

Nature of the Mathematically Gifted

Psychologists' Concern for the Nature of the Mathematically Gifted

Since the beginning of a recorded interest in man, much has been written about the differences and similarities between individuals with respect to many well defined and not so well defined characteristics. It has been speculated that similarities naturally indicate similar talents and that any special interest group can be defined by the common characteristics of the individuals within the group.

Psychologists have utilized the idea of common characteristics or common behaviors to recognize special talent. Common characteristics have sometimes been interpreted as similarities of performance among members of a representative group known to be gifted. Tests, or questionnaires, or performance, with respect to certain tasks have been used so that any individual performance could be compared to the representative group performance. Similarities by an individual to the representative group performance were then scrutinized and interpreted.

Prior to the 1900's the studies of the gifted were studies of characteristics of individuals after public recognition of their achievements. The search for common characteristics was conducted after the giftedness of the individuals could not be disputed. In his article Characteristics of the Gifted, MacDonald Discussed the typical study written before 1900:

> A representative study of this kind (retrospective) was by Yoder (Yoder, 1894). Yoder identified fifty outstanding individuals, men whose eminence and achievements were widely, if not universally, recognized in the western world. He determined what sources of information about these individuals were the most authoritative. His next

> step was to cull the sources for specific information about each of these individuals to determine the frequency with which certain events appeared in their lives, and the extent to which certain characteristics were common among them. Some of the individuals whom he studied were Edison, Bismarck Beecher, George Eliot, Darwin, Tennyson, Emerson, John Stuart Mill, Garabaldi... The most extensive studies of the kind I have been describing were conducted by Sir Francis Galton in two volumes, and appeared in Hereditary Genius (Galton, 1869) and English Men of Science (Galton, 1874). Galton's contribution to the study of talent or genius is one of the most significant even though he was limited by not having quantitative tools. Galton's contribution was a new conception of what constituted genius. Prior to Galton's time, and in much popular thinking today, talent or genius is treated as a qualitative difference between individuals. That is, the gifted person is thought of as being in a special category, as somehow uniquely different from ordinary individuals. Galton defined this uniqueness not in terms of qualitative differences among individuals but in terms of quantitative differences. The gifted individual was the person who had an unusual combination of personal traits which placed him at the extreme on a scale of differences... He began by grading individuals into categories "according to their natural gifts." Then, using tables devised by Quetelet, he estimated the number of individuals in these categories, achieving a distribution familiar to us as the normal distribution of ability. Few individuals fell in the superior categories, genius was defined as the infrequent occurrence of a combination of traits... His study of Hereditary Genius indicated that men of eminence tended to come from the same families. This fact was for Galton sufficient evidence to argue for a hereditary basis to genius... These traits which comprised genius were intellect, zeal, and the power of work, characteristics which Galton had identified as salient in the lives of the men of eminence. (MacDonald, 1967, p.3)

The principle of a normal distribution of ability, with "gifted" individuals placing at the extreme of a "normal scale," was carried into the twentieth century by the psychologists who worked in the area of psychometric theory.

Psychometric theory studies methods for the measurement of individual differences. This theory developed written tests in such a way that a score defined "mental age" or "achievement level" or "aptitude level." Many of the tests developed by the psychologists in the twentieth century were based on the assumption that "intelligence" or "achievement" or "aptitude" could be scored and the scores would fall into a normal distribution.

In the early 1900's, Binet developed a test for describing what he called "mental ability." He defined what he meant by mental ability and he developed a test for identifying it. Binet's definition of intelligence centered around the general faculty of judgement (Binet and Simon, 1916). This definition determined the criteria of success (such as success in school) against which his methods were validated. Louis Terman at Stanford University refined the Binet test for use in America (Terman, 1916). Essentially the mental tests scaled individuals according to their ability to profit from schooling. The Stanford-Binet test is still commonly used to identify gifted children. It defines a gifted child as one who scores at or above the 130 IQ level. In the mental tests, the psychologist finally had a quantitative method for identifying gifted children (Terman, 1926).

In his paper entitled <u>A Piagetian Approach to Intellectual Precocity</u>: Keating address the issue, of what it is that psychometric tests really test, with the following comment:

> It has been suggested that high-scorers on psychometric tests are not necessarily precocious, but just "good test takers." Evaluation of the cognitive developmental level of psychometrically defined bright and average students through a

> separate methodology would seem to resolve this question... whether similar aspects of "intelligent behavior" are being tapped by the differing methodologies of the two traditions. (Psychometric and Piagetian)... If the same or essentially similar aspects are observed by the two methods then similar results in terms of individual differences would be expected. Even if, as frequently suggested, the psychometric evaluation looks at products and the clinical Piagetian evaluation at processes, it is to be expected that a close relationship exists between product and process of the same global behavior. (Keating, 1976, p. 91)

In recent years, there have been many contributions made to the "nature" of the mathematics gifted by the Study of Mathematically Precocious Youth (SMPY) directed by Julian Stanley. SMPY studied psychometric methods and refined them for use in the recognition of the mathematically gifted among very young junior high school children. (Stanley, 1979, p.3)

In a paper, A Historical Step Beyond Terman, Page discussed the special use of the College Entrance Examination Board's Scholastic Aptitude Test-Mathematical (SAT-M by SMPY to separate the very young mathematics gifted child from the average child. In the following comment he justified the need for early recognition of mathematical talent:

> The biographies of important Mathematicians teach us that the greatest contributions commonly come early. In fact, there is a kind of ordering of disciplines by age of greatest productivity, with an apparent loading toward youth of certain fields, and an apparent loading toward middle age of certain others (Lehman 1954). In general, the early-peaking fields appear to be those which are not dependent on the accumulation of large data bases, of large informational store; but are, rather, dependen on the facile manipulation of symbols, on long strings of transformations. Thus, math and chemistry, romantic poetry and sonatas, are alike in being ripe fields for youthful accomplishment... Notwithstanding the existence of math prodigies, and notwithstanding youth surge of math-experience in math is apparently essential to productivity.

> Math is not commonly learned at home, beyond the most elementary level. And it is not commonly picked up on one's own. (Page, 1976, p.297)

Page goes on to clarify the reasons for the use of the SAT-M on a population much younger than the seventeen year old population for whom the test was designed:

> Once one has decided to identify the gifted, especially out of a huge population, then how may one go about it? On the other hand, all the extremely gifted will bump their heads on the ceiling of the tests designed for their agemates. On the other hand, it is sometimes daid, a bright ten-year-old is not an average fifteen; he is a bright ten. From this viewpoint, different ages are not suitably measured with the same instrument. Indeed, when this viewpoint is pushed far enough some are led to say (truthfully but misleadingly) that every one is an individual. SMPY rightly pushed this objection aside and took the much sounder, historical line that intelligence is linear, and additive, and identifiable no matter at what age it is exhibited. There is a still stronger assumption, again the historical one in the measurement of intelligence: that intelligence will continue to grow, much as one's own individual rate, at least as long as physical growth continues. (Page, 1976, p.299)

In contrast to these efforts to identify giftedness by selecting individuals who place high in a normal distribution, phsychologists have again been looking at the mathematically gifted person as a member of a "special group." However, in recent studies, instead of a search of common traits from information gathered <u>in retrospect</u> as in many of the studies before 1900, the psychologist now asks questions directly of a chosen group with known talent. The answers to the questions are (as in the early studies) then examined (but with modern statistical methods) for common responses hoped to reveal common traits among members of the special group. (Renzulli, Smith, White, Callahan, Hartment, 1976)

In a study by Linsemeier, the 35 top scoring students from a 1972 mathematics talent search were compared with students chosen at random. Values were measured by the questionnaire form of the Allport Vernon-Lindzey test. Mathematics ability was measured by the Math Level I Achievement test or the Sequential Test (Series II) of the SAT-M. In the following statement, Linsenmeier explained the purpose of the questionnaire, and then described the results of the questionnaire:

> The Study of Values is designed to measure the relative prominence of six basic motives in an individual, corresponding to the six types posited by Eduard Spanger in his Types of Men (1966): Theoretical, economic, aesthetic, social, political, and religious. The theoretical man values above all else the discovery of truth. Economic man is practical, placing paramount value on utility. Aesthetic man values form and harmony highly. Social man values love of people. Power is the chief value of political man. And religious man is mystical, valuing unity, the oneness of the universe... The means (of the top 35 scoring students as compared with the randomly selected students) on the theoretical, economic, aesthetic and religious scales are significantly different from the expected means... Scores on the theoretical scale are extremely high, and those on the aesthetic scale extremely low. Economic scale scores are moderately high and religious scores moderately low. (Linsenmeier, 1976, p.286)

In another study called The Values of Gifted Youth, Fox (1976) discusses the association found between certain values chosen by students and mathematical precocity. She said in her abstract of the study:

> The Allport-Vernon-Lindzey Study of Values was administered to 655 gifted boys and girls who participated in the 1973 mathematics talent search. It was also given to boys who were winners or near-winners in the 1972 and 1974 contests. Over half the highly precocious boys in all three years scored highest on the theoretical scale. Only one-third of the less precocious boys in the 1973 contest were highest in that value. Girls in the contest were most likely to score highest on the social scale, and only fifteen percent

> scored highest on the theoretical value. Thus a theoretical value orientation appears to be related to mathematical precocity and may in part explain some of the differences found between the sexes with respect to eagerness for acceleration in mathematics (Fox, 1976, p.273)

In another psychological study in 1976, the characteristics of the mathematically gifted as a special group were examined. In that paper, A Summary Profile of the Nonintellectual Correlates of Mathematical Precocity, Denham and Haier compared the profiles of creative adult scientists and mathematicians with profiles of psychometrically tested youth who were identified as mathematically precocious. In discussing personality traits exhibited by adult mathematicians and similar traits exhibited by mathematically precocious youth, Denham and Haier wrote:

> Personality characteristics and differences among creative male and female mathematicians were examined by Helson and Crutchfield (1970) and Helson (1971). The first of these studies indicated that, compared with "average Ph.D's" creative adult male mathematicians showed lower scores on the California Psychological Inventory Scale for Self-Control and higher scores on the Flexibility scale. Apparently the ability to "let go", to change mental set, and to be highly flexible are advantageous qualities for persons seeking careers in mathematics. As noted (in this study), MG (Mathematics Gifted) boys and the MG (Mathematcs Gifted) girls show significantly higher Flexibility scale scores than non-IQ-gifted reference groups, and are lower on the Self-Controle scale (though not necessarily significantly so) than all but one reference group. Thus the sample, both male and female, is similar to creative mathematicians on a limited, but apparently important, set of dimensions. (Denham and Haier, 1976, p.232)

There was some evidence that academic achievement was not a completely cognitive function (Demos and Weijola, 1966). There were also folklore associations of the "immature misfit" with the gifted child, even by educators. (Haier and Solano, 1974) The unfavorable adjectives most frequently checked by the educators in that study were argumentative, opinionated, and impatient. If gifted students were to be given special accelerated or enriched programs that required "maturity" and "social adjustment" it was necessary to determine whether their interpersonal abilities thought necessary for adequate progress in the special programs (such as well being, responsibility, socialization, self-control, tolerance, communality, intellectual efficiency, flexiblity, etc.) compared favorably with those of older normal student groups.

Weiss, Haier and Keating in a study Personality Characteristics of Mathematically Precocious Boys, used The California Psychological Inventory Scales (CPI) to show that there were no "significant deficits" in the interpersonal abilities of mathematically gifted children when compared with older students. Weiss, Haier, and Keating made clear that the results of their study did not support the association of the "gifted child" with the "misfit" in the following statement:

> The most salient and important finding to be extracted from the analyses is that the mathematics gifted students as a group are not interpersonally ineffective or maladjusted... This finding should help lay to rest the notion that gifted children are by definition misfits and maladjusted... The two most salient CPI scales associated with the Mathematics Gifted group are Achievement via Independence and Flexibility. (Weiss, Haier, and Keating, 1974, p.135)

Other psychologists such as Guilford (1964, 1967) and Torrence (1969) looked for behaviors that would describe and separate out the creative individual, because creativity was believed to be a prime characteristic of the mathematically gifted individual. In the following, Gallegher discussed the Guilford model for the recognition of creativity in the gifted individual:

> Guilford's "structure of the intellect" are seen as ways of providing an analytical approach to creativity... In the Guilford model, for example, divergent production has been identified by a number of writers as closely equivalent to creativity. Many exercises and activities that would enhance these skills have been produced in the last decade. (Gallegher, 1979, p.34)

In work that is somewhat independent of western scholarship, Russian psychologists in recent years have also been concerned with the recognition of the mathematically gifted, and "optimal ways of forming and devloping mathematical abilities at school age."

In a study, The Psychology of Mathematical Abilities in Schoolchildren, begun in Russia in 1955 and concluded in 1966, V. A. Krutetskii explored the nature and structure of mathematical abilities; questions of their growth and development though instruction were left for later research. In the study he asks the question "can any one become a mathematician or must one be born one?" In answer to this question he says,

> "Anyone can become an ordinary mathematician; one must be born an outstanding talented mathematician." In the course of the study he reminds the reader that "mathematical abilities are not innate, but are properties acquired in life that are formed on the basis of certain inclination. The role of inclinations varies according to which abilities are involved - it is minimal in cases of the development of ordinary abilities in mathematics, and it is exceptionally great when it is a matter of the outstanding giftedness of research mathematicians... some persons have inborn characteristics in the structure and functional features of their brains which are extremely favorable (or quite unfavorable) to the development of mathematical abilities." (Krutetskii, 1976, p.361)

Mathematicians' Concern for the Nature of the Mathematically Gifted

The mathematician has not been in disagreement with the psychologist in recognizing first the general intelligence of the individual and then looking for special traits indicating a special propensity for mathematics. However, mathematicians have disagreed among themselves as to whether mathematicians are born (heredity) or developed (culturally and within a certain environment.)

Hadamard, the accomplished French mathematician noted in his book, The Psychology of Invention in the Mathematical Field (1945), that the Phrenologist Gall in the early 1800's defined mathematics ability as a certain bump on the head and the psychologist Sourian maintained that invention occurs by pure chance. Actually, Gall was a forerunner of the notion of cerebral localization. Phrenology claimed a connection of every mental aptitude with a greater development not only of some part of the brain but also of the corresponding part of the skull - thus there was a special bump for mathematical ability. Moebius (the grandson of the mathematician) did a study which contained data on the hereditary traits of families of mathematicians and confirmed the somewhat classic opinion that most mathematicians are fond of music, and asserts that they are also interested in other arts. Hadamard himself said:

> It is more than doubtful that there exists one definite "mathematical aptitude." Mathematical creation and mathematical intelligence are not without connection with creation in general and with general intelligence. It rarely happens, in high schools, that the pupil who is first in

> mathematics is the last in other branches of learning; and, to consider a higher level, a great proportion of prominent mathematicians have been creators in other fields. (Hadamard, 1945, p.5)

Therefore, Hadamard is in agreement with the psychologist who looks for mathematically gifted youngsters amongst those of general high "intelligence."

Later in his book, Hadamard also described various mathematicians' personal habits that were important to their own creative thinking. For instance, he mentions his own habit of pacing a room when he is thinking, and the chemist Teeple's habit of taking two baths in a row to encourage creative thinking. However, he did not claim these to be habits that either define the creative scientific personality, or necessarily habits that would produce creativity even in the talented individual. As a matter of fact, he gave the impression that each creative personality develops his own personal environment conducive to creative expression.

Many mathematicians believe that the potentially creative mathematician is the individual of high intelligence who needs to express his intelligence, and due to strong cultural influences becomes interested in mathematics.

Wilder, in particular, maintains that invention and mathematical creation are not accidental, but the result of a cultural evolution acting upon the general intelligence of certain individuals who are the natural tools of scientific evolution.

In a book entitled The Foundations of Mathematics, Wilder known as a successful teacher and researcher of mathematics at the University of Michigan, talked about a general intelligence stimulated and developed into mathematical creativity by a suitable environment. He said:

> Like any other cultural element, mathematics grows by ev olution and diffusion. Given a suitable juxtaposition of ideas, either in the mind of an individual or in the minds of a group, certain syntheses take place and new concepts come forth. Thus, one can trace the evolution of the concept of an abstract group in the writings of nineteenth century mathematicians in the same manner as one can trace the evolution of the theory of biological evolution in the works of Darwin's predecessors and his own, and contemporary writings. Every research mathematician is familiar with the numerous instances of the simultaneous announcement of new theorems and theories by two or more mathematicians. This is an entirely similar phenomenon to that which occurs when a new invention is about to be made; when the cultural conditions are right, it will appear without fail... Mathematics does not grow because a Newton, a Riemann, or a Guuss happened to be born at a certain time; rather, great mathematicians were made because the cultural conditions - and this includes the mathematical materials - were conducive to developing them. (Wilder, 1952, p.277)

In his book on The Nature of Mathematics (1914) reprinted Philip Jourdain, the mathematics historian, supported the idea of the strong environmental influence (superimposed upon an independent and purely intellectual need of mathematicians), guiding the nature of mathematics and the nature of mathematicians. He commented:

> In the history of the human race, inventions like those of the wheel, the lever, and the wedge were made very early - judging from the pictures on ancient egyption and Assyrian monuments. These inventions were made on the basis of an instinctive and unreflecting knowledge of the processes of nature, and with the sole end satisfaction of bodily needs... We can well imagine that this pursuit of science is attractive in itself; besides helping us to communicate facts in a comprehensive,

> compact and reasonably connected way, it arouses a purely intellectual interest. It would be foolish to deny the obvious importance to us of our bodily needs; but we must clearly realize two things:
> 1) The intellectual need is very strong, and is as much a fact as hunger or thirst; sometimes it is even stronger than bodily needs - Newton, for instance, forgot to take food when he was engaged with his discoveries.
> 2) Practical results of value often follow from the satisfaction of intellectual needs. It was the satisfaction of certain intellectual needs in the cases of Maxwell and Hertz that led to wireless telegraphy: it was the satisfaction of some of Faraday's intellectual needs that made the dynamo and the electric telegraph possible. But many of the results of strivings after intellectual satisfaction have as yet no obvious bearing on the satisfaction of our bodily needs. However, it is impossible to tell whether or not they will always be barren in this way... The economical function appears most plainly in very ancient and modern science. In the beginning, all economy had in immediate view the satisfaction simply of bodily wants. With the artisan, and still more so with the investigator, the most concise and simplest possible knowledge of a given province of natural phenomena - a knowledge that is attained with the least intellectual expenditure - naturally becomes in itself an aim; but though knowledge was at first a means to an end, yet, when the mental motives connected therewith are developed and demand their satisfaction all thought of its original purpose disappears... The generally accepted account of the origin and early development of geometry is that the ancient Egyptians were obliged to invent it in order to restore the landmarks which had been destroyed by the periodical inundations of the Nile... For the Greek geometricians, as we shall see, seem always to have dealt with geometry as an abstract science. (Jourdain, 1914, p.8)

If it is true that mathematical creativity is a matter of contact conditions and cultural experience, might it not be possible to develop "creativity" by exposing the "highly intelligent" individual to the full mathematical experience? That is, should we assume that creativity or mathematical talent is a "trait" to be discovered in some special individuals, or "the proper cultural and environmental experience" acting upon the "highly intelligent" human being!

May we then conclude that the scientifically or mathematically talented individual is a generally intelligent individual with a strong intellectual need that is maybe culturally developed or maybe inborn?

One of the foremost mathematicians of all time, Henri Poincare, wrote down his beliefs about creation and creators of mathematics. His writings led one to believe that mathematical creation is a kind of emotional sensiblity on the part of the creator. In his article entitled Mathematical Creation (1880) he commented on "mathematical intelligence." He said:

> A first fact should surprise us, or rather would surprise us if we were not so used to it. How does it happen there are people who do not understand mathematics? If mathematics invokes only the rules of logic, such as are accepted by normal minds; its evidence is based on principles common to all men, and that non could deny without being mad, how does it come about that so many persons are here refractory? That not every one can invent is nowise mysterious. That not every one can retain a demonstration once learned may also pass. But that not every one can understand mathematical reasoning when explained appears very surprising when we think of it. And yet those who can follow this reasoning only with difficulty are in the majority: that is undeniable, and will surely not be gainsaid by the experience of secondary school teachers?... A mathematical demonstration is not a juxtaposition of syllogisms, it is syllogisms placed in a certain order, and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition, to speak, of this order, so as to perceive at a glance the reasoning as a whole, I no longer fear lest I forget one of the elements, for each of them will take its allotted place in the array, and that without any effort of memory on my part. It seems to me then in repeating a reasoning learned, that I could have invented it. This is often only an illusion; but even then, even if I am not as gifted as to create it by myself, I myself re-invent it is so far as I repeat it... In fact, what is mathematical creation? It does not consist in making new combinations with mathematical entities already known... To create consists precisely in

> not making useless combinations and in making those which are only a small minority...all the combinations would be formed in consequence of the automatism of the sublimal self, but only the interesting ones would break into the domain of consciousness... More generally the priveleged unconscious phenomena, those susceptible of becoming conscious, are those which, directly or indirectly affect most profoundly our emotional sensibility. It may be surprising to see emotional sensibility invoked a propos of mathematical demonstration which, it would seem, can interest only the intellect. This would be to forget the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance. This is a true aesthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility. Now, what are the mathematical entities to which we attribute this character of beauty and elegance, and which are capable of developing in us a sort of aesthetic emotion? They are those whose elements are harmoniously disposed so that the mind without effort can embrace their totality while realizing the details. This harmony is at once a satisfaction of our aesthetic needs and an aid to the mind, sustaining and guiding. And at the same time, in putting under our eyes a well-ordered whole, it makes us foresee a mathematical law. Now, as we said above, the only mathematical facts worthy of fixing our attention and capable of being useful are those which can teach us a mathematical law. So we reach the following conclusion: The useful combinations are precisely the most beautiful, I mean those best able to charm this special sensiblity that all mathematicians know, but of which the profane are so ignorant as often to be tempted to smile at it. (Poincare, 1880, p.2041)

The same strong emotional and aesthetic appeal of mathematics to the mathematician was also emphasized by the British mathematician E.H. Hardy, in his essay entitled. A <u>Mathematicians Apology</u>. He said:

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas... The mathematician's patterns, like a painter's or the poet's must be <u>beautiful</u>; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics... The best mathematics is <u>serious</u> as well as beautiful... Thus a serious mathematical theorem, a theorem which connects significant ideas, is likely to lead to important advances in

> mathematics itself and even in other sciences. No chess problem has ever affected the general development of scientific thought: Pythagoras, Newton, Einstein have in their times changed its whole direction. (Hardy, 1910, p.2027)

This would suggest that the mathematically gifted would show a set of strong aesthetic values as well as a strong feeling for serious scientific truth. This is contrary to the profiles of the youngsters labeled "mathematically precocious" (see Linsenmeier, 1976. This study was discussed earlier under Psychologists Concern for the Nature of the Mathematically Gifted).

Summary

Thus we see that psychologists and mathematicians differ substantially in their discussions of the nature of mathematically gifted people. Psychologists tend to focus on traits - preferably those which can be measured by testing. Mathematicians on the other hand tend to focus on creative potential and the ability to seize opportunities from cultural and aesthetic cues. There is some evidence that a synthesis of both viewpoints may be possible.

The recent discussion of mathematically gifted children (in The National Council of Teachers of Mathematics professional series) suggested the following guidelines for a system of identification.

> A comprehensive and systematic plan should attempt to take account of as many of the following guidelines as possible. They are adapted from an unpublished paper prepared by Marshall Sanborn, Research and Guidance Laboratory for Superior Students, University of Wisconsin, and are used with permission.
>
> A. An adequate plan for the identification of gifted and talented individuals requires the use of a variety of techniques over a long period of time.

1. The gifted and talented express themselves in many ways.
2. Gifted and talented performances may emerge at certain times in life and under certain opportune conditions.

B. Identification of gifted and talented individuals should be based on a knowledge of the individual, the cultural - experiential context in which the individual has developed, and the fields of activity in which he or she performs.
   1. At least some methods of identification should be individualized, yielding case study data unlikely to be obtained by standardized group methods.
   2. Identification techniques can be locally developed with methods and criteria that are appropriate for the particular population to be studied.
   3. The identification process should call for systematic involvement of those professionals who are acquainted with the individuals through direct observation of behavior and performances.
   4. The identification process should call for the involvement of those persons best qualified to judge the quality of a performance or product. This is especially important in areas such as the visual and performing arts.
   5. The identification process should involve persons best qualified to understand the culture and the specific circumstances of individuals whose performances are being assessed.

C. Both self-chosen and required performances should be assessed.

D. Considerable freedom of expression of response should be allowed.

E. The adequacy of the identification program should be continually reassessed.
   1. Follow-up study should involve both those who were identified as gifted or talented and those who were not.
   2. Methods and procedures for identification should be modified by refining, adding, and deleting on the basis of evidence obtained.
   3. Those individuals who have been misclassified can be appropriately reclassified on the basis of evidence.

F. Evidence obtained during the identification process should provide a primary basis for programming experiences and opportunities.

(Ridge and Renzulli, 1981, p.204)

Nurture of the Mathematically Gifted

Educators' Concern for the Nurture of the Mathematically Gifted

> We have organized special classes for, built special schools for, trained specially prepared teachers for, and awarded honors for success in the teaching of the weak or handicapped. But we have largely left the student at the other end of the spectrum of ability to take care of himself, ignoring the fact that the deviates at either end of the scale have equally peculiar and pressing needs. If many gifted students have turned out to be average in performance, one reason is that we have trained them with the average and by methods designed for the average. MHA 1953 (Ahrendt, 1953, p.285)

M. H. Ahrendt was Executive Secretary of The Nation Council of Teachers of Mathematics, Washington, D.C. in 1953, when he shared his concern for the lack of attention for educating the mathematically gifted student. He suggested that when one expects no more than average accomplishment, one takes a loss for the difference in capability and accomplishment and the consequent contribution to society that the difference would make. He went on to explain that to educate the gifted, we must have a definition of "mathematical aptitude." He discussed mathematical aptitude in terms of what he believed to be levels of mathematical learning (a common belief among mathematicians). On this latter point, we note that Klein had (before Ahrendt) prompted the consideration of "stages of development" in learning mathematics even into University mathematics. Klein, the German mathematician and mathematical educator, as the teacher of teachers said in Elementary Mathematics from an Advanced Standpoint: (1936):

> The teacher so to speak, must be a diplomat. He must take account of the psychic processes in the boy in order to grip his interest; and he will succeed only if he presents things in a form intuitively comprehensible. A more abstract

> presentation will be possible only in the upper classes. For example: the child cannot possibly understand if numbers are explained axiomatically as abstract things devoid of content, with which one can operate according to formal rules. On the contrary, he associates numbers with concrete images. They are numbers of nuts, apples, and other good things, and in the beginning they can be and should be put before him only in such tangible form. While this goes without saying, one should-mutatis mutandis-take it to heart, that in all instruction, even in the university, mathematics should be associated with everything that is seriously interesting to the pupil at that particular stage of his development and that can in any way be brought into relation with mathematics. (Klein, 1932, p.3)

M. H. Ahrendt, in his discussion of the meaning of "mathematical aptitude", submitted that no one really knows what mathematical aptitude is, but he suggested that it might be best defined by the "levels of mathematical learning" that a student passes through. He explained his position with the following:

> No one knows at present just what mathematical aptitude is. No doubt it is a complex of traits, but what the individual traits are is a question. It is likely that the specific traits needed vary somewhat from one branch of mathematics to another, with a certain core of traits needed for success in any branch of mathematics. Since we cannot define "mathematical aptitude" in terms of its components, we shall define it in terms of its manifestations. In this paper the term "mathematical aptitude" means the ability to succeed at progressively more and more advanced levels of mathematical learning." The above definition infers that there are various levels of aptitude in mathematics or that individuals possess ability in mathematics in varying degrees. Such a notion agrees with experience. There are many persons who are able to succeed in elementary mathematics courses who cannot succeed in courses on a higher level... Whatever traits give one ability in mathematics, it is certain the computational ability is not necessarily one of them... there are scores of competent mathematicians, capable of dealing skillfully with the abstractions of higher mathematics who have trouble in balancing their own bank accounts... Most bright students can succeed in high school courses in mathematics. Instruments which are of value of identifying students with ability on the lower levels of mathematics are: intelligence tests, especially the scores in the quantative parts; aptitude

> tests; grades in previous courses in mathematics; interest tests; achievement tests; cumulative records; and personal observation of the teacher... However, the devices which pick out the students who can succeed in lower level courses do not necessarily tell whether these students will do well at the higher levels...we can tell pretty well by the instruments available today whether a student can succeed in college algebra. But the same instruments do not predict his success in a more specialized field such as matrices, for example. he may reach his saturation point in college algebra. (Ahrendt, 1953, p.285)

M. H. Ahrendt gives the distinct impression that nurture of the mathematically gifted should incorporate levels of learning and the talented student should be given the opportunity to climb the learning levels, special to mathematics, whenever he is ready. If the educator accepts Ahrendt's definition of "mathematical aptitude," then he must also accept nurture as dependent upon the mathematical level of the student. Does provision of nurture at the student's level of development mean acceleration through the school system for the gifted student, or enrichment outside the school curriculum? We shall discuss how acceleration or enrichment has been interpreted in the past, and how these interpretations have formed the methods of nurture utilized by educators.

Methods used by educators to nurture mathematical talent

Ruderman outlines the devices used by teachers to develop mathematical talent in a report written by the School Mathematics Study Group of A CONFERENCE ON MATHEMATICS FOR GIFTED STUDENTS (1967).

His article is entitled Comments on Acceleration and Enrichment. Under enrichment he lists: Clubs and seminars, visiting lecturers, and specialists, contests and fairs, projects and reports, inserting selected topics for the selected class, visits to computer centers, and computer oriented topics, publishing a school mathematics paper, Saturday and Summer Institutes, differentiated assignments, selected readings, submitting solutions to problems in the NCTM publication The Mathematics Student Journal, harder verbal problems, and computational problems. Under acceleration he lists: moving algebra down into the eighth year, consolidating plane and solid geometry, consolidating eleventh year mathematics with advanced algebra, permitting children to take final examinations for the next term without taking the course, permitting children to move ahead at their own rates in ungraded classes, consolidating three years into two years work, moving two years of college mathematics into the high school as, for example, with the Fehr Project.

Ruderman went on to give his impressions regarding the acceleration and enrichment methods used by the teachers. The following is his list of the important points following his observation of the methods employed:

1. For the most part teachers are familiar with what acceleration is but are not competent to judge what significant enrichment is.
2. Teachers need guidance in the selection of such materials judged as significant and welcome assistance.
3. There is a feeling that significant enrichment is acceleration, perhaps out of order. (Ruderman p.25)

The last point is certainly a matter of interpretation. Summer Institutes for gifted high school students at Universities could be considered acceleration out of order. They have included mathematical materials and content not normally taught until undergraduate college courses in mathematics. Sometimes mathematical content has even extended into graduate school mathematics. The six week summer institutes from 1956 to 1960 at the State University of South Dakota reported:

> The course work in mathematics and English does not duplicate that in any standard high school or college. It is not the purpose of the institute to give credit for courses that the student would take in the future, but to give him, through the challenging university environment, a foundation that will make future courses more meaningful... The instructional program in mathematics consisted of two hours daily of classroom instruction. One hour each day was devoted to material in Finite Mathematics by Kennedy, Snell, and Thompson. The sections in this text on compound statements, sets and subsets, partitions and counting, and probability theory were covered in detail... The other daily lecture was aimed at increasing the student's interest in mathematics... some of the topics covered were: Number Systems; Concept of a Function; Limits; Continuity; Definition of a Derivative and its Application to Maximum; Minimum Problems; Differentials; Rate Problems; Fundamental Theorem of Integral Calculus. (Gutzman, 1962, p.279)

The State University of South Dakota may not have been duplicating any of its own standard college courses but Finite Mathematics from the book by Kennedy, Snell, and Thompson, was a standard college course at Tulane University, and probably other colleges at that time. The lecture topics covered were also included in most college Calculus courses at that time.

In commenting generally about the summer mathematics training programs for high ability secondary-school students (1962) Bell, of The University of California at Los Angeles, named the basic objective to be "to bring outstanding students into direct contact with college teachers and research scientists of recognized competence so that these students may be inspired to gain an understanding in science beyond that usually acquired in the high school." (emphasis added) He went on to describe one such program at The University of California at Los Angeles from 1959 to 1961. He said:

> The course included the development of the number system, definition and operation with sets, postulational development of algebra, equations and inequalities, logical analysis of statements and truth tables, functions, relations, and various other materials. The teacher for this course was Professor Kenneth Rogers of our mathematics department... Although no actual mathematical research is likely to be produced by students in a program such as ours, nevertheless the methods of research in mathematics can be demonstrated. Thus Professor Rogers required each student to select an interesting topic (which was new to the student) for the purpose of searching the library for pertinent material. All of this material was studied and then assembled in the form of a paper. Expansion of ideas was encouraged as well as extension proofs if possible... A few topics selected were: Game Theory; Godel's Theorems on Completedness of Arithmetic and Related Topics; Infinite Cardinals; Introduction to Probability Theory; Linear Programming; Multi-valued Logic; Probability Theory as Applied to Genetics; Symbolic Logic; The Mathematics of Voting; Zero Sum, Two person Game Theory. (Bell, 1962, p.276)

A close scrutiny of the topics covered by the Summer Institutes suggested that mathematical acceleration out of order is not an unreasonable description of the programs encountered by the mathematically gifted high school students in those summer programs, although the non duplication of high school level mathematics could be interpreted as non-acceleration within the secondary school system.

Another article, <u>Extra-Curricular Activities in Mathematics for Mathematically Gifted Secondary School Students</u>, (1967) by Robert Kalin, separated and summarized the mathematics extra-curricular activities into:

1. Contests

2. Group Study

3. Independent Study

4. Tutoring

Contests. Kalin noted the many different kinds of contests in which American students could participate from local, to state, to national, to international. The problems varied from "a large sample of fairly easy multiple-choice questions administered in an hour on the one extreme, to a small number of quite difficult problems administered on an open-book or take-home basis on the other extreme. In at least three states (Alabama, California, Wisconsin) mathematical competitions have been used as a means of searching out mathematical talent among high school students... The Wisconsin and Stanford examinations owe much of their concept and their emphasis upon original problem-solving to the well-known Hungarian Eotvos competition." Kalin also discussed Olympiads held in the U.S.S.R. and the International Mathematical Olympiads for students of European Communist Countries and The Eotvos Competition in Hungary. Two of the questions raised concerning the competitions were:

1. Is it possible to test for creative mathematical ability with competetive problems?
2. Do they have a bad effect on a truly able student who cannot solve anything within a few hours time limit and in strange surroundings?

Group Study. Kalin listed summer programs, Saturday programs, mathematics clubs and seminars or lectures. He said that among the purposes of the summer programs are "the identification of mathematically gifted students and the subsequent attempt to give them insights into higher mathematics (science) and the work of mathematicians (scientists). For these reasons, the programs are usually administered by universities and held on their campuses... Among the great variety of topics taught, the following have been chosen most often: some form of linear-algebra, algebraic structures, probability and statistics, computer programming and computer-related mathematics, logic and set theory, number systems, number theory, foundations of geometry. An attempt has been made to expose students to key mathematical ideas and techniques without giving them a complete course from the regular curriculum." (Kalin, 1967, p.63)

He included among the activities at mathematics clubs: puzzle solving, problem solving, listening to lectures on mathematics or mathematical careers, planning contests, independent projects with reporting of results to other students, and general socializing. According to Gnedenko (1957), a variety of programs was also typical of the school mathematics clubs of the U.S.S.R. Some math clubs (that have no analog in the United States) used graduate and undergraduate students to teach school students how to solve problems and were related to the Russian Olympiads.

Independent Study. Kalin reported that aside from its occurence in some summer programs, independent study by gifted school students had been promoted through science fairs and some other enrichment programs. However, records showed that independent research at science fairs such as the Westinghouse Science Talent Search was mostly in science with only a small fraction devoted to mathematics.

<u>Tutoring</u>. Kalin commented that "One finds occasional side references in the literature to the idea of having mathematically gifted students serve as tutors or teachers. But there is no indication that anyone has explored this technique in any detail. What are its possibilities? Should gifted students be paid as teacher aides? Would there be enough intellectual return to the gifted student for this investment in his time?" (Kalin, 1967, p.76)

Kalin concluded that there were many extra-curricular activities available in the United States for mathematically talented secondary students." But with the exception of the (SSTP)programs and the (MAA) contest, few have been sufficiently well organized and promoted with the real urgency these gifted students deserve." (Kalin, 1967, p.77)

<u>Ability Grouping</u> could be listed under the heading of enrichment or acceleration and has been used by some school systems to compensate for the differences of abilities of the students within a grade. Mainly three types of grouping have been tried by school systems:

1) Grouping within a heterogeneous class.
2) Grouping of students by ability into several classes and thus forming <u>honors</u> classes at the top of the scale of ability.
3) Grouping of students of high ability or special talent in a special <u>honors</u> school.

Grouping in contrast to acceleration, avoids the separation of the gifted youngster from his age mates. Grouping often involves developing a special <u>accelerated curriculum</u>, special faculty, special procedures, and special standards of admission for the high ability groups.

Elective Courses have also provided enrichment for students, and gifted students in particular have the ability to carry the extra electives of their choice.

Enrichment Versus Acceleration as Methods of Nurture for Gifted Children.

Many teachers feel that the heterogeneous class is difficult to teach and tends to level the instruction to the average ability of the class, shortchanging the children of ability at the extreme ends of the class (both slow learners and the gifted).

Enrichment is one of the ways of giving compensatory educational opportunities to the gifted. In a paper called "Enrichment," Worcester gave the following description of enrichment:

> Enrichment as a way of giving better opportunities to the mentally advanced child implies providing experiences for which the average or below average child lacks either the time, the interest, or the ability to understand. Clearly, these experiences, if they are to be enriching, will be such as to widen and deepen the child's understanding... To be enriching in the best sense, the added work should be integrated with the general curriculum activities. Too frequently, enrichment has been assumed to result from casual museum trips, a foreign language taken for a semester or two in the early grades and then dropped, turning on television for some national or political event, answering the telephone for the principal, helping the teacher with her records, helping the less gifted with his work, making unsystematic collections of almost any kind of material, and the like... The line between enrichment and busywork is sometimes a thin one... Enrichment requires of the teacher much ingenuity and understanding. It is not enough just to say, "Go find something to do." Necessary is the same discerning guidance needed in the teaching of any subject. (Worcester, 1979, p.98)

Worcester goes on to say that enrichment without acceleration is favored by most administrators and most teachers because it is in keeping with the tradition that it is good for the child to remain a child and still enjoy a wide range of experiences. However he notes that, "Enrichment is really very rarely encountered except in special classes in which that is a definite aim."

The last comment is really a statement about the quality of many of the enrichment programs organized by the individual teacher and is an echo of the first point made by Ruderman. (see Ruderman, 1967) In his observations of mathematical enrichment practiced in the schools, in which he said that for the most part teachers "are not competent to judge significant enrichment." Enrichment programs obviously need much thought, much ingenuity, very good organization, and a good teacher with an advanced depth of knowledge of the subject being enriched.

Renzulli, put all education either beyond or out of step with the regular curriculum in the schools under a large heading of curricular enrichment (for instance the introduction of advanced courses early or the supplementation of the regular curriculum). He said:

> First, there are indeed certain basic competencies that all students should master in order to adapt effectively to the culture in which they are growing; and second, the mastery of these competencies should be made as streamlined as possible... It is, however, precisely because the regular curriculum (even in its most excellent manifestations) fails to meet the needs of all students that we require special provisions for some youngsters, and in the case of the gifted and talented those special provisions almost always take the form of some type of curricular enrichment. In its simplest form, enrichment may be merely a matter of introducing gifted students to advanced courses early. This practice sometimes referred to as vertical enrichment or acceleration, usually consists of allowing students to enroll in courses to which they would not ordinarily have access until later years. Although this approach lacks imagination so far as curricular reconstruction is concerned, it may very well be appropriate in subjects such as mathematics, physics, and computer science that are highly structured and sequential in concept complexity... The learner has two other dimensions that must be respected in an enrichment situation, and even an advanced course may fail to take account of 1) the student's specific content interests and 2) his or her preferred style(s) of learning. (Renzulli, 1979, p.86).

In Provisions for Gifted Students (1979), Meister and Odell agreed that acceleration had "the advantage of challenging the more able student, and it permits him to avoid frustration and growth of bad habits." Although this method of nurture was the least expensive and the easiest to utilize, Meister and Odell pointed out that it carried with it the danger of the social maladjustment of the gifted youngster.

Gaps in mathematical development, real or imagined, of the accelerated student is another often repeated argument against the method of acceleration for the gifted student. The following statement from Glenadine Gibb, a recent past president of the National Council of Teachers of Mathematics, supports that argument against acceleration:

> Programs that the gifted can be expected to encounter if accelerated, are designed for the mainstream student at that level. Although evidence from research supports the success of talented students in such courses, one must be reminded that for the most part these evaluations reflect a student's ability to perform on traditional, convergent-production tasks. These evaluations can be expected to neglect an important component of giftedness, that of divergent-production tasks... Furthermore, students who have experienced programs of acceleration have been found to have superficial mathematical understandings and insights and to have gaps in their programs of study. They also have been denied the opportunity to develop their innate abilities of divergent thinking and creativity, abilities that are characteristic of talented people and particularly of mathematicians and scientists. Indeed, they have had the opportunity to study "average" material sooner, only to have an "average" education. (Gibb, 1979, p.220)

To the above statement, (recorded on the same page as Gibb's statement above) Stanley replied that "in SMPY's experience, the holes in the background theory doesn't hold up empirically at all."

In an article entitled "Intellectual Precocity," Stanley supported the argument of acceleration of mathematically gifted children with examples of success such as the following:

> For example, many seventh graders who score in the upper one percent on a test of mathematical aptitude can master a year's algebra course in a couple of months, working on their own. We have had six 13 year old boys happily taking high school chestry and twelfth-grade calculus. They had learned the rest of high school mathematics well part-time at college or in our special course... A number of accelerated ninth and tenth graders handled twelfth-grade honors advanced placement calculus well at age 13 in competition with the mathematically ablest twelfth graders. A boy completed 23 college credits in computer science, mathematics, and chemistry shortly after his fourteenth birthday, being the best student in the Calculus I class at a selective college, and went on to Calculus II and III. A boy earned his master's degree in computer Science and while still 17 years old, began work for the doctorate at a major university. The list of examples could go on and on. It seems uncomfortably probably that much of the intellectual alienation of brilliant high school graduates is due to their having been educated at a snail's pace too many years. (Stanley, 1974, p.18)

Finally, a recent discussion on <u>nurture</u> of gifted children (in the National Council of Teachers of Mathematics professional series) summarized opinion in the mathematics education community concerning methods of acceleration and enrichment.

> Renzulli (1977) cites acceleration in the form of taking advanced courses early (a la SMPY) as the simplest kind of enrichment... Varying proportions of acceleration and enrichment are found in situations such as the following:
>
> * Highly rigorous programs such as the Secondary School Mathematics Curriculum Improvement Study (SSMCIS) and the Comprehensive School Mathematics Program (CSMP) at the secondary level
>
> * More theoretical or comprehensive treatment of regular course content at any level, whether for individuals or specially formed groups or classes

* Extensions both in breadth and depth at any level through clubs or preparation for competition and independent study (with no formal classes) done either under the aegis of a school program or strictly independently, perhaps by correspondence or supported by a personal study of the literature and occasional interviews.

* Saturday morning programs, usually sponsored by school boards or area jurisdictional groups

* Summer institutes such as that offered by the University of Chicago

(Ridge and Renzulli, NCTM, 1981, p.216)

In that same discussion, three types of enrichment were examined:

Type I. General exploratory activities to stimulate interest in specific subject areas. That is, experiences designed to bring the student in contact with areas of study in which he/she may have interest, such as demonstrations by experts or visitations to interest centers where the student could explore or experiment.

Type II. Group training activities to develop processes related to the areas of interest developed through Type I experiences. That is, training exercises to develop processes such as critical thinking or problem solving, and based on a "logical outgrowth" of interests generated through type I experience.

Type III. Individual and small group investigations of real problems. That is, involvement in independent investigative activities. Experiences should involve problem finding as well as problem solving.

It was suggested that Type I and Type II enrichment was appropriate for most students, but Type III was appropriate mainly for the gifted.

CHAPTER III

THE SUMMER INSTITUTES OF 1979 AND 1980

Introduction

This chapter will acquaint the reader with the summer institute programs for 1979 and 1980. The case studies that follow in Chapter IV will, it is hoped, be more meaningful in light of familiarity with the history and organization of the summer institute.

Chapter III is in three parts:

1. History of the Summer Institute
2. Important Components of the Summer Institute
3. Overview of the 1979 and 1980 Summer Institutes

History of the Summer Institute

The Summer Institute in Mathematics for Gifted High School Students, directed by Arnold E. Ross, began in the summer of 1957 as an extension of a special teachers program at Notre Dame University. The program grew out of a desire of some mathematicians, (in particular Professor Ross, the chairman of the Mathematics Department of Notre Dame University in 1957), to "transmit a dedication and a mastery of mathematics to another generation."

The communication with the younger generation started with a secondary school program. The teachers' program looked for the best possible forms of instruction that could be applied to the teaching of young students. In Ross' words, "The program sought forms of instruction capable of developing a curiosity, a sense of adventure, a sensitivity of perception, and a mastery of the basic skills of mathematics in the secondary school student." Very soon after the teachers' program started, it became clear to Ross that a "teacher's mastery of the art of teaching was not an obvious consequence of a certain degree of familiarity with factual information." (The Mathematics Teacher, Oct. 1961) As a result, the program shifted its emphasis to a direct participation of the teachers in the work with gifted pupils. Observation and participation in the instruction of half a dozen gifted children in mathematics provided the secondary teachers with teaching models as well as experience with methods. The summer project at Notre Dame University shifted again, almost imperceptably, from an academic interest in the mathematical instruction of school children to a personal involvement in their mathematical development. Although the program retained close association with the teachers' program at Notre Dame, by the summer of 1962, the program for gifted high school children in mathematics was autonomous. It's participants came from grades 10, 11, and 12.

The children's interests ranged over all of the sciences, engineering, and mathematics. Ross explains his choice of number theory as a vehicle for developing scientific thought processes with the following:

> At the outset, therefore, one must confront the question of what purpose should be served by a mathematics program for a collection of young individuals who have in common

> only eagerness, curiosity, an unbounded (and hitherto undirected) supply of vitality and, possibly, an ultimate destiny in science. One may decide upon the selection of some mathematical skills generally considered useful in the sciences. One may choose some applications falling within the range of the student's experience and jointly treat the mathematical tools and the ideas in this selected field of application. One may, on the other hand, take advantage of the fact that with proper choice of material one can remain within the field of mathematics and yet exhibit a whole gamut of problems which will confront every scientist. Even though we did not intend to neglect any of the above alternatives entirely, we subordinated everything to the aim of providing our young charges with the opportunity of acquiring experience in scientific thinking, while remaining within the field of mathematics... Since 1957, we have used number theory as the basic vehicle for the development of the student's capacity for observation, invention, the use of language, and all those traits of character which constitute intellectual discipline. We have chosen number theory because of it's wealth of accessible, yet fundamental and deep mathematical ideas and for it's wealth of challenging but tractable problems. (Ross, 1978, p.2)

Number theory seemed to provide the student with a <u>need</u> to develop the facility for an effective, precise and concise use of language. The students played an active part in the subsequent development of the subject into a deductive science. They shared their problems continually with instructors, counsellors, and each other, in seminars and in the dormitory residence provided for both students and counsellors together. Under Ross' direction, the summer program for gifted high school children in mathematics operated on the campus of Notre Dame University until 1963. Then it moved to the campus of The Ohio State University until 1975. It moved again to the University of Chicago in the summer of 1975, where it remained through the summer of 1978. Late in the Spring of 1979, when it was discovered that the University of

Chicago would not have a Summer Institute, Ross' program was quickly reorganized to relocate again on the campus of The Ohio State University.

Important Components of the Summer Institute in Mathematics for Gifted High School Students

There were several important components that blended together to give the Summer Institute it's character, and none of them was independent of the others.

Lectures were the most conspicuous components of the summer mathematics program. Professor Ross gave eight weeks of daily lectures, Monday through Friday, in the first number theory course. These were prepared for the first year participant, but they were attended by all of the summer participants: advanced participants, counsellors, seminar instructors, and visiting faculty. Ross' number theory lectures were the unifying thread of the program. Everyone gathered together before and after the morning lectures to discuss the various problems students could or could't solve the night before. There were also lectures in other mathematical topics taught by other professors, organized within the eight week structure of the summer institute. The first year number theory student sometimes attended the lectures of one or two advanced courses for two or three weeks until the depth and volume of work became too much to carry.

Seminars in number theory for first year participants were in session for one hour, three days a week for eight weeks and were taught by mathematics faculty to supplement Ross' lectures. They were designed to encourage discussion of the questions raised in lectures and on problem sets.

Problem Sets were handed out daily at the end of each number theory lecture. They were also given to advanced students at regular intervals in the advanced course seminars. The number theory problem sets were made up by Professor Ross. They guided the problem oriented seminar and directed the first year participant's (as Ross would say) "undisciplined curiosity" in many different carefully considered directions. (Copies of all number theory problem sets used in 1979 are found in appendix A.)

It is also important to note that problem sets compelled the student to give his/her attention to the communication of solutions to seminar instructors, counsellors, and peers.

Tests were made up by Ross to test progress of the first year number theory students at three, six, and eight week points during the eight week program. The three tests allowed flexibility in the program in the sense that they gave Ross the opportunity to, as he often said, "play it by ear." That is, the tests apprised him of the progress made by students at regular intervals, and suggested what direction the problem sets and lectures should take. It should, however, be noted that an effort to slow down the learning process was never an acceptable solution (to Ross), if a test showed slower than expected progress. Instead he looked for something more that could be done for the students such as special topics in extra lecture-seminars.

(Extra lecture-seminars were given by Ross three times a week for three weeks after test I in 1980. At another time the counsellors gave dormitory group seminars.)

Advanced participants/counsellors were in at least their second year of participation in the summer institute. Participants were asked to return as counsellors only when Ross thought both the participants and the program would benefit from such an arrangement. Each counsellor was assigned to a small number of first year students with whom he would live (in the same dormitory), indirectly guide, tutor, nag and nurture through the problem sets. Later he/she graded problem sets and reported the results to Professor Ross at the weekly counsellor meeting. Professor Ross prepared three examinations during the eight week program. Counsellors graded the three examinations (and the rewrites of those examinations) in number theory. However, this was not the counsellors' only responsibility. The counsellor attended and studied advanced courses made available within the summer institute. The counsellor had to work out the solutions to problem sets and examinations designed for the advanced course that would later be graded by the seminar instructor or sometimes the lecturer of that course. The more advanced a participant was, the more mathematics he/she was expected to master during the summer.

First Year Participants and number theory students had one responsiblity only, to learn number theory and show their knowledge to the counsellors and instructors in the form of verbal and written ideas,and possibly solutions to mathematical questions raised by:

problem sets, examinations, instructors, counsellors, classmates, or themselves.

Most of the participants (first year, advanced participant/counsellor) lived and ate in the same dormitory for the eight weeks of the summer institute. Some participants were commuters.

Tradition and folklore of the summer institute was carried and passed on from summer to summer. The threads of continuity of the reconstructed old tales and the active construction of new tales to be passed on in the "next" summer depended on the enthusiastic participation of each "freshman" group.

As an example, the following is one of the legends of the Summer Institute. It was written so many years ago that no one (not even Ross) remembers when, but it passes on from counsellor to student, from summer to summer (names of seminar instructors change each year), and it never fails to amuse the new students. They recite it and hang it up on the walls, because it expresses their own experience and they have become a part of the tradition it expresses. The observer can feel the structure of the summer program within the lines of the following text:

GENESIS

In the beginning, Gauss created Number Theory.
And Number Theory was without form and comprehension.
And ignorance was on the face of engineers.
And Gauss said, let there be light; and there was ROSS
And the evening and the morning were the first day.

And Gauss said, let the proofs bring forth problem sets,
and the numericals yielding conjectures, and
the conjectures yielding theorems of a kind; and it was so.
And the evening and the morning were the third day.

And Gauss made two great lights; the greater light
to rule the problem seminars and the lesser light
to rule the second-year students; he made Rally also.
And the greater light he called Woods and the lesser
light he called Leonard. And the evening and the
morning were the fourth day.

And Gauss said, let us make counsellors in our image,
after our own likeness, and let them have dominion
over students and over proofs and over every
cretin wombat that creepeth within the program.
So Gauss created Counsellors in his own image; from
his preimage he mapped them, and Gauss blessed them.
And Gauss said unto them, "Be fruitful and multiply
and check the students' arithmetic and have dominion
over the students and subdue them." And Gauss saw
that everything that he had made and beheld was consistent.
And the evening and the morning were the sixth day.

And on the seventh day, Gauss ended his complete works,
And Gauss blessed the seventh day and sactified it,
because henceforth all the students would do the work.

****

Overview of the 1979 and 1980 Summer Institutes

The Summer of 1979

Several groups of people were involved in the 1979 program (1) first year participants, (2) second year participants/counsellors, and (3) faculty.

A. FIRST YEAR PARTICIPANTS

The subjects of this study were the first year participants. In June of 1979, twenty young people started at The Ohio State University as first year participants in the 8 week summer program in mathematics for high ability high school children. Seven were non-resident student participants who lived and studied at home in Columbus, Ohio. They commuted to campus to attend lectures and seminars. Thirteen were resident students who lived and studied in a dormitory on the Ohio State campus. They ranged in age from fourteen to eighteen.

In the spring of 1979, a summer institute for gifted high school children in mathematics at Hampshire College was already in it's selection stage of operation when David Kelly, director of that program, made contact with Ross. The Hampshire program had more very good applicants than available places at their summer institute. Kelly agreed with Ross to keep the <u>older</u> applicants and send the <u>younger</u> applicants to the newly revived summer institute at The Ohio State University. There were six participants at the Ohio State Summer Institute who were referred by the Hampshire program. These students had already been slected through the Hampshire program's selection system and were recommended to The Ohio State University summer institute. All but one were fifteen years of age. The sixth student recommended by the Hampshire Program was a late applicant who was seventeen years old.

The other fourteen first year applicants to the summer institute of 1979 were chosen mostly on the basis of recommendations from teachers, university colleagues, and other interested parties, since sudden reorganization and change of location of the Ross program did not leav e time for a long questionnaire, or submission of a set of problem solutions as in previous years. First year participants came from high schools in New York, New Jersey, Illinois, Pennsylvania, and Ohio. Ten were sophomores, seven were juniors, two were seniors, and one was in the eighth grade. All the participants were judged to be of equivalent experience, except for the one eighth grade commuter who was supposed to attend the summer institute for only two or three weeks, and one participant who had attended the summer mathematics program at Hampshire the summer before. The participants' background ranged from courses in Algebra I and II (except the eighth grader who had only Algebra I) to courses on mathematical topics on the same level as Algebra II (for instance, Trigonometry, Analystical Geometry, Probability). Some students had taken an introductory year of calculus, but were still judged by Ross to be of equivalent experience.

B. LECTURES

Although the last minute reorganization and shift of location caused the selection process of participants to be abbreviated, it did not significantly modify the content of the Ross summer program. First year students studied topics from number theory during eight weeks of daily presentations lasting approximately one hour each day. Topics presented in number theory lectures during the summer of 1979 (as described in Ross' article, Talent Search and Development), were:

> Euclid's algorithm, continued fractions and their uses in constructing algorithms for the solution of linear Diaphantine equations in Euclidean rings; divisibility theory; residue class rings $\mathbb{Z}_m$ and in particular the structure of the group of units

> in $\mathbb{Z}_m$; quadratic congruences, including the law of reciprocity; basic properties of arithmetic functions, including the Moebius inversion formula and some questions regarding $\pi(x)$. A reasonably detailed study was made of other arithmetics such as the arithmetic of polynomials, the domain of Gaussian integers and some other quadratic domains, Euclidean or not; finite fields appear as residue class rings in $\mathbb{Z}_p[x]$; the Pell equation and questions of best approximation arising in the task of approximating irrationals by rationals; construction of the group of units in the arithmetic of some real quadratic fields; geometric methods; Minkowski's theorem on lattice points and symmetric convex bodies; representation of integers by the sum of two and by the sum of four squares; Diophantine equations.
>
> The students were expected to make use of relationships between arithmetic on the one hand and algebra, geometry and analysis on the other.
>
> The participant was provided with a very large number of problems, many of which called for considerable ingenuity. Extension of some basic results in the arithmetic in $\mathbb{Z}$ to polynomials and to some quadratic fields was undertaken as an exercise of insights and methods put together under the title 'techniques of generalization'. Whenever possible, constructive methods were used, and the student was encouraged to master available algorithms. (Ross, 1978, p4)

Every morning at 9 a.m. Monday through Friday, all mathematics summer program participants in 1979 attended one hour of number theory lecture given by Professor Ross. The 9 a.m. time was set by Ross to ensure an early rise for all students. He said the early morning lecture had to be set for the benefit of the resident dormitory students so they would not stay up all night. He believed the young program participants were not experienced enough to make good and efficient use of those night hours and they would be too tired to get any work done if they were left to make up their own sleeping schedule. Students knew they had to get up for a 9 a.m. class so they managed to get some sleep before morning class.

Ross did not give an expository type of lecture. He lectured in a discovery mode. Lectures sometimes opened with one or more numerical examples and other times they would chase around a seemingly nebulous (to the inexperienced number theory student) conjecture to be "discov ered" by the first year participant during or after class. The students were expected to participate in lectures, to complete sentences, to answer questions, to give conjectures, to relate (out loud) new or old concepts to the current lecture, to suggest new directions, to carry out calculations in unison, to point out errors (often deliberate) verbal or written, and to shout out encouragement whenever appropriate.

C. PROBLEM SETS

The daily lectures dropped the number theory first year students into the depths of the problem sets that were meant to take them even further than lectures into mathematical conjecture and discovery. A section of Ross' paper to the International Congress on Mathematics Education, IV in 1980 at The University of California explains the purpose of the problem sets.

> For many years we have used suitably designed problems to develop also (Cf. Arthur Engal, 1979) the students' capacity for observation, to encourage the spirit of adventure in conjecturing and to develop their staying power in the exploration of what have been to them (as well as to the historical originators) some very difficult questions. Abstract reasoning is a very important tool in creative work in all the sciences and in technology. To master the use of this tool, it is important, I believe, to give the student an

> opportunity to participate in the process of generalization. The design of problems must respond to this need as well. (Ross, 1980, p5)

The first year number theory students received one problem set daily at 10 a.m. (see appendix A for problem sets) at the end of every number theory lecture. The problem sets asked many questions and opened up pathways to conjectures. The first year students answered them <u>in writing</u>. For each problem set received, the first year number theory student was required to give one set of solutions to his/her assigned counsellor.

D. TESTS

Three tests and three sample tests were written by Professor Ross in 1979 to examine the progress made by the first year participants in number theory at points of three weeks, six weeks, and eight weeks into the program. The first year number theory students took the one hour tests on the mornings of July 5, July 25, and August 8. (see appendix A for tests and test results). A sample test was distributed to each student in lecture just before each test to help organize his/her study.

E. THE ADVANCED PARTICIPANT AS (a) A STUDENT (b) A COUNSELLOR

<u>(a) The advanced participant as a student.</u> The six advanced participants lived and studied in the same dormitory with the resident first year participants. The organizational system of the mathematics

summer program was pyramidal in the sense that each of the advanced participants had been a student in the program at least one previous summer. His/her own program of mathematical development had been advanced and intensified with each successive summer. If a student made very good progress in the first summer as a number theory student, he/she was asked back a second summer and then sometimes a third summer as a student-counsellor, provided progress continued to be satisfactory. Occasionally a participant was asked back as an advanced student with no counsellor duties, although all six advanced participants in 1979 were also junior counsellors. We shall refer to them as student-counsellors. The student-counsellor was expected to participate in advanced mathematical studies planned especially for advanced participants. The seminar instructors of the advanced courses in mathematics constructed the appropriate problem sets and examinations for the advanced summer institute student. They were designed to test mathematical intuition and to extend the mathematical ideas developed in advanced lectures. The advanced participant was responsible for writing down his conjectures and problem solutions. They were graded by the seminar instructors of the summer institute advanced courses.

(b) The advanced participant as a counsellor. However, each student-counsellor was also assigned to one or more first year number theory participants and his/her major responsibility was to look after the first year participant.

In 1979 there were two counsellors and four junior counsellors. The counsellor was considered a more advanced mathematics student (and usually was somewhat older) than the junior counsellor and was therefore given more responsibility in the summer institute. That is, the counsellor was responsible for more students than the junior counsellor and had seniority. The counsellor advised the junior counsellor on all student affairs. The counsellor helped the first year number theory students understand the new mathematical concepts and helped them develop their expression of mathematical ideas. He/she conducted discussions of number theory and corrected written problem solutions to the daily assigned problem sets. The counsellor was required to grade the solutions promptly and give them back with suggestions. The counsellor coached all the first year participants , but particularly his/her assigned number theory student or students. The counsellor tried to coach away the weaknesses he/she recognized in the first year student, and Professor Ross coached the counsellor. The older and more advanced counsellor also assisted the younger and newer counsellor.

Once a week, in 1979, Professor Ross met with the group of counsellors to discuss questions about the behaviors, mathematical or otherwise, of the first year participants of the summer institute. Each counsellor gave a detailed report with examples of behavior (accented by an interesting tale describing the behavior in complete detail), and also examples of the quality of mathematical work of every student assigned to him. Guided by Ross, mathematical behaviors suggesting levels of progress were discussed informally and philosophically, but with mathematical examples when appropriate and in much detail. These discussions naturally extended to deal with appropriate responsive behaviors for the counsellor.

Three tests were given by Professor Ross in number theory during the eight week long summer program in 1979. Immediately after each test, the counsellors distributed the grading duties amongst themselves and graded all tests together so that the results would be uniform. Each of these tests was as important to the counsellor as to the first year participant taking the test. The counsellor could see the results of his/her coaching efforts and could plan a new coaching strategy, if necessary. In the weekly meeting following the test, the counsellors gave a progress report to Ross on each of their assigned students Using the results of the test, recent solution sets, and verbal contact, the counsellors made judgements.

Each number theory student was later asked to put together all the solutions he/she could work out to the test questions (i.e. rewrite the test) in a few days and give them to his/her counsellor for grading. Professor Ross looked over the tests and test rewrites carefully, to get a feel for the progress of the first year students.

F. SEMINAR

First year number theory students attended one-hour number theory problem seminars three times a week, on Monday, Wednesday, and Friday. Day and night and in-between sleep, food, lectures, and seminars, the students worked problem sets with the encouragement, hints, and sometimes necessary push from the resident counsellors. Then, in seminar, they discussed what they found to be the possible or impossible solutions to their problems. The seminar instructor's role was to be the depositor of hints and a guide, not a master of complete solutions. The seminar instructor examined the questions raised by the student and the proofs presented by the student. He probed the student's background with questions and tried to fill gaps if possible.

A group of ten first year participants was assigned to each number theory seminar. In 1979, the two seminars were conducted by Woods and Leonard, members of the mathematics faculty.

G. FACULTY OF THE 1979 SUMMER INSTITUTE

Faculty of the 1979 program were Professors Ross, Woods, Leonard, Rally, Saltzer, Yaqub and Gupta.

Professor Arnold Ross, whose research area is number theory, served as both administrator and teacher for the program. He organized the program and the faculty for the eight week program. He taught Number Theory for the first year participants, Monday through Friday, for one hour. He also taught Linear Algebra for advanced participants on Monday, Wednesday, and Friday. He made up the problem sets for Number Theory and Linear Algebra. He presided over all meetings of the counsellors (usually once a week). He kept in touch with daily events in and out of class, and he looked out for the appearance of difficulties that might affect any member of the institute.

Professor Alan Woods' research area is number theory, and he taught one Number Theory Seminar for first year participants attending

Ross' number theory lectures.

Professor Douglas Leonard's research area is combinatorics, and he taught the other Number Theory Seminar. (Professor Leonard is a graduate of one of the earlier Ross summer programs.)

The number theory seminar instructors were responsible for discussing the questions brought up in Ross' lectures and on problem sets. The seminar instructors also observed the first year participants' reaction to new ideas and their mathematical contributions in the form of proofs or questions and gave opinions to Ross concerning the quality of work in seminar of students attending their seminar.

Professor Thomas Ralley's research area is algebra, and he taught a course in Modern Algebra for advanced students, Monday, Wednesday, and Friday, for one hour. He also made up and graded problem sets for that course.

Professor Charles Saltzer's research area is applied analysis, and he taught a course in Discrete Probability for advanced participants, Monday, Wednesday, and Friday for one hour.

Professor Jil Yaqub's research area is number theory, and she taught a course, (available to advanced participants) in Projective Geometry Monday, Wednesday, and Friday, for one hour.

Professor Ram Gupta's research area is combinatorics, and he taught a course, (available to advanced participants) in Combinatorial Mathematics, Monday, Wednesday, and Friday, for one hour.

Professor Douglas Leonard made up combinatorics problem sets, and conducted a seminar supplementing lectures (Tuesday and Thursday) for summer institute participants.

The Summer of 1980

The structure of the 1980 Summer Institute was much the same as that of the 1979 Summer Institute. Ross taught Number Theory for first year participants at 9 a.m., as usual, but there were two more number theory seminars because there were thirty-one first year participants in 1980. The seminars were directed by Professors Baishanski, Woods, and Yaqub of Ohio State and by Professor Leonard of Michigan State. The Summer Institute of 1980 also had an expanded number of advanced courses taught by Professors Baishanski, Bojanic, Dowling, Mahler, Robertson, Saltzer, Yaqub, and Zassenhaus. The advanced participants (eight of whom were first year participants in 1979) were expected to attend one or more of the following courses:

A triple track experimental course in Analysis consisting of:

(1) Real Calculus (Track I MWF) taught by Professor Baishanski

(2) P-adic Calculus (Track II) taught by Professor Mahler of the Institute of Advanced Studies, A.N.U., Canberra, Australia. The course dealt with much new elementary material from the second edition of Professor Mahler's Cambridge Tract on P-adic Analysis.

(3) Analysis Seminar taught by Professor Bojanic (Tu, Th, S) where problems from both Track I and Track II were discussed and where the students could observe the similarities and the essential differences between these two important versions of the calculus.

A course in Algorithmic Mathematics (MWF) was taught by Professor Zassenhaus of The Ohio State University with special lecturers, Dr. Yun of the IBM Watson Research Laboratory and Dr. Pohst of the University of Koln (West Germany). Each lecturer discussed topics that pertained to his own special interests in the subject.

A course in Geometry was taught by Professors Yaqub and Zassenhaus of The Ohio State University.

A course in Combinatorics (Monday through Friday) taught by professors Dowling and Robertson was open to the advanced participants although this was a regular graduate mathematics course.

In the summer of 1980 there were fifteen advanced participants. Six were full time counsellors (third summer), eight were first year participants from 1979 returned as second year participants and junior counsellors, and there was one additional second year junior counsellor from Germany who had attended an Institute in Mathematics for Gifted High School Children in Germany at Gottingham University. It was taught by Professor Ross (in the Fall of 1979). In 1980, there was also a class of first quarter Ohio State mathematics graduate students (called Headstart Mathematics Graduate Students) who were required to attend Ross' number theory lecture daily and a number theory seminar taught by Woods daily. The graduate students were required to do the same problem sets and examinations as the high school participants, but they were graded by Woods.

The total number of participants in the summer institute of 1980 was larger than in 1979, but the essential character of the institute was the same.

CHAPTER IV

CASE STUDIES

Introduction

Questionnaires were at first considered for the purpose of collecting information from the summer institute participants at various points in the mathematics program. However, participants in previous summers were known to have had no interest in anyone or anything during the eight weeks program not directly concerned with the solution of their mathematical problems. For eight weeks most participants would be consumed by the mathematics. They would be a single-minded and self-centered group and would probably have little time for strangers; in particular for a stranger asking time consuming questions and carrying a questionnaire Therefore, it seemed more appropriate for the analyst to accept a counsellor role and become a part of the mathematical summer institute. She went to lectures and seminars, answered the problems on problem sets, took notes and talked to counsellors and participants about the mathematics in lectures and seminars. She spent many hours in cars, in hospital emergency rooms and in dormitory rooms with the participants. And so, naturally, observations on and interviews with the participants took place in cars, in classrooms, in dormitories and during long waits in emergency rooms, without notebook and pencil and in-between snatches of conversation about an interesting mathematical concept or an interesting piece of program gossip.

The background of all the first year participants in 1979 was judged by Ross to be about equivalent. All participants had completed grade 10 and Algebra II except for one commuter (Lorraine) who had just completed Algebra I - accelerated and was considered a special student; and four students, Bob, Sol, Al, and Benjamin, had taken a course in Calculus. One participant, (Al) had come through the Hampshire mathematics summer institute the previous year, and he was the only first year student considered to have an advantage because of more than usual mathematical experience. Six students, Mark, Sam, Jerry, Elaine, Ronald, and Benjamin had applied to the Hampshire program and had been recommended to Ross. (see chapter III, Overview A.)

From reading applications of summer institute applicants, one could see that many of the first year participants had taken competetive tests and had been in competitions and had placed rather well.

Dropouts:

> H. (grade 10 - a commuter who dropped out of the program before test 1) had an MAA score that was in place 102 of 25, 190 participants and an OTC competition score that was in place 157 of 1891 scores.

> S. (grade 12 - a dormitory student who dropped out of the program before test 1) placed tenth in the district and received an honorable mention in the state, on the Ohio Tests of Scholastic Achievement. On the National Mathematics Examination, she received the Ohio Certificate of Superior Achievement. (She ranked 296 of 25,910 students.)

Participants of one summer 1979:

> Mark (grade 10) scored a 770 on the PSAT examination, and he had the top score on the MAA test of 1500 students in his school.

Maureen (grade 10) was first in her class in the district, second for all classes, second in her class in the state and thirteenth overall in algebra on the Ohio Tests of Scholastic Achievement.

Dick (grade 11) received an honorable mention in Geometry on the Ohio Tests of Scholastic Achievement.

Ronald (grade 10) had an MAA score of 79, and a score of 78 in the Atlantic Regional Mathematics League Competition (New York).

Participants of Two Summers 1979 and 1980:

Jerry (grade 10) had an MAA score of 81 (second in his school of 800 students). He was first in his school and received an honorable mention in the state in the Second Annual New Jersey Mathematics Teachers Contest. He was in the top fifteen in the county in the Bergen County Mathematics League Competition.

Al (grade 11) placed first in the Bergen County Mathematics League Competition.

Sam (grade 10) had an MAA score of 72 and a PSAT score of 760.

Bob (grade 11) had an MAA score of 96 that was top score in his Cleveland high school, and he placed third in the Jets Mathematics Test (at Lakeland Community College).

Todd (grade 10) placed third in a regional competition in Chicago and he had the top score in the Illinois Institute of Technology Mathematics Contest. On the National Mathematics examination his score was 80 points.

Organization

The observations and interviews of the analyst were condensed into case studies.

(1) The first report consists of case studies of each of the eight participants who arrived for their first Summer Institute of mathematics in 1979 and were successful enough to be returned for a second summer in the Summer Institute of 1980 as advanced student-counsellors.

The names of these eight participants, who form the basic subjects for this study, have been changed. They will be referred to here as Elaine, Jerry, Bob, Todd, Benjamin, Sam, Al, and Rich.

Within this group there were important differences in study habits. Some students worked with others in group study. Others were "loners." So the eight two years students are described as members of subgroups based upon study patterns.

(2) Following the eight case studies above there is a briefer report on each of the eight participants who attended the Summer Institute as first year participants in 1979 and did not return in 1980. These students' names have also been changed. They will be referred to as Paul, Stella, Maureen, Mark, Ronald, Dick, Sol, and Lorraine.

Some of these students influenced, during the 1979 Summer Institute, students in group (1). Therefore, their names arise in the case studies of group (1) students.

In preparing case studies for each of these participants, eight components of a nurture process for gifted students were identified as possible foci. These nurture components were: teaching procedures, counselling techniques, problem set challenges, group interaction, living conditions, home support, changes in behavior either during or after the summer program, and books-as-resources. Chapter 5 will consider these components as they related to all participants.

Study Groupings

In the following case studies, it is often necessary to refer to clusters of students who developed special interactions and relationships for studying or working on problem sets.

These study groups occasionally changed membership on a temporary basis. Nevertheless, the following groups seemed to form during the summer of 1979 and continue in modified form during the 1980 program.

The studies will refer to names of groups such as inner circle study group, outer circle study group, loners, and commuters. These names are not meant to indicate a preferred study group, or a preferred method of study, or the approval or disapproval by the analyst of students within any grouping. These names were used by the counsellors during the summer of 1979 and referred to a physical arrangement of some kind. During a general mathematical discussion in the dormitory, a group would often form in Elaine's room with members of the inner circle arranged in an inner semi circle and with the members of the outer circle forming an outer semi circle. Occasionally one would see a "loner" join the group for a very short period of time, but habitually the loner worked alone in his room. Although the commuters did sometimes join study groups, they did not study in the dormitory for as long a period of time as the resident students.

The inner circle study group in 1979 consisted of Elaine, Jerry, Maureen, and Bob, and sometimes the counsellor, Jack. They tended to work together during the more intensive periods of study and often very late into the night. When problems really got "tough" they shut out everyone not in inner circle. They were successful at cracking the more difficult problems together.

The inner circle of students was joined by members of the outer circle in 1979 during the less intensive periods of study. They indulged in more general mathematical discussions that took place more often in the earlier evening and often formed just before examinations. The outer circle group was successful in pacing it's members with respect to the problem sets, giving an overall view of past and present problems, and indicating general directions for possible proof.

Finally, it should be noted that Todd, Sam, Benjamin, Dick and Al were not habitual members of any study group, and so they were referred to as the "loners." A conscientious and talented loner had the advantage of "doing it all himself" and becoming "independent" of everyone else.

The commuters, Paul, Stella, Rich, and Lorraine, did not form a special study group. Instead, they were distributed among the various groups already mentioned. However, their study difficulties often appeared to be associated with their being commuters rather than their talent or the way in which they studied. Therefore, the commuters had to be considered apart from the resident participants.

Two-Summer (1979 and 1980) Students

GROUP STUDY STUDENTS

First Summer, 1979 (Elaine)

Elaine is barely 17 years old. She steps off the plane from New York with jeans, long straight hair, a duffel bag, and a friendly smile. Elaine is one week late and she must spend the second week catching up. Elaine is outgoing and the other students immediately volunteer to help her catch up. An inner mathematics study group forms around Elaine. The inner circle contains the students Elaine, Jerry, Bob, and Maureen. The second year students and counsellors come and go from Elaine's room where all the overt mathematical and social activities take place.

Elaine is a hard worker and she leads much of student activity toward discussion and investigation of mathematical problems. Elaine and combinations of both inner circle and outer circle members work into the early morning hours daily. She likes to be thorough and this slows her down. She must work more hours than anyone else to cover the same problem sets. The counsellors think she writes "super good" problem set solutions and that she has no trouble grasping new ideas. The counsellors and the other students think highly of Elaine.

Elaine is active in seminar (Elaine is in Woods' seminar) and often volunteers to put solutions on the board. They are always good solutions. Elaine knows when she has a solution and she never tries to skim over difficult points or cover up holes.

On exam #1, Elaine does not do as well as everyone expects her to. Her score (106) is tenth highest among sixteen scores with a mean of 116.5. The counsellors say that she is slow and thorough and therefore does not get enough done during a time limit examination. Professor Ross says that thoroughness is no excuse for so little done on an examination. The counsellors point out that what she did on the examination was done extremely well and with full instead of partial credit. Professor Ross answers with the suggestion that the examination only shows that she does not have the mathematics at her fingertips, (possibly because she was a late arrival) and she must work a little harder.

Elaine's rewrite of test #1 is better than any other rewrite of test #1 (except for Ed's). She has received the message from Professor Ross and she acts upon it immediately. Elaine's rewrite of test solutions is easy to read because she uses ordered lemmas to make her arguments clear.

After test #1, Elaine is more "caught up in the mathematics" than she had been before. She does work even harder than before, although she still does not work quickly.

On test #2, Elaine starts working with Jack, a second year participant and part time counsellor/junior counsellor. She never lets up on the hard work. She is very tired, but she never takes the time to rest. She now occasionally skips seminar (unusual for an enthusiastic student) so that she can keep up with her work on new problem sets. She also cannot find time to come to extra seminars (attended by almost all students weak and strong) on the sample test problems.

Elaine looks worn out when the third test comes, and her score of 108 is only seventh highest out of fifteen scores with a mean of 99. The score alone is no measure of the good work she has been doing on the solution sets since test #2.

The counsellors say, "maybe Elaine works <u>too</u> hard!"

From the start, Elaine was immersed in mathematical proof, and although she tried numerical problems, she did not limit her time to numerical problems only. She tried to develop an ability to communicate her ideas, clearly and precisely. Her goals went further than the completion of problem solution sets. She always looked for the general interpretation of particular problems. To this end, she enjoyed the full advantage of the live-in dormitory system and she used the friendship of the young mathematical community to accelerate her learning process. She also took advice, support, and help from her counsellors whenever she needed it. Her fault was in her method of "keeping up" by working too late into the night and skipping classes. She lost the advantage of discussions in seminars that were on a level different from dormitory discussion.

Second Summer, 1980 (Elaine)

Elaine came back to OSU as a second year participant with eyes shining and excited about spending a second summer "doing mathematics". Reflecting upon her first summer, she was embarrassed to admit that fear was one of the emotions that kept her upright and working hard instead of flat on her back sleeping more hours. "When I arrived last summer, one week late, I took one look at the imposing figure of Professor Ross and a second look at the problem sets, and I immediately sat down to work, thinking all the time that I would never catch up."

She recalled that working on problem sets with another student like Jerry was very important for her, since they had a good exchange of ideas. They would really get going after first talking about the problems with each other. She was careful to note that even together they could not have done all the mathematics that was expected of them without the counsellors.

In 1979, books were not of much use to either Elaine or Jerry, but the counsellors filled the mathematical resouces gap. It was not that the counsellors always told her what to do; often they gave little or no information other than to tell her she could do the problem if she looked at it a certain way. Elaine pointed to a problem and said, "A counsellor telling me that I could do it on my own suddenly made the problem no longer look impossible. A stream of ideas suddenly popped into my head and I found that I could do it on my own. I would have been intimidated by many of the problems that I actually solved if I hadn't had the counsellors to lean on, and if they hadn't made me believe that I was capable of solving them."

She described her first encounter with a Ross number theory lecture as a strange experience.

"If the Rossian lecture legend were not explained to you by the counsellors, you would think that either something is wrong with you because you are supposed to understand the lecture with some kind of complete picture formed in your mind, or something is wrong with Ross because he thinks you should see a complete mathematical picture from the lecture. What you see at first, is this older man making errors upon errors, and you wonder if he really knows what he is doing. Then you realize that the Rossian Legend demands that you must participate in the correction of the errors. That is, you must shout out the corrections to the errors (which I know now he may pretend not to hear if he doesn't like or want the correction - or he may deliberately misunderstand the correction and compound the error) before the errors completely gum up the mathematical works."

She explained that working the problem sets was not only important but stimulating because she was cracking problems that she knew mathematicians (maybe even Gauss) had worked on many years before her. She tried to talk to her parents about what she was doing, but they only worried about her working too hard and wanted her to let up. This she couldn't do and she was glad that they were not there to put pressure on her to rest.

Elaine said she went home at the end of the 1979 program bubbling over with all the new mathematics she had learned. She looked around for someone to talk to - anyone - about mathematics. She found a student who had been to another program in mathematics who claimed she had studied number theory. Elaine excitedly poured out all her new thoughts about quadratic reciprocity until she noticed that the student looked uncomfortable. The other program only gave lectures, not problem sets, she was told. By December in 1979, Elaine missed doing mathematics enough to write a paper on what she called Epsilon (E is for Elaine) Functions, a function that counts the number of primes that divide an integer. She said she knew then that she wanted to return to the mathematics program for a second summer.

Elaine's high school had a special program which arranged for her to work in a microbiology laboratory during the school year. Her approach to solving problems in mathematics during the summer of 1979 transferred to her new role as microbiologist. She explained that "one day while examining the variables to figure why what I was doing wasn't working, my laboratory professor said it was the way I observed and not the results that made me a scientiest. I learned good observation, patience, and tenacity from working with the mathematics problem sets."

The professor that Elaine worked for in the microbiology lab talked to her about education. He said that education at most universities consisted mostly of lectures and they were much the same throughout the country. But occasionally one would find the exceptional professor. He then described an exceptional class in religion at Yeshiva University. The class of students studied in teams of two and discussed each line of a religious passage out loud. The partners would discuss each line and try to crack the meaning of the text in question. When they were stuck on the meaning of a particular passage they would form lines in front of the teacher. The teacher could easily have given them the answers because he knew it all; but he would send them back to the passage with more questions than he had answered. He said the meaning would come to them only if they thought through the passages on their own.

Elaine said, "This reminded me of the summer program. We had that same feeling about a deep meaning of mathematics that came only from our own hard work. Thinking and doing mathematics does not solve a particular problem; it leads to another problem and so it snowballs into many thoughts that are open ended and you never really finish thinking about what you uncover along the way."

During the second year, Elaine was a very conscientious counsellor. She took her role as teacher very seriously, and she tried to give back all the nurture that she had received. She described the following learning process as if she were seeing it for the first time. She said of her students, "It is interesting that the first year students whose papers I am grading now, express themselves intuitively in words. They write as if they were doing an essay instead of a mathematical problem.

There are lots of words and lots of good ideas, but there are great big holes in their logical argument. I do not like to discourage them from exploring the problems in their own way and they are thinking about mathematics. Eventually I hope to guide them into tighter arguments. Intuition seems to come first, followed by ideas expressed rather loosely. Expressing oneself in a mathematically tight argument seems to develop later with more experience."

In her second summer, Elaine's counsellor duties were unusually heavy. Elaine worked very hard and was very intense. She did not sleep enough. This was her second summer in the program and she wore herself out before the summer program was over, just as she had done in her first summer. Five weeks into the program she had to drop the problem seminar in p-adic analysis because she did not have the energy to continue working problems late into the night and also grade the solution sets of her first year number theory students. The problem seminar demanded more of her time than lectures and she continued to attend the lectures in p-adic analysis. She performed her counsellor duties mainly though verbal communications because she had to catch up with solution sets after she dropped her seminar. Verbal communications with first year students were faster than written communications though less detailed. Elaine found time to work in the real analysis course (with seminars) for advanced participants. She, Jerry, and Jack worked together on real analysis. All the members of that study group did a "respectable examination" and had equivalent scores in real analysis. In her second year, Elaine was at the stage where she could research her own problems, even though she had a tendency to take on more work than she could comfortably handle. She could read mathematics books and she knew enough to know what to look for. She no longer needed a counsellor to translate "symbols into plain language"; but now it was her turn to give back the help that she had enjoyed the previous year.

Elaine was accepted at Harvard as a freshman in the fall of 1980. She did not believe that the mathematical experiences of her two summers were necessarily meant to direct her into science. She saw the summer program as an extension of a "truly liberal arts" background.

Elaine, as a student of the Summer Institute in 1979 expressed her experience with a poem from the depths of her "liberal arts" background.

Summer 1979 Two Poems in a Finite series of N(1+i)

I. Number Theory, Number Theory!
Mind goes blank and head grows weary.
Wheels in brain that click and turn
In a ceaseless strain to learn.

Problem Sets, Oh Problem Sets;
Burn them with Bernd's cigarettes.
Prove the Disproof, Justify,
Sigma, z^2, 2 + i

Dr. Ross, Oh Dr. Ross,
I am no Carl Friedrich Gauss.
Eyes asleep in half-dazed wonder
Cannot sparkle at a number.

Summer 1979 Two Poems In A Finite Series of N(1+i)

II. OSU, Oh OSU!
Now that we're here, what shall we do?
Pinball, Larkins, Drackett Tower;
High Street, Bernies, waste an hour.

Hit the books-awake 'til two,
Captain Crunch will get us through
An hour of class and sleep 'til noon-
North Commons lunch comes all too soon!

Counsellors one, and Counsellors all,
Incline thine ear, oh hear my call!
When I die inscribe o'er me
Quadratic Reciprocity.
Q.E.D.

2. (Jerry) First Summer 1979

Jerry is 16 years old when he comes to the summer program from New Jersey in 1979. In observing Jerry, one tends to believe that his place of residence is not significant. Jerry could have come from any part of the country and he would have behaved in exactly the same way. He is immature and he is intimidated by the mathematics he sees. He is highly emotional and he finds it difficult to settle down to work. He has to be assured by the counsellors that he will understand the mathematics if he works at it. Elaine arrives one week late and Jerry offers to help her catch up. Actually, he derives as much comfort and help from the study relationship as Elaine does. Jerry is an instant member of the mathematics inner circle. Jerry and Todd are roommates and Todd is so competetive that he draws Jerry into a competetive relationship with him. Jerry works with the inner circle mathematics study group, and so he works at a slower pace than Todd who hardly sleeps. Jerry's problem sets are good, and the inner circle study group cracks many of the hard problems with Jerry's help. Jerry is in Woods' seminar. He participates in seminar and asks questions. He will write a complete and competent solution on the board, although with not as much detail as Elaine, when asked to do so, but unlike Todd, when the bell rings he leaves seminar immediately.

On test #1, Jerry's score (127) is sixth highest out of sixteen scores and above the mean of 116.5. Jerry's rewrite of test #1 mentions all the important ideas but not always in clear language. Occasionally a proof is incomplete.

After test #1, Jerry gains some confidence and independence and he works sometimes alone and sometimes with the group.

On test #2, Jerry keeps the same position with a score of 117 on a test with a mean of 101.5. His problem set solutions are now quite good, and no one is worried about Jerry's progress-not even Jerry. He has grown into a tranquil person as well as a confident one.

On test #3, Jerry's score (107) is eighth highest out of fifteen scores with a mean of 99. Jerry distributes his solutions equally among the numerical problems, the non-numerical problems, and the problems with extensive notation.

Jerry seemed to enjoy both the academic and the social aspects of the summer program of 1979. He worked well with his peers in small homogeneous groups and he derived great pleasure from the communal younger brother relationship he had with the counsellors. He attended all lectures and seminars and participated, if necessary. Jerry was not a work-a-holic like Todd. He worked and learned on a slow, steady, and regular schedule. He kept up with problem sets and he also managed to get enough sleep.

Second Summer, 1980 (Jerry)

Jerry was back in 1980 as a second year participant in the summer mathematics program. When I talked with him away from the other students, Jerry's first comment was, "I love my mathematics classes this summer (1980) and I would like to come back again next year." Having registered his request, and the thought uppermost in his mind, he settled down to discuss his previous summer in the mathematics program.

Jerry's first thoughts were about the metamorphosis that takes place in the summer participant from "green first year student" to "seasoned second year student." He recalled his own experience as a first year participant as he said, "I liked thinking about a problem and I liked putting all the symbols down on a blank piece of paper and moving them around until they made sense. I really enjoyed learning to use the mathematical symbols to express myself, and when I was done it gave me great pleasure to end my work with a QED. Although I was on my school's mathematics team and I had prepared for competitions, I hadn't had contact with many of the symbols, and I didn't have the experience of writing down mathematical ideas for someone else to read. No one told me 'this is how you write mathematics.' In the summer math program, you wrote and you wrote and you wrote, and someone had to read it to make sense of it. From the many discussions and comments that came back from your counsellors on your problem solutions and tests, you learned how to comunicate."

He described the learning process in mathematics as both a visual and physical contact with words and symbols. He said, "Now, if I hear a lecture or I read some mathematics, I write it down so I can say - Oh! - that's what it means. I can follow an argument when I think about how I would go about explaining it myself."

Although Jerry worked in a group when he was a first year number theory student, he thought a first year student learned in a group only if the student was "active" in the group. "This summer, I had to keep my first year student in my room while I was working. She was working with other students in a passive way. She would listen to their explanations and copy them down. You can only work well with other students when they are working on the same level and they have the same experience as yourself. My student was much 'greener' than the students she chose to work with and she could not learn enough from the group discussion to make it worthwhile. I see that she works, but I do not show her how to do the problems. She is improving and she is becoming more independent."

He thought there was a point past which one no longer needed a counsellor because there were books one could read, and one could conduct his own research. He admitted that this would not have been possible for him last year in '79. He reflected on this by commenting, "I am just learning how to read books. Although I can figure many of the mathematics problems on my own if I write them out and think about them, some of the mathematics is getting too complicated to do without expert help. I didn't read mathematics books at all last year except when a problem involved a reading search."

He thought his 1979-1980 high school year was "O.K." although it was not very stimulating. He expanded the thought by saying, "Last year I went back to my high school and took calculus. I wish that I had learned more about limits and ε, δ proofs. My teacher taught us how to take derivatives, but we did not study the general ideas behind derivatives and we never proved anything. I took the SAT examination and I scored 600 in English and 760 in mathematics. I will be a senior next year and I will take the SAT examination again. I will try for 800 in mathematics and I suppose I will also need to work on my English.

Jerry took p'adic analysis and real analysis classes in the summer of 1980. Again that summer, Jerry often worked with the small homogeneous group made up of Elaine, Jack (a third year participant) and himself. Jerry wrote a "respectable examination" (said his seminar instructor) in both courses. Jerry was also a very good counsellor.

Jerry won honors for a mathematics paper submitted to the 1981 Science Talent Search for the Westinghouse Science Scholarships and Awards. Jerry also has been awarded a $5800 a year scholarship to Washington University starting in the Fall of 1981.

3. (Bob) First Summer 1979

Bob is 17 years old when he comes to the Summer Institute from Cleveland, Ohio. He has had some mathematical experience with Number

Theory. Bob's high school offered a course in Number Theory in the previous school year. He loses no time and he starts working very hard from the very first day. It takes Bob three weeks to discover the library to study in and Maureen (see Maureen's case study in the One Summer Only Students section) to study with, the two divergent forces of his summer. Maureen is floundering and he knows it, and he comes to her rescue with notebook and pencil in hand. This cooperation is beneficial to Maureen. At the beginning, the extra time spent with Maureen is learning time for both of them and teaching Maureen how to speak, read, and write number theory is constructive work for Bob. But, Maureen also gives Bob something-she gives him instant entrance to the inner circle study group at the dormitory. Bob and Maureen often like to work as part of the inner circle study group. Bob not only works on problem sets with the other students of the inner circle and the counsellors, he is also one of the few first year students who has enough background to try to read mathematics books on his own (with very limited success). When the day of the first test arrives, Bob is certainly ready for it and he achieves the highest score of all on test #1. The view from the top of the mountain is disturbing to Bob because there is nowhere to go (he thinks). Bob immediately goes into a tail spin and doesn't even bother to rewrite well, the parts of the examination he didn't complete. (Examinations were never completed, even by the top scorers.) His proofs on the test rewrite are sketchy and vague. At best they are pointed in the right direction and at worst not even clear enough to indicate knowledge of the sketch of a correct proof. This is considered a very bad sign by his counsellor. His counsellor says Bob is disillusioned because he did so much better than anyone else on the first test. Bob feels there is little competition from the other participants. Bob even stops going to his seminar immediately after test #1. He no longer works hard on problem sets after the first test. His problem sets are "O.K.," but counsellors think they should be better. Bob gets restless enough to think about going home on the fourth weekend into the program. He is talked out of making the trip home by his counsellor. Bob becomes very restless, but his working relationship with Maureen and inner circle keep both him and Maureen learning mathematics. Bob and Maureen start working harder just before test #2, although Bob never again shows as much interest in mathematics as he did before test #1. Just before test 2, Bob attends seminar again, but only as a spectator, since he does not contribute to seminar discussion as he did before test #1. Bob was the main contributor to mathematical discussions in seminar before test #1. In 1979, there were two seminars, one taught by Woods and one taught by Leonard. Bob's seminar (taught by Leonard) contains mostly commuters to the program. Bob, Ronald, and Dick are the only resident participants in the seminar. The seminar never produces any sparks, only an occasional question already seen by Bob at least a few hours before the seminar focuses on it. This confirms his feeling of lack of competition from other participants. On test #2, Bob's score slips to fifth place.

Bob is doing "well enough" without a great deal of effort. Bob's seminar attendance continues to be sporadic. However, the spirit and comeraderie of the inner circle dormitory study group keep him operational if not at top performance. When test #3 comes around, Bob places fourth among fifteen students. This means that Bob is holding his position from test #2. Bob and Elaine have interchanged places on test #3, but not because Bob is learning more, but because Elaine is overtired. Bob is comfortable riding the study waves made by the inner circle dormitory study group and Elaine. He is a rider not a driver. Elaine's solutions on problem sets are better than Bob's. He is a good listener, he enjoys the ride and he seems to have no more interest in driving. Bob came to the program well prepared and the study group helps to carry him scorewise over Elaine. Bob and Maureen ride with the dormitory inner circle study group to the end of the summer and Bob maintains his position (as fourth highest scorer) to the finish, but he never does the most difficult questions on the problem sets.

(Bob) Second Summer, 1980

Bob was not scheduled to come back in the summer of 1980. Because Bob was to enter Dartmouth College in the fall of 1980, he had agreed with his parents that he should work that summer and earn money for school expenses. Bob left home without telling his parents he was going to Columbus. He visited the summer institute on its orientation day, stayed another day to visit classes and then found himself sucked in by the force of the old friendships and the feeling of comeraderie. He decided to stay on with only the clothes on his back and no money. Because his previous summer's record was good and his enthusiasm appeared to be interest for learning more mathematics, Bob's parents were contacted and it was agreed that he could stay on as a second year participant and part-time counsellor in the summer institute.

This was a good beginning, but the story becomes complicated by the fact that Bob's interest, upon closer inspection, appears to be in becoming a part of the dormitory group again. It appears that Bob's lack of dedication to mathematics observed at the end of the first summer carries into the second summer. The mathematics of the second year requires much more individual dedication and talent than that of the first summer, and riding the waves made by a second year study group would not be possible without a large amount of individual effort on the part of each member of the study group. Bob would have to immediately decide to drive hard to survive the second year with accomplishment. Bob appears to know this when he voices his good intentions and complains about the first year students who are "uncommitted" to the task of learning number theory and who don't study "hard enough." He seems to remember well the first four weeks of his strong committment to the study of mathematics in the summer of 1979, but he seems unable to recapture that feeling again. However, Bob becomes more interested in learning more number theory the "second time around" and he becomes more interested in increasing the efforts of the first year participants in number theory.

His first test score is 15 in p-adic analysis and this is low enough to completely "turn off" his efforts in the direction of advanced studies in mathematics. Bob stops attending all advanced courses, but still does not appear to be unhappy. Instead he shows more enthusiasm for his counsellor duties and he puts more effort into "guiding" students through the first year number theory which he is making effort to learn in "more depth."

The following comments are Bob's impressions of the summer institute:

> "Last year I did not attend seminars at the beginning of the program because there were mostly commuters in my seminar and they were behind on the problem sets and the questions would center around problems that I had already done. Sometimes there would be questions on the problem set handed out that very morning and I wanted more time to think about those problems before discussing them. I had experience with very elementary number theory before coming to the program last summer (1979) and so I was able to do the first few weeks of problem sets without seminar if I worked hard at it. I did not even realize the value of the counsellors, although they were there when I needed help or advice. Somehow, I expected most of the other students to be super bright and better than me right from the beginning. When I topped the first test, it was a shock to me, and I felt almost disappointed, and then I met Maureen. We worked together but not always."

> "It is surprising how different the lectures look to me this year, Ross' lectures last year seemed disconnected and different from the problem sets. I often did not see where they were going. Now, I see they are filled with many different roads going in different directions and the listener must work to find his own direction, and travel down the road a long way before it makes sense to him. The counsellor can be very useful in pointing out some directions, but the counsellor cannot travel the road for the student. I couldn't read mathematics books last year. I wouldn't have known where to look and what to look for and I'm not sure I would have been able to understand the connections anyway. The books are hard to read because they are written in an abstract general form. For instance, last year I looked up the Chinese Remainder Theorem, but I couldn't understand the two pages of general theory surrounding the theorem. I asked John (counsellor) and he said, 'look at it this way; you want these two numbers to be 0 with respect to the first modulus and those second two numbers to be 0 with the respect to the second modulus ... etc.', then it was clear what had to be done and I did it!"

"Lectures might be easier to follow if there was only one road and one topic with everything pointing in one direction, but I am not sure that you would call that discovery. Maybe first year students need to find their own way and maybe it takes a year to see where they have been."

"I took calculus in high school last winter and it was easy. I finished all my school had to offer in mathematics by the end of the first quarter. I took the SAT exams, and I scored 780 on the math part."

Inner Group Study Students

Summary

The four students, Elaine, Jerry, Bob and Maureen were the "Group Study Students". Elaine and Jerry worked together both in their first summer in 1979 as number theory students and again in their second summer of 1980 as second year participants. They attended the same classes and they worked on the same problems at the same time so that it was almost impossible in the second year to separate the quality of their work. Although they were very different in personality, they agreed on many things including the importance of the continuation of the program and the counsellors. Elaine, Jerry, Jack (a third year participant) Todd, and Dieter (a second year student from the Ross program in Germany), were staples of the 1980 program and provided a tradition and stability in counselling.

Bob was also a "Group Study Student" but for one summer only. Bob's friendship and work with the inner circle study group of 1979 (that included Elaine, Jerry, Bob, and Maureen) appeared to be an important part of Bob's nurture. In 1980, Bob did not study with the same group or (as it seemed) very much alone, because Maureen (see case study in one summer 1979 only students) was no longer at the institute and his interest turned to counselling and away from advanced study.

LONERS

1. (Todd) First Summer, 1979

Commandment 4. Forget about the sabbath day; seven days thou shalt labor and do all thy number theory.

Commandment 5. Honor thy counsellors and seminar instructors that thy days be long.

Commandment 6. Thou shalt not sleep.

Commandment 7. Thou shalt not covet thy neighbor's problem sets, nor his axioms, nor his proofs, nor anything else that is thy neighbor's.

No student lived by these program commandments more completely than Todd. He is from Chicago, Illinois; he is 15 years old, and he looks like a young fifteen both physically and socially. Todd's beginning proofs show he is keen but immature. Todd is so keen that he is not shy about leaning on everyone and anyone to extract solutions to problems on problem sets. Seminar instructors, counsellors, books, and even other students are of interest to Todd. Anyone for some mathematical conversation and a hint! He is extremely competetive and his burning ambition is to learn the solution to every problem on every problem set. He thinks it would be "great fun" to come back next year as a second year student and junior counsellor. He understands that some solutions will not come easy and he is willing to work for those solutions both day and night.

During the first three weeks, Todd talks about his existence outside the "inner mathematics circle" formed at the dormitory as a study group. He says they ignore him! He appears to be jealous of the close relationship among members of the inner circle. But, Todd's ambition and competetiveness allow him to become independent of that study group without becoming isolated.

He has Jerry, his roommate, wake him up at 2 A.M. when everyone else (except one of the counsellors) goes to bed. He often works through the night. Although Todd feels himself outside the inner and outer math study group, this does not stop him from hanging around long enough to absorb some support and information. He then hides out in his room and works out his problem solutions alone. He is regarded as somewhat of a pest by the other students, because he puts his solutions together after talking to everyone else and then he hides them under the seventh commandment. His problem sets are "very good" and he has managed to crack some of the more difficult problems on his own.

When test #1 arrives, Todd has almost everything he has worked on memorized. Todd's score (171) is third highest among sixteen scores with a mean of 116.5. On the examination, he successfully answers both numerical and non-numerical problems.

The results of the examination excite him and he works even harder. His rewrite of test #1 is good, although he is still not clear on some details of the use of the well ordering principle in a proof.

After the examination and four weeks into the program, Todd starts working with Richard. Rich is a commuter and can only work with Todd a few hours during the day when he is not in class. No one knows exactly what conversation goes between them, but the mathematical results of this union are written down only by Todd.

Todd is now the most active student in Woods' seminar. He listens carefully to every word mentioned in seminar, and he stays after seminar to ask his own special questions. In fact, it is hard to get him to break away from seminar. He stays and talks and talks and talks... and he follows the seminar instructor down the stairs talking mathematics and related topics all the way down.

On test #2, his score (197) is second highest among sixteen scores with a mean of 101.5. Todd answers numerical problems, but also does many proofs using involved notation on the examination. Although he may depend somewhat on memory, he must also understand the problem well to handle the complicated notation on a time limit examination. He says that his association with Rich has really helped improve both his communication of mathematical concepts and his understanding of the concepts.

Todd is even more excited about the results of the second examination and he is also more frustrated than before by anything he doesn't understand. Todd finds solutions to almost all the problems except the most difficult ones. He cannot accept the possibility that any problem on any set is beyond his comprehension. In seminar, he may ask for an explanation over and over again of the same problem, until he thinks he can handle it. If he can't work a problem at home alone, he comes back again with a different set of questions about the same problem. He may be frustrated by a problem, but he refuses to let go. He says he will complete the problems in his folder during the winter if he must leave with an incomplete problem.

Todd now works in the daytime with Rich and in the night alone. He wants to be #1 but he cannot catch up to Ed who has had the experience of one previous summer in a mathematics program and who is also working hard.

On test #3, Todd maintains his second place position with a score of 174 points out of fifteen scores with a mean of 99.

Todd is a student who engaged in mathematical discussion on each and every level available-from the talks with other first year participants to discussion with his professor after seminar. He was a loner, but he used this to his advantage because he rearranged his sleep schedule from 8 P.M. to 2 A.M. so he could use his time and energy more efficiently. He socialized only when it was convenient for his work schedule. He enjoyed competition and his competetive spirit made him work hard enough to win a personal victory.

<u>(Todd) Second Summer, 1980</u>

Todd returned to the program in 1980 for a second summer. Todd has matured. The "pest" image has disappeared. In its place and in spite of his young age, Todd in 1980 was a dependable junior counsellor. The first year students could see that Todd knew number theory well, and more importantly, he could communicate easily with them. He was sensitive to the students' learning problems and enjoyed working with them. He said that the summer of 1979 changed his life. When he was asked to think back about what was important to him in the summer of 1979, he first recalled that the counsellors were invaluable. "Many definitions and some mathematics could not be found in books," he said, "but the counsellors were always there to fill in the holes, or to tell you that with more work you could fill in the holes yourself. There were always a lot of holes in Ross' lecture and we had to find the answers ourselves (the first year student). I liked that. In seminar, Woods gave many answers - but he always managed to get tangled up in some discussion at least once every seminar so that we would go back and try to work it out. That was fun! The counsellors were around to encourage you to work harder, give you hints, or let you know that the problem was trivial. They graded the problem sets and made it quite clear what they did or didn't think of your work."

Todd comments that nine second year students in the summer program of 1980 must serve as part-time counsellors in addition to attending courses in mathematics. He says it is a heavy load but he "enjoys it". He behaves as if he is very proud of his counsellor status. He reminds me, "It is not easy to be a counsellor at 16. You have to know a lot more than the students in number theory." He goes on to describe his many duties as counsellor. "I make sure my students do their work", he continues, "you know we (counsellors) are responsible for grading solution sets from our first year number theory students." Todd tells me he wrote the comment "garbage soup" behind a nonsense statement on the solution set his first year charge had just given him.

Todd wanted him to know that he would not tolerate a nonsense statement or a wishy-washy solution.

Todd said that in his first year he liked to work with other students because "sometimes they see things from a different point of view". He said talking about problems made him think about mathematics surrounding the problems even when solutions were not forthcoming from the discussions. Todd pointed out that the number of first year participants in 1979 was small. Todd supposed that a larger group might not have the same feeling of comraderie. However, when he thought about it some more he said he didn't see why a large group of first year participants couldn't break up into smaller groups and work together with the same feeling of comraderie.

Todd said he liked competition almost as much as working with other students. He found it "stimulating," but thought other students did not.

Todd's high school allowed him to take an honors calculus course at The University of Chicago during the 79-80 academic year. He did not have calculus in high school previously like many of the other students in his class. Many of the students in this calculus class complained about the rigor, but he thought that was the best part of the course. After the rigor he had been subjected to in number theory, the calculus presented no problems he could not solve with a little bit of work.

Immediately after the summer 1980 number theory test #2, it was clear that one of Todd's students needed help. He felt his odd sleeping arrangement was in conflict with her working schedule. He felt so responsible for her bad performance that he changed his working schedule to accommodate her schedule. Todd's work suffered. It was not worth the exchange because Todd lost more than his first year charge gained. He attended both real analysis and p-adic analysis. He wrote "very good" examinations in both classes but his professors thought he was capable of even more.

Todd had a scholarship to Swarthmore College for the fall of 1981.

2. (Benjamin) First Summer, 1979

Benjamin is from a suburb of Chicago. He is fifteen years old when he first enters the program. Upon first meeting Benjamin, one gets the impression that he is a very shy, quiet, and withdrawn young man. He is shy, you are told, because he has a communication problem. He is a twin and his twin sister is reputed to be very good at the communication skills. Benjamin has been placed at about a fourth grade level of competency in reading, writing, and speaking. He is in a special school and has been tested and retested by educational psychologists because of his "communication problems." However, it is noted that his mathematical skills are better than average. Since Benjamin's twin is reputed to be "not very good at mathematics," Benjamin's interest in mathematics gives him some prestige in the family. Ben speaks with a heavy lisp and leaves out every other word in a sentence so that it sounds like a sentence put together by a very young child just learning how to talk. To understand him, you must first get used to the lisp and then learn to fill in the missing words.

Benjamin was discovered by Dr. Boas, Professor of Mathematics at Northwestern University. Professor Boas said that Benjamin just "turned up" in the winter quarter of 1979 at the mathematics department of Northwestern University and asked to take an Advanced Placement test for a calculus course he studied on his own. He did reasonably well on that test, and in the spring quarter he was allowed to sit in another calculus course, and did well in that class. Benjamin was immediately accepted into the summer institute of 1979 upon receipt of a letter and personal recommendation from Boas to Ross.

Benjamin is carefully protected at home, but it is decided that Benjamin should not be protected from mathematics and he should not be treated differently from any other participant in the mathematics summer program.

At the beginning of the summer in 1979, Benjamin hardly speaks, and when he does it is hard to understand him. At the dormitory, the counsellors make sure that Benjamin is working and they ask for and receive written problem solution sets from him. Benjamin is locked up in his room most of the time. The problem sets from Benjamin are very hard to read because of his poor sentence structure, grammar, and spelling. With great effort, Benjamin's mathematical ideas become discernible. The problem sets open the door to discussion and some necessary verbal explanations concerning his written words. Benjamin cannot speak clearly, and the counsellors must listen very carefully to understand what Benjamin is saying. For instance, if he tells them the proof of a certain theorem is trivial, he will say, "ith trifial

It is hard to tell which comes first. The counsellors' increased ability to communicate with Benjamin or Benjamin's increased ability to communicate with the counsellors. In any case, Benjamin starts to talk not only to the counsellors, but to everyone else involved in the summer program. After the initial strangeness of Benjamin's speech patterns, everyone becomes used to his use of language. The counsellors say if they don't count the spelling or grammar and odd use of language the "mathematics in his problem sets is good."

Benjamin complains about how hard he has to work, but he works hard anyway and appears to enjoy it. He begins to ask questions in seminar and he laughs at the mathematical jokes. He still will not write a solution on the board in seminar, but he sometimes volunteers a verbal solution. If one listens carefully, the solution is all there in "Ben's language."

On test #1, Benjamin's score (164) is fourth highest among sixteen scores with a mean of 116.5. He now is acknowledged as a strong member of the student group and this makes communication easier between him and the other young participants. The conversations between Benjamin and Todd before and after seminar make it clear that they are now competing mathematically with each other.

Benjamin never did become part of the mathematics inner or outer circle that summer of 1979, but he was communicating with some of the other students.

Six weeks into the program, the counsellors note, "a real improvement in his capacity to communicate." Benjamin still says he would not have come if he knew how hard he would have to work, but he always says it with a smile and it is clear he is proud of his mathematical ability and he wants the counsellors to recognize this by assuring him his hard work is worth the pain.

Benjamin, Todd, and Sam talk to each other. It is interesting to note that Benjamin, Todd, and Sam, the three hard working students outside the inner and outer mathematical study groups have formed a spontaneous support group. Without realizing it, they derive mutual support and respect from their spontaneous mathematical discussions. They are all in competition with each other and they are all basically "loners" who work alone at night.

Benjamin talks about his communication problem and says that he wishes he was "better at English" and he wants to learn more about it. He is no longer embarrassed by his obvious handicap in English expression when he is among the mathematics participants.

Benjamin, like Todd, still has some vague ideas concerning the use of the well ordering property or induction to produce a rigorous proof. However, his latest solution sets show increased understanding of many of the new mathematical concepts and their proofs. He is now doing the non-numerical problems and many of the ingenuity problems. Benjamin can, for instance, handle the more complicated questions on order in a finite commutative group. Benjamin has also learned to work well with Gaussian Integers. Benjamin's score on the second test (156) is fourth highest out of sixteen scores with a mean of 101.5, and it reflects his increased ability to communicate his mathematical ideas.

Benjamin's score (148) on test #3 is third highest out of fifteen scores with a mean of 99, and he is even talking about coming back next summer.

It appears that Benjamin needed not to be treated in a special way. The counsellors demanded that he communicate his solutions daily, both verbally and in writing. His written solution sets, like every other participant's solution sets, were graded and returned with corrections and comments. This forced him to fraternize mathematically and the social fraternizing became a by-product of the "good mathematician" standing he earned from his fast developing communication skills.

After the summer of 1979, Benjamin went back home to live in a suburb of Chicago. Benjamin was a special student in high school because of his communication problems, but he had a very successful summer in mathematics at O.S.U. and he was allowed to take three mathematics courses at Northwestern University as a non-credit special student during the 1979-1980 academic year. He earned all A's and one B in 3 quarters of topology, linear algebra and complex analysis, all three quarter courses that require calculus as a prerequisite.

Jack, one of the summer '79 counsellors, lived in the same area as Benjamin and was in contact with him throughout the 1979-1980 academic year. He said "Benjamin ate up mathematics books like they were hamburgers."

<u>(Benjamin) Second Summer, 1980</u>

Benjamin came back to O.S.U. as a second year student in the 1980 summer mathematics program for high ability secondary school children. In his second summer at the dormitory, Ben roomed with Al. They lived together in a <u>mess</u> of papers and stacks of books scattered around their room. They were roommates but they were not friends. They studied independently but often in the same room.

Todd said, "Last year Benjamin stayed in his room most of the time. This year he walks around the halls of the dormitory asking the first year students why they are not busier doing their number theory and he generally nags them to work if they look like they are goofing off. He likes to tell the second year students how easy things are that they are struggling with. He is usually right, although sometimes he will let everyone struggle rather than show them the solution and 'spoil their fun'."

In the Real Analysis class for advanced participants, Ben would very often have a solution or a counter-example to problems posed in class, even when the intuition of most other students broke down.

Elaine said, "Benjamin's intuition is different from ours because he works from a background of tons of examples in his head."

Todd told the following story about Ben's experience as an advanced participant in 1980,

"At the beginning of the summer program in Real Analysis, B. was worried about overfacing us (high school students). He threw out five or six little problems to test us, and Ben got every one of them in a few seconds, before B. could write them on the board. It just blew his mind!"

Again, it was decided by Professor Ross that in the 1980 summer program, Benjamin must not be treated differently from any other participant. He had to pull his weight as a second year student and part-time counsellor, like every other second year participant. He was still hard to understand unless one made great effort to interpret his odd speech patterns. Now all the new summer students he was counselling also had to get used to him. Ironically, the first year student assigned to him gave him pages and pages of solutions in essay form for every problem on every problem set. Benjamin struggled to read all the wordy solutions, a strain for any counsellor, but particularly for Benjamin who did not read words very fast. The mathematics was no problem for Benjamin, and he rose above the words to the mathematical ideas.

Ben had no problem in 1980 reading books containing difficult and generalized mathematical ideas. Benjamin said of his first summer in the program, "I had books, but I only used them to look up something like a definition. I didn't read the proofs, I did those myself. I didn't need books in number theory. I didn't like to read. Now I read my student's papers. He used to write six pages on every problem, but the last set had six problems on a page. I'm really proud of him! The mathematics was mostly O.K.... and nice and short."

Benjamin made it clear by his attitude toward the first year students that he believed writing down mathematics during long hard hours at work was the key to learning. "I have a new student, Rich. He should be a second year student, but he doesn't work hard enough. I worked eight hours a day last summer, and that wasn't hard enough. He doesn't work more than four hours a day. He doesn't write enough down. Maybe twelve hours a day would be enough; three hours to eat, three hours of class, six hours of sleep. Maybe he doesn't need all that sleep because fifteen hours of work might catch him up to number theory."

Benjamin did not believe that first year students needed to work in groups, although he admitted that working in a homogeneous group was sometimes fruitful. "Groups are not always good for students. Often there is someone who is ahead of everyone else, and he does the work. One student has all the ideas and everyone else listens. If a student can find one or two students who work at the same level as himself, then it is good for everyone in that group and more mathematics gets done than if each worked alone but not 2^p times as much (you know p = the # of people in the group, $p \neq 1$). I couldn't find any group to work with last summer, although I would sometimes talk to my counsellor if I could find him. Sometimes I would talk with other students but wouldn't work with them. I could ask questions in seminar if I had any. I worked alone in my room."

Even in the second summer of 1980, Benjamin was not totally free from communication difficulties in mathematics. One example in particular showed that Benjamin still needed careful nurturing. In the summer of 1980, he attended a graduate course in p-adic analysis with many of the other advanced participants. Two tests were given to students in p-adic analysis so each student could have two chances, if necessary, to show what he/she knew about p-adic analysis. Benjamin was fifteen minutes late to the first two hour test (he overslept). He was upset and he had trouble writing solutions. His performance was below average. After a talk with Professor Ross, he worked harder for the second makeup test, and he wrote a completely correct set of solutions to the second test. He scored 100 points out of a possible 100 points.

Benjamin was recognized as a truly gifted student by all five of the professors who taught him mathematics in the summer of 1980. On the strength of his performance, he was accepted by The University of Illinois, at seventeen years of age, as a non-degree graduate student and he was allowed to take graduate mathematics courses for credit in the Spring of 1981. At the end of the Spring quarter at the University of Illinois, Benjamin was allowed to take the Ph.D. qualifying examination in Mathematics. Ben placed in the top third of the eighteen students who passed that qualifying examination.

3. (Sam) First Summer, 1979

Sam is from New York City, where he attends a "good" city high school. Sam, 15 years old, is short overweight, wears glasses, and has the look of a bright student who lives in his own little world. At first, Sam is not very willing to work. He dwells on a rejection (because of his age) from another mathematics summer program for high school students. He imagines that the other program would have been easier and more fun. The O.S.U. program was his second choice and he lets everyone know about it.

He is not in the inner mathematics circle or the outer circle and no one wants to work with Sam because he is so moody. He tells bad jokes and he doesn't take baths. The other participants ignore him and he sulks. He becomes absorbed with the computer, he plays computer games and he spends much of his time away from the dormitory and at the computer center. He says he thinks that the computer takes on the personality of the person who programs it. He talks about the computer like it is a good friend. Sam also talks about his not being a part of the mathematics inner group. He feels isolated and it bothers him. At the dormitory he works alone, and he never seems to get past the numerical problems on each problem set because he gets frustrated easily. He tries to get information from different people, but he doesn't know how to do this without annoying the other students. They always end up telling him to "bug off" before he makes any progress with his "questions." In self defense, he insults other students which doesn't make them like him any better.

Sam is a willing participant in Woods' seminar, and he obviously enjoys it. His solutions show he understands the problems he chooses to talk about in seminar. However, he makes it clear that he only likes to talk about problems he knows something about. It appears that to Sam, "the competition is the whole thing." In any case, he is beginning to talk about problems other than numericals and he is making some progress.

On test #1, his score (121) is eighth highest among sixteen scores with a mean of 116.5. On his rewrite of test #1, he proves one of the problems using "Induction" where everyone else uses "The Well Ordering Principle." The counsellors do not like to have the first year students use induction (which is why he uses it). A reader must read through his whole proof to decide what his induction hypothesis is. The general idea of proof is there, but his induction hypothesis is confused and the reader must rewrite it.

Six weeks into the program, and Sam spends less time at the computer center. He continues to work alone, but the other students are more tolerant of him and they accept him as "just Sam" although he still likes to pick arguments. He no longer complains about his rejection from the "other program." Sam and Todd compete in seminar and he also begins to hang around after seminar with Todd.

On test #2, Sam's score (112) is seventh highest among sixteen scores with a mean of 101.5. He answered mostly numerical problems although he did do three other problems rather well. Sam writes in tiny little boxes like he is saving paper and giving the reader a little summary snapshot of a solution. Sam has good intuition and good ideas, but he tries to put it all in a 2" by 3" box, and if it doesn't fit, he doesn't write it down.

Immediately after the second test, first year number theory students are paired off one above the median with one below the median. Sam (above the median) is paired off with another student (Dick) who works alone, but Sam will not cooperate. He says he gets more work done by himself and he goes his own way.

On the seventh week, Sam decides he has done enough work, and he stops working. He says he is happy with his progress and the counsellors say "he has made progress and he has gained maybe more than Todd."

Sam's score (89) on the third test is ninth highest out of fifteen scores, but this time it is below the mean of 99. This is not a surprise, since the mathematics part of the program ended for Sam over one week earlier.

In the last week of the program, the other students are very tolerant of Sam, and they allow him a part in their non-mathematical activities. The counsellors say all the first year participants believe they have earned their membership into the "Ross Summer Institute Brotherhood," and Sam is afterall a "brother." Sam takes advantage of this "brotherhood," but he feels he has already paid his academic dues and he does not resume study for the final.

Sam was always too busy sulking about one rejection or another to get enough help in his mathematics. He fought everyone who could have helped him and he was deliberately obnoxious and contrary even when it hurt his mathematical progress. He seemed to take pride in "doing his own thing." Potentially (the counsellors believed) he had talent, but he could not develop it on his own, and although he did make progress, the strain was so great that he had to stop work one week early.

(Sam) Second Summer, 1980

Sam returned to the program in the summer of 1980 as a second year participant. He said he was not confident with what he knew about number theory, particularly when faced with questions from first year students. He described his difficulties with trying to counsel his first year student. He explained, "This year I need to learn much more about number theory because I am trying to explain it to my first year student. You must understand something well before you can teach it to someone else. I am pleased when I can get him (my student) to put some mathematical solutions together with the information that I can give him. I sometimes think I give him too much information, because it is hard for me to give direction from a hint and without giving away all of the solution. Someone like Professor Ross can do that because he has more experience in mathematics, but I do not know enough to follow someone else's thinking in a way different from mine. All I can do is try to lead him into my thoughts about the problem. This is particularly true about those problems I copied from seminar instead of doing my own."

He talked about how he fell behind some other second year students because he made no progress during the 1979-1980 year in high school. He said about high school, "I think it set me backwards because I did not have anything challenging to think about all year. The school would not let me take calculus because I was a junior and only seniors were allowed to take calculus. I think this is why I am less confident than other students like Todd and Benjamin. They took mathematics courses at a university and learned more mathematics during the school year instead of forgetting what they learned in the program last summer. I took unified math which was mostly a repeat of the previous two years. There were some new things to learn, but I could have learned them in three weeks if they had allowed it. For me, mathematics is a 'doing it yourself' subject. I am not good at memorizing. If I am forced to memorize facts about a subject, like history for instance, I never really learn it in any depth and I forget it quickly. That was the trouble in high school. I got very good grades in mathematics (95-97) and I couldn't have done better than that as far as grades are concerned, but I could learn more if it were taught differently. The teacher put statements up on the board with no explanation and no proof, and then told the class to memorize the facts so they could do the exercises. Everyone went home and tried to use facts they never needed to think about. That's why so many high school students don't like mathematics. It's a long list of facts that have no meaning. It would take only a few minutes to go into a proof or explanation and the class would understand more about the mathematics. I would sit in my seat and try to derive the theorems and play around with numerical examples to get a feel for what was happening mathematically. Of course, I didn't do any better on a test because we weren't tested on anything but trivial applications of the facts we memorized. I took a SAT examination and scored 760 on the mathematics part and 576 on the English part. The mathematics questions are not very deep, and even if you don't know the answers offhand, you can figure them out. The english part requires a good memory, I don't know if I want to take the time out to work on the english part again and try to improve my score. My history teacher says it is a matter of priorities, and if I want to go to a good college (even if I want to study math) I must get good scores on all parts of the SAT examination and achievement tests, and memorize many facts about other things besides mathematics.

I was on the school mathematics team. We prepared for mathematics competitions and I was disappointed because I learned only a bag of tricks to use in the competitions."

Sam said that as a first year number theory student, he didn't mind working alone but as a second year student he wished he was working in a group. "Last year I would think about problems in the morning, but I waited until evening to write anything down," he commented. "I would work alone and continuously for about five hours. It was easy to work like that when you lived in a dormitory where every one else was also doing mathematics, and that seemed to be enough then. I am working alone again this year. I am not quick, and I need time to think about problems. Elaine, Jack, and Jerry work together on p-adic analysis and I am outside that group. Mathematics solutions come much slower when you work alone. You may sit for a long time and not have many ideas. I find numericals very helpful. Numerical examples give me intuition and I can make conjectures based on them. I find if I have no examples to explore that I have no intuition. I can't do mathematics without intuition. Before I entered the program, I was not aware that it was possible to solve a mathematics problem abstractly and that without intuition, one could logically write out a solution. I can't do that yet. Last year even after I got hints from the counsellors, I had to explore the examples on my own before I could make a conjecture or give a proof. The second year courses go much faster, and I need more time to go through all the problems on my own. I can't do it any other way!"

In the summer of 1980 as a second year student, Sam could hardly keep up with his counsellor duties. Professor Ross suggested to him that he drop all second year courses and concentrate on number theory only. Sam did not listen to this advice. He talked about needing the second year courses so he could enter a Westinghouse competition. Again in 1980, as before in 1979, he pouted over not being a part of the "study group." Todd showed that a second year student did not need to work in a group to do well. Sam fell further and further behind in the second year courses, and eventually he had to drop out of every one of them, but he did learn a little more number theory. Unlike Bob, he was not a good counsellor. He appeared to be more concerned with "looking good He said he could not compete "well enough" with the other counsellors. He also said the first year students "didn't like him" because he "didn't know enough." Sam improved his counsellor image in the last two weeks of the summer institute in 1980 by just trying to help first year students with their mathematics problems. During the winter of 1981, Sam accepted a place as a freshman at Stanford University starting in the fall of 1981.

4. (Al) First Summer, 1979

Al is 17 years old when he arrives on the Ohio State campus. A high school student from New Jersey, he is a tall, nice looking young man with an "eastern city" accent. He projects a touch of arrogance. He prides himself on having been to a previous "summer mathematics institute." He continually reminds anyone who will listen. Most first year participants develop a dislike for Al almost immediately and that isolates him from the others. He is considered by the other participants to be a bully both physically and verbally, and they see him as constantly having to prove his superiority at their expense. He tells everyone how much "more attractive" the situation was at the other institute where "opportunities for social, cultural, and leisure activities abounded," that the program was "less demanding," that "afternoons were free"; that "the lecturers were not GODS," but "ordinary college people" or graduate/undergraduate students who happened to be Putnam winners or the like!" Everything Al has done appears to be better in retrospect than what he is doing now. He picks on every other participant he regards as weaker than himself. If he sees the other student weaker mathematically, he insults him and picks on his mathematical weaknesses. He behaves humbly with the counsellor he regards as "brilliant" and he refuses to wrestle with the other first year student he regards as "stronger" physically than himself.

Al does not attend his assigned seminar from the very first meeting because he thinks he "doesn't need it" and he likes his "afternoons free." Al is the only first year participant with formal mathematical experience (particularly in number theory) beyond the high school level, and he appears to be the only first year participant truly able to benefit from reading mathematics books. These assets, together with the help of his counsellor, with whom he rooms, make it possible for Al to isolate himself from the other participants and the seminar and still keep up with the problems on the problem sets.

On test #1, Al scores a 199, a score that is only the second highest score on the test and quite a bit below the top score (220). This appears to upset him because he says he "knows more than everyone else" but he just doesn't "work as hard."

The following is a counsellor's comment on Al's progress at the fourth week of the summer program:

> Al is doing well, but not as well as he could be. He is not turning in problem sets, but instead just looks over the problem set and attempts thos problems which he feels he does not already know. Unfortunately, he often thinks he knows how to do problems which he does not actually know how to do, or he does not look deeply enough into problems on the sets. He does not spend enough time on sets to see the conjectures and generalizations beneath them. I think his score on the first test spurred him to spend more time on problem sets. He is very smart, of course, and very quickly solves thos know previously.

Al proceeds to prove his point by working harder and moving into first place on the next examination. He does not impress the other first year participants who point out that "since (as he keeps reminding us) this is his second year in a summer institute learning number theory, he should know it as well as the counsellors and he doens't know it that well."

Al must maintain first place to support his act of superiority, and he does! Al continues to be irritating to all first year students, particularly the weaker ones. His need to prove superiority gives Al a reason to learn and to excel. The rather large "Let's hate Al movement" also gives many students a reason and a will to compete with Al and top him if possible, although his previous experience gives him an edge and no one succeeds in topping Al on the remaining examinations in the summer of 1979.

Second Summer, 1980 (Al)

To the other students' surprise Al returns to the summer institute for a second summer. He arrives on orientation day with news on the tip of his tongue of his college scholarship of $5800 for the next academic year. He is happy about his scholarship until he hears that another counsellor, his own age, has been awarded a slightly better college scholarship. The reaction is typical of Al and no one, certainly not the other counsellor, takes his disgruntled remarks seriously.

Five weeks into the summer program, it is clear that Al has made no more friends this summer than he had the previous summer. He is still a bully and he is also subject to fits of temper which he displays often. The younger students and the other counsellors do their best to neutralize Al's influence on the first year participants. Al is active in only <u>one</u> advanced course in Geometry (participation in one course is fewer than expected of a good second year participant).

The following is Al's account of his current and recent past experience in the summer institute:

> I went to a six week mathematics summer institute for high school students at another college before I came here. It was like a summer camp. You learn mathematics but you also do many other things unrelated to mathematics and the program has the college to itself. For the first three weeks, the students are in different small groups doing background work, and the last three weeks, students have free choice to attend any or all of the lectures. There is a one standard course which has been number theory and a maxi-course 1½ hours a day for three weeks plus mini-courses. The program is less demanding than the Columbus program. Although the problems are taken seriously, there is not the same emphasis and insistence on them. There are counsellors who participate in all aspects of the program, both socially and academically, but they do not insist that the participants hand in a daily quota of problems like they do in the Columbus program.
>
> I found that when I came here I was able to do the problem sets by attending lectures, but without attending seminars. I roomed with one of the counsellors, and I could discuss problems with him when I wanted to and I learned to read mathematics at the other summer institute.
>
> At the beginning of the summer, I did not enjoy counselling because I did not see myself as a nagger, but now it's better because with a little explanation, Rich (my student) works on his own.

Shortly after the conversation above, Al lost his counsellorship of Rich. Al also lost both of the first year number theory students assigned to him. Rich (a second year student doing number theory for a second year) was reassigned to Al after Al lost his second first year student five weeks into the program. One by one, each of Al's assigned students started working with other counsellors because (they complained) of his neglect and/or bad temper. None of the students assigned to Al liked him as a counsellor because he would either call them stupid or make them feel stupid for asking what he thought were trivial questions. Al had trouble with the personal relationships between himself and every other participant and he had "bad" relationships with many. However, not <u>all</u> impressions of Al were negative. Al gave some "very good talks" (acknowledged to be very interesting even by the other advanced participants) in the Advanced Geometry course. Al was also very active in class discussions and the other students complained that the lectures were often discussions between Al and the lecturer and sometimes the lecturer appeared to be talking directly and solely to Al. Al impressed his professors in the geometry as "enthusiastic and a very bright student with good ideas." It appeared that Al was impressed with the "importance" of the impression he made in that course and he was even happy enough to ask if he could return to the summer institute in 1981.

In the fall of 1980, Al was a freshman at Washington University with a $5800 a year scholarship.

LONERS

Summary

The previous four students, Todd, Benjamin, Sam, and Al were the "Loners" in the program either by choice or because of their "difficult" personalities.

In the case of Todd, he chose a difficult working schedule which naturally isolated him from the other students at his level.

Benjamin's physical handicaps made him difficult to understand and therefore difficult to work with.

Sam separated himself, but not by choice, by fighting with everyone at every level and then sulking over his isolation.

Al was a thorn in the program. Some students disliked Al and some students were afraid of Al, but it was hard to ignore Al and even harder to work with him. Al seemed deliberately to push everyone away and therefore he was a loner by choice.

For Todd and Benjamin, working alone did not interfere with their progress. They accepted help from counsellors, teachers, and occasionally other students and developed their independence into a virtue.

Sam's obstinance and inability to accept help interfered ultimately with his progress.

Al was able to work alone because he had a good previous mathematics experience, was able to read mathematics books on his own and was aggressive enough to go to an instructor or Ross or the advanced counsellors for advice whenever he needed it.

COMMUTERS

1. (Richard) First Summer, 1979

Richard is 16 years old. He is a commuter. He attends the alternative school in Worthington, Ohio. Rich comes to the program highly recommended by his high school. Rich does not participate in the first couple of weeks of Woods' seminar. He is shy and talks enough to the counsellors to let them know that he understands the mathematics. He also talks to other students about the current problems and solutions. No one is worried about Rich. "How are his solution sets?" the counsellors are asked by Ross. "Well," the counsellors say, "He doesn't like to write things down - it is a trait of his personality to hide away and do his own thing." One hears vague bits and pieces of mathematical conversations from Rich but never any details and everyone thinks Rich is ready for test #1.

Test #1 comes and his score (80) is eleventh out of sixteen scores with a mean of 116.5. He's done better than any other commuter. He is a commuter, but does this make a difference?

Rich does not prepare a rewrite of test #1. Rich begins to work with Todd. Both students are happy with the arrangement. Todd gives the counsellors good solution sets, but Rich gives them nothing. He is going to write them up, but he never seems to have the time. Now, everyone begins to worry. The counsellors are talking to him more carefully now. He cannot or will not answer questions put to him in seminar. Does he really understand what is happening mathematically? Todd says he does, but nothing written comes from Rich.

It is a few days before test #2 and Professor Ross wants to see what Rich is doing with his problem and set solutions. Ross suspects that Rich might have deliberately withheld the written solution sets from the counsellors. He decides to ask Rich personally for as many problem solutions as he can put together overnight. The next morning, Rich arrives with a few sets under his arm. They look as though he threw them together quickly and without much thought. Much of his work is nonsense and the rest is vague at best. It is too late to paddle backwards now. Test #2 is close at hand.

Rich's score (27) is fourteenth among the sixteen scores with a mean of 101.5.

"What can be done to help Rich?" the counsellors ask. Since Todd and Rich are already working together, it is decided that Todd must watch over Rich. Todd is much further ahead than Rich and it is Todd's responsibility to see that Rich writes out all the solutions when they work together. Todd sees that Rich hands in solutions and that they are well written. The counsellors ponder, "how much of the written work is Todd's?"

Rich begins to ask questions and answer some questions in seminar, so Professor Ross is informed that <u>something positive</u> is happening to Rich.

On test #3, Rich's score (44) is fourteenth out of fifteen scores with a mean of 99. His position is unchanged, but his score is better.

Rich was a commuter. Commuters seemed to be at a disadvantage, perhaps because they had to work at home with many distractions, no help from other students, and no help from the counsellors. Two of the important jobs of the counsellor were: 1.) to serve as a mathematical resource and translate the mathematical symbols and words into plain language, and 2.) to help the first year student communicate, on paper, precisely with new words and symbols.

Rich was bright and spoke well and knew enough mathematical jargon to make his instructors answer their own mathematical questions. When his secret was discovered, he was too far behind to catch up.

(Richard) Second Summer, 1980

Richard came back to the 1980 summer mathematics program as a second year live-in dormitory student. He remembers his first summer in the program as difficult. He says, "I was a commuter last year and I had to force myself to work at home alone because there were so many distractions at home and no one to talk mathematics to. At the beginning I hardly talked to anyone else in the summer program. I tried to listen carefully to lectures and seeing some of the problems done in seminar was helpful."

He was very much aware of his inability to write proofs clearly. "I tried to read about the mathematics whenever I could find something relevant to the problems I was thinking about. I thought I understood what was going on in class, but I had trouble expressing myself mathematically," he recalls.

Rich recognized the effort made by Todd to help him over his difficulties with communicating mathematics solutions in writing. Concerning Todd's help, he said, "Toward the end of the summer I spent more time at the dormitories and with counsellors and I worked with Todd. He helped me write my mathematical thoughts down on paper instead of keeping them scattered around in my head. Working with someone my own age was really good for me. I knew I wasn't communicating my ideas in seminar. I knew what I was saying, but no one else did. Todd understood what I meant, and he showed me how to write it down."

Rich was made to take a precalculus course in high school in the 1979-1980 school year. The summer program pointed out to Rich that he was not ready for another advanced mathematics course at The Ohio State University that year.

The summer of '79 mathematical experience did not change his attitude enough toward learning to make him write things down. He said, "The class in high school was very uninteresting and I did exactly four hours work during the four hours of examinations. I had A's and B's on the tests and my teacher told me at the end of the year that I had not worked hard enough to earn more than a B. Of course that was true, but he never encouraged me during the year to work harder. He said that it was up to me to do the homework when I had the need to do it and so I didn't do it! I read a book on mathematical Game theory instead."

Rich carried nothing new mathematically into the summer program of 1980 and it began badly, "I started to write things down very late in the summer of '79, and there are many things I didn't do. Now, my first year student wants answers to questions I never thought about and do not know how to do. I don't feel good about this, but I am learning the number theory I didn't learn last year," he said, explaining why things did not go well for him at the start of the summer of '80.

Rich had to give up his counsellor duties and his p-adic analysis class, the only advanced class he attended. He could help no one nor learn more advanced mathematics without learning more number theory first. Benjamin became his counsellor for part of the summer, and Al was his counsellor for the last three weeks of the summer. Rich swallowed his pride and stopped covering up what he didn't understand by using mathematical doubletalk he had developed the previous year. He sat down and worked solution sets on his own and the later ones made sense to the reader. He did minimally well on the number theory final at the end of the summer of 1980.

COMMUTER

Summary

Richard is in a category all by himself, because he was the only commuter (of the seven who started in the summer of 1979) who survived the number theory course. His progress was rather limited. His growth was slowed down by the small number of hours of contact with counsellors and other students. Consequently, although his ideas were good (the other counsellors said) he never did develop the ability to write his ideas down so his instructors could understand them.

ONE SUMMER (1979) ONLY STUDENTS

1. (Paul) Summer, 1979

Paul is 18 years old and he is a commuter (from Columbus, Ohio) in the summer mathematics program. He is very friendly and outgoing and does not have to be coaxed into spending time at the dormitory with the resident students.

The counsellors say that he doesn't work hard enough, but this can be translated into the fact that he doesn't give them as many solutions to the problems as the dormitory resident students. The counsellors admit that, since Paul does not live in the dormitory, they do not know how hard he studies after he leaves the campus.

Paul appears very keen to learn and he asks many questions in seminar. Paul is in Leonard's seminar. There are only three resident students; of these Bob hardly attends and Dick is too shy to speak up. This means that each remaining student has the opportunity to ask many questions but it also means that the discussion is dominated by a couple of students and there is very little interaction between students. From Paul's many questions, one gets the impression that he often does not understand the problem about which he is asking. The result is confusion about what he means to ask. He tries to understand the answers to his questions and he is not shy about asking for more details if he does not follow a discussion, Paul exhibits good humor at all times and the seminar instructor never minds giving him more detail. Paul understands the numerical techniques and he will volunteer to share his solutions with the rest of the seminar. He makes it clear to the seminar instructor that he is trying to understand the assigned problems and that he is always having trouble keeping up with the most current problem set. Although this is common among the first year students, Paul is still very worried about it.

Paul's score (72) on test #1 is fourteenth among the sixteen scores with a mean of 116.5 and a median of 117. By this time, everyone (counsellors, teachers, students) notices the extra strain put on the commuters by the difference in treatment and Paul's score is not considered bad for a commuter.

Paul does not have time to prepare a rewrite of test #1 for his counsellor. After test #1, Paul tries to catch up to the more recent problem sets by spending more time at the dormitory, but he still must go home every evening. Everyone likes Paul and tries to help him. He seems to thoroughly enjoy the friendship of the other program participants, but he falls further behind the resident students with each passing problem set. He now does a problem here and there in an attempt to keep (in his words) "systematically behind". Just before test #2, his questions in seminar reflect his frustration with the new problems that depend on concepts learned in the problems on the older problem sets left undone! The gap grows in size as the problem sets build on one another. Paul starts attending both seminars in an effort to "get more solutions" for his problems. Paul now spends many of his evenings working at the dormitory with the resident students.

On test #2, Paul's score (78) is eleventh among sixteen scores and only seven points below the median of 85 and twenty-three and one-half points below the mean of 101.5. Paul is now best among the commuters although still below the mean.

Immediately after test #2 there is a family problem. Paul must leave town for a few days and he drops out of the program.

2. (Maureen) Summer, 1979

Maureen is 15 years old and a student at a private girl's school of good academic reputation. Maureen starts the program as a commuter. She is a pretty and shy dark haired girl. She does not talk in Woods's seminar and she does not answer any questions in seminar. She behaves as if she is overwhelmed by the mathematics when one questions her. She cannot answer if a question is asked by the instructor in seminar, about topics under discussion in number theory lecture. and on problem sets. It is suggested to her, by the analyst, that as a commuter she must seek out help from the program counsellors at the resident dormitory. She follows the suggestion and visits the dormitory after classes. However, she must be driven back and forth from home to campus and she can only spend a limited amount of time at the dormitory. By the second week, the counsellors report that she is having trouble with many new concepts and that she is becoming increasingly frustrated with her inability to work many of the problems at home. Two weeks into the program it is clear to the counsellors and instructors that the time Maureen spends away from the dormitory sends her further and further behind.

Elaine is the only female resident student in the dormitory and there is an empty bed in her suite. Arrangements are made for Maureen to move into the dormitory suite with Elaine. Maureen becomes an instant member of the mathematical inner circle study group with headquarters at the suite she shares with Elaine. Her work improves with her move.

It does not take long for Bob and Maureen to become special friends. They start working together on problem sets. Maureen moved into the dormitory with serious doubts about her ability to do the kind of mathematics expected of everyone in the program. Her friendships at the dormitory help her build self-confidence.

On test #1, her score (126) is seventh highest out of sixteen scores and above the mean of 116.5. Although most of her credit was obtained from the numerical problems, she did do some non-numerical problems with limited success. The rewrite of test #1 that she prepares for her counsellor shows that she has trouble making up a complete proof. Her proof is several isolated statements, all needed to make up a proof , but leaving to the reader the task of supplying the logic and the details.

The friendship of Maureen and Bob continues to flourish. Bob did so well on test #1 that he stops working immediately afterward. Maureen cannot afford a vacation at this point and it slows her progress.

After a week, Maureen and Bob start working again together and with the study group, but neither Bob nor Maureen work as hard as they did before test #1. Bob has proved to himself that he can do the mathematics well when he works hard but Maureen never really had the chance.

Almost five weeks into the program her counsellor reports that Maureen still finds that writing out a proof from "scratch" is difficult for her, however, "she is beginning to make associations with previous problems and useful results that can be applied to a theorem at hand."

On test #2, although Maureen works mostly on numerical problems, she does show improved understanding of what constitutes a proof of a mathematical statement. Her score (79) is tenth among sixteen scores and it is six points below the median of 85 and twenty-two and one-half points below the mean of 101.5.

On test #3, her score (57) is again tenth among fifteen scores with a mean of 99 and a median of 107. In spite of the relatively low score, her choice of problems shows some increased knowledge of proof.

(Maureen) Summer, 1980

In the summer, 1980, Bob makes contact with Maureen although she is not a second year participant in the summer institute.

Maureen said that she had to work exceptionally hard in the summer institute of 1979 just to "keep up." At the beginning she thought she would drop out because it was much more difficult than she had expected and a more different kind of mathematics than she was familiar with. She said she learned "quite a bit of mathematics." This was confirmed by her mathematics teacher at the girls' school who said she was his "best" student and he was impressed by her knowledge of mathematics. Later in the academic year, 1979-1980, Maureen took the SAT examinations and achieved a "very high score." She won an award in a statewide competition in mathematics. Maureen said she was glad she had that 1979 summer's experience in mathematics, but one summer was enough!

3. (Stella) Summer, 1979

Stella is pretty, blond, and a mature 14 years of age when she arrives on campus one week late. Although Stella is only 14, she has completed tenth grade in Columbus, Ohio at the same private girls' school as Maureen. Stella is a commuter but her previous friendship with Maureen gives her easy access to resident students' study groups, and the inner circle study group in particular, when she is on campus. However, she cannot stay on campus unless her father is on campus (he is on the faculty of The Ohio State University). Stella has double trouble: 1) in having to catch up with her late beginning and 2) in not being able to spend enough time on campus in touch with the counsellors (the usual commuter trouble).

Stella's solutions are relatively fewer than the resident students and she can answer nothing in Woods' seminar, but she appears to be enthusiastic about working the problems when she manages time on campus.

The counsellors say she is capable of learning mathematics and she would be doing much better if she were allowed to stay at the dormitory at least for more extended periods of time.

Her scores seem to be a reflection of the amount of time she has to spend on campus. She stays at the dormitory with Maureen and the inner circle many more hours in the last two weeks of the program.

Stella scores a 65, 14, and 51 respectively, on tests #1, #2, and #3 with means of 116.5, 101.5, and 99.

4. (Mark) Summer, 1979

Mark is 16 years old when he arrives on campus with his parents. He is small and sensitive and he looks frail and younger than his sixteen years. He comes to the summer institute with many allergies and his parents request that he room alone.

At first, Mark is anxious to ask questions and raise conjectures in seminar. His conjectures are sometimes true and sometimes false. When challenged to give some support for his conjectures, he tries, but it becomes apparent that he cannot express himself clearly enough to indicate the source of his ideas. This is not unusual since many students at the beginning of the program have trouble expressing their mathematical ideas to others.

From the start, Mark is in the outer circle study group and on the outside fringe of the mathematical activities at the dormitory. He is always tired and he finds it hard to work. He stays in his room much of the time, but does not appear to be working very hard there.

On test #1, Mark's score (113) is ninth among sixteen scores with a mean of 116.5 and a median of 117. Mark answered only numerical problems on the test except for two theorems with very short proofs.

Mark is accustomed to doing very well on examinations. He appears to be depressed by the first test and he voices his disappointment in the test results even though his score is only three and one-half points below the mean. Mark withdraws even further than before. He does not prepare a rewrite of test #1 for his counsellor and he gives no excuse. He stops participating in seminar and he no longer asks any questions there. Immediately after test #1, he stops giving his counsellor his solution sets to be corrected. Although he eventually starts working on problem sets again, his solutions are sporadic. He does not look well and behaves as if he is too tired to work at night like the other participants.

On test #2, Mark's score (87) is eighth among sixteen scores and two points above median and fourteen and one-half points below the mean. This time, Mark worked only the numerical problems on the test. The counsellors regard this as a bad sign. They say it indicates only a kind of surface knowledge of the mathematics.

By the seventh week into the program, Mark's allergies are very active and the counsellors say he is hardly sleeping. He is no longer giving the counsellors any problem set solutions to be corrected.

On test #3, Mark only works on the first eight numerical problems; his score (54) has slipped to eleventh place and it is forty-five points below the mean of 99 and fifty-three points below the median of 107.

5. (Ronald) Summer, 1979

Ronald is an outstanding student of mathematics in high school and he has excellent recommendations from all of his teachers.

The summer institute starts its second week and Ronald is not on campus yet; Ronald's parents are called and told he cannot stay away any longer and expect to catch up. Ronald's parents are not worried because in their experience Ronald never has any trouble with mathematics and the institute waits just a little longer for Ronald to arrive.

Ronald is over a week late when he finally shows up bragging about his mathematical talents to the other participants who are unimpressed. He is loud and he tries hard to get the other students' attention. He tries to infiltrate the already existing mathematical study groups but his pompous attitude keeps him on the outside. At the beginning he must work alone; he attends seminar but just listens at first.

Ronald is slow in catching up and on test #1 his score (75) is thirteenth among the sixteen scores and fourteen points below the median, certainly not very impressive to the other participants.

Yet, the other students are becoming accustomed to Ronald's play for attention and they are becoming more tolerant of him. He begins to work on the fringe of the study groups and he becomes a regular member of the outer circle study group. Ronald seems to work better with other students and he begins to make more progress than before test #1. The counsellors say his problem sets are getting better.

Ronald is in the small seminar (three commuters dropped out) taught by Leonard and he begins to ask questions and participate in discussion although his questions are always guarded so he can and sometimes does retract them if he senses they appear not to be "good" questions (observing the reaction to them) to the seminar instructor. If he asks a question and realizes it is something he is supposed to know, he covers it up with a "I knew that all the time" statement.

On test #2 (with a mean of 101.5 and a median of 85) Ronald has a score of 70 and he places twelfth among the sixteen participants. Ronald has learned the numerical techniques. His score appears to reflect his lack of knowledge of the more general types of problems, proof, and the use of mathematically sophisticated notation.

After test #2, Ronald works mostly on the fringe of the study groups. He continues to try to impress the other students with "bad jokes" or how far he can throw a frisbee. Ronald never does succeed in impressing the other students with his "superior intellect" because he never succeeds in catching up mathematically with the better of the first year participants.

On test #3, Ronald's score (111) moves into sixth place and is above the median (102) and the mean (99).

6. (Dick) Summer, 1979

Dick is 17 years old in 1979 when he arrives in Columbus. He comes from Marion, Ohio, a small city about fifty miles north of Columbus and he acts very self-conscious about being a "small town" boy in the "big town" summer institute with (as it looked to Dick) much "smarter kids." His self-consciousness keeps him from becoming an integral part of the dormitory group. Dick does not behave like a natural loner, he just looks lonely. He appears to walk along the edge of the study groups peering in from the outside and afraid to sit in because someone might notice that he is not as smart as he looks. His voice shakes and he becomes very uncomfortable if any attention focuses on him. The counsellors say he moves into the "sea of mathematics" with the outer circle study group only long enough to get his "feet wet" and then withdraws. Occasionally he works with Sam who also moves along the edge of the study group.

On test #1 (with a mean of 116.5 and a median of 117) he scores 132 and his score is fifth out of sixteen scores. The counsellors report, "he surprised himself on how well he did on the first exam. He has slowed down since then because he thinks it's a 'fluke." He has confidence problems. He tried working with Sam immediately after test #2 but no one can work with Sam for very long because Sam sulks and gets moody with little reason."

Eventually Dick's feeling of inferiority slows him down. He falls into a "I always knew I couldn't do it" mood and his accomplishment is lost in the emotional energy it takes for him to face the high level of competition every day. Dick's feeling of incompetence gets in the way of his working with other students although he makes it clear he would like to.

On test #2 he slips down into ninth place with a score of 83, two points below the median of 85 and eighteen and one-half points below the mean of 101.5. Dick picked up his points on the numerical problems. This seems to indicate he does not have the general understanding of the most recent mathematical content.

Dick continues to have confidence problems even after the second test, but he does get to know the other first year students well enough to break the self-imposed barriers and participate in group study at the dormitory. The counsellors work hard at boosting his confidence which they say they recognize as a major obstacle to his learning in depth.

Dick moves back into fifth place with a score of 119 on test #3, twelve points above the median of 107 and twenty points above the mean of 99.

Dick leaves the program feeling "lucky" rather than accomplished.

7. (Sol) Summer, 1979

Sol is 17 years old when he comes to the summer institute from Brooklyn, New York. Sol comes from a tough neighborhood. Sol had bad experiences in Russia a few years earlier. Sol's family recently immigrated to the United States from Russia. In his last years in Russia, Sol was treated as a non-person in school because his family applied for permission to leave Russia. Academic achievement was an important part of their cultural tradition in Russia. Sol did not make his own decision to participate in the summer institute because his family made the decision for him.

From the beginning it appears that Sol's recent experiences in a non-academic environment, together with his indifference to mathematics, make it impossible for Sol to deal with the intensity of the work at the summer institute.

Three weeks into the program his counsellor reports, "He can't get started; he doesn't make enough effort. It seems as if he doesn't know how to work. He makes it clear the program was not his idea and his parents forced him to come!"

Non-academically, Sol gets along very well. The other participants have respect for Sol in spite of the obvious fact that he is not doing well mathematically. Sol has the ability to lead the other students but he seldom chooses to do so. He can sit and work or just sit with the inner circle anytime he wishes, but he seldom chooses to work. He enjoys the friendship of the other students; he protects them from Al who is afraid of Sol, and Sol even attends seminar because he enjoys the group action. He seems to enjoy watching the other students and the seminar instructor perform in class.

Sol earns a 76, 43, and a 43 respectively on tests #1, #2, and #3 (with means of 116.5, 101.5, and 99).

Although Sol's efforts, interest and achievement appear to be minimal, Sol's interest in number theory was strong enough by the end of the summer to allow him to organize, with the help of his teacher at his high school, a special elective in number theory.

8. (Lorraine) Summer, 1979

Lorraine probably should not be included as a first year participant since she was at no time considered a regular participant in the summer institute, but...

Lorraine lives in Columbus, Ohio. She is pretty, well-developed and looks much older than her 14 years even though she behaves like an immature 14 year old. Lorraine does not apply to the program with strong recommendations from teachers. He parents call Ross because she is "interested" in mathematics and they think she will benefit from the summer program as a commuter. From the beginning, Lorraine is accepted as a special student. It is suggested by Ross that Lorraine attend the program for only the first two or three weeks.

But, Lorraine seems to enjoy both the classes and the study and social activity with "older" boys and girls. She mentions how much she enjoys telling her freinds at home what she is doing in a "mathematics summer program at Ohio State" because they are "impressed." She appears to be more interested with the cosmetic effect of the program on her lifestyle than with the cognitive effect of the program.

The following report on Lorraine's progress comes from the counsellors, four weeks into the 1979 program.

> This program was supposed to be "just an adventure for two weeks" but her parents forced her to stay. B. (counsellor) works with her a couple of hours every day, but she doesn't understand - she doesn't seem to want to understand. She can't work on her own!

Lorraine appears to be "playing" the interested mathematics student. She bounces around the dormitory, cheerful and unconcerned about all the problems she cannot do. She smiles through Woods' seminar without answering any questions. She is accepted by the other participants as a "younger sister" and she gets attention whenever she wants it both from the other participants and the counsellors. Even with all the attention she receives on problem sets, she can do only the most basic kind of numerical example.

Lorraine's scores are 41, 17, and 6 respectively on tests #1, #2, and #3 with means of 116.5, 101.5 and 99. She has the lowest scores on tests #1 and #3 and the next to lowest score on test #2.

Lorraine comes back to the summer institute in 1980 as a regular first year number theory student and again she is a commuter. She says she had so much "fun" in the summer institute of 1979, she wanted to "come back."

She is now barely 15 years of age. She appeared to learn little mathematics in 1979 (recall she was not a regular student in 1979); so she has no advantages over the other first year participants in 1980.

Lorraine's score on the Number Theory final of 1980 is 73 points. The final had a median score of 96, a high score of 200 and a low score of 46. She placed twenty-third in a class of thirty-one students.

One Summer (1979) Only Students

Summary

The previous eight students were those who attended only the summer institute of 1979. None were invited to return because it was felt they would not benefit from a second summer. The students include:

Two commuters, Paul and Stella, had all the same problems of the resident students plus the seemingly debilitating extra problem of not having enough contact with counsellors.

Maureen, started out as a commuter, floundered, and was saved by a move into the dormitory, but just for one reasonably successful summer.

Mark was frail as a result of his many allergies and too tired most of the time to keep up and "lost heart" at least temporarily.

Ronald couldn't tolerate not being "top dog" and spent too much energy trying to impress the other students rather than work on mathematics.

Dick thought he was inferior so that his lack of confidence interfered with his learning and made some of his fears come true.

Sol was pushed to come by his parents and initially had no "commitment" to learn mathematics although the other students' strong commitment transferred something of interest to Sol by the time he left in 1979.

And finally, Lorraine was not only handicapped by commuter status, but it appeared, even more by her immaturity.

It is hoped the case studies of Chapter IV have given the reader some insight into the part each of the eight categories (teaching, counsellors, problem sets, group interaction, living conditions, home support, delayed effect on changes in behavior, and books) have played in the nurture of the individual students.

In Chapter V, each of the eight categories is examined separately for its impact (as perceived by students, faculty, and the investigator) on the nurture of the collective body of students.

CHAPTER V

OBSERVATIONS BY CATEGORIES

Introduction

Recall from Chapter IV that observations of students were expected to provide data in eight categories:

1) Teaching
2) Counsellors
3) Problem sets
4) Group interaction
5) Living conditions
6) Home support
7) Delayed effects or changes
9) Books

In this chapter data from the case studies and advanced participants/counsellors are organized in these categories except for category 6 for which there was not enough information.

Strong home support could be implied because it would have been impossible for the young participant to attend the summer institute in 1979 without the permission, financial support and considerable effort of his/her guardian(s). Room, board, and the trip were not inexpensive (see appendix for information in application materials concerning room and board). The first summer participants of 1979 were generally high school children of middle class background with at least one parent and/or teacher who cared about his/her education and who actually encouraged him/her to attend. Beyond this, there was little occasion to collect information about home support.

The remaining categories are discussed in order for convenience; categories four and five are combined.

Teaching

Introduction

It is nine o'clock one morning in number theory class. The lecture hall is filled with sleepy summer institute first year participants, advanced participants, counsellors, seminar instructors, and visitors. There is agreement among seasoned participants that it is "traditional" for all institute participants to go to number theory lectures and see if they can "fill in the holes faster than Ross can dig them." Ross starts the lecture with two huge numbers to be multiplied together. He looks back at the class and there is a curious silence. I am reminded of Ross's often repeated statement to students about numericals:

> Never think of numericals as exercises to see if you understand ideas, because otherwise you condemn yourself to a life of being a follower. If you want to be a discoverer, think of a numerical as an experiment where you must observe and you must understand and maybe then you may discover some important ideas.

The sound of his voice breaks into my thoughts. "What would be your guess?" he askes the class. Only first year participants are traditionally allowed to answer. "A large number?" is the reply. Ross ignores the answer and says to the class, "What is the very ancient theorem you've been studying?" "Chinese Remainder Theorem" they reply as a class. (Although not one of its important applications, the Chinese Remainder Theorem provides a system to multiply huge numbers, but the class of first year participants does not know this yet.) Following that reply the numbers are erased and they are replaced by two much smaller numbers: (342)•(257). The first few numbers of the following calculation went up on the board with little help from the class and much confusion as to what was going on:

> 342 factors into ... $3^2 \cdot 2 \cdot 19$
>
> the factors of 257 - not 2, not 3, not 5, not 7, not 11, not 13, - not ... anything.
>
> $\cancel{2}, \cancel{3}$, 5, 7, 11, 13, 17, $1\cancel{9}$, 23, 29, 31, 37, ...
>
> ... see what primes we can use ...
>
> 11 • 13 • 17 • 23 large enough?
>
> (pause)

One or two students reply with a NO!

A student suggests replacing 13 by 29.

> 11 • 29 • 17 • 23 large enough?

One or two students reply with a YES!

Little by little, a few more students were drawn into the following calculation on the board...

primes		29	23	11	17
342	$\equiv$	−6 mod 29	− 3 mod 23	1 mod 11	2 mod 17
257	$\equiv$	−4 mod 29	4 mod 23	4 mod 11	2 mod 17
x	$\equiv$	−5 mod 29	11 mod 23	4 mod 11	4 mod 17

...until the last few numbers were chanted by most members of the class.

Many students now realize they are looking for a number x, such that $x \equiv -5 \pmod{29}$ $x \equiv 11 \pmod{23}$ $x \equiv 4 \pmod{11}$ $x \equiv 4 \pmod{17}$

since: $-5 \equiv (342) \cdot (257) \equiv x \bmod 29$

$11 \equiv (342) \cdot (257) \equiv x \bmod 23$

$4 \equiv (342) \cdot (257) \equiv x \bmod 11$

$4 \equiv (342) \cdot (257) \equiv x \bmod 17$

More calculation goes up on the board:

$$\underbrace{(\underbrace{23 \cdot 11 \cdot 17}_{9} \cdot 13)}_{1 \bmod 29} \quad \underbrace{(\underbrace{29 \cdot 11 \cdot 17}_{-5} \cdot 9)}_{1 \bmod 23} \quad \underbrace{(\underbrace{29 \cdot 23 \cdot 17}_{-2} \cdot 5)}_{1 \bmod 11} \quad \underbrace{(\underbrace{29 \cdot 23 \cdot 11}_{-7} \cdot -5)}_{1 \bmod 17}$$

The students who follow note that:

$-5 \cdot 23 \cdot 11 \cdot 17 \cdot 13 \equiv -5 \bmod 29$ $\quad$ $11 \cdot 29 \cdot 11 \cdot 17 \cdot 9 \equiv 0 \bmod 29$

$-5 \cdot 23 \cdot 11 \cdot 17 \cdot 13 \equiv 0 \bmod 23$ $\quad$ $11 \cdot 29 \cdot 11 \cdot 17 \cdot 9 \equiv 11 \bmod 23$

$-5 \cdot 23 \cdot 11 \cdot 17 \cdot 13 \equiv 0 \bmod 11$ $\quad$ $11 \cdot 29 \cdot 11 \cdot 17 \cdot 9 \equiv 0 \bmod 11$

$-5 \cdot 23 \cdot 11 \cdot 17 \cdot 13 \equiv 0 \bmod 17$ $\quad$ $11 \cdot 29 \cdot 11 \cdot 17 \cdot 9 \equiv 0 \bmod 17$

$4 \cdot 29 \cdot 23 \cdot 17 \cdot 5 \equiv 0 \bmod 29$ $\quad$ $4 \cdot 29 \cdot 23 \cdot 11 \cdot -5 \equiv 0 \bmod 29$

$4 \cdot 29 \cdot 23 \cdot 17 \cdot 5 \equiv 0 \bmod 23$ $\quad$ $4 \cdot 29 \cdot 23 \cdot 11 \cdot -5 \equiv 0 \bmod 23$

$4 \cdot 29 \cdot 23 \cdot 17 \cdot 5 \equiv 4 \bmod 11$ $\quad$ $4 \cdot 29 \cdot 23 \cdot 11 \cdot -5 \equiv 0 \bmod 11$

$4 \cdot 29 \cdot 23 \cdot 17 \cdot 5 \equiv 0 \bmod 17$ $\quad$ $4 \cdot 29 \cdot 23 \cdot 11 \cdot -5 \equiv 4 \bmod 17$

The board calculation continues:

$$x \equiv -5 \cdot 23 \cdot 11 \cdot 17 \cdot 13 + 11 \cdot 29 \cdot 11 \cdot 17 \cdot 9 + 4 \cdot 29 \cdot 23 \cdot 17 \cdot 5 + 4 \cdot 29 \cdot 23 \cdot 11 \cdot -5 \pmod{29 \cdot 23 \cdot 11 \cdot 17}$$

$\equiv 337352$

$\equiv 87894$

At the end of the calculation the following conversation took place.

Ross: What have we found?

Class: We have found the product (342) • (257), but...wouldn't it have been easier to just multiply the two numbers?

Ross: Yes, in this case the numbers were not too large, but if the numbers had been very, very large so that the computer did not have the capacity to multiply them...

Role of Teaching (first year)

Why does Ross use this seemingly confusing (to the uninformed student) immediately less informative and incomplete method of lecture?

A. ROSS USES THE STUDENT DISCOVERY LECTURE METHOD DELIBERATELY WHERE THE STUDENT MUST DISCOVER AND SOLVE PROBLEMS SUGGESTED BY WELL THOUGHT OUT INCOMPLETE PIECES OF INFORMATION.

Ross says that he uses discovery teaching not because it is a personal preference but because he believes it gives young students the mathematical experience they need and because it concentrates on recognizing problems and solving them. He explains the reason behind his particular style of teaching in the number theory course for the first year participants by saying, "Expository lecturing for the very young and inexperienced student of mathematics in inappropriate. If the student has experience similar to the lecturer then language is his friend. Otherwise language must be used by the lecturer with extreme caution."

B. NUMERICAL AND CONCRETE EXAMPLES SUPPLY EXPERIENCE UNDERLYING MATHEMATICAL IDEAS TO BE DISCOVERED.

He believes that inexperienced students must be allowed to experiment with mathematical examples so that they may gain the experience that will develop their knowledge and use of language. The numericals and concrete examples will supply the experience underlying the concepts they are to "discover." The "experience" must be given one or two weeks before the "ideas" are developed in lecture, so the ideas are meaningful to the student. He says that too often the ideas are formally developed and the numericals and examples are given afterward, simply to illustrate what the lecturer has been talking about. Ross goes on to emphasize that numericals and the background examples in general must be very well thought out since they are most important. They may or may not be complicated or use cumbersome notation but they must be meaningful and pointed in the direction the lecturer wants the student to look. When the student has gained some experience, then the lecturer can introduce language a little at a time.

C. THE COUNSELLOR MUST KEEP THE STUDENTS WORKING ON THE PROBLEMS SO THEY MAY ACCUMULATE THE NECESSARY EXPERIENCE TO MAKE SENSE OF LECTURES.

The counsellor is part of the discovery teaching method used by Ross. He believes the counsellor must keep the student working on the problems on the sets so the student has the necessary experience for the lecture. Ross emphasizes the importance of the numerical problem when he tells the counsellors, "The first year student must not be allowed to give up on numericals because they are very important."

D. NUMERICALS ARE NOT TO BE REGARDED AS FINGER EXERCISES.

Ross continues, "In the past, (on exercises at the end of a chapter in a high school text) students have regarded numericals as finger exercises but this is not the case here in number theory. The scientist (including mathematician) has to experiment with messy calculations often before he even knows what or where the problem is. Too often in a contest or on a test, the student does problems that are well formulated and the student needs only to supply a solution -- but that is not the way it is in the real life of a scientist or a mathematician. He often does not know if there is a problem until he does many calculations, and maybe they lead nowhere, but he is not to be discouraged easily since he cannot know if he does not first try."

E. THE STUDENT WHOSE MATHEMATICAL EXPERIENCE IS BUILT AROUND COMPETITIONS HAS EXPERIENCE ONLY WITH THE WELL FORMULATED PROBLEM.

Ross is concerned with the stress on competition in the school. He believes that, "Students whose mathematical experiences are built around competitions or olympiads can work a problem that is very well formulated but may not be able to formulate conjectures based on observation of untreated data. Numericals are not finger exercises, they are meant to provide the student with food for thought and conjecture."

F. THERE MAY BE A SHORT PERIOD OF CONFUSION AT THE BEGINNING OF THE PROGRAM WHILE THE STUDENT ADJUSTS (WITH THE HELP OF THE COUNSELLOR) TO THE DISCOVERY STYLE OF TEACHING.

The need for adjustment to the teaching method was described by Elaine who was surrounded by a group of second year student-counsellors when she said:

> Coming straight from high school into one of Ross' lectures is a strange experience. If the Rossian legend were not explained to you by the counsellors you would think that either something is wrong with you because you are supposed to understand the lecture with some kind of complete picture or something is wrong with Ross because he thinks you should see a complete mathematical picture from the lecture... To begin with, one does not see the patterns. Patterns are very important. A student works on interesting problems which are seemingly disconnected -- SUDDENTLY -- the problem will come together into patterns of shape and form and into a beautiful mathematical result... One does basic problems to start with and from the simple things one builds up some extraordinary and difficult concepts.

There were smiles and vigorous nods of agreement from the surrounding group. The impressions just expressed by Elaine seemed to agree with those of many other participants. The analyst has heard a similar description many times in different words from different students. The period of adjustment to the teaching method may take longer for some students than for others. Bob did not make good use of either lectures or seminars in his first year as a participant. One should recall that he did not attend seminars and so he brought less of the recent numerical or experimental experiences into the lecture than other students who attended seminar. Bob comments:

> It is surprising how different the lectures look to me this year. Ross's lectures last year as a first year participant seemed disconnected and different from the problem sets; I often did not see where they were going. Now in my second year, I see they are filled with many roads going in different directions and the listener must work to find his own direction and travel the road a long way before it makes sense to him. The counsellor can be very useful in pointing out some directions but the counsellor cannot travel the road for the student. Lectures might have been easier to follow if there was only one road and one topic with everything pointing in one direction but I am not sure one could call this discovery.

G. STUDENTS MUST TAKE AN ACTIVE PART IN FINDING BOTH THE QUESTIONS AND ANSWERS IN LECTURES.

The number theory lecture may introduce a numerical example that must be worked by the audience or the lecture may remind the audience about previously worked problems that might suggest some related underlying principles into which the students may look more deeply. Or, the lecture may vaguely suggest some new ideas which look promising and must be checked out because they may be true, or true in part, or false. Each participant in the audience is expected to come back to the next lecture with suggestions of new ideas related to the old and, if possible, insight into or solutions to discussions in previous lectues, or an explanation or conjecture for why the numerical example could or couldn't be solved the way it was suggested in lecture.

Role of Teaching (second year)

Professor Ross expects the second year student to adjust to the advanced course lecturer independent of his/her style of teaching. He does not expect the inexperienced first year student to be able to follow formal expository-type lectures. However, it is a fact that every second year participant has the experience from the first year in number theory. Therefore, Ross says the advanced student should follow any type lecture with the help of problem sets and a problem-oriented seminar.

The analyst did not find it necessary to ask second year participants in 1980 what they thought about teaching styles because comments were at the tips of their tongues about teaching methods or teachers at the summer institute, and the following comments were made without prompting or need for interview. Conversations about teaching and teachers surfaced spontaneously whenever any two or more second year students gathered together. Second summer students (more advanced students did not seem as interested) liked to discuss their reactions to methods of teaching, since they seemed to regard this as very important to their personal success in learning mathematics.

The summer of 1980 was the second summer institute for eight participants. Each of them started all the available courses for institute participants (as is customary), but then dropped all but those courses they thought they preferred and could handle in one summer. Some second year participants claimed to make this decision based on teacher or teaching method rather than on content material.

Does the second year student find it difficult to adjust to different methods of teaching?

A. <u>THE SECOND YEAR STUDENT'S ADJUSTMENTS TO TEACHING STYLES DOES SEEM TO BE RELATED TO THE AMOUNT AND KIND OF EXPERIENCE ACCUMULATED IN THE FIRST YEAR.</u>

The second year student's view of the teacher or the use of teaching methods employed in advanced topics in their second summer seemed to be related to the type of progress or lack of progress they made in their first summer of number theory.

Todd's opinion of a lecturer, focused on the lecturer's interest in his subject, not on the lecturer's style of teaching. He commented:

> If a lecturer sounds uninterested in the mathematics he is teaching, then he puts you to sleep. If a lecturer stops and asks interesting questions and he is also

> interested in your answers and the solutions are discussed, then everyone including the audience is involved and student discussions become interesting parts of the lecture. Although I am tired, I wake up to think about interesting questions... Mahler lectures in little boxes. He is very organized with each box representing a complete set of ideas. Last year (1979), I would have had trouble with that kind of lecture (expository), but this year I can write everything down and think about it again later when I do the problems given in seminar.

Sam stopped working after six weeks into his first summer (1979). He did not go very deeply into number theory. Sam's opinion of second year (1980) lectures centered around his frustration with the expository-type lectures, because it gave him no time to do it himself. About one lecturer he said:

> He writes for 48 minutes on the board. Someone asks a question and he writes for ten more minutes on the board. I like to work the questions from the lecture on my own. I don't like to see boards filled with writing and with no input from me. I can't follow a lecture immediately and without time to do and think the mathematics through. The second year courses go much faster than number theory and I need more time to go through all the problems, but I just can't do it any other way. Before I entered the program, I was not aware that it was possible to solve a mathematics problem abstractly, and that without intuition one could logically write a solution. I can't do that yet.

Elaine and Jerry made good progress with group study in their first summer of number theory. Elaine's and Jerry's description of their favorite method of teaching outlines informal group-type discussions in class in the second summer of study, and centers around the successful development of mathematical intuition through mathematical experience. Elaine said (with support from Jerry):

> We love his (Baishanski) class. He is great. He throws out five or six little problems in every lecture which we discuss and try to find solutions to right there in class. Sometimes he leads us into a problem by asking questions. Our intuition often leads us into the wrong direction and into a trap. Ben often manages to keep us out of the traps because he can come up with many examples and counter examples. He's read so much mathematics since last summer. Funny how intuition changes with experience.

It was clear from the way Al tried and succeeded in dominating the class discussion in advanced geometry, that he enjoyed a one-to-one mathematical discussion with the lecturer. He read enough and kept himself informed enough to keep it going from day to day. He also gave volunteer talks in that same class. Al said he "was impressed" with Zassenhaus's knowledge of his subject and made no reference to his style of teaching. It appeared that Al responded to special attention from the instructor rather than a particular style of teaching. Al dropped all other courses even when the lecturers were very knowledgeable in their subject.

Rich had difficulties communicating his mathematical ideas. He responded only when someone sat over him and <u>helped</u> him learn and also helped him communicate what he learned. By his own admission, he was not able to follow lectures in any of the advanced courses, expository or otherwise and he was still having trouble following number theory lectures in detail. Rich may have <u>wanted</u> to "go away and do his own thing" like the counsellors said in 1979, but he seemed to <u>need</u> an extraordinary amount of one-to-one instruction in 1980 to do a minimal amount of mathematics.

Summary

There are two aspects of the teaching within the summer institute in 1979 and 1980. One aspect focuses on the teaching of the first year participants by Ross' "discovery method" in 1979. The other aspect focuses on the teaching and the different teaching methods employed by lecturers of the advanced mathematics courses of the summer institute of 1980.

The comments of Bob and Elaine point out that their initial view as first year participants in 1979 of the discovery method used by Ross to teach number theory, was different from their view of that teaching method as student-counsellors in 1980. The general agreement and nods from other counsellors in 1980 support their explanations and approval of the discovery teaching method for first year number theory students.

Todd, in describing his reaction to expository lectures in 1980, said he wouldn't have been ready for "that kind of lecture" (expository instead of discovery) in 1979. He had spoken previously of how filling in the "holes" left by Ross in number theory lecture was part of his (and the other students in 1979) learning process. Now, he implied, he was ready for other kinds of teaching but preferred a lecturer who was interested in his subject.

Sam said he wanted to continue to learn by discovery in 1980 and in fact he claimed he couldn't do it any other way! He complained that the student participation methods of teaching were not employed in the lectures he attended in 1980.

Al said he could learn from anyone who was knowledgeable in his subject and in fact he chose a lecturer who was indeed knowledgeable. Although style of teaching did not appear to concern him, he seemed to flourish when the lecturer gave him an extraordinary amount of personal attention. Al attended only the two classes taught by the same lecturer. He said little about the teaching of number theory and its effect on him, since he believed he knew most of it before he came in 1979.

After test #1, Benjamin competed with Todd daily, in seminar or dormitory discussion, in trying to "fill in the holes" left by lectures in number theory and trying to be the first to propose solutions to difficult problem set questions. The discovery method of learning utilized in number theory gave Ben a reason and a need to communicate with other summer institute participants in 1979. In 1980, Ben seemed more concerned with the contents of the courses rather than with method of lecture. He only talked about the mathematics delivered in a lecture, not the method of delivery. Ben did well in both analysis courses. He particularly impressed his real analysis instructor with his knowledge and ability to participate in spontaneous mathematics discussion.

Jerry attributed his success as a "seasoned second year student" to the metamorphosis that took place through his experience with symbols while in the process of communicating his mathematical discoveries to others.

Rich was an example of the student commuter who didn't do very much himself in 1979 because he lacked discipline and he did not appear to have enough contact with the disciplined study of the dormitory. Rich did not make enough progress to stay in any of the advanced courses.

Counsellors

Introduction

The counsellor plays a double role: one role as a "teacher" and another as a "student." In both roles, the counsellor is learning mathematics.

As an advanced student, it is obvious he is learning because he is responsible for attending classes, producing solution sets for professors, and writing examinations.

As a teacher to first year number theory students, the learning process for the counsellor is indirect but equally as effective. The counsellor is forced to learn number theory very thoroughly because he must answer any questions the first year participant might have and he must also grade and criticize the methods of solution used by the first year student on solution sets and tests.

Many first year students said they believed they could not have accomplished eight weeks of intensive study without the counsellors. However, counsellors were more important to some students than to others.

Al probably could have learned number theory without the counsellors' help. As a first year student in number theory, he claimed he got little help from the counsellors and he did not need much help because he attended a summer mathematics program in the summer of 1978 at another college. Al was not a typical first year participant because his previous summer experience made him more like a second year participant. He said of his number theory experience in Columbus:

> I found that when I came here I was able to do the problem sets by attending lectures but without attending seminars. I roomed with one of the counsellors and I could discuss problems with him whenever I wanted to.

It is interesting to note that the other summer program Al attended also had counsellors (counsellors in that program were not graduates of the program) he considered important to that program. Al continues,

> There were counsellors who participated in all aspects of the program both socially and academically but they did not insist that the participants hand in a daily quota of problems like they did in the Columbus program. The program was less demanding than the Columbus program.

Al recognized the importance of counsellors by his own admission in that last statement but he implied he preferred help without the kind of pressure he received from counsellors at the summer institute at Ohio State University.

A student like Benjamin who started the number theory program with communication handicaps could not do what was expected of him without the help of the counsellors. The counsellors helped Benjamin develop communication skills through discussion and feedback on the written solution sets he handed in.

Role of the Counsellor

Al and Benjamin represent extremes with respect to the help derived by first year participants in 1979 from counsellors. What did the counsellor do the the average first year participant in the summer of 1979?

A. <u>THE COUNSELLOR APPLIES PRESSURE AND ENCOURAGES TENACITY -- SOMETIMES WITH A HINT.</u>

In general, the counsellor kept the first year student working on what at first glance looked like impossible problem tasks. At the beginning of the summer program Ross told his counsellors:

> Even if it looks hopeless, the students must not be allowed to give up. One must not spoil the student's fun by giving solutions. Counsellors must draw out the student's personal resources to solve problems.

B. <u>THE COUNSELLOR PROVIDES TRADITION</u>

In the words of one counsellor who sometimes worked with Elaine and Jerry in 1979:

> Tradition is very important. The older counsellors provide the tradition. They went through the same process and have made visible progress. The first year student can see counsellors are graduate students and undergraduate students at a number of different well-known and respected colleges.

C. <u>THE COUNSELLOR PROVIDES MOTIVATION</u>

Jack, another counsellor who had tremendous influence with the number theory students of 1979 expressed his role as the following:

> The first two weeks into the mathematics, the counsellors supply motivation and then it becomes internalized and the mathematics itself becomes the motivation. At the beginning, the first year students cannot see where the mathematics is going so that they can't see why they are doing what they are doing. Involvement in the doing of mathematics then becomes an experience that is intrinsically rewarding. That is, the relationship between mathematics and student changes and the counsellor no longer has to supply the motivation.

D. THE COUNSELLOR PROVIDES NECESSARY BACKGROUND KNOWLEDGE AND PRE-REQUISITES.

Elaine gave many reasons why the counsellors were important to her. But the importance of the counsellor became especially clear when she talked about her role as counsellor in the summer of 1980:

> Although the kids come with good knowledge, they have not seen the symbols. We, the counsellors must translate the symbols to simple English because of their initial inexperience. Counsellors have different levels of experience. Students have the opportunity to interact with counsellors at different levels of experience. First year students think we ask too much of them -- and they can't do it. We, the counsellors, convince them that they can do it!

E. THE COUNSELLOR INTERPRETS TEST RESULTS AND OVERCOMES DISCOURAGEMENT.

And so it was with Elaine. In her first year after her disappointment with her first test, the counsellors every-ready support carried her through the problem sets and prepared her for subsequent tests because they let her know they believed in her ability, in spite of her test results.

F. THE COUNSELLOR PROVIDES INDIVIDUALIZED TUTORING.

As a second year participant, when Bob talked about the importance of counsellors, he said he wished he'd realized their importance the previous year because he did not make proper use of them. He said:

> I had experience with very elementary number theory (in high school) before coming to the program last summer and so I was able to do the first few weeks of problem sets without seminar, if I worked hard at it. I did not even realize the value of the counsellors, although they were there when I needed help or advice... The counsellor can be very useful in pointing out some directions... For instance, last year I looked up the Chinese Remainder Theorem but

> I couldn't understand the two pages of general theory surrounding the theorem. I asked a counsellor and he said -- look at it this way; you want these two numbers to be zero with respect to the first modulus and those two numbers to be zero with respect to the second modulus, etc., then it was clear what had to be done and I did it!

G. THE COUNSELLOR MUST SHOW SUPERIOR ACADEMIC LEADERSHIP.

The counsellor role brought to the surface any weakness that a second year student had. Both Sam and Rich complained about their difficulties with counselling because of their weaknesses in number theory. It was impossible for them to hide their weakness because the first year students let everyone know when they were unhappy with the quality of a counsellor's help. In defense of himself, Sam said:

> This year I am learning much more about number theory because I am trying to explain it to my student. You must understand it well before you can teach it to someone else.

Sam blamed his slow progress on his inability to memorize and his need to explore many examples. The counsellors in 1979 said that Sam concentrated mostly on numerical examples and did not move onto the more general considerations underlying the numericals; that this delayed his progress. It is possible that it could have been different if he had discussed the more general consequences of the numerical problems with other students. The counsellors said they tried to move him in more general directions but he stubbornly did exactly what he thought he should do and then complained that ideas were slow in coming when you worked alone! Even Ross could not get through to Sam with advice.

Rich expressed his weakness in background with the blunt acknowledgement of his counselling difficulties because he wanted to be relieved of his duties. The first year participants were putting too much pressure on him and first year students are generally insensitive and even intolerant to a counsellor's difficulties. The counsellors are the first year students' fairy godparents and they complain when they see no magic performed mathematically. One student bluntly said to Rich, "You don't know any more than I do about number theory. Why are you a counsellor?" Immediately after this remark, Rich sought to retire from counselling until he had more mathematics to show. Rich himself complained:

> My role as a part time counsellor is difficult for me; I am learning many things in number theory for the first time. I was a commuter last year and I had to force myself to work at home alone because there were so many distractions and no one to talk to. ...I tried to read about the mathematics whenever I could find something relevent to the problems I was thinking about. I thought I understood what was going on in class but I had trouble expressing myself mathematically. ...I knew what I was saying but no one else did!

H. THE COUNSELLOR PROVIDES HELP WITH COMMUNICATION SKILLS.

Without the same extensive amount of help on communciation skills that the resident students received from the counsellors, Rich wrote nothing down and never did learn how to express himself so he could not communicate with the first year participant. Rich did not have the benefits of full time help from the counsellors in his first year and therefore it appeared he was handicapped towards becoming a good counsellor in his second year.

Todd, Elaine and Jerry felt the counsellor was so important to the program that they each rearranged their own working schedules to give more time to their first year students in the summer of 1980.

I. THE COUNSELLOR PROVIDES FEEDBACK AND EVALUATION OF PROGRESS

Todd changed his personally convenient sleeping hours from the efficient 8 p.m. to 2 a.m. schedule to a more normal midnight to 7 a.m. schedule so his student could easily talk to him without staying up until 2 a.m.

Todd expressed his feeling about the importance of counsellors by the following statement:

> The counsellors were invaluable in 1979. Many definitions and some mathematics could not be found in books, but the counsellors were always there to fill in the holes or to tell you that with more work you could fill in the holes yourself. They were there to encourage you to work harder, give you hints, or to let you know that the problem was trivial. They graded the problem sets and made it quite clear what they did or didn't think of your work.

J. THE COUNSELLOR MONITORS LEARNING STYLES.

Jerry also rearranged his working schedule to allow his student to work in his room without disturbance. He explained:

> This summer I had to keep my first year student in my room while I was working. She was working with other students in a passive way. She would listen to their explanations and copy them down. You can only work well with other students when they are working on the same level as yourself. She was greener than the students she chose to work with...

When Elaine's student did badly on his second number theory test, Elaine decided she needed to give him more of her time. As a result, Elaine found it necessary to give up seminar and solution sets in p-adic analysis and attend only lectures in the last three weeks of the summer institute, 1980.

K. COUNSELLING FORCES THE COUNSELLOR TO THINK MORE DEEPLY ABOUT NUMBER THEORY.

Counselling put a burden of responsiblity on Todd, Jerry, and Elaine that each of them carried in his/her own way. On the other hand, the counselling duty also forced them to "think more deeply" about old and new number theory problems that they would not have reconsidered in their second year, if not for their counsellor role.

Summary

There are two aspects of counselling, one concerning the first year student reaction to counsellors in 1979 and the other concerning the same student reaction in 1980 as a counsellor.

The following explanation of the relationship between student and counsellor by Jack (a counsellor in 1979) sets the stage for the attitude of the counsellor toward the number theory student of 1979:

> Students need to feel that counsellors care and Ross cares. This becomes part of the motivation to learn. At first, you (the first year student) do not want to disappoint the people who care about you and then the pleasure of success carries you along until you really get involved in the mathematics and it becomes a part of you.

Student reaction to counsellors in 1979 was observable even though it was not always verbalized. All the first year number theory students at different times during the summer of 1979, mumbled complaints of how counsellors drove them hard, yet the tone of their complaints seemed to indicate they were in awe of "The Counsellor" while at the same time receiving comfort and support from "The Counsellor." Al's complaint of "the aloofness of the counsellor" was different than that of any other student in 1979. He did not like there to be either a social or academic gap between any counsellor and himself although he seemed to create an insurmountable gap between him and the first year students in 1980 when he was a counsellor. Ben, for instance, continually complained of too much counsellor pressure in the summer of 1979, yet he returned for a second summer and he seemed to put even more pressure on the number theory students of 1980, commenting that they didn't work hard enough!

The 1980 second year student-counsellors said their counsellors of 1979 were important to their success that first year. In the 1980 counsellor meetings, they talked of the need to give back to their number theory students all the "good things" they had received in 1979 like the day-to-day correction of solution sets/mathematical ideas, hints, guidance, tradition, and emotional support.

Problem Sets

Introduction

The problem sets served two important purposes. The problem sets had the capacity to teach and the capacity to monitor the results.

It appeared that much of the learning in the first summer of number theory came through the work done by the student on the problem sets and the resulting feedback from counsellors and seminar discussion of solutions to problems on the sets. The student who failed to do most of the problems or restricted his attention to numerical types of problems or avoided generalizations achieved less on examinations in his first summer. More importantly, he/she appeared to be stunted mathematically in his/her second year.

If a student or a group of students failed to do certain types of problems or exhibited certain types of difficulty, this was immediately recorded on the solution sets. It told the counsellor(s) and/or Ross to make teaching adjustments.

<u>The Role of the Problem Set</u>

From the point of view of the counsellor, why were the problem sets vital to the learning process?

A. <u>THE PROBLEM SETS FORCE A STUDENT TO EXAMINE THE CONJECTURES FOR TRUTH.</u>

From Lee (an advanced counsellor, a sophomore and mathematics major at Yale taking graduate mathematics courses) one hears the following explanation for the importance of problem sets:

> The problem sets make you look at things very carefully -- to see a pattern and maybe to make conjectures that fit the existing examples. It is this looking carefully at the obvious and non-obvious for general information -- to prove or disprove your conjecture that forces you to know more about your problem. This attitude about solving problems is what you carry into other subjects. You learn not only to look at what is on top but what is hidden that may change the surface suggestion of truth and make a statement false. The conjectures given in the problem sets are designed with this in mind. The headings to the problem statements say, 'PROVE OR DISPROVE OR SALVAGE IF POSSIBLE.' This means you must think very carefully about each statement since you look for a counterexample; then if you cannot find one, you try to prove it true. If you cannot do either, then you make up examples of what you think look like reasonable statements and examine them for clues. Sometimes when you think something is true, you find it easy to prove since you can make errors and still think you had the right idea because the truth of the statement does not challenge your ideas. But if a statement is false, or false

> in part and you try to prove it, then it is interesting for you and the other students to see how convincing an argument can be and still be false. One must be very precise to do mathematics correctly. One learns not to believe statements until a good proof is given...

B. THE PROBLEM SETS FORCE A STUDENT TO DO HIS/HER OWN RESEARCH.

Counsellor Lee continues,

> ...the problem sets raises questions that are different than the kind given in high school. High school teachers are so busy that they can only give questions that are easily solved. They also give you all the background information needed to answer the question. This is a real time saver, since it saves the student the time of researching the problem. The student gains time but loses the research experience.

C. THE PROBLEM SETS SUGGEST DIFFERENT QUESTIONS TO DIFFERENT STUDENTS ON DIFFERENT LEVELS AND FORCES EACH STUDENT TO BE PRECISE IN HIS/HER LANGUAGE.

From Jack, a third year counsellor and part of the Elaine - Jerry study group in 1980, we hear the following:

> All the sciences need to use precise language to describe their problems. Writing solution sets provide good exercise for learning to be precise in language and in development of proof of one's conjectures. The very nature of conjecture is to raise more interesting questions to answer questions. In the problem sets one can answer a problem (maybe a numerical) on a certain level and then go on to another problem or one can raise another question which raises another question or maybe generalizes a generalization and then take three years to answer a problem. Obviously one does not go to this extreme as a first year student or problem set number one would take the whole summer. The depth with which one chooses to answer a question will vary and is dependent upon the individual student. Of course, the deeper one goes into a problem, the more one learns. A second year student can go through the same first year problem set for a second time and look at Ross' lecture in a slightly different way than in the

> previous year and see a totally new set of problems and again not finish with them that summer.

D. THE PROBLEM SETS TEACH A STUDENT TO EXAMINE EXPLICIT AND IMPLICIT ASSUMPTIONS.

From Jerry one hears that the problem sets teach students not to take anything for granted since they ask a student to prove what may appear obvious on the surface. He explains:

> You learn not to take things for granted. For instance, you do not take for granted that there are no integers between zero and one. This is something (on one of the problem sets) that needs to be proven about integers although it is intuitively obvious... If you take that as obvious, then you must take all the non-obvious questions dependent upon that as obvious.

E. THE PROBLEM SETS DEVELOP PATIENCE, GOOD OBSERVATION, AND TENACITY

From Elaine one hears that tracking through the problem sets is similar to tracking through a survival course. She gives the following description:

> The mathematics summer program is like an outward bound survival program. We work hard, develop patience, good observation, and tenacity. Sometimes one is purposely stymied by questions on problem sets. Problems on the sets range from very accessible to very difficult.

F. THE PROBLEM SETS PROVIDE THE STUDENT WITH OPPORTUNITIES TO EXPERIENCE THE THRILL OF DISCOVERY.

Jack adds:

> There are many things I pick up now that I hadn't seen before as I work with my first year number theory student. We give help but we do not give answers. If we gave answers there would be no discovery. You can never duplicate or describe the thrill you have of discovery by saying to someone else, 'this is how it felt.'

G. THE PROBLEM SETS PROVIDE A BLUEPRINT FOR THE GRADUAL CONSTRUCTION OF MATHEMATICAL EXPERIENCE.

It appears that each student must build his experience along with the early problem sets in order to survive the later more difficult problem sets. Commuters have difficulty "surviving" because they do not make enough contact with counsellors and fall behind on problem sets which make both lectures and seminars eventually unintelligible to them.

H. THE PROBLEM SET MONITORS THE QUANTITY AND QUALITY OF A STUDENT'S WORK.

For example, the problem sets at the beginning of the 1979 summer program told Maureen and the counsellors she didn't understand much of what was happening in lecture and she probably would never catch up without the full time help of the counsellors. By the second week of the program, she was unable to write solutions to the problem sets although her high school mathematics background (and record) were excellent. As her solutions dwindled down to a very few, she was moved into the dormitory; this seemed to revitalize her efforts and gave her enough support to stay afloat through the summer.

Paul was also a commuter. Early in the program of 1979, Paul was accused by the counsellors of 'not working hard enough.' The fact was, Paul asked more questions in his seminar than anyone else and it appeared that he could not do the problems on the sets because he often did not understand the problems he was asking about. Paul was trying to understand the problems on the sets, but he could only understand numerical techniques and could not generalize. When he realized he was behind the resident students, he said: "I now do a problem here and there in an attempt to stay systematically behind." A problem here and there was not good enough. Paul did not survive the summer program past the second test. His failure to keep up with the work was registered by the quality and quantity of solutions he gave to the counsellors.

I. <u>THE PROBLEM SETS RECORD THE INDIVIDUAL'S PATTERN OF PROGRESS.</u>

In the summer mathematics program, it was impossible to hide a number theory student's progress or lack of progress for any extended period of time because the progress report was recorded on the solution sets.

Rich managed to get away with no visible progress report because he was a commuter and he said he didn't have enough time to write down his solutions. The counsellors soon learned that a lack of visible progress on the problem sets summed up to a lack of progress, as confirmed by the written test.

Bob started in the summer of 1979 with a stronger mathematical background than most of the other first year number theory students and this showed on his solution sets and the first test. He admitted he needed little help on sets at the beginning and later didn't ask for much help because he was not in the habit of asking. The solution sets showed when Bob slowed down and some other students caught up and passed him. Bob started out so strong that he did not realize the teaching power of the counsellor and the problem set. By the end of his first summer, the quality of his solutions fell behind those of Jerry, Elaine, Todd, and Benjamin. His second summer was also not as fruitful as that of the students just mentioned.

J. THE PROBLEM SETS PACE THE STUDENT.

Al came in the summer of 1979 with a previous summer's experience in another mathematics program. Al was able to learn number theory with little help because the previous experience made it possible for Al to pace his own progress with problem sets.

K. THE PROBLEM SETS PROVIDED THE OPPORTUNITIES FOR DEVELOPING THE COMMUNICATION SKILLS.

Some of the observable results on solution sets (during the course of the summer of 1979) were the individual's developing communication skills. The translation of symbols into plain English and English into symbols and mathematical sentences was recorded on solution sets.

All number theory students had to make improvements in their communication skills in order to handle the more complicated and abstract generalization type questions on the later sets. In some students, the improvement in communication skills was extraordinary. This improvement was monitored by the solutions sets. The counsellors passed the information from the sets to Ross who was always kept informed. The solution sets made the counsellor quickly aware of unusual progress as in the case of Benjamin.

Benjamin's first communications came through his written solutions to problem sets. Later the counsellors' efforts to understand and correct his odd use of language on solutions resulted in conversations between him and counsellors and sometimes other students. The expansion of Benjamin's communications skills was described by the following statement from his case history:

> It is hard to tell which came first. The counsellors increased ability to communicate with Benjamin or Benjamin's increased ability to communicate with the counsellors. In any case, Benjamin started to talk with not only the counsellors but to everyone else involved in the summer program (of 1979). After the initial strangeness of Benjamin's speech patterns, everyone became used to his use of language. Benjamin complained about how hard he had to work (particularly on problem sets) but he worked hard anyway.

L. THE PROBLEM SETS DEVELOP GOOD WORKING HABITS.

It appeared that Todd became proficient in number theory because of his hard work and his persistence in keeping up with solutions. He worked solutions in complete detail for almost all problems on all the problem sets. He not only learned to communicate his mathematical solutions, he also developed good working habits.

This was in contrast to Sam who worked mostly numerical problems and then stopped even those after the sixth week in 1979. Sam had the experience of solving numerical problems which appeared to be an important beginning but he stopped there. Stopping short of the generalization type problems seemed to arrest the learning process almost at the level of the "finger exercise."

Summary

The problem set as a teaching device appears to have been effective.

The number theory student had to discover new techniques and new mathematics to do many problems on the sets.

The number theory student had to develop communication skills to express the new mathematics he/she discovered. The problem set experience made him/her familiar with the use of symbols. The number theory student generally started his first solution set with long essays and later became proficient with more precise symbols and fewer words.

The solution set as a monitor of the quantity and quality of work also appeared to be effective.

The problem sets could be called the "watchdogs" of the summer institute. They gave the first signs that something good or bad was happening to the student.

The solutions to problem sets described difficulties that number theory students might generally be having at some point in time. The problem set allowed the summer program to "play it (mathematics) by ear." If the problem sets showed certain mathematical areas were weak amongst many number theory students, extra lectures and/or extra help seminars were prepared to cover these weaknesses. An individual student's weaknesses was often brought out by a problem set and a counsellor could give a specific kind of help.

Group Interaction and Living Conditions

Introduction

It is not possible to separate the group interaction from the living conditions. One may observe that because the participants lived together there was more than usual group interaction.

The impact of living conditions on the learning process is emphasized by the plight of the commuter. But why did they have difficulties? Certainly not just because they lived in a different location than the resident students. It appeared that they were denied the same intensity of group interactions that the resident students enjoyed with other participants.

The counsellors were also an important part of the group interactions. The study groups had instant feedback from various counsellors who moved in and out of the groups with directions and suggestions and were sometimes themselves participants in group activities. Although the counsellor and his student often developed a strong one-to-one relationship, these relationships often expanded to include study groups or study mates because all resident participants lived together. If a particular number theory student was having difficulties, explanations or mini lectures concerning those difficulties might include whoever happened to be within hearing of that participant. Paul and Stella, although they were commuters, appeared to derive benefit from the group interactions for the time they were able to spend at the dormitory.

The Role of the Living Conditions

During an interview with some counsellors and participants in the summer of 1980, a reporter for the campus newspaper asked the following question: Question. Is there an advantage for students in a dormitory together rather than at home?

A. <u>STUDENTS WITH A COMMON INTEREST OF MATHEMATICS PRODUCE MORE QUESTIONS AND ANSWERS TOGETHER THAN THEY DO SEPARATELY.</u>

(Lee, a sophomore mathematics major at Yale University and a third year participant):

> Putting students together with a common interest in mathematics and a common set of problems to work means there is much interaction between students who give and take questions and answers about mathematics and sometimes other things.

B. <u>STUDENTS FIND IT HUMBLING TO SEE OTHERS AS GOOD AS THEY ARE AT MATHEMATICS.</u>

Elaine indicated that the initial response to dormitory is living in SHOCK.

> It is very HUMBLING because you find other people who are better after you were the best! When I first came I was awed by everyone -- then I found they were awed by me. You learn mutual respect because you must work with each other.

C. <u>SOME STUDENTS FEEL THEY SHARE AN AESTHETIC EXPERIENCE WITH SCHEMES AND IDEAS AS AN ART FORM.</u>

Elaine continues,

> You see other students interested in the same kinds of things - that makes you feel you are a part of a scheme of ideas that exist out there and need to be put together into beautiful patterns. I think it would be hard if we were commuters and could not share the experiences.

D. <u>SOME STUDENTS ENJOY THE INTENSE COMPETITION.</u>

Sam indicated that the intense competition in the dormitory kept you going on a problem because you wanted to be the first to crack it!

E. STUDENTS KEEP EACH OTHER WORKING LONG HOURS.

Jack stated that you become accustomed to working long hours because everyone else is doing it too.

The Role of Interaction

Did all participants gain from the strong interaction produced by the living conditions?

F. GAINS ARE MADE BY STUDENTS WHO USE RECIPROCAL WORK ARRANGEMENTS WISELY.

Certain counsellors like Elaine believed that enormous gains could be made by students who used group interaction wisely. She voiced disappointment that her students at a certain point in the summer of 1980 were not taking advantage of it. She remarked,

> My students this year are not working with each other; they do not seem to be able to share their ideas and draw mathematical help from each other. They work individually and in isolation although they could benefit from a study relationship. I have suggested this as strongly as I could without ordering it, but without results. They will only accept help from a counsellor when in many cases their problems would be efficiently solved with a reciprocal work arrangement with another student with difficulties - but not exactly the same difficulties.

G. GROUP STUDY CAN CARRY STUDENTS THROUGH THE FIRST YEAR PROGRAM.

One may recall that an inner study group in 1979 formed around Elaine and also contained Jerry, Bob, and Maureen. Students and counsellors came and went from the suite shared by Elaine and Maureen, where many of the mathematical and social activities were planned.

Maureen started as a commuter and probably would have been an early dropout if she had not moved into the suite with Elaine. Together, Bob and Maureen rode the study waves made by the study group even when their interest in mathematics waned.

H. GROUP STUDY IS NOT GOOD FOR ALL FIRST YEAR STUDENTS.

But it may be noted that even the student, who worked well in groups, did not believe that group study was good for all students. Recall that Jerry said group study could be destructive. He said,

> This summer I had to keep my first year student in my room while I was working. She was working with other students in a passive way. She would listen to their explanations and copy them down. You can only work well with other students when they are working on the same level and they have the same experience as yourself.

Ben said much the same thing as Jerry when he commented that,

> Groups are not always good for students. Often there is someone who is ahead of everyone else and he does the work. One student has all the ideas and everyone else listens. If a student can find one or two students who work at the same level as himself, then it is good for everyone in that group and more mathematics gets done than if they each worked alone, but not 2^p times as much.

I. THE LONERS DEVELOP SPONTANEOUS FORMS OF INTERACTION.

The "loner" derived support from certain group interactions although not so obviously. Although Todd in 1979 felt himself outside the inner study group, this did not stop him from hanging around long enough to absorb support and information and then sleeping until 2 a.m. when Jerry, his roommate woke him so he could work out his solutions. This behavior would not have been possible in a commuter program. Ben was carefully protected at home and treated as a "special student" by his school but he was not treated differently from any other participant in the mathematics summer program. Ben may not have been forced to communicate if it were not for the intimate friendships developed because of the living conditions. Al seemed to enjoy the group interaction even when it was negative interaction. He enjoyed his position as "bully" and "top dog." For position, one needs an audience that the dormitory living conditions provided. He also gave other students the will to try to "beat him." Rich derived much help from his study relationship with Todd, but Todd derived much more from trying to help Rich. Sam said he wanted to form some good study relationships but couldn't, so he met spontaneously with Ben, or Todd, or other students in hallways and discussed current problems. Same did not appear to work that well alone, and probably would have benefited from a support group.

Interaction was not always mathematical. The impact of the social interaction on the program was sometimes obvious and sometimes subtle enough to be hardly noticeable.

J. SOCIAL INTERACTION IS NOT PLANNED FOR PARTICIPANTS BUT HAPPENS ANYWAY

There were hardly any social events planned. On an occasional Friday evening, there were cultural events (for instance, a talk by a polar scientist on polar expeditions, or a performance of a musical group) planned by the counsellors for the summer institute participants. However, there were never any social events planned to take place off-campus. Actually, all off-campus excursions by any participant(s) needed the permission of Ross and most excursions were discouraged. It was quickly understood by all participants that the purpose of the summer institute was to learn mathematics.

Social interaction among participants needed no planning. Interactions among participants happened without planning and they did influence the outcomes of the summer institute. The institute students organized their own soccer games, played tennis, used the university pool, played piano, and threw frisbees whenever they found time between problems.

K. STUDY GROUPS ARE ALSO SOCIAL UNIONS.

Study group interactions such as those of the inner and outer circles were social as well as mathematical unions. Between problems, they visited with each other, ate "captain crunch", and took talks, and ate snacks together on High Street on many a hot summer night.

L. SOCIAL INTERACTION AFFECTS/SUPPORTS MATHEMATICAL ACHIEVEMENT.

The social interaction between Bob and Maureen obviously had some effect on the results of their summer institute experience.

It appeared the subtle supportive mathematical interaction between Ben and Todd before and after seminar extended to interaction between Ben and the rest of the participants and became a less subtle social interaction with discussion of his communication handicaps as part of nonchalant small talk. For example:

Ben: I think I will teach mathematics.

Todd: How will you teach when you can't speak clearly?

Ben: I can speak good mathematics!

Todd: I can see it all now...

Ben: Maybe I'll become an English major and learn how to speak good English.

Elaine: Why not?

Jack: If a mathematics professor is good enough at doing mathematics, he doesn't have to speak English!

M. MATHEMATICAL INTERACTION IS SOMETIMES DEPENDENT UPON SOCIAL INTERACTION.

Todd and Rich also formed a constructive mathematics relationship; not because they were at the same level mathematically but because both were fifteen years old and they liked each other. They seemed to give each other emotional as well as mathematical support.

Sam seemed to need more supportive friendship than anyone was willing to give him. He was a "loner" because no one liked him well enough to want to work with him. Ross tried to organize study pairs (one below the median on test two and one above the median on test two). It didn't work between Sam and Dick or between any other two participants who did not already work together, because the social aspect of the study group was missing.

Summary

The living conditions produced a stronger interaction (both mathematical and social) among students than one might expect in a commuter program. The living conditions also produced an intensity of purpose, in this case "to study mathematics" because all the energy of the participants was focused on that purpose.

Delayed Effects or Changes

Introduction

Students in the mathematics summer institute appeared to learn a process as well as a product. This process may alter or develop a personality in certain directions. Each summer, Ross gathers a group of teenagers who share "eagerness, curiosity and unbounded, though hitherto undirected, vitality." He emphasizes, "What they will experience in these eight weeks is more than a course in higher mathematics. They will spend these weeks in total dedication to something that is difficult and challenging. They will not go home unchanged."

What kind of changes did summer institute counsellors and former number theory students observe in their own behavior, or that of their first year charges?

A. FIRST YEAR STUDENTS DEVELOP THE ABILITY TO DEDICATE THEIR ENERGY TO COMPLETING A DIFFICULT TASK.

Carl, a counsellor in 1979 and a mathematics major at The California Institute of Technology said of his first year number theory students,

> They learned, for instance, the importance of dedication to one's task, whatever it might be, and the importance of a personal committment to carry out the task to the finish in the face of great difficulties.

B. STUDENTS CHANGE THE WAY THEY LOOK AT THE WORLD. THEY COME UP WITH THEIR OWN IDEAS AND TRY TO PROVE THEM.

Lee, an advanced student and counsellor, gives a personal description of changes brought on by the experience. She says,

> I do not think that the effect of the special mathematics program is a direct one. I think that <u>doing mathematics</u> alters one's personality. It makes changes in the way one looks at the world... When I came for my first summer, I wan't interested in math but I had heard the program was intellecually stimulating and I wanted to think about difficult questions and learn self control... I had never before been told that I could come up with my own idea and then prove it or not.

Lee is now a sophomore mathematics major at Yale University taking advanced graduate mathematics courses.

C. <u>THE STUDENT REMEMBERS THE EXPERIENCE AND THE THRILL OF THE "VICTORY" OF SOLUTION AFTER HARD WORK.</u>

In Elaine's experience,

> The math summer program is like an outward bound survival program -- we work hard, develop patience, good observation, and tenacity. Sometimes one is purposely stymied.

In Jack's experience,

> You can never duplicate or describe the thrill you have of discovery by saying to someone else, 'this is how it felt'...

D. <u>STUDENTS DEVELOP THEIR INTERNAL CREATIVE TALENT.</u>

Elaine continues,

> I have entrance to Harvard although I think it is too early to decide what career I will follow. I do not think that the mathematics program trains only future mathematicians or scientists. I am sure that this experience has helped the creativity in each of us and will help us accomplish our own personal goals - be they in science or art or whatever.

E. PARTICIPANTS WILL PROBABLY USE SOMEWHAT THE SAME KIND OF APPROACHES FOR CRACKING THOGHTFUL PROBLEMS IN ANY DISCIPLINE.

Jack, an undergraduate mathematics major at the University of Chicago describes his experience with changing intuition in the following,

> Mathematics seems to develop one's intuition and it employs the kinds of methods needed to solve scientific questions.

F. MATHEMATICAL INTUITION CHANGES WITH EXPERIENCE.

Jack goes on to say,

> Intuition seems to need to change with more mathematics. It becomes necessary to change the ideas you have when they seemingly contradict the mathematics and you can prove it! One learns to synthesize thoughts and develop a new kind of intuition based on mathematical experience. Things aren't always as they seem.

Recall that Elaine pointed to Ben's different mathematical intuition developed after the summer of 1979, "Ben's intuition is different from ours because he works from a background of tons of examples in his head." "You see," Jack says, "He eats up mathematics books like they were hamburgers." Yet, Ben cannot read (on standard achievement tests) at the "normal" high school level. Ben, at seventeen years of age, is now a non-degree graduate student in mathematics at the University of Illinois because he could and did read much mathematics in 1980 which developed his mathematical intuition.

Summary

Some changes were observed and described by the counsellors. Some changes observed and even discussed in counsellor meetings seemed so natural as to go unmentioned under the heading of "change." For instance, counsellors and instructors observed and discussed the first year students struggle to master the numericals first, and gradually it seemed he/she moved on to the more general abstractly formulated problem. The learning process through which the student in the program passed suggested a hierarchy of changes that will be discussed in Chapter 6. There also appeared to be a natural transition from ideas to mathematical language.

Books

Introduction

In number theory, there are many reference books and so one might expect the first year participants, who were studying number theory, to look into many books and derive much more help from books than they did. (See Appendix C for book list.)

Al was the only first year participant with a previous mathematical experience beyond the high school level because he attended a mathematics summer institute the previous summer. He appeared to be the only first year participant truly able to benefit from reading mathematics books. Indeed, the ability to derive benefit from "reading mathematics" seems to herald the passing of the young mathematics student from one stage of "mathematical maturity" into another as exemplified by Benjamin's advanced mathematical inutition associated with his "eating up mathematics books like they were hamburgers."

Role of Books

Why didn't the first year student read more mathematics?

A. <u>THE FIRST YEAR STUDENT FINDS THE MATHEMATICS BOOKS "TOO GENERALIZED" TO READ.</u>

The comment that the mathematics books were too "generalized" and difficult to read seemed to be a common complaint among first year participants and less successful second year students like Sam who claimed he had to do his own thing (write hiw own solutions) before he could understand the mathematics.

The following statement from Bob is typical of the comments made to the investigator concerning the first year participant's ability to understand mathematics from books:

> I couldn't read mathematics books last year. I wouldn't have known where to look and what to look for and I'm not sure I would have been able to understand the connections anyway. The books are hard to read because they are written in an abstract general form. For instance, last year I looked up the Chinese Remainder Theorem but I couldn't understand the two pages of general theory surrounding the theorem. I asked John (counsellor) and he said "look at it this way..." then it was clear what had to be done and I did it!

B. <u>IT APPEARS A STUDENT CAN READ MATHEMATICS IF HE/SHE HAS THE EXPERIENCE TO WRITE IT OUT AND FIGURE THE DETAILS.</u>

From Jerry there is an equally revealing statement (made in 1980) about the difficulty for first year participants of reading mathematics books:

> I am just learning how to read books. I can figure out many of the mathematics problems on my own if I write them out as I read them and think about them. I didn't read mathematics books at all last year (1979) except when a problem involved a reading search.

C. COUNSELLORS ARE MADE AWARE OF THE STUDENT TEMPTATION TO COPY SOLUTIONS FROM BOOKS WITHOUT UNDERSTANDING THEM.

Ross was well aware that the first year participant would have trouble reading mathematics books. He pointed out the difficulties the number theory students would have with trying to look up solutions to problems on problem sets; and not fully understanding the given solutions they would copy them down and look no further into their meaning. Ross regarded this as a danger the counsellor should be aware of. Many counsellors suggested that the reference books be kept away from the first year participant but Ross thought the contact could be beneficial, provided they were guided into learning to understand what they read. Part of the counsellors' job was to hold the books in their rooms and see that the first year student used them "wisely."

D. BOOK SEARCH QUESTIONS ON PROBLEM SETS GIVE THE STUDENTS A REASON TO LOOK INTO BOOKS.

There were "book search" questions on number theory problem sets, to acquaint the first year student with the availability of books on mathematical reference material. Ross said that by the second year, students and books should become "good friends."

Summary

Sometimes books and the second year participants were good friends and sometimes they weren't. But when they weren't, it appeared they had trouble keeping up with second year lectures. The ability to read mathematics books seemed to be a mark of readiness for second year lectures and a recognizable level of a learning hierarchy in higher mathematics. Todd, Ben, Jerry, Elaine, and Al could and did read reference mathematics books in their second year to supplement their studies.

It appeared that Sam and Rich would not or could not do very much reading.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

In Chapter VI the analyst separately considers observations of each of the important nurture components and she makes conclusions concerning these.

These individual conclusions are followed by more general conclusions concerning the nurture of high ability children and the chapter ends finally with some recommendations.

Chapter VI is organized in the following way:

A. Conclusions about

   1. Problem sets
   2. Counsellors
   3. Lectures
   4. Seminars
   5. Group interactions
   6. Living conditions
   7. Books.

B. Conclusions about the interaction of nurture components.

C. Conclusions about levels of mathematical maturity.

D. Recommendations for further study.

E. Recommendations for organizing and adapting other programs for high ability children.

Conclusions

1) Problem Sets

PROBLEM SETS HAD TO BE DESIGNED TO TEACH WITH LEVELS OF MATHEMATICAL DIFFICULTY (MATURITY) WRITTEN INTO THE ORDER OF QUESTIONS AND TIME OF APPEARANCE.

Problem sets brought out the creativity in the student as well as teaching him/her mathematics. Some of the problems on problem sets depended somewhat on memory of lecture, and some were exercises testing methods outlined in lecture, but the types of problems given on problem sets never ended there. Any one set was written at many levels of accessibility. A single problem set might have some numerical questions and/or some short proof theorems and/or groups of theorems to be organized into lemmas for a more general theorem and/or ingenuity and/or open ended problems with no known solutions. For instance, some topics like that of quadratic reciprocity were developed in parts, by lemmas over many problem sets, because the background for the problem was so extensive it had to be developed slowly.

The solutions (both verbal and written) developed by the student to answer the problems on sets, taught the student to use language precisely to express himself/herself. The first year student's use of precise and concise language in answer to problems on problem sets was practiced daily in conversations between counselors and students, and between instructors and students. These mathematical expressions were finally written in solution sets to be corrected by counsellors.

2) Counsellors

COUNSELLORS WERE AN EFFECTIVE GROUP OF PEER TEACHERS AND THE PYRAMIDAL NURTURE SYSTEM SUCCESSFULLY TRAINED THE JUNIOR COUNSELLOR FOR PEER TEACHING MAINLY THROUGH THE MEDIUM OF PROBLEM SETS.

Counsellors were the immediate and most convenient contact for the first year student. Much of the teaching in the summer institute came through the discussion between counsellor and student concerning probable directions for solutions to new problems on problem sets. Counsellors also contributed suggestions for the adjustments that should be made on students' previous written solutions of problems on previous problem sets.

THE PROGRAM OF NURTURE HAD TO BE FLEXIBLE AND ALLOW FOR DIFFERENCES AMONG STUDENTS.

There had to be someone close enough to first year students to notice the differences and guide the differences toward the best possible outcome (within the structure of the program) for the individual participant. In the summer institute this function was performed by the counsellors. (see chapter 5, Role of the Counsellor, I., J)

Students, whether gifted mathematically or not, are more different than they are alike. A group of mathematically gifted children never appears to be academically homogeneous even given that the children are the same age and of the same background. All students in a group may be of high ability but they are different. On student may like to study in a group (see case studies, chapter 4, Group Study students, Elaine, Jerry, or Bob) or another may prefer to study alone (see chapter 4, Loners, Todd, Sam, Al). Group study may be good for one student and bad for another and may be independent of preference (see chapter 5, Role of Living Conditions, G., H.). One student may accomplish more by working in the very early hours (Todd) and another may work more efficiently in the late evening hours (inner circle study group). One student may be verbally fluent (Elaine) and another may not (Benjamin). One student may try to read reference books immediately (Benjamin) and another may need to be coaxed (Sam).

A program of mathematical nurture cannot plan to provide equal treatment for all high ability students. A program must be aware of, and sensitive to, the many variations in personalities of gifted children. A nurture program may plan to teach the students as a group, but they must also plan to adapt to the study needs of the individual.

3) Lectures

EXPOSITORY LECTURES ARE HARD FOR THE INEXPERIENCED STUDENT TO FOLLOW

It appeared that the first year student could not follow the expository lecture; this changed in his/her second year. It seemed that if a student made good progress in his/her first year then he/she could adjust to any type of lecturer, and the form of lecture with advanced students became less significant. (see chapter 5, Role of Teaching A.)

The program of the summer institute was designed to develop the student's ability to play an active role in learning mathematics. Not only was the student expected to (i) listen and understand someone else's explanation of mathematical concepts, he/she was also expected to (ii) talk, (iii) write, and (iv) read about mathematical ideas. Actually the program seemed to need to develop the last three abilities in order to achieve the first. The first year student seemed to need much experience, with numericals or examples of concepts and background materials surrounding concepts, before the concepts were introduced. Statements of theory with complete definitions and polished elegant proofs of theorems, and with an occasional example given after the proof of a statement, did not seem to explain concepts to the inexperienced student. First year students who attended other mathematics lectures, besides number theory, usually had difficulty following the formal lectures. First year students did not survive very long in expository type lecture courses. This is particularly interesting, since the most common way of passing on information and conducting classes in higher science courses (including mathematics) is the formal expository lecture.

4) Seminars

A DYNAMIC AND EFFECTIVE PART OF THE NURTURE PROCESS WAS THE PROGRAM'S ABILITY TO PLAY IT BY EAR

A program of nurture must be flexible because students change from year to year and other things change from day to day. The summer institute had to be flexible enough and informed enough to make either (1) learning adjustments or (2) teaching and problem set adjustments suited to the students' progress.

Seminar instructors focused on general student progress/difficulties. Seminar instructors kept the summer institute and the lecturer generally informed so the program could adapt quickly. Seminars filled in holes that counsellors were unable to fill and gave direction and depth to problem solutions. Seminar instructors asked for questions from students and then asked students in seminar to explain their understanding of the questions. That is, the instructor often presented the questions for class discussion. The seminar instructor might ask individual students to discuss or write their ideas on proof of a solution to a question raised in class. A student speaker was open to criticism or suggestion from any listener including the seminar instructor. This gave the instructor a general feel for the individual as well as the group progress of seminar students on all the ideas discussed on a particular day. The lecturer was informed of the students' progress/difficulties. If the students in seminar were not making progress thought by the lecturer to be good enough for just a teaching or problem set adjustment, then special tutoring and teaching sessions were arranged for groups of first year students according to their needs. (see chapter 3, Important Components under Problem Sets.)

5) Group Interaction

GROUP INTERACTIONS WERE AN EFFECTIVE STIMULANT FOR MATHEMATICAL DISCUSSIONS AND DISCOVERIES

Planned study groups obviously stimulated discoveries. The contact of students with thoughts on the same kinds of problems (usually from problem sets) often exploded into spontaneous short term study groups, over lunch, or in hallway encounter groups, even between such loners as Ben and Todd.

6) Living Conditions

THE DORMITORY SETUP EXCLUSIVELY FOR MATHEMATICS PROGRAM PARTICIPANTS EFFECTIVELY ARRANGED FOR A PLACE WHERE METHODS OF DISCIPLINED STUDY COULD BE PREACTICED.

The dormitory environment was subtly controlled by the counsellors. They did not behave as police, but guided students mostly by their presence and their approval or disapproval of certain types of activities. Youngsters seemed to respond readily to peer pressure. The living conditions, with considerable help from the counsellors, stimulated group interactions between counsellors and students, or between students and students.

MATHEMATICAL NURTURE OF STUDENT COMMUTERS WAS MUCH MORE DIFFICULT THAN THAT OF LIVE-IN STUDENTS

Commuters appeared to be disadvantaged compared to the dormitory student. Commuters seemed to miss extensive contact with the counsellor and the disciplined study within the dormitory. Seven commuters started in 1979 as first year students and only three finished the eight week program. Only one of thirteen dormitory students dropped out of the program. Furthermore, the three commuters finished in the last four positions on test 3, and the quality of their problem sets in the opinion of the counsellors, was considerably below that of most dormitory students. (see appendix A for test scores)

7) Books

THE ABILITY AND DESIRE TO READ MATHEMATICS REFERENCE BOOKS SEEMED TO BE A NECESSITY FOR THE SECOND YEAR PARTICIPANT

The ability to read mathematics books seemed to develop slowly among the young first year participants, but it was present in those students who ultimately did well in advanced courses as second year participants. It is possible that the inability to read was the real handicap in learning advanced mathematics; or it is possible that learning to read mathematics develops the maturity needed to handle the advanced topics. Ross was well aware of the importance of reference books when he arranged a small lending library in the counsellor rooms and also provided book searches on problem sets. (chapter 5, Books D.) Maybe, as in the case of Benjamin, learning to read mathematics and then reading heavily, suitably changes one's intuition for higher mathematics.

NURTURE COMPONENTS INTERACT AND WOULD NOT BE EFFECTIVE IF ISOLATED

In observation of the case studies in assessing the impact of the various features of the program upon the first year participants' mathematical nurture and learning, it seemed natural to look for some ordering of importance of these different features in their respective contributions to mathematical maturity of the participant. In order of decreasing effectiveness, it appeared at first glance that in the first summer, students absorbed the most mathematics from:

1. problem sets
2. counsellors
3. seminars
4. lectures
5. group interaction
6. living conditions
7. books

However, upon reflection and after becoming immersed in this process of development, the appearance of an order of effectiveness of the parts of a nurture process fades into a view of a dynamic whole nurture process. The above order of effective features is an illusion, and the separation of the above nurture components might even be counterproductive. The nurture components in the Ross program can neither be separated nor ordered with respect to effectiveness.

The different components of nurture seemed to have a kind of symbiotic relationship to one another. For instance, the lecture proposed the conjecture that sent the student for more information to clarify the lecture (see chapter 5, Role of Teaching A.) The lectures were deliberately delivered in a non-expository manner with examples and hints of an underlying structure, but with only enough information to send the listener to other sources to fill in the missing parts. The student consulted with such sources as the counsellors, problem sets, books, seminars or other students. (see chapter 5, Role of Teaching G.; Role of Counsellors I.; Role of Problem Sets G.; Role of the Living Conditions A.; Role of Books C. and D.) If the listener did not supply the missing parts then the next lecture may have been incomprehensible or at best incomplete. The lectures were not directed at the inactive listener. (see Chapter 5, Role of Teaching B - G; Chapter 3, Overview B.)

Although problem sets were closely related to the lectures, the relationships were not always clear to the student unless he or she worked at both simultaneously. (see Chapter 5, Role of the Problem Set C., D; Chapter 3, Overview C.)

The problem sets organized communication most importantly between the counsellors and the first year participants, but also between the first year participants and everyone else in the summer institute - even visitors. (see chapter 5, Role of Problem Sets H., K; The Role of Living Conditions F., G.; The Role of the Counsellor A., B., C., D., H.; chapter 3, Overview E. part (b)).

The seminars discussed problems that students were thinking about from lectures and problem sets. Seminars gave probable direction for solutions but demanded that the student check it out and fill in the missing detail. (see chapter 3, Overview F.)

Although reading mathematics books to find solutions was deliberately encouraged by Ross, the counsellor was forewarned of the danger of reading without comprehension. The counsellor looked over the first year student's shoulder and made it clear that he thought it should be a matter of pride for the student to discover a solution rather than just copy a book's solution. If the student chose to read about a problem, then the counsellor asked the student to translate a solution and rewrite it to make it clearly the student's own. This was an important part of the process of learning how to read mathematics books since many of the better students became desperate for solutions on the very hard problems, and resorted to reading heavily to find a way out of their difficulties. (see chapter 5, Books C.)

<u>THERE WERE RECOGNIZABLE LEVELS OF MATURITY OF A FIRST YEAR PARTICIPANT</u>

There seemed to be developmental stages of mathematical maturity observed in the progress of the summer institute participant.

Inexperienced first year participants started working the numerical examples and at first could do no other type of problem. First they developed a mastery of numerical problems in the sense of finding a particular solution to the particular numerical. The numerical problem suggested no conjectures and no generalizations to first year students at the early stage of maturity, (as in the case of Lorraine), and the counsellor had to lead the student into the conjectures suggested by the numerical. That is, the student treated the numerical as a "finger exercise." The participant at the second level of maturity and making good progress began to work successfully (withou extensive help from the counsellor) on the well-formulated problem asking for short proof. Later that same student might start to solve well-formulated abstract problems with a non-obvious solution. The ability to find a generalization, or to derive theorems based on underlying concepts from experimentation with numericals, came only after successful experience with both proof of the well-formulated theorem and the successful completion of many numericals.

RECOMMENDATIONS

Certainly both the nature and nurture of gifted students involve variables whose impact is not fully assessed in this study. Based upon her observations, the investigator would entertain strong doubts about the value of research in which efforts are made to separate one variable from others to try to rank it's importance relative to other variables. What she hopes for, then, are other case studies; case studies that involve rather direct observation of students identified by some carefully described procedure; case studies that describe progress or lack of progress toward well defined aspects of mathematical maturity or productivity.

Some examples of investigations this analyst would recommend are:

1. An in depth investigation into methods of training peer teachers such as counsellors.

2. Investigation into the effect of an increased mathematics reading requirement on mathematical maturity.

3. An investigation into the levels of mathematical maturity climbed by a successful student. A study may start with assumptions of probable levels, defined by solutions to certain problems on problem sets.

4. An investigation into the possibility of designing a commuter program for high ability children much the same as the summer institute, but with a planned study environment for program students (such as a student lounge with counsellors in attendance and with extended hours of student access) as a substitute for living conditions.

RECOMMENDATIONS FOR ORGANIZING AND ADAPTING OTHER PROGRAMS FOR HIGH ABILITY CHILDREN

For anyone who wants to establish programs for high ability children, I would recommend:

1. Informal Experience Oriented Teaching

   The experiences of the youngsters in the summer institute seemed to indicate that a formal type of expository lecture is in general not appropriate for youngsters without a great deal of mathematical experience. Teaching should take the form of student experimentation with examples or illustrations or demonstrations (in the particular) of ideas to be introduced later. There should be a buildup of experience from which a student might discover the pattern of the concept before he/she comes to the formal statement of that concept.

2. Problem Sets Designed with Problems Representing Different Levels of Difficulty

   Numericals, book search, and discovery type problems can be designed by the teacher even in the early grades. A student should be given many opportunities (emanating from problem sets) to discover the particular patterns that precede a concept before it is introduced formally. Lectures need not be restricted to the problems on sets once the student has acquired experience to understand a lecture of considerable depth. Some very difficult ideas could be explored with the teacher leading the search and the students doing the research.

3. Training of Counsellors

Older student-counsellors can be trained by working through a program first. The teacher may then observe the students, who would most benefit from extended study, and who would be most suitable as counsellors. This can be done in seminars before the start of a new program. Counsellors should later sit in all classes with students, and they should assist in the grading of student solutions.

4. A Special Study Environment

A special study environment, such as a study lounge for program only, should be arranged and made available to students for extended school hours. There should be counsellors in attendance most of the time, and a small carefully chosen reference library should be set up in the lounge.

5. Seminars

Problem seminars should be conducted by either the teacher, or another adult, or possibly a mature and very experienced counsellor.

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APPENDIX A

PROBLEM SETS, SAMPLE TESTS, AND TESTS

OSU Number Theory **Problem Set #0** AE Ross Columbus 6/17/79

Reading Search. a number which is = to the sum of its proper divisors

Q1. What is a perfect number? What is a Mersenne prime?

Q2 What is the meaning of $\sum_{\substack{d|n \\ d>0}} \frac{1}{d}$?

Exploration.

P1. A permutation on a set $S=\{1,2,\dots,n\}$ is a one-to-one (bijective) map $f: a \mapsto af$ of S onto itself. We shall write $f=\binom{a}{af}=\binom{1,\ 2,\ \dots,\ n}{1f,\ 2f,\ \dots,\ nf}$. Let $f=\binom{1\,2\,3\,4\,5\,6}{2\,3\,1\,5\,4\,6}$. We observe that $1f=2, 2f=3, 3f=1, 4f=5, 5f=4, 6f=6$. Thus f can be described in terms of cycles (trajectories) as follows: $f=(1,1f,1f^2)(4,4f)(6)=(1,2,3)(4,5)(6)$. If $g=\binom{1\,2\,3\,4\,5\,6}{2\,3\,4\,5\,6\,1}$, then the product $f\circ g$ is $\binom{1\,2\,3\,4\,5\,6}{3\,4\,2\,6\,5\,1}$. Write $f\circ g$ in terms of its cycles (trajectories)

P2. Consider the non-zero elements U_7 of $\mathbb{Z}_7$. Write down all the permutations of U_7 of the form $\binom{a}{ba}$ with $b\in U_7$. Write down each of these six permutations in terms of its cycles. Any interesting conjectures?

Numerical Problems (Some Food for Thought).

P4. Let $N=10723$ to base ten. Write N to base 2, to base 3, to base eleven. In this last case introduce, if necessary, the new digit τ=ten

P5. Without change of base (a) add $(6513)_7$ and $(3545)_7$, (b) subtract $(2436)_7$ from $(4534)_7$, (c) multiply $(562)_7$ by $(345)_7$, (d) divide $(5062)_7$ by $(34)_7$. Here the base is 7 throughout.

P6. Using division to base 7, write $N=(34652)_7$ to base 2, to base 5, to base 11.

P7. Write each of the following numbers to base 3: $3, 9, 27, 243, \frac{1}{3}, \frac{1}{81}$. Write each of these numbers to base 2.

P8. Find the following elements in $\mathbb{Z}_5$: $-1, \frac{1}{2}, \frac{2}{3}, \sqrt{-1}$. How many of these elements can you find in $\mathbb{Z}_6$? In $\mathbb{Z}_{10}$? In $\mathbb{Z}_{11}$? In $\mathbb{Z}_{13}$?

P9. Calculate each of the following summations: $\sum_{\substack{d|6 \\ d>0}}\frac{1}{d}$, $\sum_{\substack{d|28 \\ d>0}}\frac{1}{d}$, $\sum_{\substack{d|496 \\ d>0}}\frac{1}{d}$. Can you make an interesting conjecture?

OSU Number Theory Problem Set #1 AERoss Columbus 6/18/79

Reading Search.

Q1. What are matrices? How does one add and multiply them?

Q2. What is a Farey sequence of order 3? Of order 7? Of order n? How many elements are there in the Farey sequence of order 3? Of order 7? Of order 11? Any conjectures? Represent the fraction $\frac{m}{n}$ by the point (m, n) in the plane. Draw all the entries of the Farey sequence of order 7.

Asking Good Questions.

P1. Compare the following mathematical systems with each other

$$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, 2\mathbb{Z}, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_{11}.$$

There are two operations (addition and multiplication) in each of these systems. Which of these systems resemble each other in regard to the essential properties of these operations?

Exploration.

P2. Consider a two by two matrix $\alpha = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ whose elements are in $\mathbb{Z}_3$. Next consider the mathematical system

$$\mathbb{Z}[\alpha] = \{a\,I,\ b_0 I + b_1 \alpha,\ c_0 I + c_1 \alpha + c_2 \alpha^2, \ldots\}$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the a's, the b's, the c's, etc. are in $\mathbb{Z}_3$. Thus our system consists of all the polynomials in α with coefficients in $\mathbb{Z}_3$. How many distinct elements are there in our system? How does it compare with $\mathbb{Z}$? with $\mathbb{Q}$? With $\mathbb{Z}_{11}$? With $\mathbb{Z}_6$?

Numerical Problems (Some Food for Thought).

P3. If u_1 and u_2 are in $\mathbb{Z}_m$ and $u_1 u_2 = 1$ we say that u_1 and u_2 are units in $\mathbb{Z}_m$. The set of all units in $\mathbb{Z}_m$ we denote by U_m. Study the behavior under multiplication of the system (group!) of units in $\mathbb{Z}_{15}$, in $\mathbb{Z}_{17}$, in $\mathbb{Z}_{21}$, in $\mathbb{Z}_{22}$, in $\mathbb{Z}_{25}$, in $\mathbb{Z}_{18}$, in $\mathbb{Z}_{105}$. Which of these are cyclic groups? Can you suggest, perhaps, for which m would U_m be a cyclic group?

P4. Find all the roots of the equation $x^2 - 6x + 8 = 0$ in $\mathbb{Z}_{15}$. Find all the roots in $\mathbb{Z}_{15}$ of the equation $x^2 - 6x + 10 = 0$. Find all the roots in $\mathbb{Z}_{105}$ of the equation $x^2 - 6x + 8 = 0$. Any conjectures?

Prove or Disprove and Salvage if Possible.

In resolving questions and problems posed in our problem sets one has to make use of many properties which one associates with integers. Make a list (an inventory) of such properties and see if this inventory suffices for every discussion of questions of arithmetic which we undertake. If this inventory will prove to be adequate for such discussion, then we shall consider it as an acceptable description of integers.

P5. $a|a$ for every a in $\mathbb{Z}$. P6 $a|b \Rightarrow b|a$. Here a, b are in $\mathbb{Z}$.

P7. $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $c \in \mathbb{Z}$, then $a|b$ and $a|c \Rightarrow a|b+c$.

P8. a, b, c in $\mathbb{Z}$, then $a|b$ and $b|c \Rightarrow a|c$. P9. a, b, c in $\mathbb{Z}$, then $a|b \Rightarrow a|bc$.

P10. a, b, c in $\mathbb{Z}$, then $a|bc \Rightarrow a|b$ or $a|c$.

P11. p a positive prime in $\mathbb{Z} \Rightarrow 2^p - 1$ is a prime.

P12. If a and b are positive integers and $a|b$, then $a \le b$.

OSU Number Theory Problem Set #2 A.E. Ross Columbus 6/19/79

Reading Search.

Q1. If $\alpha \in \mathbb{R}$, what is "the greatest integer function $[\alpha]$? Calculate $[\sqrt{5}]$, $[-\sqrt{2}]$, $[\frac{\sqrt{7}}{2}]$ $[2+\sqrt{3}]$.

Q2. What is meant by the canonical factorization of a positive integer into prime factors.

Exploration.

P1. Consider complex numbers $a+bi$ with $a \in \mathbb{Z}$ and b in $\mathbb{Z}$. These numbers (Gaussian integers) form under addition and multiplication a mathematical system (denoted by $\mathbb{Z}[i]$) not unlike $\mathbb{Z}$. Which of the properties in P5–P10 of Set #1 hold true also in $\mathbb{Z}[i]$.

Prove or disprove and salvage if possible.

P2. $n \in \mathbb{Z}$ and $2 \nmid n \Rightarrow 8 \mid n^2-1$.

P3. $a|b$ and $a|c \Rightarrow a \mid br+cs$ for every r and s. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$.

P4. Let a and n be positive integers greater than 1. If a^n-1 is a prime then $a=2$ and n is a prime.

Numerical Problems (Some Food for Thought).

P5. Construct a table of "logarithms" (indices) for U_{17}.

P6. Use the table of "logarithms" in P5 to find all the solutions of each of the following equations in $\mathbb{Z}_{17}$: a) $x^2=2$; b) $7x^2=6$; c) $x^3=3$.

P7. Find all the generators in U_{19}. Find all the non-zero elements in $\mathbb{Z}_{19}$ which are perfect squares in $\mathbb{Z}_{19}$. Here U_{19} denotes the group of units in $\mathbb{Z}_{19}$.

P8. Use Euclid's algorithm to find the g.c.d. of 29 and 11.

P9. Use the results of P8 to show that

$$\frac{29}{11} = 2 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{3}}}}$$ (The simple continued fraction for $\frac{29}{11}$.)

P10. Any fixed real number α may be approximated by a rational fraction $\frac{a}{b}$ with any desired degree of accuracy if we allow the denominator b become large enough. The size of the denominator is the price we pay for a good approximation. In the light of these observations, how high is the price (i.e. the size of the denominator vs. the accuracy) which we pay as we approximate $\frac{29}{11}$ by the fractions

$$2,\ 2+\frac{1}{1},\ 2+\cfrac{1}{1+\frac{1}{1}},\ 2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}$$ (The convergents of the simple continued fraction in P9.)

P11. The polynomial $x^3-9x^2+26x-24$ has roots 2, 3, 4. Write down a polynomial whose roots are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

Ingenuity.

P12. Consider a rectangular Cartesian coordinate system in a plane. Points with integral coordinates we shall call lattice points. Show that no triangle whose vertices are lattice points can be an equilateral triangle.

OSU Number Theory Problem Set #3 A.E. Ross Columbus 6/20/79

Reading Search.

Q1. What are the arithmetic functions $\varphi(n)$, $\mu(n)$, $\sigma(n)$, $\tau(n)$?

Numerical Problems (Some Food for Thought).

P1. Describe (and perhaps justify) the process represented by the following table for calculating successive convergents in P10 of the continued fraction in P9, Set #2:

		2	1	1	1	3
	1	2	3	5	8	29
	0	1	1	2	3	11

Have you noticed anything of interest about the two-by-two determinants

$$\begin{vmatrix}1&2\\0&1\end{vmatrix},\ \begin{vmatrix}2&3\\1&1\end{vmatrix},\ \begin{vmatrix}3&5\\1&2\end{vmatrix},\ \begin{vmatrix}5&8\\2&3\end{vmatrix},\ \begin{vmatrix}8&29\\3&11\end{vmatrix}?$$

P2. Proceed with the fraction

$$\frac{7469}{2463}$$

as in P8, P9, P10, Set #2. Plot the values of all the convergents on the real line and observe carefully relations between the convergents and the fraction itself as well as the relations of the convergents to one another. In approximating the fraction by the convergents of its continued fraction expansion, is the price for accuracy in each case unexpectedly high? low?

P3. Use the results in P2 and your observations in P1, to find an integral solution $\{x, y\}$ for the following Diophantine equations: $29x + 11y = 1$ and $7469x + 2463y = 1$.

Prove or disprove and salvage if possible.

P4. Given two rational integers a, b there exist integers q, r such that

$$a = bq + r, \quad 0 \le r < |b|.$$

P5. $c \neq 0$, $ac = bc \Rightarrow a = b$. True in $\mathbb{Z}$. True in $\mathbb{Z}_m$. True in $\mathbb{Z}[i]$.

P6. Every integer >1 has a positive prime divisor.

P7. If $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \dots, p_n$ are the first n positive primes then $P_n = p_1 p_2 \cdots p_n + 1$ is a prime for every n.

P8. If $m \in \mathbb{Z}$ and $m > 0$, then $m \ge 1$.

Technique of Generalization.

P9. Describe Euclid's algorithm in a way which would make it possible to apply the process in P8, P9, P10 Set #2 to $\sqrt{3}$ and to other real irrationals.

Ingenuity.

P10. Find three digits a_0, a_1, a_2 and a base $b > 1$ such that

$$(a_0 a_1 a_2)_b = 2(a_0 a_1 a_2)_{ten} \quad \text{(Sierpinski)}$$

P11. The sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for $n > 1$, is never an integer.

The Art of Counting.

P12. Write $|A|$ for the number of elements in a finite set A. Then

$|A \cup B| = |A| + |B| - |A \cap B|$.

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$|A_1 \cup A_2 \cup A_3 \cup A_4| = ?$

OSU Number Theory Problem Set #4 A.E. Ross Columbus 6/21/79

<u>Reading Search.</u>

<u>Q1.</u> What is the highest power of a prime contained in a factorial?

<u>Exploration.</u>

<u>P1.</u> We have considered a rational function of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}}$$

of the n variables

$$a_1, a_2, a_3, \ldots, a_n$$

and we have called this function a <u>finite</u> <u>continued</u> <u>fraction</u>. For this continued fraction we use the convenient notation

$$[a_1, a_2, a_3, \ldots, a_n].$$

Let $P_1 = a_1$, $P_2 = a_2 a_1 + 1, \ldots, P_\ell = a_\ell P_{\ell-1} + P_{\ell-2}, \ldots$

$Q_1 = 1$, $Q_2 = a_2, \ldots, Q_\ell = a_\ell Q_{\ell-1} + Q_{\ell-2}, \ldots$

Then $[a_1, a_2, \ldots, a_\ell] = \frac{P_\ell}{Q_\ell}$ (The ℓ'th convergent!)

for $\ell = 1, 2, \ldots, n$.

Hint $[a_1, a_2, \ldots, a_{\ell-1}, a_\ell] = [a_1, a_2, \ldots, a_{\ell-2}, a_{\ell-1} + \frac{1}{a_\ell}]$.

<u>P2.</u> Using the relations in P1, show that

(1) $P_{\ell-1}Q_\ell - Q_{\ell-1}P_\ell = (-1)^{\ell-1}$ for $\ell > 1$ and (2) $P_{\ell-2}Q_\ell - Q_{\ell-2}P_\ell = (-1)^\ell a_\ell$ for $\ell > 2$.

<u>Prove or disprove and salvage if possible.</u>

<u>P3.</u> $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ and $(a,b) = 1 \Rightarrow ax + by = 1$ has a solution in integers x, y.

<u>P5</u> $a \mid bc$, $(a,b) = 1 \Rightarrow a \mid c$. True in $\mathbb{Z}$.

<u>A Reminder.</u>

<u>P6.</u> A convenient description of $\mathbb{Z}$!

<u>Numerical Problems</u> (Some Food for Thought).

<u>P7.</u> Use Euclid's algorithm to calculate $(7469, 2464)$.

<u>P8.</u> Find an integral solution $\{x, y\}$ of the Diophantine equation

$$7469x + 2464y = 210.$$

<u>P9.</u> Find an integral solution $\{x, y\}$ of the Diophantine equation

$$2689x + 4001y = 17.$$

<u>P10.</u> What is the order of 28 in U_{29}? Of 16 in U_{29}? Of $28 \cdot 16$ in U_{29}? Next consider U_{71}. What is the order of 7, of 2, of $7 \cdot 2 = 14$, of 54, of 51, of $54 \cdot 51$? Any conjectures? To facilitate calculations one can use the tables of "logarithms" (Indices!).

<u>Technique of generalization.</u>

<u>P11.</u> Consider the set of all polynomials in x with coefficients in $\mathbb{Z}_3$. Denote such a set of polynomials by $\mathbb{Z}_3[x]$. Is $\mathbb{Z}_3[x]$ a ring under the addition and the multiplication of polynomials? Is $\mathbb{Z}_3[x]$ a commutative ring? With (multiplicative) identity? With the cancellation law? When should we say that one polynomial divides another in $\mathbb{Z}_3[x]$? What are the (multiplicative!) units in $\mathbb{Z}_3[x]$? Which of the following polynomials are primes in $\mathbb{Z}_3[x]$: x^2+2x+2, x^3+x^2+1, x^3+x^2+x+2, x^3-x^2+1, x^4+1, x^4+x+2?

OSU Number Theory Problem Set #5 A.E. Ross Columbus 6/22/79

Prove or disprove and salvage if possible.

P1. d is the smallest positive value of $ax+by$ for integral values of x and $y \Rightarrow d=(a,b)$. True in $\mathbb{Z}$.

P2. p a prime and $p|ab \Rightarrow p|a$ or $p|b$. True in $\mathbb{Z}$.

P3. If m and n are positive rational integers and if $m|n$, then $m \leq n$.

P4. If $\{x_0, y_0\}$ is an integral solution of the equation $ax+by=c$, where a,b,c are in $\mathbb{Z}$, then all the integral solutions $\{x,y\}$ of this equation are given by the formulas $x=x_0+bt$, $y=y_0-at$ for $t\in\mathbb{Z}$.

P5. The parallelogram in Fig 1 is determined by the vector $\mathfrak{a}$ with components a_1, a_2 and by the vector $\mathfrak{b}$ with components b_1, b_2. The area of this parallelogram is equal to the absolute value of the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

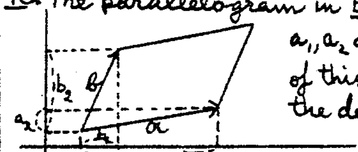


Fig 1.

P6. Every rational integer >1 is a product of positive primes.

P7. $m|a-b \Rightarrow (a,m)=(b,m)$.

Numerical Problems (Some Food for Thought).

P8. Find an integral solution $\{x,y\}$ of the Diophantine equation
$$5391x+3976y=11$$
with the least positive value of x and also one with the least positive y.

P9. Expand $\frac{5391}{3976}$ into a simple continued fraction and make use of the process given in P1, Set #3 (and also in P1, Set #4) for calculating the values of successive convergents. Compare the actual value of the difference between each convergent $\frac{P_n}{Q_n}$ and the given fraction, with $\frac{1}{Q_n^2}$ and with $\frac{1}{Q_nQ_{n+1}}$. Any conjectures?

P10. Find all the positive integral solutions $\{x,y\}$ of the Diophantine equation
$$158x+57y=20000.$$

P11. Find all the solutions of
$$2017x=532$$
in $\mathbb{Z}_{4001}$. Explain.

P12. Expand $\sqrt{3}$ into a simple continued fraction using the results of P9, Set #3. How much do you know about the continued fraction for $\sqrt{3}$ after calculating, say, the first five partial quotients? Any conjectures?

P13. Multiply $2x^3+3x^2+x+4$ by $3x^2+2x+2$ in $\mathbb{Z}_5[x]$. Multiply the same two polynomials in $\mathbb{Z}_6[x]$. In each case compare the degrees of the factors with the degree of the product. Explain.

Technique of generalization.

P14. Factor those polynomials in P11, Set #4 which are not primes, into prime factors. Construct a table of primes of degree 2 in $\mathbb{Z}_3[x]$. Does some form of division algorithm hold true in $\mathbb{Z}_3[x]$? In $\mathbb{Z}_{17}[x]$? In $\mathbb{Z}_p[x]$?

OSU Number Theory Problem Set #6 A.E. Ross Columbus 6/25/74

Reasoning.

P1. We propose that we should accept the following properties as a description of $\mathbb{Z}$: We assume that $\mathbb{Z}$ is a commutative ring with identity and cancellation law, that $\mathbb{Z}$ is totally ordered under "$\leq$" and that its positive elements are well-ordered. Do these properties of $\mathbb{Z}$ suffice to resolve all the problems concerning $\mathbb{Z}$ with which we have dealt to date?

Technique of generalization.

P2. Does some form of division algorithm which leads to Euclid's algorithm exist in $\mathbb{Z}[i]$?

Prove or disprove and salvage if possible.

P3. If a, b, c are in $\mathbb{Z}$, then the Diophantine equation $ax + by = c$ has at least one solution in integers x, y.

P4. U_m has exactly $\varphi(m)$ elements.

P5. U_m is a group under multiplication in $\mathbb{Z}_m$.

P6. Use Euclid's algorithm to calculate the g.c.d. of two successive Fibonacci numbers F_m, F_{m+1}. What is the number of steps required? Explain.

P7. $n \in \mathbb{Z} \Rightarrow \prod_{\substack{d \mid n \\ d>0}} d = n^{\frac{1}{2}\tau(n)}$.

P8. If m is a positive integer $\neq 3$, then $2^m + 1$ is not a perfect square.

P9. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}_m[x]$ and $g(x) = b_\ell x^\ell + b_{\ell-1}x^{\ell-1} + \cdots + b_0 \in \mathbb{Z}_m[x]$ and the degrees of $f(x)$ and $g(x)$ are n and ℓ respectively $\Rightarrow f(x)\cdot g(x)$ is of degree $n+\ell$.

P10. There exist infinitely many distinct positive primes in $\mathbb{Z}$.

P11. If $u_1 \in U_m$ has the order n_1 and $u_2 \in U_m$ has the order n_2, then the order of $u_1 u_2$ is $n_1 \cdot n_2$.

P12. $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$ is the canonical decomposition of n into prime factors. Then $\sigma(n) = \prod_j \frac{p_j^{\alpha_j+1} - 1}{p_j - 1}$ and $\tau(n) = \prod_{j=1}^{\ell} (\alpha_j + 1)$.

True in $\mathbb{Z}$, True in $\mathbb{Z}[i]$, True in $\mathbb{Z}_5[x]$.

Numerical Problems (Some Food for Thought).

P13. The polynomial $2x^3 + 3x^2 + x + 4$ is written to base x. Write this polynomial to base $x-2$ (Cf. P6, Set #0).

P14. Calculate $\tau(15!)$. Explain.

P15. Find all the perfect squares in U_{29}.

Ingenuity.

P16. $m \in \mathbb{Z}$, $m > 0$, $m \neq 3 \Rightarrow m^4 + m^3 + m^2 + m + 1$ is not a perfect square.

Exploration.

P17. Let $\frac{P_1}{Q_1}, \frac{P_2}{Q_2}, \frac{P_3}{Q_3}, \frac{P_4}{Q_4}, \ldots$ be the convergents of the simple continued fraction expansion of $\sqrt{7}$. Calculate the value of $P_j^2 - 7Q_j^2$ for $j = 1, 2, 3, 4, \ldots$. Calculate the value $P_j^2 - dQ_j^2$ for the simple continued fraction expansion of $\sqrt{d}$ when $d = 10$ and also when $d = 13$. Do you care to make any conjectures about solving the Diophantine equation

$$x^2 - dy^2 = 1, \quad d \neq \square.$$

P18. Problems 1 and 2 in the Towards the Abstract paper.

OSU Number Theory Problem Set #7. A.E. Ross Columbus 6/26/79

Exploration.

P1. Accept the inventory of the fundamental properties of $\mathbb{Z}$ as given in P1, Set #6, as a description of $\mathbb{Z}$. Indicate a chain of reasoning which starts with this basic inventory and yields Euclid's algorithm, the fundamental theorem of arithmetic (i.e. $a|bc$, $(a,b)=1 \Rightarrow a|c$), and the unique factorization theorem.

P2. Suppose that the continued fraction of P1, Set #4 is simple, i.e. all the quotients $a_2, a_3, \dots$ are positive integers and a_1 is a non-negative integer. Then

(1) $Q_{n+1} \geq Q_n$ for $n \geq 1$, with strict inequality when $n > 1$;

(2) $Q_1 = 1$, $Q_2 \geq 1$, $Q_3 \geq 2$, $Q_4 \geq 3$, $Q_5 \geq 5$, $Q_n > n$ for $n > 5$;

(3) $(P_n, Q_n) = 1$.

P3. What do you mean when you say that an infinite simple continued fraction

$$[a_1, a_2, a_3, \dots, a_n, \dots]$$

represents a real number α?

Numerical Problems (Some Food for Thought).

P4. What is the relation between Fibonacci numbers and the convergents in the continued fraction expansion of $\frac{\sqrt{5}+1}{2}$?

P5 Use simple continued fraction expansion and find the convergent with the smallest denominator which would approximate $\sqrt{5}$ to within one part in 1000.

P6. Find the g.c.d. (f,g) of $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$ and $g(x) = x^2 - 2x + 3$ in $\mathbb{Z}_5[x]$ (Euclid?!)

P7. See if the method we used in $\mathbb{Z}$ suggest how we can use the results in P6 to find a solution $\{X, Y\}$ of the "Diophantine" equation

$$f(x)X(x) + g(x)Y(x) = (f(x), g(x))$$

in $\mathbb{Z}_5[x]$. Does our algorith in $\mathbb{Z}$ apply here?

P8. Find the g.c.d. of $7+11i$ and $3+5i$ in $\mathbb{Z}[i]$. Does our algorithm in $\mathbb{Z}$ apply here?

Counting techniques.

P9. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_e^{\alpha_e}$ be the canonical factorization of a positive integer n into positive primes in $\mathbb{Z}$. Use the formulas in P12, Set #3 to obtain a formula for $\varphi(n)$.

Towards the Abstract.

P10. "Towards the Abstract" paper; Problems 3 and 4.

Technique of Generalization.

P11. Does there exist a division algorithm in $\mathbb{Z}_3[x]$? In $\mathbb{Z}_p[x]$? In $\mathbb{Q}[x]$? In $\mathbb{Z}[x]$?

P12. U_{15} is not a cyclic group. It is howevever a "product" of cyclic groups. That is $U_{15} = \{1,2,4,7,8,11,13,14\} = \{2,4,8,1\} \cdot \{14,1\}$. Here we multiply cyclic groups $\{2,4,8,1\}$ and $\{14,1\}$ as we multiply complexes—that is we multiply every element of $\{2,4,8,1\}$ by every element of $\{14,1\}$. Does U_{21} "factor into a "product" of cyclic groups? What about U_{105}? Exhibit such factorizations if they exist.

OSU Number Theory Problem Set #8 A.E. Ross Columbus 6/27/79

Prove or disprove and salvage if possible.

P1. $\mu(ab) = \mu(a)\mu(b)$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_p[X]$. True in $Q[X]$.

P2. Let $a = a_1 + a_2 i$ be a Gaussian integer. Its norm is $N(a) = a_1^2 + a_2^2 = a\bar{a}$, where $\bar{a} = a_1 - a_2 i$. We have $N(ab) = N(a)N(b)$ for every two Gaussian integers a and b.

P3. $u \in U_m \Rightarrow u^{\varphi(m)} = 1$ in $\mathbb{Z}_m$.

P4. Let $\alpha \in \mathbb{R}$ and let $\frac{P_n}{Q_n}$ be the n-th convergent in the expansion of α into a simple continued fraction. Then

$$\left|\frac{P_n}{Q_n} - \alpha\right| < \frac{1}{Q_n^2}.$$

P5. One form of the remainder theorem in $\mathbb{Z}$ says that for positive a, p in $\mathbb{Z}$, $a = pq + \varepsilon r'$ where $0 \le r' \le \frac{p}{2}$ and $\varepsilon = +1$ or -1. We assert that

$$\varepsilon = (-1)^{\left[\frac{2a}{p}\right]}.$$

Exploration.

P6. Do as in P5, Set #7 for $\sqrt[3]{5}$.

P7. Out of the ring $\mathbb{Z}_3[X]$ construct a ring $(\mathbb{Z}_3[X])_{x^2+x+2}$ of remainders modulo x^2+x+2. Do this in a manner similar to that used to construct $\mathbb{Z}_m$ out of $\mathbb{Z}$. How many distinct elements are there in $(\mathbb{Z}_3[X])_{x^2+x+2}$? Calculate the number of units in $(\mathbb{Z}_3[X])_{x^2+x+2}$. Is the group of units cyclic? Is our new ring a field? Is $\sqrt{-1}$ in our "ring of remainders"? Justify your assertions.

Numerical Problems (Some Food for Thought).

P8. Compare the roots of the polynomial $2x^5 - 7x^4 + x^3 + 3x^2 + 4x + 5$ with the roots of the polynomial $5x^5 + 4x^4 + 3x^3 + x^2 - 7x + 2$.

P9. Describe all the "integral" multiples $(1+2i)a$ of $(1+2i)$. Here a runs through all the Gaussian integers.

P10. Do the following examples show that the theorem on unique factorization into primes does not hold in the ring $\mathbb{Z}[i]$?

$$2 = (1+i)(1-i) = i(1-i)^2;\quad 5 = (2+i)(2-i) = (1-2i)(1+2i);\quad 4+7i = (2+i)(3+2i) = (2-3i)(-1+2i)$$

Towards the Abstract.

P11. "Towards the Abstract" paper: Do as in Problem 2 for the sets

$E = \{2, 3, 4, 6, 8, 12, 18, 24, 30\}$,

$F = \{1, 2, 3, 5, 8, 13, 21, 34\}$,

$G = \{3, 5, 17, 257\}$.

Reasoning.

P12. $a < b$ and $b < c \Rightarrow a < c$. True in $\mathbb{Z}$. True in Q.

OSU Number Theory Problem Set #9 A.E. Ross Columbus 6/28/79

Exploration.

P1. Suppose that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + x^4 = (x-r_1)(x-r_2)(x-r_3)(x-r_4)$. Express the coefficients a_0, a_1, a_2, a_3 in terms of r_1, r_2, r_3, r_4. Why, do you think, we speak of a_0, a_1, a_2, a_3 as (the elementary) symmetric functions of r_1, r_2, r_3, r_4.

P2. Consider the polynomial $f(x) = x^{p-1} - 1 - (x-1)(x-2)\cdots(x-p+1)$ in $\mathbb{Z}_p[x]$ where p is a positive prime. What are its coefficients? What is its degree in $\mathbb{Z}$? For how many values of x in $\mathbb{Z}_p$ does it vanish? What are the values in $\mathbb{Z}_p$ of the coefficients of $f(x)$? What does this last information tell you about the elementary symmetric functions of $1, 2, 3, \ldots p-1$?

Prove or disprove and salvage if possible.

P3. $[x+y] \geq [x]+[y]$. Here x, y are real numbers and $[x]$ is the greatest integer in x.

P4. $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$. Here n is a positive integer and $x \in \mathbb{R}$.

P5. $u \in U_m$, ℓ is the order of u in $U_m \Rightarrow \ell \mid \varphi(m)$.

P6. u a unit in $\mathbb{Z}[i] \Leftrightarrow N(u) = 1$.

P7. $(-1)a = -a$. True in $\mathbb{Z}$, True in $\mathbb{Q}$. True in $\mathbb{Z}[i]$. True in (?).

P8. $[n+x] = n + [x]$. Here $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

P9. Let $f(x) \in \mathbb{Z}_p[x]$. If $a \in \mathbb{Z}_p$ and $f(a) = 0$, then $f(x) = (x-a)q(x)$ where $q(x) \in \mathbb{Z}_p[x]$. Would our assertion be true if we replace $\mathbb{Z}_p$ by $\mathbb{Z}_m$, by $\mathbb{Q}$, by $\mathbb{R}$, by $\mathbb{C}$, by any ring?

P10 A polynomial of degree n with coefficients in $\mathbb{Z}_m$ cannot have more than n distinct roots.

Numerical Problems (Some Food for Thought).

P11. How many elements are there in the ring $(\mathbb{Z}[i])_{1+2i}$ of remainders modulo $1+2i$. What are the units in $(\mathbb{Z}[i])_{1+2i}$. Is the group of units cyclic? Is our ring of remainders a field? Justify your assertions.

P12. Factor $x^2 - 6x + 8$ in $\mathbb{Z}_{105}[x]$ in every possible way into a product of linear factors. (See P4, Set #1).

P13. $(a,b) = 1 \Rightarrow (a+b, a-b) = 1$ or 2. $a, b \in \mathbb{Z}$.

Problems from the Text

P14. Uspensky and Heaslet: Page 103, Problems 2 and 3.

Towards the Abstract.

P15 "Towards the Abstract" paper: Problem 5.

OSU Number Theory Problem Set #10 A.E. Ross Columbus 6/29/79

Numerical Problems (Some Food for Thought).

P1. Calculate the g.c.d. (f,g) of $f(x) = x^4+x^3+2x^2+x+1$ and $g(x) = x^3-1$ in $\mathbb{Z}_3[x]$.

P2. Solve the "Diophantine" equation

$$f(x)\,X(x) + g(x)\,Y(x) = x^3+2x^2+2x+1$$

in $\mathbb{Z}_3[x]$ using the results of P1.

P3. Write down all the elements in the ring $(\mathbb{Z}_3[x])_{x^3-1}$ of remainders modulo x^3-1. Is this ring a field? Is $2x^2+2x+2$ a unit in our ring?

P4. Write down all the distinct elements in the ring $(\mathbb{Z}[i])_{2+3i}$ of remainders modulo $2+3i$. What are the units in this ring? Is the group of units cyclic? Is our ring of remainders a field?

P5. Find all the positive integral solutions $\{x,y\}$, i.e. solutions with $x>0$ and $y>0$, of the Diophantine equation

$$606x + 786y = 3390.$$

Problems from the Text.

P6. Uspensky and Heaslet: Page 40, Problems 1, 2, 3. Page 41, Problems 4, 5, 6, 7.

Towards the Abstract.

P7. "Towards the Abstract" paper: Problem 6.

Prove or disprove and salvage if possible.

P8. If neither a nor b divide the other then $[a,b] > a$ and $[a,b] > b$. Here a and b are positive rational integers and $[a,b] = $ l.c.m. of a and b.

P9. $a = \prod_{p|ab} p^{\alpha_p}$ $(\alpha_p \geq 0)$ and $b = \prod_{p|ab} p^{\beta_p}$ $(\beta_p \geq 0) \Rightarrow \begin{cases} (a,b) = \prod_{p|ab} p^{\min\{\alpha_p,\beta_p\}} \text{ and} \\ [a,b] = \prod_{p|ab} p^{\max\{\alpha_p,\beta_p\}}. \end{cases}$

True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_p[x]$. True in $F[x]$, where F is a field

P10. Let l be the order of u in U_m. If d is a positive divisor of l, then there exists $u_1 \in U_m$ such that the order of u_1 is d.

P11. Recapitulation and Analysis.

OSU Number Theory Problem Set #11 A.E. Ross Columbus 7/2/79

Test #1: 8-10 am. Thursday, July 5 in our classroom - Room

Rational Arithmetic and its Generalizations.

Recapitulation and Analysis.

Investigate in detail the chain of reasoning, technique of generalization and related ideas used in the development of each of the following arithmetics:

The arithmetic in $\mathbb{Z}$.	$\mathbb{Z}[i]$	$\mathbb{Z}_p[x]$	$\mathbb{Q}[x]$	$\mathbb{Z}[x]$			
Definitions: divisibility, primes, units, g.c.d., l.c.m., etc..	?	?	?	?			
Th1. $a\mid b \Rightarrow \lvert a\rvert \le \lvert b\rvert$. Th2. u a unit in $\mathbb{Z} \Leftrightarrow u=+1$ or $u=-1$. Th3 $a\mid b, a\mid c \Rightarrow a\mid bg+ch$ for $g,h\in\mathbb{Z}$.	? ? ?	? ? ?	? ? ?	? ? ?			
Th4. a, b with $b\neq 0 \Rightarrow \exists\lvert q, r$ such that $a=bq+r$, $0\le r<\lvert b\rvert$. Th4'. a, b with $b\neq 0 \Rightarrow \exists\, q, r$ such that $a=bq+r$, $\lvert r\rvert \le \frac{1}{2}\lvert b\rvert$. Th5. (a,b) is the last non-vanishing remainder in Euclid's algorithm.							
Th6. $ax+by=(a,b)$ has solutions $\{x,y\}$ with x, y in $\mathbb{Z}$. Th7. Euclid's algorithm yields a continue fraction expansion which in turn yields an algorithm for the solution of the Diophantine equation in Th6. Th8. $a\mid bc$, $(a,b)=1 \Rightarrow a\mid c$.							
Th9. Every integer >1 has a prime factor. Th10. The unique factorization into primes theorem. Th11, $\lvert\mathbb{Z}_m\rvert = m$.							
Th12. The ring $\mathbb{Z}_p$ of remainders modulo a prime p, is a field of p elements. Th13. The number of elements in U_m is $\varphi(m)$ where $\varphi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k})$. Here $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the canonical decomposition of m into prime factors. Th14. $u\in U_m \Rightarrow u^{\varphi(m)}=1$ in U_m. Th14'. p a prime, $p\nmid a \Rightarrow a^{p-1}\equiv 1 \pmod p$.							

Ingenuity.

P1. Consider a polygon whose vertices are lattice points in the sense of P12, Set #2. Find a simple formula relating the area of such a polygon and the number of lattice points within and on the boundary of this polygon.

OSU Number Theory SAMPLE TEST #1 A.E. Ross Columbus 7/3/79

<u>Numerical Problems</u> (Some Food for Thought) Please display your calculations neatly! Calculate efficiently.

<u>P1.</u> Obtain the required results without changing to base ten: $(456)_7 \cdot (342)_7 = (?)_7$; $(5242)_7 - (645)_7 = (?)_7$; $(13641)_7 \div (36)_7 = (?)_7$; $(6435)_7 = (?)_5$

<u>P2.</u> Calculate the following elements in $\mathbb{Z}_{29}$: $-1, -7, \frac{1}{2}, \frac{1}{5}, \frac{7}{3}, \sqrt{-1}, \sqrt{-13}$. Explain.

<u>P3.</u> Construct a table of "logarithms" (indices) for $\mathbb{Z}_{17}$ and use it to solve the following equations in $\mathbb{Z}_{17}$: (a) $x^3 = 5$; (b) $7x = 11$.

<u>P4.</u> Find all the positive $(x>0, y>0)$ integral solutions $\{x,y\}$ of the Diophantine equation $127x + 52y = 641$. Illustrate by drawing a picture.

<u>P5.</u> Expand $\alpha = \sqrt{14}$ into a simple continued fraction and find the convergent with the smallest denominator which approximates $\sqrt{14}$ within $\frac{1}{100}$. Explain.

<u>P6.</u> Find the smallest positive integer with exactly 15 positive divisors.

<u>P7.</u> Find a g.c.d. $(33+17i, 16-2i)$ in $\mathbb{Z}[i]$.

<u>P8.</u> Calculate $\mu(130)$, $\tau(130)$, $\sigma(130)$. Also $\varphi(130)$.

<u>P9.</u> Find all the positive integers a, b for which $(a,b)=10$ and the l.c.m. $[a,b]=100$

<u>P10.</u> The polynomial $f(x) = 3x^4 + x^3 + 2x^2 + 4x + 5$ is written to base x. Write $f(x)$ to base $x-2$ in $\mathbb{Z}[x]$.

<u>P11.</u> Use the table in <u>P3</u> to find all the generators in U_{17}. Find all the elements of order 4 in U_{17}. Find all the perfect squares in U_{17}. Explain.

<u>P12.</u> Is $1+2i$ a unit in $(\mathbb{Z}[i])_{3+5i}$? Justify your assertion.

<u>Prove or disprove and salvage if possible.</u> Proofs here may be based on important results which have been derived from our basic inventory of properties of $\mathbb{Z}$. State carefully which derived properties you use in your proof. Do please be both precise and concise.

<u>P13.</u> $a \in \mathbb{Z}, b \in \mathbb{Z} \Rightarrow \tau(ab) = \tau(a)\tau(b)$.

<u>P14.</u> $a \mid bc$, $(a,b)=1 \Rightarrow a \mid c$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_3[x]$. True in $\mathbb{Q}[x]$.

<u>P15.</u> $a \in \mathbb{Z}_m$, then a is a unit in $\mathbb{Z}_m$ if and only if $(a,m)=1$.

<u>P16.</u> $n \in \mathbb{Z}$, n positive, $n \mid (n-1)!+1 \Rightarrow n$ is a prime.

<u>P17.</u> If $N(a)$ is a prime in $\mathbb{Z}$, then a is a prime in $\mathbb{Z}[i]$.

<u>P18.</u> If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_e^{\alpha_e}$ is the canonical decomposition of n into prime factors and if $d \mid n$, then d is not divisible by any prime other than $p_1, p_2, \ldots, p_e$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_p[x]$. True in $\mathbb{Q}[x]$.

<u>P19.</u> There exist arbitrarily large gaps between consecutive primes in $\mathbb{Z}$.

<u>P20.</u> Replace $\mathbb{Z}_m$ in <u>P15</u> by $(\mathbb{Z}[i])_{m_1+m_2 i}$, also by $(\mathbb{Z}_5[x])_{m(x)}$. What can you say about the units in these systems? Justify your assertions.

<u>P21.</u> $n \in \mathbb{Z} \Rightarrow 42 \mid n^7 - n$.

<u>P22.</u> $(n-1)^2 \mid n^\ell - 1 \Leftrightarrow (n-1) \mid \ell$.

<u>P23.</u> If q_ν is the last quotient in the Euclid's algorithm for (a,b), then $q_\nu > 1$. Here $a, b \in \mathbb{Z}$.

<u>P24.</u> With how many zeros does $37!$ end?

<u>P25.</u> $2^p - 1$ a prime and $n = 2^{p-1}(2^p - 1) \Leftrightarrow \sigma(n) = 2n$.

<u>P26.</u> $(a^n - 1, a^m - 1) = a^{(n,m)} - 1$. Here a, n, m are positive integers.

page 2

OSU Number Theory SAMPLE TEST#1 A.E.Ross Columbus 7/3/79

Reasoning. Please indicate which properties in our inventory of basic properties of $\mathbb{Z}$ you use in each argument. Do please be both precise and concise.

P27. Every rational integer $m > 1$ has a prime divisor.

P28. If $m > 0$, then $m \geq 1$. True in $\mathbb{Z}$.

P29. If a and b are in $\mathbb{Z}$, then there exists a pair of integers q and r such that $a = bq + r$ and $0 \leq r < |b|$.

P30. What form does this last result (in P29) take in $F[X]$, F a field.

P31. $a < b$ and $b < c \Rightarrow a < c$. True in $\mathbb{Z}$. True in $\mathbb{Q}$.

P32. $a|b$ and $b|c \Rightarrow a|c$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_p[X]$. True in $\mathbb{Z}[X]$.

P33. $a \leq b \Rightarrow a^2 \leq b^2$. True in $\mathbb{Z}$. True in $\mathbb{Q}$. True in $\mathbb{R}$.

P34. $a^2 \leq b^2 \Rightarrow a \leq b$. True in $\mathbb{Z}$:

P35 $d|a$ and $d|b \Rightarrow d|ax+by$ for every x, y. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}[X]$.

P36. $0 \cdot a = 0$ for every a in $\mathbb{Z}$.

P37. $(-a)b = a(-b) = -(ab)$. True in $\mathbb{Z}$. True in $\mathbb{Z}[X]$.

P38. $a|b$ and $b|a \Rightarrow a = bu$ where u is a unit. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$, in $\mathbb{Z}_p[X]$.

P39. $a > 0$, $c > b \Rightarrow ac > ab$. True in $\mathbb{Z}$. True in $\mathbb{Q}$.

Prove or disprove and salvage if possible. {Here proofs may be based on important results which have been derived from a reduced inventory of properties of $\mathbb{Z}$.

P40. $u_1, u_2 \in U_m$, m_1, m_2 are the orders of u_1, u_2 respectively, $(m_1, m_2) = 1 \Rightarrow$ the order of u_1u_2 is m_1m_2.

P41. If a and b are in $\mathbb{Z}[i]$, then there exist q and r in $\mathbb{Z}[i]$ such that
$$a = bq + r \text{ and } N(r) \leq \tfrac{1}{2}N(b).$$

P42. For every real $\alpha > 0$ there exist infinitely many rational fractions $\frac{P}{Q}$ for which
$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{Q^2}.$$

P43. How many distinct elements are there in the ring $(\mathbb{Z}[i])_{a_1+a_2 i}$. Here $a = a_1 + a_2 i \in \mathbb{Z}[i]$. Justify your assertion.

P44. $\varphi(ab) = \varphi(a)\varphi(b)$. True in $\mathbb{Z}$.

P45. For $n = 1, 2, \ldots$ none of the integers of the form $2^{2^n} + 5$ is a prime.

P46. If $a_1, a_2, \ldots, a_\ell$ are positive integers and $a_1 + a_2 + \cdots + a_\ell = n$, then
$$a_1!\, a_2! \cdots a_\ell! \mid n!$$

Test #1: 8–10 am. Thursday, July 5 in our classroom.

OSU Number Theory TEST #1 A.E. Ross Columbus 7/5/79

Numerical Problems (Some Food for Thought) Please display your calculations neatly! Calculate efficiently.

P1. Obtain the required results without changing to base ten: $(654)_8 \cdot (423)_8 = (?)_8$; $(4252)_8 - (546)_8 = (?)_8$; $(45230)_8 \div (525)_8 = (?)$; $(5463)_8 = (?)_7$.

P2. Construct a table of "logarithms" (indices) for $\mathbb{Z}_{19}$ and use it to solve the following equations in $\mathbb{Z}_{19}$: $x^5 = 13$; $7x = 11$.

P3. Use the table in P2 to find all the generators in U_{19}. Find all the perfect squares in U_{19}. Justify your results.

P4. Calculate the following elements in $\mathbb{Z}_{19}$: $-1, -7, \frac{1}{2}, \frac{1}{5}, \frac{7}{3}, \sqrt{-2}, \sqrt{7}$. Explain.

P5. Find all the positive $(x>0, y>0)$ integral solutions $\{x, y\}$ of the Diophantine equation $223x + 91y = 901$

P6. Expand $\sqrt{41}$ into a simple continued fraction and find the convergent with the smallest denominator which approximates $\sqrt{41}$ within $\frac{1}{7000}$. Explain.

P7. Calculate $\mu(245)$, $\tau(245)$, $\sigma(245)$, $\varphi(245)$.

P8. The polynomial $f(x) = 3x^4 + x^3 + 5x^2 + 4x + 2$ is written to base x. Write $f(x)$ to base $x-3$ in $\mathbb{Z}[x]$

P9. Is $1+i$ a unit in $(\mathbb{Z}[i])_{3+5i}$? Justify your assertion.

P10. Show that $35 \mid 2^{123} + 3^{123}$.

P11. Construct the (finite) ring $(\mathbb{Z}_2[x])_{x^3+x^2+1}$. Is this ring a field? How many distinct elements are there in this ring? Justify your assertions.

P12. Is $4+5i$ a prime in $\mathbb{Z}[i]$? Justify.

Reasoning. {Please indicate which properties in our inventory of basic properties you use in each argument. Do please be both precise and concise.

P13. If $m > 0$, then $m \geq 1$. True in $\mathbb{Z}$.

P14. $a(x), b(x) \in \mathbb{Z}_p[x] \Rightarrow$ there exist $q(x), r(x)$ in $\mathbb{Z}_p[x]$ such that $a(x) = b(x)q(x) + r(x)$ and $\deg r(x) < \deg b(x)$ or $r(x) = 0$. Here p is a rational prime.

P15. $a(x) \mid b(x)$ and $b(x) \mid c(x) \Rightarrow a(x) \mid c(x)$. True in $\mathbb{Z}_p[x]$. Here p is a rational prime.

P16. $0 \cdot a = 0$ for every a in $\mathbb{Z}[i]$.

P17. $a > 0$, $c > b \Rightarrow ac > ab$. True in $\mathbb{Z}$.

P18. $a < b$ and $b < c \Rightarrow a < c$. True in $\mathbb{Z}$.

P19. $a \mid b$ and $b \mid a \Rightarrow a = bu$ where u is a unit. True in $\mathbb{Z}_p[x]$, where p is a rational prime.

Prove or disprove and salvage if possible. {Here proofs may be based on important results derived from a reduced inventory of properties of $\mathbb{Z}$ or from generalizations of such results.

P20. $a \mid bc$, $(a, b) = 1 \Rightarrow a \mid c$. True in $\mathbb{Z}[i]$.

P21. $a(x) \in (\mathbb{Z}_p[x])_{m(x)}$, then $a(x)$ is a unit in $(\mathbb{Z}_p[x])_{m(x)}$ if and only if $(a(x), b(x)) = 1$.

P22. $n \in \mathbb{Z}$, n positive, $n \mid (n-1)! + 1 \Rightarrow n$ is a prime.

P23. If $N(a)$ is a prime in $\mathbb{Z}$ then a is a prime in $\mathbb{Z}[i]$.

P24. $(n-1)^2 \mid n^k - 1 \Leftrightarrow (n-1) \mid k$.

P25. $(a^n - 1, a^m - 1) = a^{(n,m)} - 1$. Here a, n, m are positive integers.

P26. $a, b \in \mathbb{Z}[i] \Rightarrow$ there exist $q, r \in \mathbb{Z}[i]$ such that $a = bq + r$ and $N(r) = \frac{1}{2}N(b)$.

P27. $\varphi(ab) = \varphi(a)\varphi(b)$. True in $\mathbb{Z}$. May not assume here as known any formula for φ.

P28. $a = a_1 + a_2 i \in \mathbb{Z}[i] \Rightarrow |(\mathbb{Z}[i])_a| = N(a)$.

P29. If p is a positive prime in $\mathbb{Z}$, then every elementary symmetric function of $1, 2, \ldots, p-1$ is divisible by p.

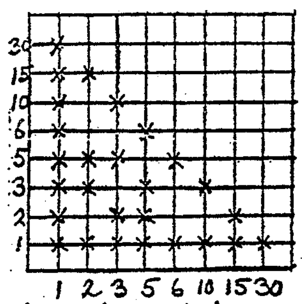
Problem Set #12

P1. Do please complete the discussion of the problems in Test #1 which you did not get to during the test period.

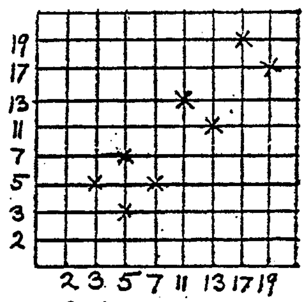
OSU Number Theory Problem Set #13 A.E. Ross Columbus 7/6/79

Exploration. A binary relation ρ in a set S is completely described by a subset A of the Cartesian product $S\times S$. This subset A consists exactly of those pairs (a,b) which are in relation ρ to each other. If $(a,b)\in A$ we write $a\rho b$.

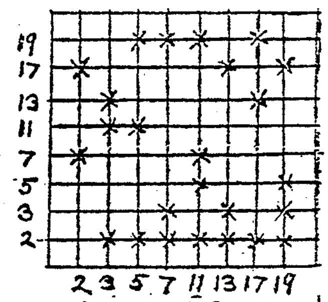
P1. Let $S=\{d \mid d>0, d\mid 30\}$. Describe in words the binary relation given by the $*$ subset of $S\times S$ as indicated in Graph #1.



Graph #1.



Graph #2.



Graph #3.

P2. As in P1 for $S=\{p \mid p \text{ a positive prime} <20\}$ and for the binary relation described by Graph #2.

P3. Same S as in P2. Describe in words the binary relation given by Graph #3.

P4. (Reading search.) List the properties which characterize each of the following binary relations: (1) Relation of equivalence; (2) Partial Ordering; (3) Linear Order.

P5. Consider the set $S=\{n \mid n\in\mathbb{Z}, 1\le n\le 15\}$. Draw the graph of the equivalence relation given by the following partition of S: $\{2,4,6,8,10,12,14\}$, $\{3,9,15\}$, $\{1,7,13\}$, $\{5,11\}$. Check the graph for all the expected properties.

Prove or disprove and salvage if possible.

P6. We say that a is congruent to b modulo m and we write $a\equiv b \pmod m$ if and only if $m\mid b-a$.

1) The binary relation "$a\equiv b \pmod m$" is a relation of equivalence (for a fixed m).
2) $a\equiv b \pmod m \Leftrightarrow a+c\equiv b+c \pmod m$.
3) $a\equiv b \pmod m \Leftrightarrow ac\equiv bc \pmod m$
4) $a\equiv b \pmod m$ and $c\equiv d \pmod m \Rightarrow a+c\equiv b+d \pmod m$ and $ac\equiv bd \pmod m$.

True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_3[x]$. True in $\mathbb{Z}_p[X]$. True in $F[X]$, where F is a field.

P7. We note that each equivalence class ("residue class") in P9 is completely described by giving one element (a "representative") which belongs to it. We write $\mathcal{C}_a$ for the residue class containing a. Is it true in the examples given in P9 that $\mathcal{C}_a\oplus\mathcal{C}_b=\mathcal{C}_{a+b}$ and $\mathcal{C}_a\odot\mathcal{C}_b=\mathcal{C}_{ab}$? Is this true for the binary relation in P6 in general? Here $\oplus$ and $\odot$ are "The addition and the multiplication of complexes".

P8. Residue classes $\{\mathcal{C}_a \mid a\in\mathbb{Z}\}$ in P6 under induced (?!?!) operations form a ring. Same holds for $\{\mathcal{C}_a \mid a\in\mathbb{Z}[i]\}$ and for $\{\mathcal{C}_a \mid a\in F[X]\}$.

Numerical Problems (Some Food for Thought).

P9. Describe the partition induced in $\mathbb{Z}$ by the equivalence relation "$a\equiv b \pmod 5$". Do the same for $\mathbb{Z}[i]$ and "$a\equiv b \pmod{2+i}$". Also for $\mathbb{Q}[x]$ and "$a\equiv b \pmod{x^2+1}$". In each case give all the equivalence (residue) classes and determine their number.

OSU Number Theory Problem Set #14 A.E. Ross Columbus 7/9/79

Numerical Problems (Some Food for Thought).

P1. What is the value of the infinite periodic continued fraction $[2,\dot{1}]$? Of $[3,\dot{2},\dot{6}]$? Show that in each case this value is a root of a quadratic polynomial in $\mathbb{Z}[x]$.

P2. Find all the positive $(x>0, y>0)$ integral solutions $\{x,y\}$ of the Diophantine equation $3x+6y=45$. Draw a picture.

P3. $f(x)=2x^4+3x^2+4x+1 \in \mathbb{Z}_5[x]$. Calculate $f(x)^5$ in $\mathbb{Z}_5[x]$. Also calculate $f(x)^{25}$.

P4. Find an integral solution x,y of the Diophantine equation $x^2-41y^2=1$. Will your observations in P17, Set #6 suggest a method of solution?

Exploration.

P5. If $u_1, u_2 \in U_m$ and if n_1 and n_2 are the orders of u_1 and u_2 respectively what can you say about the order of the product $u_1 \cdot u_2$? Do you think that there is a simple clear cut answer to this question in general? Does your experience with related problems in the preceding sets suggest a more useful question to ask?

P6. Can one use our beautiful geometrical construction of the elements of $(\mathbb{Z}[i])_{3+5i}$ in the complex plane, to determine all the units in this ring of remainders? Can you suggest a counting procedure to determine the number of units in our ring $(\mathbb{Z}[i])_{3+5i}$ without actually constructing all of them? Could this procedure, perhaps, be similar to the one used to count the number of units in $\mathbb{Z}_m$?

Prove or disprove and salvage if possible.

P7. $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + b^n$, where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k} \text{ for } k=1,\dots,n-1.$$

Here n is a positive integer. Note that $\binom{n}{k}$ is also an integer.

P8. p a positive prime in $\mathbb{Z} \Longrightarrow (a+b)^p = a^p + b^p$, $(a+b)^{p^2} = a^{p^2} + b^{p^2}, \dots$ in $\mathbb{Z}_p[x]$. Same is true in $(\mathbb{Z}_p[x])_{q(x)}$, where $q(x) \in \mathbb{Z}_p[x]$.

P9. $\sum_{\substack{d|n \\ d>0}} \mu(d) = \begin{cases} 1 \text{ if } n=1, \\ 0 \text{ if } n>1. \end{cases}$

P10. The number of perfect squares in U_p is $\frac{p-1}{2}$. Here p is a positive rational prime.

P11. If a is a prime in $\mathbb{Z}[i]$, then $N(a)$ is a prime.

Towards the Abstract.

P12. "Towards the Abstract" paper: Problems 7 and 8.

Please complete every problem in Test #1 and hand in this corrected version of Test #1 to your counsellor who will share your papers with me. Do try to complete this write up by Wednesday, July 11.

OSU Number Theory Problem Set #15 A.E. Ross Columbus 7/10/79

Prove or disprove and salvage if possible.

P1. Let $\frac{P_n}{Q_n}$ and $\frac{P_{n+1}}{Q_{n+1}}$ be two consecutive convergents to α. Then at least one of them satisfies the inequality

$$\left|\frac{P}{Q} - \alpha\right| < \frac{1}{2Q^2}.$$

Thus an infinite number of convergents $\frac{P}{Q}$ satisfy the last inequality.

P2. If U_m is cyclic and you know one of its generators, say g_1, how can you find all the other generators? How many generators would a cyclic U_m have?

P3. Justify the process of long division of polynomials.

P4. $u \in U_p$ then u is a perfect square in U_p if and only if $u^{\frac{p-1}{2}} = 1$ in U_p.

P5. 6 is the only square free perfect integer.

P6. If π is a prime in $\mathbb{Z}[i]$, then π divides some prime p in $\mathbb{Z}$.

P7. Consider the ring $(\mathbb{Z}_5[x])_{x^3+2x+4}$. Find all the roots in this ring of the polynomial y^3+2y+4. Hint: See P8, Set #14.

Exploration.

P8. Explore the arithmetic in the ring $\mathbb{Z}[\sqrt{2}] = \{a_1 + a_2\sqrt{2} \mid a_1, a_2 \in \mathbb{Z}\}$.

Technique of generalization.

P9. Generalize the Euler's function $\varphi(n)$ from $\mathbb{Z}$ to $\mathbb{Z}[i]$.

Numerical Problems (Some Food for Thought).

P10. Consider U_{31}. The order of 15 is 10 and the order of 7 is 15. Find an element of U_{31} of order $[10, 15] = 30$. Check your result and explain.

P11. How many "integral" multiples $(1+2i)(a_1+a_2 i)$ are incongruent modulo $1+7i$?

P12. Does $\mathbb{Z}_{105}[x]$ possess the unique factorization property?

P13. The Diophantine equation $x^2+y^2=719$ has no solution in integers x, y.

P14. Find a solution x, y of the Diophantine equation $x^2-41y^2=1$. Do your observations in P17, Set #6 suggest a method of solution?

Towards the Abstract.

P15. "Towards the Abstract" paper: The game of a 'reduced inventory'.

OSU Number Theory Problem Set #16 A.E. Ross Columbus 7/11/74

Numerical Problems (Some Food for Thought).

P1. Consider the remainder ring $(\mathbb{Z}_5[x])_{x^3+2x+4}$. Is this ring a field? Factor the polynomial y^3+2y+4 into the product of linear factors in this ring.

P2. Show that $(1+2i) \mid (1+i)^{N(1+2i)-1} - 1$ in $\mathbb{Z}[i]$. What is the corresponding relation in $\mathbb{Z}$?

P3. Calculate the value of $\varphi(3+i)$.

P4. Find positive rational integers n for which $\varphi(n)$ is not divisible by 4.

P5. Is 5 a square in $\mathbb{Z}_{11}$? Is 7 a square in $\mathbb{Z}_{19}$? Is -1 a square in $\mathbb{Z}_{2141}$? Justify your conclusions.

Prove or disprove and salvage if possible.

P6. Suppose that π is a Gaussian prime and that $\pi \mid p$ where p is a prime in $\mathbb{Z}$. Then $p = \pi \cdot \bar{\pi}$.

P7. $u_1, u_2 \in U_m$ and m_j is the order of $u_j \Rightarrow U_m$ contains an element of order $[m_1, m_2]$.

P8. A prime of the form $4n+3$ is not a sum of two integral squares.

Exploration.

P9. Compare $(\mathbb{Z}[i])_{1+2i}$ with $\mathbb{Z}_5$. Any conjectures?

P10. Find all the "solutions" of the congruence $35x \equiv 14 \pmod{182}$. How many distinct "solutions" does this congruence have? Explain.

P11. Find all the "solutions" in $\mathbb{Z}[i]$ of the congruence

$$(3+5i)g \equiv 2 \pmod{7+11i}.$$

How many distinct solutions does this congruence have?

Technique of generalization.

P12. Calculate $\tau(19+77i)$ and $\mu(19+77i)$.

OSU Number Theory Problem Set #17 A.E. Ross Columbus 7/12/79

Reading search.

Q1. In our discussion in the classroom we considered maps h from one mathematical system onto another such that $h(a+b)=h(a)+h(b)$ and $h(ab)=h(a)h(b)$. At that time we were interested in such maps from $\mathbb{Z}$ onto $\mathbb{Z}_m$. We observed that many algebraic properties of $\mathbb{Z}$ were "carried over" to $\mathbb{Z}_m$ by such maps. The maps $h: \mathbb{Z} \to \mathbb{Z}_m$ which we considered were many-to-one maps. Another very important case is when maps such as the above are one-to-one and onto (bijective). Such bijective maps are called isomorphisms. They define a relation of equivalence (isomorphism) among algebraic systems.

Prove or disprove and salvage if possible.

P1. $a = p_1 p_2 \cdots p_k$ is the canonical decomposition of a, k is even. Then

$$\sum_{\substack{d|a \\ 0<d<\sqrt{a}}} \mu(d) = 0.$$

P2. $[a,b](a,b)=ab$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_5[x]$. True in $F[x]$ where F is a field.

P3. $(N(g_1),N(g_2))=1 \iff (g_1,g_2)=1$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}[\sqrt{2}]$.

P4. Let $U_m=\{u_1,u_2,\dots,u_{\varphi(m)}\}$ and let n_j be the order of u_j. If $n=\max_j n_j$, then $n_i|n$ for every i.

P5. $f(x)=a_n x^n + a_{n-1}x^{n-1}+\dots+a_1 x+a_0 \in F[x]$ where F is a field. If $f(x)$ is of degree n, then $f(x)$ cannot have more than n roots in F.

P6. a,b,a',b' are all positive and $\frac{a}{b}<\frac{a'}{b'} \Rightarrow \frac{a}{b}<\frac{a+a'}{b+b'}<\frac{a'}{b'}$.

OSU Number Theory Problem Set #18 A.E.Ross Columbus 7/13/79

<u>Technique of generalization.</u>

<u>P1.</u> Generalize the result in <u>P4</u>, Set #17 to any finite commutative group.

<u>P2.</u> Generalize the result in <u>Th.4</u>, Set #11 to any finite commutative group.

<u>Numerical Problems</u> (Some Food for Thought).

<u>P3.</u> What are all the units in $\mathbb{Z}[\sqrt{2}]$? Justify your assertions.

<u>P4.</u> What is the multiplicity of 5 as a divisor of $11!$ Explain.

<u>P5.</u> Which of the following are primes in the ring $\mathbb{Z}[i]$ of Gaussian integers:

$$2, 3, 1+i, 2+3i, 3+4i, 7, 17?$$

Factor those which are not primes into prime factors in $\mathbb{Z}[i]$.

<u>P6.</u> Find all the distinct solutions of the congruence $35x \equiv 24 \pmod{182}$. Explain.

<u>P7.</u> Find all the distinct solutions of the congruence $35x \equiv 14 \pmod{182}$. Explain.

<u>P8.</u> Find all integers x such that $x \equiv 1 \pmod 4$, $x \equiv 3 \pmod 5$, $x \equiv 2 \pmod 7$. Explain.

<u>P9.</u> Find a formula giving all integers x such that

$$x \equiv b_1 \pmod 4,\ x \equiv b_2 \pmod 5,\ x \equiv b_3 \pmod 7.$$

Here b_1, b_2, b_3 are in $\mathbb{Z}$.

<u>P10.</u> Find all the solutions in $\mathbb{Z}_5[x]$ of the congruence $(x^2-1)\underline{X}(x) \equiv x^2+x-2 \pmod{x^3-1}$. Also! Count the number of distinct solutions of this congruence without solving it. Explain.

<u>Prove or disprove and salvage if possible.</u>

<u>P11.</u> $\sum_{\substack{d|n \\ d>0}} \varphi(d) = n.$

<u>P12.</u> The group U_p is cyclic.

<u>P13.</u> $\left[\frac{2a}{p}\right] = 2\left[\frac{a}{p}\right] + \left[2\left(\frac{a}{p}\right)\right]$. We recall that if $\alpha \in \mathbb{R}$, then $\alpha = [\alpha] + (\alpha)$ with $0 \leq (\alpha) < 1$. Here a and p are in $\mathbb{Z}$.

One of the participants requested that some difficult and challenging problems be given in our Number Theory Problem Sets. Such problems will be marked by * or ** or *** to indicate their relative degree of difficulty. They will not be discussed in seminars and counsellors will not provide any help in their solution. I shall be very happy to discuss his solution with any participant who will present me with a written solution of any of these specially selected problems.

* <u>P14.</u> Find all the integral solutions of the Diophantine equation

$$y^2+2 = x^3.$$

OSU Number Theory Problem Set #19 A.E. Ross Columbus 7/16/79

Exploration.

P1. Compare U_7 and the system of bijective maps $U_7 \to U_7$ given by
$$f_n : a \mapsto an \quad (n = 1, 2, 3, 4, 5, 6_0)$$
under the composition (product) of maps.

P2. Consider two functions f and g such that $f(n) = \sum_{\substack{d \mid n \\ d>0}} g(d)$ for every positive integer n. Express g in terms of f.

P3. We note that $\sqrt{-1} \notin \mathbb{Z}_7$. Construct a finite field $F \supset \mathbb{Z}_7$ and such that $\sqrt{-1} \in F$. Explain.

Prove or disprove and salvage if possible.

P4. Use a property of the arithmetic function $[x]$ to prove that the binomial coefficient $\binom{m}{k}$ is an integer.

P5. $[x] + [-x] = -1$.

P6. The group of units U_π in $(\mathbb{Z}[i])_\pi$ is cyclic. Here π is a prime in $\mathbb{Z}[i]$.

P7. If $(a, m) = 1$, the congruence $ax \equiv b \pmod{m}$ has one and only one "solution" (residue class!). True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}[\sqrt{2}]$. True in $\mathbb{Z}_p[x]$. True in $\mathbb{Q}[x]$. True in $F[x]$, where F is a field.

P8. If u is a unit in $\mathbb{Z}_p$ (p a prime ≥ 7) whose order is 3, then $u+1$ is a unit of order 6.

P9. p is a positive prime in $\mathbb{Z}$, $\pi(x)$ a prime of degree n in $\mathbb{Z}_p[x] \Rightarrow \pi(y) \mid y^{p^n} - y \Rightarrow$ $\Rightarrow \pi(y)$ splits into linear factors in $\mathbb{Z}_p[x]/\pi(x)$.

P10. If u is a generator of the group U_p of units in $\mathbb{Z}_p$ and p is a positive prime, then u is not a square in $\mathbb{Z}_p$.

Numerical Problems (Some Food for Thought).

P11. Find the number of units in $(\mathbb{Z}[i])_{3+5i}$ by calculating $\varphi(3+5i)$.

P12. Find all the solutions of the congruence $49x \equiv 15 \pmod{847}$.

P13. Find all the solutions of the congruence $49x \equiv 14 \pmod{847}$.

P14. Expand $\alpha = \sqrt{2}$ into a simple continued fraction $\alpha = [a_1, a_2, a_3, \dots]$ and calculate the following

		a_1	a_2	a_3	a_4
P_k	1	a_1	P_2	$\cdots$	
Q_k	0	1	Q_2	$\cdots$	
$P_k^2 - 2Q_k^2$		$P_1^2 - 2Q_1^2$	$P_2^2 - 2Q_2^2$	$\cdots$	

What can you say about the norm of $3+2\sqrt{2}$ in $\mathbb{Z}[\sqrt{2}]$? About the norm of $17+12\sqrt{2}$? Compare $3+2\sqrt{2}$ and $17+12\sqrt{2}$ with the powers of $1+\sqrt{2}$. Do you find much food for thought here?

OSU Number Theory Problem Set #20 A.E.Ross Columbus 7/17/79

Exploration.

P1. We recall (define?!) that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then
$$f'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$
and
$$f(x+y) = f(x) + f'(x)y + \frac{f''(x)}{2!}y^2 + \frac{f^{(3)}(x)}{3!}y^3 + \cdots .$$

Numerical Problems (Some Food for Thought).

P2. Find all the solutions of the congruence
$$7x^4 + 19x + 25 \equiv 0 \pmod{27}.$$

P3. Find all the solutions of the congruence
$$x^3 + 2x + 2 \equiv 0 \pmod{125}.$$

P4. Is the group U_{7-6i} of units in $(\mathbb{Z}[i])_{7-6i}$ cyclic? Justify your assertion. What is the order of the group U_{7-6i}?

P5. Find all the solutions in positive integers n of $\varphi(n) = 24$.

Prove or disprove and salvage if possible.

P6. Let $a = pq + r$, $0 \le r < p$. Then $[\frac{2a}{p}]$ is even if $0 \le r < \frac{p}{2}$ and $[\frac{2a}{p}]$ is odd if $\frac{p}{2} \le r < p$. (Compare with P13, Set #18).

P7. Given a prime each of whose (decimal) digits is equal to 1. Then the number of its digits is a prime. (It is not known whether there is an infinite number of such primes).

P8. $f(n) = \sum_{\substack{d|n \\ d>0}} g(d)$. Then if $g(n)$ is a multiplicative arithmetic function so is $f(n)$.
We say that $g(n)$ is multiplicative if $g(n_1 n_2) = g(n_1)g(n_2)$ whenever $(n_1, n_2) = 1$.

P9. Consider a congruence $ax \equiv b \pmod m$. If $(a,m) | b$ this congruence has exactly (a,m) distinct solutions. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}_p[x]$.

P10. p a positive prime in $\mathbb{Z}$ and $p \equiv 1 \pmod 4 \iff p | a^2+1, a \in \mathbb{Z} \iff p$ is not a prime in $\mathbb{Z}[i]$.

OSU Number Theory Problem Set #21 A.E. Ross Columbus 7/18/79

Prove or disprove and salvage if possible.

P1. $f(x) \in \mathbb{Z}[x]$. If $x_1 \equiv x_2 \pmod{m}$, then $f(x_1) \equiv f(x_2) \pmod{m}$.

P2. If G is a finite subgroup of the multiplicative group of a (commutative) field, then G is cyclic.

P3. If $f(n) = \sum_{\substack{d \mid n \\ d>0}} g(d)$, Then $g(n) = \sum_{\substack{d \mid n \\ d>0}} \mu(\frac{n}{d}) f(d) = \sum_{\substack{d \mid n \\ d>0}} \mu(d) f(\frac{n}{d})$.

P4. $a'b - ab' = 1 \Rightarrow (a+a', b+b') = 1$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $F[x]$, F a field.

P5. Use P11, Set #13 and the Möbius's inversion formula in P3 to obtain the familiar formula for $\varphi(n)$.

P6. Consider the products $a\cdot 1, a \cdot 2, \ldots, a\frac{p-1}{2}$. We have

$$al \equiv \varepsilon_l r_l \pmod{p}, \quad 0 < r_l < \frac{p}{2}, \quad l = 1, 2, \ldots, \frac{p-1}{2},$$

$$\varepsilon_l = (-1)^{\left[\frac{2al}{p}\right]}$$

and hence

$$a^{\frac{p-1}{2}} \equiv (-1)^{\sum_{l=1}^{\frac{p-1}{2}} \left[\frac{2al}{p}\right]} \pmod{p}.$$

Here p is a positive odd prime in $\mathbb{Z}$ and $(a,p)=1$.

Numerical Problems (Some Food for Thought).

P7. Find all the solutions in $\mathbb{Z}[i]$ of the congruence

$$(16-2i)g \equiv -1+5i \pmod{33+17i}.$$

Also! Count the number of distinct solutions of this congruence without solving it. Explain.

P8. Which a is a perfect square in $\mathbb{Z}_p$?

a	2	2	2	2	11	3	150
p	7	17	31	101	101	103	151

P9. Let each fraction $\frac{m}{n}$ be represented by the point (m,n) in the plane. Plot all the fractions in the Farey sequence of order 5. For every pair of consecutive entries $\frac{a}{b}$ and $\frac{a'}{b'}$ calculate $a'b - ab'$. Is the result of this experiment completely unexpected in view of P5, Set #5 and P1, Set #11?

P10. Construct the table of all primes π in $\mathbb{Z}[i]$ with $N(\pi) < 50$ and plot them in the complex plane.

P11. Find three units $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{Z}[\sqrt{2}]$ such that $N(\alpha_i) = 1$ and $\alpha_i > 1$. Similarly find three units with a positive norm in $\mathbb{Z}[\sqrt{2}]$ and lying between 0 and 1.

OSU Number Theory Problem Set #22 A.E. Ross Columbus 7/19/79

Recapitulation.

P1. When is a Gaussian integer a prime in $\mathbb{Z}[i]$? Explain.

Exploration.

P2. What can you say about primes in $\mathbb{Z}[\sqrt{-2}]$.

Numerical Problems (Some Food for Thought).

P3. Construct the residue class ring modulo $x^4+x^3+x^2+x+1$ in $\mathbb{Z}_2[x]$. Is x a generator of the group of units of our ring (field?)

$$F = (\mathbb{Z}_2[x])_{x^4+x^3+x^2+x+1} = \mathbb{Z}_2[x]/_{x^4+x^3+x^2+x+1}.$$

Calculate successive powers of a generator and construct a table of logarithms (indices) with this generator as a base, for the units of F. Use this table to calculate the (multiplicative!) inverse of $x+1$, of x^2+x+1. Use the same table to find (the unique?) solution $\bar{X}(x)$ of the congruence

$$(x^2+1)\bar{X}(x) \equiv x+1 \pmod{x^4+x^3+x^2+x+1}$$

or, what is equivalent, find the solution of the equation

$$(x^2+1)\bar{X}(x) = x+1 \quad \text{in } F.$$

Here as is usual we write $x+1$ for the "residue class" $\mathcal{C}_{x+1}$, i.e. each residue class is characterized by one of its representatives.

P4. Find all the roots of $f(y) = y^4+y^3+y^2+y+1$ in the field F of P3. Compare these roots with the powers x, x^2, x^4, x^8 of x. Explain.

P5. Find $(3+2\sqrt{-2}, 4-\sqrt{-2})$ in $\mathbb{Z}[\sqrt{-2}]$.

Prove or disprove and salvage if possible.

P6. $a, b \in \mathbb{Z}$ are positive, odd, relatively prime $\Rightarrow \sum_{\substack{0<x<\frac{b}{2} \\ x\in\mathbb{Z}}} \left[\frac{ax}{b}\right] + \sum_{\substack{0<y<\frac{a}{2} \\ x\in\mathbb{Z}}} \left[\frac{by}{a}\right] = \frac{a-1}{2}\cdot\frac{b-1}{2}$

Hint: Draw a picture.

P7. $\frac{a}{b}, \frac{a''}{b''}, \frac{a'}{b'}$ are three successive entries in a Farey sequence of order n, then

$$\frac{a''}{b''} = \frac{a+a'}{b+b'}.$$

P8. $d\in\mathbb{Z}, d>0, d\neq\square$. Then $|P_n^2 - dQ_n^2| < 2\sqrt{d}+1$ for every convergent $\frac{P_n}{Q_n}$ to $\sqrt{d}$.

OSU Number Theory Problem Set #23 A. E. Ross Columbus 7/20/79

Prove or disprove and salvage if possible.

P1. Define the symbol $\left(\frac{a}{p}\right)$ by the congruence

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p} \text{ and } \left(\frac{a}{p}\right) = +1 \text{ or } -1 \quad \text{(Legendre!)}.$$

Here p is a positive odd prime in $\mathbb{Z}$ and $(a,p)=1$. If also $(b,p)=1$, we have

(1) $a \equiv \square \pmod{p} \Leftrightarrow \left(\frac{a}{p}\right)=1$;

(2) $a \equiv b \pmod{p} \Rightarrow \left(\frac{a}{p}\right)=\left(\frac{b}{p}\right)$;

(3) $\left(\frac{ab}{p}\right)=\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$;

(4) $\left(\frac{a^2}{p}\right)=1$;

(5) $\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}$.

P2. p a positive prime $\Rightarrow \sum_{u \in U_p} u = 0$

P3. $a \equiv b \pmod{m} \Rightarrow (a,m)=(b,m)$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $F[x]$, F a field.

P4. The product of any k consecutive integers is divisible by $k!$.

* P5 If F is a finite field then $|F| = p^n$ for some positive rational prime p and a suitable positive rational integer n. Note that this is true for every finite field F however arrived at.

P6. Use the arithmetic in $\mathbb{Z}[i]$ to show that if p is a positive prime of the form $4n+1$ then the Diophantine equation $x^2+y^2=p$ has an integral solution $\{x,y\}$.

Numerical Problems (Some Food for Thought).

P7. Draw the picture used in the geometric proof of

$$\sum_{0<x<\frac{b}{2}} \left[\frac{ax}{b}\right] + \sum_{0<y<\frac{a}{2}} \left[\frac{by}{a}\right] = \frac{a-1}{2}\cdot\frac{b-1}{2}, \quad x,y \in \mathbb{Z},\ (a,b)=1 \qquad (\text{P6, Set \#22})$$

for the case $a=7$, $b=5$. Also for the case $a=17$, $b=11$.

P8. The congruence $x^2 \equiv 10 \pmod{13}$ has exactly two distinct solutions. Why? How many distinct solutions does the congruence $x^2 \equiv 10 \pmod{169}$ have? How many distinct solutions does the congruence $x^2 \equiv 10 \pmod{2197}$ have? Find all the solutions of each of the above congruences. Explain and generalize.

P9. Find all the solutions of each of the following congruences:

(1) $x^2 \equiv 2 \pmod{2737}$, (2) $x^2 \equiv 3 \pmod{1771}$.

P10. Does the Diophantine equation $x^2+y^2+z^2=95$ have an integral solution $\{x,y,z\}$?

P11. Calculate $\sum_{d \mid 3+5i} \varphi(d)$. We take $\varphi(1)=1$ and select only one d among its associates in $\mathbb{Z}[i]$.

Towards the Abstract.

P12. "Towards the Abstract" paper: Section 7, Problems 10 and 11.

OSU Number Theory SAMPLE TEST #2 Part I A.E. Ross 7/23/79

Exploration.

P1. Explore the arithmetic of $\mathbb{Z}[\sqrt{-2}]$.

Numerical Problems (Some Food for Thought)

P2. Let $\alpha = [3, \dot{2}, 1, 2, \dot{6}]$. Find the quadratic polynomial in $\mathbb{Z}[x]$ of which α is a root and express α in closed form.

P3. Find an integral solution $\{x, y\}$ of $x^2 - 22y^2 = 1$ with $y \neq 0$.

P4. Find all the distinct solutions in $\mathbb{Z}[i]$ of the congruence $(6-17i)x \equiv 5+i \pmod{13}$. Check by calculating the number of distinct solutions directly without solving.

P5. Find all integers x such that $x \equiv 3 \pmod 8$, $x \equiv 11 \pmod{20}$, $x \equiv 16 \pmod{75}$.

P6. With how many zeros does $49!$ end?

P7. Construct the residue class ring modulo $x^4 + x^2 + 2$ in $\mathbb{Z}_3[x]$. Is this ring a field? What is the order of x in the group of units? Is $x+1$ a unit? Calculate $\frac{1}{x+1}$. Find all the roots of $y^4 + y^2 + 2$ in $\mathbb{Z}_3[x]/x^4+x^2+2$.

P8. Find $(2+3\sqrt{-2}, 4-\sqrt{-2})$ in $\mathbb{Z}[\sqrt{-2}]$.

Prove or disprove and salvage if possible.

P9. a, b in $\mathbb{Z}[i] \Rightarrow \varphi(ab) = \varphi(a)\varphi(b)$.

P10. If $\pi(x)$ is a prime in $\mathbb{Z}_p[x]$, then $\pi(y)$ factors into linear factors in $F[y]$, where $F = \mathbb{Z}_p[x]/\pi(x)$.

P11. If $\frac{a}{b}$, $\frac{a'}{b'}$ are two successive entries in a Farey sequence of order n, then $a'b - ab' =$

P12. If π is a prime in $\mathbb{Z}[i]$, then if $u \in U_{\pi^3}$, we have $u^{N(\pi)^2(N(\pi)-1)} = 1$ in U_{π^3}.

P13. The congruence $g_1(x)\, y \equiv g_2(x) \pmod{m(x)}$, where $g_1(x), g_2(x)$ are in $\mathbb{Z}_p[x]$, has a solution y in $\mathbb{Z}_p[x]$ if and only if $(g_1(x), m(x)) \mid g_2(x)$. The number of distinct solutions of this congruence is (?).

P14. If $\mathcal{C}_{a(x)}$ and $\mathcal{C}_{b(x)}$ are two residue classes modulo $m(x)$ in $\mathbb{Z}_p[x]$, then
$$\mathcal{C}_{a(x)} + \mathcal{C}_{b(x)} = \mathcal{C}_{a(x)+b(x)} \text{ and } \mathcal{C}_{a(x)}\mathcal{C}_{b(x)} \subset \mathcal{C}_{a(x)b(x)}$$
(We note that $a(x) \in \mathcal{C}_{a(x)}$)

P15. The fundamental theorem of arithmetic holds true in $\mathbb{Z}[\sqrt{2}]$.

P16. If $f(n) = \sum_{\substack{d \mid n \\ d>0}} g(d)$, then $g(n) = \sum_{\substack{d \mid n \\ d>0}} \mu(d) f\left(\frac{n}{d}\right)$.

P17. Let $r > 0$. The number of lattice points on the circular disk $x^2 + y^2 \leq r^2$ is
$$1 + 4[r] + \sum_{0 < x \leq \frac{r}{\sqrt{2}}} [\sqrt{r^2 - x^2}] - 4\left[\frac{r}{\sqrt{2}}\right]^2.$$

P18. $\pi \in \mathbb{Z}[i]$ is a prime in $\mathbb{Z}[i]$ if and only if (1) $\pi = 1+i$ or $\pi = 1-i$. These are associates (2) $\pi =$ a rational prime $p \equiv 3 \pmod 4$. (3) $N(\pi) = p \equiv 1 \pmod 4$ where p is a rational prime and π and $\bar{\pi}$ are not associates.

P19. Consider a commutative group G. Let n_1 be the order of $g_1 \in G$ and let n_2 be the order of $g_2 \in G$. Then there exists an element $g \in G$ whose order is $[n_1, n_2]$.

P20. There exist infinitely many primes in $\mathbb{Z}$ of the form $8n-1$.

P21. $ab \equiv ac \pmod m \Rightarrow b \equiv c \pmod m$. True in $\mathbb{Z}$. True in $\mathbb{Z}[i]$. True in $F[x]$.

P22. Let $f(n) = \sum_{\substack{d \mid n \\ d>0}} g(d)$. Then if $f(n)$ is multiplicative so is $g(n)$.

P23. Let H be a finite subgroup of the multiplicative group F^* of a field F. Then H is cyclic.

P24. If p is a positive odd prime in $\mathbb{Z}$, then $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

P25. If p and q are two distinct positive odd primes in $\mathbb{Z}$, then $\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \left(\frac{q}{p}\right)$.

P26. π is a prime in $\mathbb{Z}[i] \Rightarrow \sum_{u \in U_\pi} u = 0$ in $(\mathbb{Z}[i])_\pi$.

OSU Number Theory SAMPLE TEST #2 Part II. A.E. Ross 7/24/79

Numerical Problems (Some Food for Thought). Please display your calculations neatly! Calculate efficiently!

P1. Factor $22+7i$ into prime factors in $\mathbb{Z}[i]$.
P2. Find $(2+3i, 2-3i)$.
P3. Calculate $\varphi(17)$, $\varphi(19)$, $\varphi(637)$ in $\mathbb{Z}$. Calculate $\varphi(17)$, $\varphi(19)$, $\varphi(6+7i)$ in $\mathbb{Z}[i]$.
P4. Calculate the number of solutions of the congruence $x^2 \equiv 11 \pmod{1365}$. Explain.
P5. Find all the solutions of the congruence $x^2 \equiv 5 \pmod{551}$.
P6. Consider the largest root α of x^2-4x+2. Calculate the first three convergents of the continued fraction expansion of α.
P7. For what primes p is 6 a square in U_p? Here p is a positive odd prime in $\mathbb{Z}$.
P8. Find all the solutions of the congruence $x^2 \equiv 2 \pmod{343}$.
P9. Find an integral solution of $x^2-23y^2=1$ with $y \neq 0$. Also, solve $x^2-23y^2=-1$. Explain.
P10. Find an integral solution of $x^2-58y^2=-1$.
P11. Calculate the number of prime polynomials of degree 3 in $\mathbb{Z}_3[x]$.
P12. Which of the following integers is a square modulo 991: 231, 783, 563, 773?
P13. $2730 \mid n^{13}-n$ for every positive integer n.
P14. Find a formula giving all integers x such that $x \equiv b_1 \pmod 4$, $x \equiv b_2 \pmod 7$, $x \equiv b_3 \pmod{11}$. Here b_1, b_2, b_3 are in $\mathbb{Z}$. Use a similar formula in P5.

Prove or disprove and salvage if possible.

P15. If d is the smallest positive integer of the form $ax+by$, then $d=(a,b)$.
P16. π is a prime in $\mathbb{Z}[i]$. If $\pi \mid a^l$ where $a \in \mathbb{Z}[i]$ and l is a positive integer then $\pi \mid a$.
P17. Let $u \in U_p$. Then u is a square in U_p if and only if $\left(\frac{u}{p}\right)=1$. Here $\left(\frac{u}{p}\right)$ is Legendre symbol.
P18. a, b in $\mathbb{Z}[i] \Rightarrow \varphi(ab)=\varphi(a)\varphi(b)$.
P19. p is a positive prime in $\mathbb{Z}$, $\pi(x)$ a prime of degree n in $\mathbb{Z}_p[x] \Rightarrow \pi(y)$ splits into linear factors in $F[y]$ where $F = \mathbb{Z}_p[x]/\pi(x)$.
P20. Units in $\mathbb{Z}[\sqrt{2}]$ form a cyclic group.
P21. Let $G=\{g_1, \dots, g_n\}$ be a finite commutative group. Let n_j be the order of g_j. If $n = \max_j n_j$ then $n_j \mid n$ for every j.

OSU Number Theory TEST#2 A.E.Ross Columbus 7/25/79

Numerical Problems (Some Food for Thought). Please display your calculations neatly! Calculate efficiently!

P1. Let $\alpha = [3, \dot{2}, \dot{6}]$. Find the quadratic polynomial in $\mathbb{Z}[x]$ of which α is a root and express α in closed form.

P2. Calculate $\varphi(13)$, $\varphi(11)$, $\varphi(637)$ in $\mathbb{Z}$. Calculate $\varphi(13)$, $\varphi(11)$, and $\varphi(6+7i)$ in $\mathbb{Z}[i]$.

P3. Solve the congruence $x^2 \equiv 17 \pmod{77}$.

P4. Calculate $\sum_{d|4+7i} \varphi(d)$. We take $\varphi(1)=1$ and select only one d among its associates in $\mathbb{Z}[i]$.

P5. Calculate the number of generators of U_{19}.

P6. Find all integers x such that $x \equiv 3 \pmod 8$, $x \equiv 11 \pmod{20}$, $x \equiv 1 \pmod{15}$.

P7. Find all the solutions of the congruence $x^2 \equiv 2 \pmod{119}$.

P8. Find all the solutions of the congruence $x^2 \equiv 3 \pmod{121}$.

P9. Solve the Diophantine equation $x^2 - 19y^2 = -1$.

P10. Find an integral solution of $x^2 - 17y^2 = 1$ with $y \neq 0$.

P11. Does the congruence $x^2 \equiv 19 \pmod{191}$ have a solution x in $\mathbb{Z}$.

Prove or disprove and salvage if possible.

P12. Let G be a finite commutative group. If $g_1, g_2 \in G$, if n_1 is the order of g_1 and n_2 is the order of g_2 and if $(n_1, n_2) = 1$, then $n_1 n_2$ is the order of $g_1 g_2$.

P13. $\pi \in \mathbb{Z}[i]$ is a prime in $\mathbb{Z}[i]$. Let p be a positive rational prime such that $\pi | p$. Then either $p = \pi \cdot \bar{\pi}$ or p is an associate of π.

P14. $G = \{g_1, g_2, \cdots, g_e\}$ is a finite commutative group. Let n_j be the order of g_j and let $n = \max n_j$. Then $n_j | n$ for every j.

P15. If G is a finite commutative group, if $g_1 \in G$, $g_2 \in G$ and if the order of g_1 is n_1 and the order of g_2 is n_2, then G has an element g of order $[n_1, n_2]$.

P16. $2730 \mid n^{13} - n$ for every positive integer n.

P17. If $d(x)$ is a polynomial of the smallest degree in x of the form $a(x)X(x) + b(x)Y(x)$ then $d(x) = (a(x), b(x))$. True in $F[x]$, F a field.

P18. If p is a positive odd prime, then $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0$. Here $\left(\frac{a}{p}\right)$ is the Legendre symbol.

P19. Let $f(n) = \sum_{\substack{d|n \\ d>0}} g(d)$. Then if $f(n)$ is multiplicative so is $g(n)$.

P20. Let H be a finite subgroup of the multiplicative group F^* of a field F. Then H is cyclic.

P21. If p is a positive odd prime in $\mathbb{Z}$, then $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

P22. If p and q are two distinct positive odd primes in $\mathbb{Z}$, then $\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}\left(\frac{q}{p}\right)$.

P23. $\sum_{0<x<\frac{b}{2}} \left[\frac{ax}{b}\right] + \sum_{0<y<\frac{a}{2}} \left[\frac{by}{a}\right] = \frac{a-1}{2}\cdot\frac{b-1}{2}$, $x, y \in \mathbb{Z}$, $(a,b) = 1$.

P24. $a \equiv b \pmod p \Rightarrow \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$. Also $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. Here p is a positive odd prime and $\left(\frac{a}{p}\right)$ is the Legendre symbol.

P25. $d \in \mathbb{Z}$, $d > 0$, $d \neq \square$. Then $|P_n^2 - dQ_n^2| < 2\sqrt{d} + 1$ for every convergent $\frac{P_n}{Q_n}$ to $\sqrt{d}$.

P26. If $f(n) = \sum_{\substack{d|n \\ d>0}} g(d)$, then $g(n) = \sum_{\substack{d|n \\ d>0}} \mu\left(\frac{n}{d}\right) f(d)$.

P27. p is a positive prime in $\mathbb{Z}$, $\pi(x)$ a prime of degree n in $\mathbb{Z}_p[x] \Rightarrow \pi(y) \mid y^{p^n} - y$.

P28. $\sum_{\substack{d|n \\ d>0}} \varphi(n) = n$.

P29. $(N(g_1), N(g_2)) = 1 \Leftrightarrow (g_1, g_2) = 1$. True in $\mathbb{Z}[i]$.

P30. Use the formula for $\varphi(m)$ to prove that there exist infinitely many primes in $\mathbb{Z}$.

OSU Number Theory Problem Set #24 A.E. Ross Columbus 7/27/79

Exploration.

P1. Let $f(n) = \prod_{\substack{d|n \\ d>0}} g(d)$. Express the function g in terms of the function f.

Prove or disprove and salvage if possible.

Given a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

we define(!) the derivative of $f(x)$ as

$$f'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

P2. If $f(x)$ and $g(x)$ are polynomials then $(f(x)+g(x))' = f'(x) + g'(x)$.

P3. If a is a constant, $f(x)$ a polynomial then $(a f(x))' = a f'(x)$.

P4. Let $f_n(x) = x^n$. Then $f_n'(x) = nx^{n-1}$, $f_n''(x) = n(n-1)x^{n-2}$, $\ldots$, $f_n^{(\ell)}(x) = n(n-1)\cdots(n-\ell+1)x^{n-\ell}$

Observe that $\frac{1}{\ell!} f_n^{(\ell)}(x) = \binom{n}{\ell} x^{n-\ell}$.

P5. $f_n(x+y) = (x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$

$$= f_n(x) + \frac{f_n'(x)}{1!}y + \frac{f_n''(x)}{2!}y^2 + \frac{f_n^{(3)}(x)}{3!}y^3 + \cdots \frac{1}{n!}f_n^{(n)}(x)y^n = \sum_{\ell=0}^{n} \frac{1}{\ell!} f_n^{(\ell)}(x) y^\ell$$

P6. If $m > n$ we have $f_n(x+y) = \sum_{\ell=0}^{m} \frac{1}{\ell!} f^{(\ell)}(x) y^\ell$.

P7. Use the results in P2 and P5 to show that if $f(x)$ is a polynomial of degree m, then

$$f(x+y) = \sum_{\ell=0}^{m} \frac{1}{\ell!} f^{(\ell)}(x) y^\ell \qquad \text{(The Taylor formula.)}$$

P8. Use the result in P7 to show that

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

What if we have the product of more than two factors? What about higher derivatives?

P9. Use the result in P8 to show that if α is a root of multiplicity k of a polynomial $f(x)$, then α is a root of multiplicity $k-1$ of $f'(x)$.

P10. If $\nu_q(m) = \frac{1}{m} \sum_{\substack{d|m \\ d>0}} \mu\left(\frac{m}{d}\right) q^d$, then $\nu_q(m) > 0$.

Numerical Problems (Some Food for Thought)

P11. Find all the solutions of the congruence

$$x^3 + 19x^2 - x + 23 \equiv 0 \pmod{42}$$

P12. Find all the solutions of the congruence

$$x^3 - 2x + 6 \equiv 0 \pmod{125}.$$

Hint: Use the results in P7.

OSU Number Theory Problem Set #25 A.E. Ross Columbus 7/30/79

Numerical Problems (Some Food for Thought).

P1. Construct a polynomial $g(x)$ in $\mathbb{Z}_5[x]$, which has the same roots as
$$f(x) = x^4 + 3x^3 + 2x + 4 \in \mathbb{Z}_5[x]$$
but in which each root has multiplicity 1. Do this without calculating the roots themselves.

P2. What are all the automorphisms of the field $\mathbb{Z}_5[x]/x^3+3x+2$?

P3. Consider a finite field $F = \mathbb{Z}_2[x]/x^4+x+1$. All the roots of $\pi(x) = x^2+x+1$ are in this field. Let $\alpha = x^5 \in F$. Then α is a root of $\pi(y)$. Construct the smallest field in F which contains $\mathbb{Z}_2$ and α.

P4. How many prime polynomials of degree 4 are there in $\mathbb{Z}_2[x]$? What does P10, Set #24 tell you?

Prove or disprove and salvage if possible.

P5. For every positive rational integer n there is at least one prime polynomial $\pi(x)$ of degree n in $\mathbb{Z}_p[x]$.

P6. Summarise all that we have learned about finite fields to date and arrange all this information in a logical order. Indicate which problems contained pertinent information.

P7. All the primitive n-th roots of 1 in $\mathbb{C}$ (i.e. all the generators of the group of n-th roots of 1 in $\mathbb{C}$) are the roots of the polynomial
$$\Phi_n(x) = \prod_{\substack{d \mid n \\ d > 0}} (x^d - 1)^{\mu(n/d)} \in \mathbb{Z}[x]$$
The degree of $\Phi_n(x)$ is (?). Hint: compare with P1, Set #24 when we let $f(n) = x^n - 1$ and $g(d) = \Phi_d(x)$.

Exploration

P8. $(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2) = (?)^2+(?)^2+(?)^2+(?)^2$.
Consider a quaternion $q = x_1 + x_2 i + x_3 j + x_4 k$ and note that the distributive law and the equalities $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$ provide us with the rules for multiplication. Observe that the norm $N(q) = q\bar{q}$ is multiplicative. Here $\bar{q} = x_1 - x_2 i - x_3 j - x_4 k$ and $q\bar{q} = x_1^2 + x_2^2 + x_3^2 + x_4$.

OSU Number Theory Problem Set #26 A.E. Ross Columbus 7/31/79

Prove or disprove and salvage if possible.

P1. Let P be a positive odd integer in $\mathbb{Z}$. Following Jacobi we generalize Legendre symbol as follows: We suppose that $P = p_1 p_2 \cdots p_\ell$ (the primes p_j are not necessarily distinct!) and for $a \in \mathbb{Z}$ such that $(a, P) = 1$ we set

Jacobi $\left(\frac{a}{P}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right)\cdots\left(\frac{a}{p_\ell}\right)$. Legendre

Then (1) $\left(\frac{a}{P_1 P_2}\right) = \left(\frac{a}{P_1}\right)\left(\frac{a}{P_2}\right)$; (2) $\left(\frac{a_1 a_2}{P}\right) = \left(\frac{a_1}{P}\right)\left(\frac{a_2}{P}\right)$; (3) $a_1 \equiv a_2 \pmod{P} \Rightarrow \left(\frac{a_1}{P}\right) = \left(\frac{a_2}{P}\right)$;

(4) $\left(\frac{-1}{P}\right) = (-1)^{\frac{P-1}{2}}$; (5) $\left(\frac{2}{P}\right) = (-1)^{\frac{P^2-1}{8}}$.

P2. P and Q positive and odd and $(P, Q) = 1 \Rightarrow \left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = (-1)^{\frac{P-1}{2}\cdot\frac{Q-1}{2}}$.

P3. A set S of positive integers which contains 1 and contains $n+1$ whenever it contains n, contains every positive integer. (The basis of proofs by induction.)

P4. A set S of positive integers which contains 1 and contains n whenever it contains every positive integer ~~k~~ smaller than n, contains every positive integer. (The basis of proofs by descent.)

* P5 Using a geometrical argument show that if

$$\left|\frac{P}{Q} - \alpha\right| < \frac{1}{2Q^2} \quad \text{for } \alpha \in \mathbb{R}, \alpha > 0,$$

then $\frac{P}{Q}$ is a convergent of the continued fraction expansion of α.

P6. We know that

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Here $f(x) \in F[X]$ and $g(x) \in F[X]$, F a field. Show that

(!) $\quad (f(x)g(x))'' = f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x).$

What convention must one accept in order to be able to write the right hand side of (!) symbolically as $(f(x)+g(x))^2$? What can you say about $(f(x)g(x))'''$, about $(f(x)g(x))^{(4)}$, about $(f(x)g(x))^{(n)}$?

OSU Number Theory Problem Set #27 A.E. Ross Columbus 8/1/79

Numerical Problems (Some Food for Thought).

P1. Find the number of solutions of each of the following congruences:

(1) $x^2 \equiv 3 \pmod{31}$; (2) $x^2 \equiv 226 \pmod{563}$

(3) $x^2 \equiv 429 \pmod{563}$; (4) $x^2 \equiv 3149 \pmod{5987}$

(5) $x^2 \equiv 2 \pmod{9047}$ (6) $x^2 \equiv 35 \pmod{6887}$

Here one may use the results of P1, Set #26.

P2. Find all the solutions of the congruence

$$x^4 + 4x^3 + 2x^2 + 2x + 12 \equiv 0 \pmod{625}.$$

P3. Consider the field $F = (\mathbb{Z}[i])_7 = \mathbb{Z}[i]/_7$. How many prime polynomials $\pi(x)$ of degree 2 are there in $F[x]$? Of degree 3? Of degree 4? How can you modify the formula in P10, Set #24 so that it would apply here?

Exploration.

P4. $\mathbb{Z}[\sqrt{-3}]$ does not give all the "integers" in $\mathbb{Q}(\sqrt{-3})$. Here to get all the integers we must also consider $\frac{a_1}{2} + \frac{a_2}{2}\sqrt{-3}$ where both a_1 and a_2 are odd. Thus all our integers in $\mathbb{Q}(\sqrt{-3})$ are given by $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Plot these integers in the complex (Gauss) plane. Use a geometrical argument to establish Euclidean algorithm in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.

Prove or disprove and salvage if possible.

P5. If a prime $p = 2^{2^n}+1$, then 2 is not a generator of U_p.

P6. $(a(x), b(x)) = 1 \Rightarrow a(x)\bar{X}(x) + b(x)Y(x) = 1$ has a solution $\{\bar{X}(x), Y(x)\}$ such that $\deg \bar{X}(x) < \deg b(x)$ and $\deg Y(x) < \deg a(x)$. True in $F[x]$ where F is a field.

P7. Let $n = p_1 p_2 \cdots p_k$ be the canonical decomposition of n (i.e. $p_j \neq p_\ell$ for $j \neq \ell$). Then

$$\sum_{\substack{d|n \\ d>0}} \mu(d)\tau(d) = (-1)^k.$$

P8. Let $f(x), g_1(x), g_2(x)$ be in $F[x]$, F a field, and let $(g_1(x), g_2(x)) = 1$. Then

$$\frac{f(x)}{g_1(x)g_2(x)} = \frac{h_1(x)}{g_1(x)} + \frac{h_2(x)}{g_2(x)} \quad \text{with } h_1(x), h_2(x) \in F[x].$$

OSU Number Theory Problem Set #28 A.E. Ross Columbus 8/2/79

Numerical Problems (Some Food for Thought).

P1. Find all the solutions of the congruence
$$x^5+x^4-5x^3+2x^2+x+1 \equiv 0 \pmod{2401}$$

P2. Construct an isomorphism between the following two finite fields:
$$\mathbb{Z}_3[x]/x^3+2x+2 \quad \text{and} \quad \mathbb{Z}_3[x]/x^3+x^2+2.$$

P3. Decompose into "partial fractions" in $\mathbb{R}[x]$":
(1) $\dfrac{3x+4}{x^2+3x+2}$ (2) $\dfrac{3x-7}{(x-2)^2}$

Prove or disprove and salvage if possible.

P4. If $p(x) \in F[x]$, then
$$\frac{f(x)}{p(x)^n} = f_1(x) + \frac{a_n(x)}{p(x)^n} + \frac{a_{n-1}(x)}{p(x)^{n-1}} + \cdots + \frac{a_1(x)}{p(x)}$$
with $a_j(x)$, $f_1(x)$ in $F[x]$ and with the degree of $a_j(x)$ less than the degree of $p(x)$ or $a_j(x) = 0$ for $j = 1, \ldots, n$.
Comment. In the calculus the above is used in the case of $\mathbb{R}[x]$ with $p(x)$ either a linear or a prime quadratic polynomial. The proces (including P8, Set #27) is called there "the partial fraction decomposition".

P5. $f(x) \in \mathbb{Z}[x]$. Use the results in the Problem Sets to show that each solution of $f(x) \equiv 0 \pmod{p^{n-1}}$ yields either one or none or p solutions of $f(x) \equiv 0 \pmod{p^n}$.

P6. $\varphi(m^2) = m\varphi(m)$

Reading search.

P7. Zermelo proved the unique factorization theorem directly by induction. How does that proof go?

Exploration

P8. Can one use the method in P7 to show that the unique factorization (into primes) theorem holds true in $\mathbb{Z}[x]$?

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OSU Number Theory SAMPLE TEST #3 A.E. Ross. Columbus 8/3/74

Numerical Problems (Some Food for Thought).

P1. Consider $\alpha = [\dot{10}, 3, 2, \dot{3}]$. Find a polynomial with integral coefficients of which α is a root. Do the same for $\beta = [\dot{3}, 2, 3, \dot{10}]$. What can you say about $-\frac{1}{\beta}$? Thus what would you say about irrational numbers α whose continued fraction is purely periodic?

P2. Factor $7+11i$ into prime factors in $\mathbb{Z}[i]$. Explain.

P3. Calculate the number of units in the residue class ring $\mathbb{Z}[i]/(4+i)$. Explain.

P4. Find the two integral solutions $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of $x^2 - 21y^2 = 1$ for which $\alpha_j = x_j + y_j\sqrt{21}$ are closest to -1.

P5. Find all the solutions of the congruence $x^x \equiv 3 \pmod{10}$ in positive inters x.

P6. For what positive odd primes p in $\mathbb{Z}$ is 22 a square modulo p?

P7. Calculate the (multiplicative!) inverse of $x^2 - 2$ in $\mathbb{Z}_3[x]/(x^3+x^2+x+2)$.

P8. Determine the exact number of distinct solutions of each of the following congruences: (a) $x^2 \equiv 1 \pmod{1155}$; (b) $x^2 \equiv 2 \pmod{1155}$; (c) $x^2 \equiv 2 \pmod{2737}$; (d) $x^2 \equiv 11 \pmod{137}$. Explain.

P9. Construct a field of exactly 9 elements. Of exactly 27 elements. In each case find all the generators of the multiplicative group of the field.

P10. Calculate the number of primes of degree 4 in $\mathbb{Z}_5[x]$.

P11. $f(x) \in F[x]$. Without factoring $f(x)$ into linear factors determine which of the following polynomials have multiple roots:
(a) $f(x) = x^5 + 3x^4 - x^3 - 15x^2 - 8x + 4$; (b) $f(x) = x^3 - 3x^2 + 4$; (c) $f(x) = x^3 + 2x^2 - x - 2$;
(d) $f(x) = x^4 - 1$; (e) $f(x) = x^{10} - 1$; (f) $f(x) = x^{15} + x^{10} + x^5 + 3$.
Take $F = \mathbb{Q}$ and then take $F = \mathbb{Z}_5$.

P12. Factor $x^{15} - 1$ into prime factors in $\mathbb{Z}_2[x]$. Do this as neatly as possible.

P13. Decompose into "partial fractions" in $\mathbb{R}[x]$:

$$(1)\ \frac{3}{x^4+5x^2+4}\ ;\quad (2)\ \frac{x^3+x+1}{x^4+2x^2+1}.$$

Prove or disprove and salvage if possible.

P14. If $p > 3$, the sum of all the squares in U_p is divisible by p.

P15. $a \in \mathbb{Z} \Rightarrow a^2$ is positive.

P16. g is a unit if and only if $N(g) = 1$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}[\sqrt{3}]$. True in $\mathbb{Z}[\sqrt{2}]$.

P17. 2 is the only rational prime which is (an associate of) a perfect square in $\mathbb{Z}[i]$

P18. The group of units in the arithmetic of $\mathbb{Z}[\sqrt{3}]$ is cyclic.

P19. Let $g \in U_p$, p a prime. If g is a generator of U_p then $\left(\frac{g}{p}\right) = -1$.

P20. Let π be a prime. Then $\pi | ab \Rightarrow \pi | a$ or $\pi | b$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, in $\mathbb{Z}[x]$.

P21. $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ is Euclidean.

* P22. Let $x^2 - dy^2 = 1$, $d \neq \square$, $d > 0$. Then $\frac{x}{y}$ is a convergent of the continued fraction expansion of $\sqrt{d}$.

P23. If $x^2 \equiv a \pmod{p}$ has exactly two solutions, then $x^2 \equiv a \pmod{p^n}$ has exactly two solutions for every positive integer $n \geq 2$.

P24. The Diophantine equation $15x^2 - 7y^2 = 9$ has no solution $\{x, y\}$ with $x, y \in \mathbb{Z}$.

P25. For every prime $p \in \mathbb{Z}$ the congruence $1 + x^2 + y^2 \equiv 0 \pmod{p}$ has a solution $\{x, y\}$ in $\mathbb{Z}$.

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OSU Number Theory Problem Set #29 A.E. Ross Columbus 8/3/79

Prove or disprove and salvage if possible

P1. Choose a prime p in $\mathbb{Z}$. If $a \in \mathbb{Z}$ and $a = p^{\alpha}a_1$, $(a_1, p) = 1$ then we set
$$|a|_p = \left(\frac{1}{p}\right)^{\alpha}.$$
We let $|a|_p = 0$ if and only if $a = 0$. Then
(1) $|ab|_p = |a|_p \cdot |b|_p$ and (2) $|a+b|_p \le \max\{|a|_p, |b|_p\}$
with (2′) $|a+b|_p = \max\{|a|_p, |b|_p\}$ in case $|a|_p \ne |b|_p$.

P2. If $x = \frac{a}{b} \in \mathbb{Q}$, we let $|x|_p = \frac{|a|_p}{|b|_p}$. Then $|\ |_p$ in $\mathbb{Q}$ has the properties (1), (2), (2′) above for all elements of $\mathbb{Q}$. Here $a, b \in \mathbb{Z}$.

P3. We define the distance $d(x,y)$ between two elements x, y of $\mathbb{Q}$ by setting
$$d(x,y) = |x-y|_p.$$
Then, for every three rational "points" x, y, z we have
II $\quad d(x,z) \le \max\{d(x,y), d(y,z)\}$
and II′ $\quad d(x,z) = \max\{d(x,y), d(y,z)\}$ if $d(x,y) \ne d(y,z)$.

P4. In the geometry of $\mathbb{Q}$ with the distance in P3 we have:
(a) Every triangle is isosceles.
(b) Every interior point of a circle is its center.

P5. It follows from II in P3 that the triangular inequality holds for our distance $d(x,y)$ in $\mathbb{Q}$.

Exploration.

P6. Let $f(x) \in \mathbb{Z}[x]$. Assume that $f(x)$ has leading coefficient 1. Suppose that for $x_1 \in \mathbb{Z}$ we have
$$|f(x_1)|_p < 1 \text{ and } |f'(x_1)|_p = 1.$$
Then the sequence
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \ldots$$
approximates a root of $f(x)$ to any desired degree of accuracy. Apply this method to the congruence in P1, Set #28 starting with $x_1 = -1$ and choosing $p = 7$. Compare with the solution obtained by the method used before to solve P1.

Technique of generalization.

P7. The set of all integral quaternions is given by
$$a = a_0 + a_1 i + a_2 j + a_3 k$$
where either a_0, a_1, a_2, a_3 are all integers or a_0, a_1, a_2, a_3 are halfs of odd integers. Then: given integral quaternions a, b there exist integral quaternions q, r such that
$$a = qb + r \text{ and } N(r) < N(b).$$
Similarly for the division by b on the left (we have lost commutativity!)
Do we have the right handed (left handed) Euclidean algorithm?
Would every quaternion Diophantine equation
$$xa + yb = d \text{ where } d = (a,b)_{right}$$
have a solution in integral quaternions x, y?
Could we generalize the method used in $\mathbb{Z}[i]$ to show that every positive odd prime p in $\mathbb{Z}$ is a norm of an integral quaternion, i.e.
$$p = N(a) = a\bar{a}\ ?$$

OSU Number Theory Problem Set #30 A.E. Ross Columbus 8/6/79

Numerical Problems (Some Food for Thought).

P1. Find all the essentially distinct divisors of $7+11i$ in $\mathbb{Z}[i]$. Check by calculating the number of such divisors. Explain.

P2. Calculate continued fraction expansion of $\sqrt[3]{11}$. Find the convergent of smallest denominator which approximates $\sqrt[3]{11}$ to within $\frac{1}{300}$.

P3. For what positive odd primes p in $\mathbb{Z}$ is 21 a square modulo p?

P4. Calculate 3^{140} modulo 281 using as few multiplications as possible. Explain.

Exploration.

P5.

In calculating the numerators of successive convergents of the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

we may observe with Euler that

$$P_1 = P_1(a_1) = a_1,\ P_2 = P_2(a_1,a_2) = a_1a_2+1,\ P_3 = P_3(a_1,a_2,a_3) = a_1a_2a_3 + a_1 + a_3$$

$$P_4 = P_4(a_1,a_2,a_3,a_4) = a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1 = \{a_1,a_2,a_3,a_4\}$$

in the new notation $\{\ " \ " \ " \ "\}$, that

$$Q_1 = 1,\ Q_2(a_1) = P_1(a_1) = \{a_1\},\ Q_3(a_1,a_2) = P_2(a_1,a_2) = \{a_1,a_2\} \text{ etc.,}$$

and that the process of formation of each new numerator yields the formula

$$\{a_1,a_2,\ldots,a_n\} = a_1\{a_2,a_3,\ldots,a_n\} + \{a_3,a_4,\ldots,a_n\}.$$

Euler observed that the following simple rule for the formation of the polynomial

$$P_n(a_1,a_2,\ldots,a_n) = \{a_1,a_2,\ldots,a_n\} \text{ holds true.}$$

Form the first term by taking the product $a_1a_2\cdots a_n$. Then form every product which is obtained by omitting every pair of consecutive factors (a_i, a_{i+1}). Then form every product which is obtained by omitting any two disjoint pairs of consecutive terms, and so on.

It follows then that

$$\{a_1,a_2,\ldots,a_n\} = \{a_n,a_{n-1},\ldots,a_2,a_1\}$$

from which we obtain the recurrence relations in P1, Set #4.

Prove or disprove and salvage if possible.

P6. $d \neq \square,\ d>0 \Rightarrow x^2-dy^2=1$ has a solution $\{x,y\}$ with $x,y\in\mathbb{Z}$ and $y\neq 0$.

OSU Number Theory SAMPLE TEST #3 Part II AERoss Columbus 8/7/79

Recapitulation.

P1. Obtain a formula for $\varphi(n)$ when $n \in \mathbb{Z}$ by at least three different methods.

P2. Obtain a formula for $\varphi(g)$ when $g \in \mathbb{Z}[i]$ by any method.

P3. What do you mean when you say that an infinite simple continued fraction

$$[a_1, a_2, a_3, \dots, a_n, \dots]$$

represents a real number α? Please be precise in your explanation.

P4. Describe all the "integral" multiples $(1+2\vartheta)a$ of $(1+2\vartheta)$. Here $\vartheta = \frac{1+\sqrt{-3}}{2}$ and a runs through all integers in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.

P5. Show by at least two different methods that: p an odd positive prime $\Longleftrightarrow (p-1)! \equiv -1 \pmod{p}$

P6. Solve the congruence $x^3+2x+2 \equiv 0 \pmod{125}$ by two methods:
method 1: Use the Taylor formula.
Method 2: Use p-adic calculations.

Prove or disprove and salvage if possible.

P7. $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$. Here n is a positive integer and $x \in \mathbb{R}$.

P8. A polynomial of degree n with coefficients in $\mathbb{Z}_m$ cannot have more than n distinct roots.

P9. The group of units in the ring $(\mathbb{Z}[i])_{2+3i}$ is cyclic.

P10. Every element $c \in \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is in $\mathbb{C}$. Its norm $N(c)$ is a positive integer.

P11. The norm of an integral quaternion is an integer.

P12. $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is Euclidean.

P13. $|a+b|_p \le \max\{|a|_p, |b|_p\}$.

P14. In the geometry of $\mathbb{Q}$ with the p-adic distance every interior point of a circle is its center.

P15. P and Q positive odd integers such that $(P,Q)=1$. Then $\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = (-1)^{\frac{P-1}{2}\cdot\frac{Q-1}{2}}$.

P16. Use the Taylor formula to show that

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Test #3: 8-10 am. Wednesday, August 8. Our Caldwell classroom.

OSU Number Theory TEST#3 A.E.Ross Columbus 8/8/79

Numerical Problems (Some Food for Thought).

P1. Calculate the number of the essentially distinct divisors of $3+5i$ in $\mathbb{Z}[i]$

P2. Calculate continued fraction expansion of $\sqrt[3]{7}$. Find the convergent of the smallest denominator which approximates $\sqrt[3]{7}$ to within .005

P3. Calculate 11^{114} modulo 229 using as few multiplications as possible.

P4. For what positive odd primes p in $\mathbb{Z}$ is 11 a square modulo p?

P5. Find all integers x satisfying the following system of congruences; $x \equiv 3 (8)$, $x \equiv 1 (20)$, $x \equiv 1 (15)$

P6. Calculate the number of primes of degree 4 in $\mathbb{Z}_7[x]$.

P7. Determine the number of distinct solutions of each of the following congruences:
(1) $x^2 \equiv 3 \pmod{625}$; (2) $x^2 \equiv 5 \pmod{119}$

P8. As in P7 for the congruences:
(1) $x^2 \equiv 429 \pmod{563}$; (2) $x^2 \equiv 226 \pmod{563}$

Prove or disprove and salvage if possible.

P9. Let π be a prime. Then $\pi | a \cdot b \Rightarrow \pi | a$ or $\pi | b$. True in $\mathbb{Z}[i]$, True in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.

P10 If $x^2 \equiv a \pmod{p}$ has exactly two solutions, then $x^2 \equiv a \pmod{p^n}$ has exactly two solutions for every positive integer $n \geq 2$.

P11. For every prime $p \in \mathbb{Z}$ the congruence $1+x^2+y^2 \equiv 0 \pmod{p}$ has a solution $\{x,y\}$ in $\mathbb{Z}$.

P12 g is a unit if and only if $N(g)=1$. True in $\mathbb{Z}[i]$. True in $\mathbb{Z}[\sqrt{3}]$.

P13. The group of units in the ring $(\mathbb{Z}[i])_{4+i}$ is cyclic.

P14. $|a+b|_p \leq \max\{|a|_p, |b|_p\}$.

P15 In the geometry of $\mathbb{Q}$ with the p-adic distance every interior point of a circle is its center.

P16. P and Q positive odd integers such that $(P,Q)=1$. Then (Jacobi!) $\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right)=(-1)^{\frac{P-1}{2}\cdot\frac{Q-1}{2}}$.

P17. p a prime in $\mathbb{Z}$, $\pi(x)$ a prime in $\mathbb{Z}_p[x]$, $\pi(x)$ of degree $n \Rightarrow \pi(x) | x^{p^n} - x$.

P18 If $F(n) = \sum_{d|n, d>0} f(d)$, then $f(n) = \sum_{d|n, d>0} \mu(\frac{n}{d}) F(d)$.

P19. $\alpha = [a_1, a_2, \ldots, a_{n-1}, \alpha_n] = \frac{P_{n-1}\alpha_n + P_{n-2}}{Q_{n-1}\alpha_n + Q_{n-2}}$. Explain the notation and prove the formula.

P20. Division algorithm exists in $\mathbb{Z}[i]$.

P21. a is a square mod p or not a square mod p according as $a^{\frac{p-1}{2}} \equiv 1$ or $-1 \pmod{p}$.

P22. If $\{x,y\}$ is a solution of the Diophantine equation $x^2-dy^2=1$, then $\left|\frac{x}{y}-\sqrt{d}\right| < \frac{1}{2y^2}$.

P23. A set S of positive integers which contains 1 and contains $n+1$ whenever it contains n, contains every positive integer. (The basis of proofs by induction)

P24. If $\nu_q(m) = \frac{1}{m}\sum_{d|m, d>0} \mu(\frac{m}{d}) q^d$, then $\nu_q(m) > 0$.

P25 Use the Taylor formula to show that $(f(x)\cdot g(x))' = f'(x)g(x) + f(x)g'(x)$.

P26 Let $\pi(x)$ be a prime in $\mathbb{Z}_p[x]$. Then $\pi(y)$ splits into linear factors in $F[y]$, where $F = \mathbb{Z}_p[x]/\pi(x)$. What are all the roots of $\pi(y)$ in F?

P27 If F is a finite field, then $|F| = p^n$ where p is a positive rational prime and n is a positive rational integer.

APPENDIX B

TEST SCORES FOR 1979

TEST 1

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
Bob	221	12	12	6	12	12	12	9	11	12	5	12	12	12		12	10	10	12	12	0	8	6	12	0					
Al	199		4	12	10			10	8		12	.	12	11		5	12	0	12	9	12	12	12	12				12		6
Todd	171	4	12	12	8	12	3	12	5	8		3		12			10	12	12		12		10	12			0			
Ben	164	10	0			8	0	10	12	4				9	5	5	10	10	12	3	12	8	12	12	12	4		6		
Dick	132	12	12	6	8	11	10	8	3			12	12	10	0	3	8	2	12	3	12									
Jerry	127	9			12	11	1	10	8					12	9	12	0	6	12	3	12		10							
Maureen	126	7	12	5	10	3	7	4	12	6		4	3	5	0	12	0	9	10	3	12									
Sam	121	8	8	0	8	12	10	11	1			0		12		12	0	3	12		12		12					0		
Mark	113	12	12	12	12	12	10	11		12	5	2		10			3													
Elaine	106	9	12		8	10	0				0	12		10		12	12		12	10							11			
Rich*	80	5	4	0	8	4	1	0	10		0			0		12	12	12	0	10	0	0	2	0						
Sol	76	2			2	5		0	6	4			10	6		4	8	0	12		7			10						
Ronald	75	9	8	0	12	2	5	10	8					8		5	8													
Paul*	72	9	10	6	6	8	4	3	12	1	0	0	0				0	0	0	3	10									
Stella*	65	4	1		6	3		6		12				0		10	8	0	12	3										
Lorraine*	41		8	6	6	5		6				0		10																

MEAN 116.5 MEDIAN 117

TEST 2

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
Al	255	12	12	8	12	12	12	6	12	12	12	8	6		11		12	12	12	6		10			12	12	12	10	10	12
Todd	197	0	12	8	12	4	0	8	12	5	12	8			12	10	12		12		11	0	8	11	12			12		12
Elaine	153	10	4	12	0	4	0			12		10	12	12	12	12	12				12	5		12			12			
Ben	156	4	9	12	12	12	11	12	7			0	0	6	12	4	12				.4		2	6	12	0		0	12	
Bob	136	4	10	12	12	4	12	6	12	3	12	8	5	10			6				0				6				10	4
Jerry	117	10	12							12			9	6	12				11		8	0	7	12	12		0			6
Sam	112	10	0	1	0	12	0	6	12	3	12	8			11		8				9	6						0		4
Mark	87	9	6	7	2	12	12	12	12	3	12	0																		
Dick	83	10	6	12			12	6	0			8	1		12	0			12		0				12		0			4
Maureen	79	4	10	0	4	4	6	10	12	3		8	6				12													
Paul*	78	10	8	0			3				12	0	0	8	10		10				11						0			6
Ronald	70	7	8	12	0	4	12	0	0	3			1	8	4	11									6		0			
Sol	43	12	2	0				0	0	12	12	0					3					0					0			
Rich*	27	0	8	9	0	0	0	0	0	3	3	0		0			0			0		2	2	0	0		0			0
Lorraine*	17	6	4	0		4				3		0					0													
Stella*	14	0	6			4	0		0	0	0	0	0	0	0	0		0									0			4

MEAN 101.5 MEDIAN 85

* Commuter

TEST 3

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Al	223	0	9	12	12	12	12	12	12	12	12	12	12	12	11		6	12				12		9	10	12		12
Todd	174	6	2	12	0	12	12	12	12	10	12	12	12	8	12			8				8		12		12	0	
Ben	148		0	0	10	12	10	6	6	10	0	10		10	12	12		0	9		0	6		9	12	5	3	6
Bob	140	12		0	10	12	12	12	12	2			12	10			4			4	0	4		9	4	10		
Dick	119	10		0	0	12	12	6	6	2			6		12	12		12	3					12	12			
Ronald	111	8		12	6	12	10	11	7	2			10	2					0		0	12			11	8		
Elaine	108	2		12		0	10	12	12	2		11		11	0		0		8		1	3		12	12		0	
Jerry	107	6		12	9	0	10	6	9	4			10						0	12	0	10			1	12		6
Sam	89			12	0	0	6	3			0			10				0	12		2	10		4	12	8		1
Maureen	57	10		0		0	10	0									9	7	0					9		12		
Mark	54	6	0	0	0	12	12	12	12																			
Stella*	51	6			1	0	12	12	9				0											6	0	5		
Sol	48			0		0		6	12				10						12	6		2				0		
Rich*	44	6	0		0	0	0	0	6	2	0		12	8		4	4			2		0		0			0	
Lorraine*	6	6		0		0																						

MEAN 99 MEDIAN 107

* COMMUTER

APPENDIX C

Book List

A CRITICAL READING LIST

NUMBER THEORY

Uspensky and Heaslet: *Elementary Number Theory*

Vinogradov: *Elements of Number Theory* -- These two books were formerly the texts for our summer course.

Gelfond: *The Solution of Equations in Integers* -- A good review of some of the topics covered this summer, plus a glimpse of what lies beyond.

LeVeque: *Topics in Number Theory*, Vol. I -- An extension of this summer course.

Stark: *An Introduction to Number Theory*

Niven and Zuckerman: *An Introduction to the Theory of Numbers* -- Two standard elementary treatments.

Dickson: *Introduction to the Theory of Numbers* -- Difficult to read, but includes a good treatment of binary quadratic forms and certain topics in the theory of diophantine equations.

Hardy and Wright: *An Introduction to the Theory of Numbers* -- A thorough and classical treatment. A good reference work.

Khinchin: *Three Pearls of Number Theory* -- A very beautiful little book, but difficult.

Khinchin: *Continued Fractions* -- An extensive treatment, elementary in the beginning, but difficult toward the end.

Pollard: *The Theory of Algebraic Numbers* -- Readable; a first text on algebraic number theory.

Cohn: *A Second Course in Number Theory*

Borevich-Shafarevich: *Number Theory*

Ireland and Rosen: *Elements of Number Theory*

LeVeque: *Topics in Number Theory*, Vol. II

Serre: *A Course in Arithmetic*

Bachman: *Introduction to p-adic Numbers and Valuation Theory* -- These six books provide good introductions to some more advanced topics.

Mahler: *Introduction to p-adic Numbers and Their Functions* -- A delightful exposition of there arithmetics by an old friend of the program.

Gauss: *Disquisitiones Arithmeticae* -- Now in English. Difficult but delightful to see the original work by the master.

Weiss: *Algebraic Number Theory*

Roberts: Number Theory - A Problem Oriented Approach -- Many good problems.

Sierpinski: A Selection of Problems in the Theory of Numbers

ALGEBRA

Birkhoff and MacLane: A Survey of Modern Algebra -- Very accessible; rather superficial.

Herstein: Topics in Algebra -- The place to start -- excellent on groups, rings, fields, linear algebra. Very good problems.

Halmos: Finite Dimensional Vector Spaces -- Excellent abstract treatment of linear algebra.

Lederman: Introduction to the Theory of Finite Groups -- Accessible, though not especially well-written. A decent place to start group theory.

Rotman: The Theory of Groups -- A bit sophisticated but a nice follow-up to Mr. Herstein.

Artin: Galois Theory -- A Beautiful presentation. Somewhat slick; there are no problems, so take care to redo all the proofs.

Kaplansky: Fields and Rings -- Next step beyond Herstein.

Kaplansky: Infinite Abelian Groups -- Very good introduction to topic; begins slowly but becomes difficult. Very readable.

Jacobson: Basic Algebra -- Rigorous but difficult.

Huppert: Finite Groups -- The classic reference work in group theory.

Van der Waerden: Modern Algebra -- A classic, if somewhat old-fashioned treatment. Know Herstein first.

Aitken: Determinants and Matrices -- Good introductory text to linear algebra.

Artin: Geometric Algebra -- Fine discussion of geometric representations of elementary algebra.

ANALYSIS, ETC.

Apostol: Calculus, Vol. I and II

Spivak: Calculus -- Apostol and Spivak vie for honors as best introductory textbook; choose the one that fits your taste best.

Ostrowski: Differential and Integral Calculus -- Excellent problems.

Courant: Differential and Integral Calculus, Vol. I and II -- This is calculus as it used to be taught; although intended as an introduction, it is a rather difficult book.

Courant and John: Introduction to Calculus and Analysis -- Many good physical applications.

Bers: Calculus -- Good problems and some interesting proofs.

Knopp: Sequences and Series -- A beautiful little book.

Buck: Advanced Calculus -- A Classic; highly recommended.

Apostol: Mathematical Analysis -- Difficult and abstract, but valuable and readable. It continues where the two volumes of Calculus left off, but should be accessible after any good elementary course.

Landau: Grundlagen der Analysis -- A nice development of the real numbers and introduction to mathematical German.

Boas: A Primer of Real Functions -- Concise, accessible introduction to real analysis.

Ahlfors: Complex Analysis -- A standard introductory text.

Rudin: Principles of Mathematical Analysis -- An accessible, traditional introduction to hard analysis.

Rudin: Real and Complex Analysis -- A continuation of his Principles. Analysis in earnest; difficult but rewarding.

Royden: Real Analysis -- A readable, but difficult introduction. The problems are a must.

Knopp: Theory of Functions, three volumes and two problem books -- An introduction to functions of a complex variable; requires a high degree of sophistication; the problems books are especially good.

Spivak: Calculus on Manifolds -- A nice introduction for the mathematically sophisticated.

Hardy, Littlewood, and Polya: Inequalities -- Classic reference work.

Feller: An Introduction to Probability Theory and Its Applications -- The classic work on the subject.

TOPOLOGY

Arnold: Intuitive Concepts in Elementary Topology -- Very intuitive; very accessible.

Hocking and Young: Topology -- A good place to begin; it gives a fair overview of point set and algebraic topology.

Willard: General Topology -- More readable than Dugundji; good problems.

Dugundji: Topology -- Becoming a standard text; rather difficult with good problems.

Thron: Topological Structures -- Well-organized and readable.

Kelley: General Topology -- A standard text; it's difficult but the problems are excellent.

Greenberg: Lectures on Algebraic Topology

Massey: Algebraic Topology -- Both of these books are for second year students who know some topology. Massey is more readable than Greenberg.

SET THEORY AND LOGIC

Halmos: Naive Set Theory

Wilder: Foundations of Mathematics

Mendelson: Introduction to Mathematical Logic -- Best all-around introductory logic text.

Rosenbloom: The Elements of Mathematical Logic -- Difficult to read, but crammed full of interesting material. The first two chapters are a good follow-up to Towards the Abstract.

COMPUTERS

Trakhtenbrot: Algorithms and Automatic Computing Machines -- Very accessible.

Davis: Computability and Undecidability -- Covers Turing machines, recursive functions and more. Requires some logic.

Knuth: The Art of Computer Programming -- An encyclopedic work with excellent problems. Volume I is especially useful. Highly recommended.

GEOMETRY

Moise: Elementary Geometry From an Advanced Standpoint -- Fine work on classical geometry with historical perspectives.

Coxeter: Introduction to Geometry -- This is geometry as it ought to be done. Coxeter's other books on projective geometry and regular polytopes are also excellent.

Dembowski: Finite Geometries -- A geometry bible. The classic reference work.

Hilbert and Cohen-Vossen: Geometry and the Imagination

Bumeroft: Modern Projective Geometry

Hughes and Piper: Projective Planes -- Very difficult but stimulating; involves many combinatorial ideas.

Yaglom: Geometric Transformations, Volume I - III -- Delightful books.

Lekkerkerker: Geometry of Numbers

Yaglom and Boltyaskii: Convex Figures -- All theorems stated as problems for reader! Solutions in the back of the book.

COMBINATORICS

Erdos and Spencer: Probabilistic Methods in Combinatorics

Liu: Introduction to Combinatorial Mathematics -- Includes a chapter on Polya's theory of counting.

Ryser: Combinatorial Mathematics -- A standard work.

Hall: Combinatorial Theory -- Also a standard work. Chapter 2 generalizes Moebius Inversion.

Harary: Graph Theory

Berge: Graphs and Hypergraphs -- More readable than Harary, though more advanced.

Berge: Principles of Combinatorics -- Extremely fine exposition of combinatorics; recommended.

Cameron and Van Lint: Graph Theory, Coding Theory and Designs

Cameron: Parallelisms of Complete Designs

Van Lint: Coding Theory -- Quite advanced.

MISCELLANEOUS

Bell: Men of Mathematics -- Readable and interesting.

Smith: A Sourcebook in Mathematics

Benaceraf and Putman: Philosophy of Mathematics, Selected Readings -- A good sampling.

Courant and Robbins: What is Mathematics? -- Interesting and stimulating.

Newman: The World of Mathematics -- Ditto.

Hardy: A Mathematicians Apology -- Thought provoking. The introduction by C. P. Snow in the Cambridge University; paperback edition is very good.

Polya: How to Solve It -- A wonderful little book on the problem of how to solve problems. Along similar lines are Polya's Mathematical Discovery and Mathematics and Plausible Reasoning.

Steinhaus: Mathematical Snapshots

Littlewood: A Mathematicians Miscellaney -- Chatty.

Eves: In Mathematical Circles, Mathematical Circles Revisited, Mathematical Circles Squared and Mathematical Circles Adieu -- Anecdotal.

Knuth: Surreal Numbers -- With a few jabs at Landau.

Hadamard: The Psychology of Invention in the Mathematical Field -- An interesting exposition by a leading mathematician. Read the appendix by Einstein.

Poincare: Science and Hypothesis -- Well written philosophical work by one of the foremost figures of the late 19th century.

Eves: An Introduction to the History of Mathematics -- Best overall survey; very readable and stimulating.

APPENDIX D

Application Materials for 1979 and 1980

OSU The Ohio State University

Department of Mathematics
231 West 18th Avenue
Columbus, Ohio 43210
Phone 614 422-4975

We are glad that you are interested in our Summer Program in Mathematics. I am glad to share with you reasonably detailed information about our Program by sending you the enclosed application materials as well as a description of our plans for the summer of 1980.

In addition to a request for usual routine personal information contained on the first page, we ask you to tell us about your interests, your experience outside of formal classroom studies, and in particular about opportunities you had to strike out on your own in doing science and mathematics.

We look for ways of involving our participants as deeply as possible in the work of our Program and for ways of helping them to acquire the mastery of the ideas and skills with which we concern ourselves. We have found that problem solving is a very important aid to learning. Properly designed problems can not only test the mastery of newly acquired skills and challenge ingenuity but such problems can serve also as an effective introduction to new mathematical ideas. The problem set which is a part of our application materials (Item Number 18) is meant to give us some idea of your experience in attcking problems and to provide for you an incentive to think of such matters. When working on this set please do put down in writing thoughts that appeared to you in connection with each problem which you have attempted even though you may not obtain a full solution of the problem. You will find some very interesting observations about the art of problem solving in a little book by Professor George Polya entitled "How to Solve It" published by Doubleday in Garden City, New York, 1957 (second edition).

We would like to learn as much as possible about what you particularly like to do so that we could tell if you could enjoy working with us and benefit

by what we have to offer. Please feel free therefore to tell us about your interests and your activities which cause you to be interested in a program such as ours even if we have failed asking about them in our questionnaire.

It is desirable not to rush through our problem set. It is not unreasonable to take four to eight weeks thinking about our problems.

If you are seriously interested in joining us next summer, do complete all of the application material in a reasonable time. This would make it easier for us to review your application promptly and respond early.

All good wishes.

Sincerely yours,

Arnold E. Ross
Arnold E. Ross
Department of Mathematics
The Ohio State University

AER/pw

Enc.

THE OHIO STATE UNIVERSITY

DEPARTMENT OF MATHEMATICS

STUDENT SCIENCE TRAINING PROGRAM FOR HIGH-ABILITY
SECONDARY SCHOOL STUDENTS

June 22 - August 16, 1980

TO BE FILLED OUT BY THE APPLICANT

(Each application should be accompanied by the
official transcript of credit.)

1. Name: ______________________________

2. Social Security Number: ______________________________

3. Permanent Address (including ZIP Code): ______________________________

____

4. Telephone Number: ______________________________ Area Code: ________

5. Date of Birth: ______________________________

6. Names of Parents or Guardian: ______________________________

7. Name of Your High School: ______________________________

_____ Public _____ Parochial _____ Private (non-denominational)

8. Address of High School: ______________________________

9. Name (in full) of Principal of High School: ______________________________

10. Name of Your Sponsoring Teacher: ______________________________
(The letter of recommendation should be sent by the sponsoring teacher directly to Dr. Arnold E. Ross, Director SSTP, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210.)

11. What grade will you complete by June, 1980? ______________________________

12. How many pupils attend your high school? ______________________________

13. Completed applications should be sent to: Professor Arnold E. Ross
The Ohio State University
Department of Mathematics
231 West 18th Avenue
Columbus, OH 43210

Use additional space if needed to answer questions 14 - 20.

14. Have you ever participated in a special summer program before this year?

______ Yes ______ No

If yes, briefly identify and describe the program(s).

15. Did you participate in any school, regional, or national mathematics competitions? Please describe the kind of competition you took part in and your standing in it.

16. Did you undertake a special mathematics project for your mathematics club, a science fair, etc.? If so, please describe the project (or projects).

17. Where does mathematics fit into your overall interests? Tell in your own words about your plans and ambitions.

18. Is there a mathematical question or a mathematical problem which you found particularly interesting or challenging? Tell of this in your own words.

19. Is there anything which you would like to say in regard to your interest in joining us this summer?

PROBLEMS

20. Each applicant should attempt a solution of each of the following problems. No one is expected to be able to complete all of the problems. They are designed as much to test the applicant's intuition and ability to communicate as his experience. Indeed, all of the problems can be solved with no more experience than high geometry (roughly, if you can understand the problem you can solve it). For each partial or complete solution the applicant should prepare a clear and readable presentation. Only the most exceptional applicants will be able to complete all of the problems. Even if you are not able to give a solution, you should write down and send us any observations concerning the problems.

THE OHIO STATE UNIVERSITY

DEPARTMENT OF MATHEMATICS

STUDENT SCIENCE TRAINING PROGRAM FOR HIGH-ABILITY
SECONDARY SCHOOL STUDENTS

June 22 - August 16, 1980

TO BE FILLED OUT BY THE APPLICANT

(Each application should be accompanied by the
official transcript of credit.)

1. Name: ______________________________

2. Social Security Number: ______________________________

3. Permanent Address (including ZIP Code): ______________________________
____

4. Telephone Number: ______________________ Area Code: ________

5. Date of Birth: ______________________________

6. Names of Parents or Guardian: ______________________________

7. Name of Your High School: ______________________________

   _____ Public _____ Parochial _____ Private (non-denominational)

8. Address of High School: ______________________________

9. Name (in full) of Principal of High School: ______________________________

10. Name of Your Sponsoring Teacher: ______________________________
   (The letter of recommendation should be sent by the sponsoring teacher directly to Dr. Arnold E. Ross, Director SSTP, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210.)

11. What grade will you complete by June, 1980? ______________________________

12. How many pupils attend your high school? ______________________________

13. Completed applications should be sent to: Professor Arnold E. Ross
    The Ohio State University
    Department of Mathematics
    231 West 18th Avenue
    Columbus, OH 43210

Use additional space if needed to answer questions 14 - 21.

14. Have you ever participated in a special summer program before this year?

_____ Yes _____ No

If yes, briefly identify and describe the program(s).

15. Did you participate in any school, regional, or national mathematics competitions? Please describe the kind of competition you took part in and your standing in it.

16. Did you undertake a special mathematics project for your mathematics club, a science fair, etc.? If so, please describe the project (or projects).

17. Where does mathematics fit into your overall interests? Tell in your own words about your plans and ambitions.

18. Is there a mathematical question or a mathematical problem which you found particularly interesting or challenging? Tell of this in your own words.

19. Below are two problems that challenge the applicant's intuition and ability to communicate. We would like for you to spend a short time on these problems and to send us for each problem your solution if you can find one or any observations you can make about the problem.

Observation, Discovery, Testing (a possible counter-example), Justification

I. Let p and q be odd integers. Show that the quadratic equation

$$x^2 + px + q = 0$$

has no rational roots.

II. Consider n points and all the lines connecting them. If each line is colored either blue or red what is the minimum number of points for which you're guaranteed to have either a blue or red triangle. (How many guarantee a blue or red tetrahedron?)

20. Is there anything which you would like to say in regard to your interest in joining us this summer?

21. Expenses for students amount to approximately $496.00 for a dormitory room and twenty meals per week for the duration of the program. One should also consider laundry, books (about $30.00), a small insurance fee, and other incidental expenses.

    Excluding travel, <u>the cost to a student-participant for attending this project is</u>:

Subsistence: room and meals	$466.00
Other: laundry and books	30.00
Total	$496.00

22. Although participants are encouraged to live in dormitories if possible, a limited number of commuters will be accepted. Arrangements can be made for commuting students to eat lunches in the dormitory.

23. During the duration of the Summer Program there will be counselors available to aid and assist participants at all times. The counselors will also have room and board in the dormitories and be available to assist/supervise in any matters which may arise.

Observation, Discovery, Testing (a possible counterexample), Justification

I. Let p and q be odd integers. Show that the quadratic equation

$$x^2 + px + q = 0$$

has no rational roots.

II. A painter is required to paint a 3 ft. x 3 ft. wall. He starts at square 1, completes painting that square, and goes on to a horizontally or vertically adjacent square. At any point he is allowed to go to an adjacent square.

1	2	3
4	5	6
7	8	9

Since the painter will want to apply a second coat of paint, it is desirable for him to end the first coat at a square adjacent to square 1.

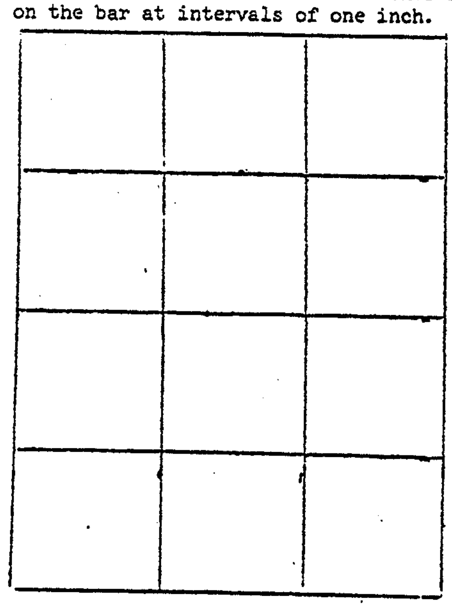
Is this possible? What will be the answer for a 4 ft. x 4 ft. wall? Give proofs for your assertions.

III. If $\epsilon_i = \pm 1$ for $i = 1, 2, \ldots, n$ and

$$\epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_3 + \ldots + \epsilon_1 \epsilon_n + \epsilon_2 \epsilon_3 + \ldots + \epsilon_2 \epsilon_n + \ldots + \epsilon_{n-1} \epsilon_n = 0$$

then n is a perfect square.

IV. Consider a chocolate bar 3 inches wide and 4 inches long with horizontal and vertical lines on the bar at intervals of one inch.



If you are allowed to break the bar along these lines, find the minimum number of breaks required to divide the bar into 12 1-inch squares. Can you solve the same problem for a bar m inches wide and n inches long?

V. Eleven coins are given, and it is known that, by removing any one of them arbitrarily, the remaining ones can be divided into two 5-element sets with equal total weights. Prove that all of the coins have the same weight.

VI. Consider n points and all the lines connecting them. If each line is colored either blue or red what is the minimum number of points for which you're guaranteed to have either a blue or red triangle. (How many guarantee a blue or red tetrahedron?)

VII. A worm is at the center of a piece of cheese 3 inches x 3 inches x 3 inches. On the first day he eats the centermost cubic inch and the next day he eats an adjacent cubic inch. If our voracious friend continues from adjacent cube to adjacent cube can he successfully devour the entire piece of cheese?

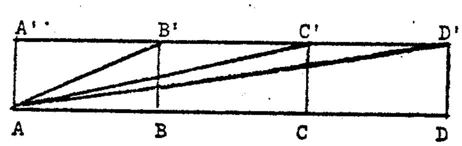
VIII. A book has eight chapters, each having at least twenty pages. On each page there is at least one misprint, but the number of misprints in any chapter does not exceed 33. Prove that there is a sequence of consecutive pages, which contains exactly 55 misprints.

IX. The resident director of a dorm wishes to set up a student committee that he can meet at lunch. He wants to invite three students to lunch each day, but he wants to make sure every possible pair of students on the committee is invited exactly once (which may very well necessitate each student being invited many times).

What is the largest committee for which such a luncheon schedule can be worked out?

X. In normal basketball a team gets 2 points for a field goal and 1 point for a free throw. It's obvious that a team can attain any positive number as a score (within the bounds of the length of a game). In ancient Byzantium the game was played with the same rules, but a team received a different number of points for a field goal then today and a different number of points for a free throw. What these numbers were have been lost to history, but it is known that exactly 35 positive numbers could not possibly be scores in Byzantine Basketball, and that one of these was 58. (For example, suppose a team got 8 points for a field goal and 3 points for a free throw. Then one of the scores a team could never reach was 5 points. However, we see these were not the values in Byzantine Basketball, since with 5 field goals and 6 free throws a team could get 58 points).

What were the values used in Byzantine Basketball?

XI. 

Given the three equal squares juxtaposed as above show without using trigonometry

$$\measuredangle B'AD + \measuredangle C'AD + \measuredangle D'AD = 90^\circ$$

21. The major part of the cost of instruction in our program is provided by our University and in some years by the National Science Foundation. The student is expected to pay only his (her) own expenses for room, board, and travel in case the National Science Foundation provides partial support. If this will not be the case, then our University will ask each participant to pay $175.00 toward the cost of the Program. In return for this the University will allow four (4) hours of college academic credit.

Expenses for students amount to approximately $576.00 for a dormitory room and twenty (20) meals per week for the duration of the program. One should also consider laundry, books (about $100.00), and other incidental expenses.

Exclusing travel, <u>the cost to the student participant for attending our program we estimate as</u>:

Subsistence: room and meals	$576.00	
Other: laundry and books	100.00	
Total Subsistence:	$676.00	
	+ 175.00	In case the NSF provides no support.
TOTAL	$851.00	

22. If, as we hope, the National Science Foundation will aid our Program then it will provide limited funds to help meet the subsistence costs for participants who would otherwise be unable to attend.

Would you be able to come with a full instructional scholarship and no other financial aid? _____ Yes _____ No

If you answer no, then please answer questions (a) and (b).

(a). Would you be able to come if given a full instructional scholarship and $275.00 which is approximately one-half of the cost of subsistence? _____ Yes _____ No

If no, please indicate the amount which in your opinion would enable you to come. $_______________

(b). Could you do with less than $275.00? _____ Yes _____ No

If yes, please indicate the amount which you would consider as adequate. $_______________

PROGRAM DESCRIPTION

SUMMER - 1980

Our aim in our summer program has been to provide the participants with a rich experience in scientific thinking while remaining almost entirely within mathematics. This choice has not limited the selection of a career by our former participants. Our program alumni have made a very impressive record beyond a Ph.D. in a large number of diverse fields of specialization.

All students participate in a program of systematic study arranged in depth. All beginning students study Number Theory. Number Theory can serve as a very effective medium for a deep involvement of eager, interested, and able students with the basic concerns of science. (See page 4 of the attached essay in the Mathematical Scientist.) Those students whose energies are not completely absorbed by their work in Number Theory also study algebra and the most experienced among the program participants also take part in courses or seminars which pre-suppose the kind of mathematical experience which is acquired in the Number Theory and Algebra courses. All students participate in small groups in problem seminars associated with their courses. Resident counsellors (a group of gifted undergraduates and graduate students) provide the needed help and an opportunity for discussion around the clock.

In the summer of 1979, in addition to our Number Theory, taught by me, we provided a course in Algebra, both basic and computer oriented, taught by Professor Thomas Ralley, a course on Discrete Probability, taught by Professor Charles Saltzer, and a course in Combinatorics, taught by Professor Ram P. Gupta with the assistance of Mr. Doug Leonard. Professor Alan Woods and Mr. Doug Leonard directed the Number Theory seminars.

In the summer of 1980, we plan to do again Number Theory, Algebra, Combinatorics, and Discrete Probability.

For advanced participants (usually those who return for the second or third summer) we plan to revive an interesting modern geometry course offered a number of years ago. This course will discuss ideas of transformation geometry, groups and geometry, convexity, isaperimetric problems, finite projective planes, combinatorial aspects of finite geometries, and other fundamental ideas of geometry. Geometry will again be taught by Professors Hans Zassenhaus, Jill Yaqub, and Arno Cronheim. Also, thanks to the availability of much interesting and very fundamental new material in the second edition of Professor Kurt Mahler's "Introduction to P-adic Numbers and Their Functions" (Cambridge University Press), we plan to give an experimental new course in analysis in three parallel tracks. One of these tracks will deal with P-adic analysis, the second with the fundamentals of real analysis, and the third will be a problem seminar dealing with problems pointing up parallelisms as well as contrasts between the two theories treated in the first two tracks. We expect Professor Mahler to join us again and do the first track. Professors Bogdan Baishanski and Ranko Bojanic will return to the program and do the second and the third tracks.

An account of the history of our program and of its international ramifications is contained in the attached report printed in the "Mathematical Scientist".