Independence number of generalized Petersen graphs

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Abstract

Determining the size of a maximum independent set of a graph G, denoted by $\alpha(G)$, is an NP-hard problem. Therefore, many attempts are made to find upper and lower bounds, or exact values of $\alpha(G)$ for special classes of graphs.

This paper is aimed toward studying this problem for the class of generalized Petersen graphs. For each n and k $(n > 2k)$, a generalized Petersen graph $P(n, k)$, is defined by vertex set $\{u_i, v_i\}$ and edge set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k}\}$; where $i = 1, 2, \dots, n$ and subscripts are reduced modulo n. We find new upper and lower bounds and some exact values for $\alpha(P(n, k))$. With a computer program we have obtained exact values for each $n < 78$. In [1] it is conjectured that $\beta(P(n, k)) \leq n + \lceil \frac{n}{5} \rceil$, for all n and k. We prove this conjecture for same cases, but for other cases it is still open. We checked the conjecture with our table for $n < 78$ and it had no inconsistency.

1 Introduction and preliminaries

In a graph $G = (V, E)$, an independent set $I(G)$ is a subset of the vertices of G such that no two vertices in $I(G)$ are adjacent. The independence number $\alpha(G)$ is the cardinality of a largest set of independent vertices. The maximum independent set problem is to find an independent set with the largest number of vertices in a given graph. It is well-known that this problem is NP-hard [5]. Therefore, many attempts are made to find an upper and lower bounds, or exact values of $\alpha(G)$ for special classes of graphs. This paper is aimed toward studying this problem for the generalized Petersen graphs.

For each n and k $(n > 2k)$, a generalized Petersen graph $P(n, k)$, is defined by vertex set $\{u_i, v_i\}$ and edge set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k}\}$; where $i = 1, 2, \dots, n$ and subscripts are reduced modulo n. An induced subgraph on v-vertices is called the inner subgraph, and an induced subgraph on u-vertices is called the outer cycle.
In addition, we call two vertices u_i and v_i as twin of each other and the edge between them as a spoke.

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It seems that Watkins [7] was the first who introduced the class of generalized Petersen graphs $P(n,k)$, and conjectured that they have a Tait coloring, except $P(5,2)$. This conjecture later was proved in [3]. Since then this class of graphs have been studied widely because of its interesting traits. There are papers discussing concepts such as tough sets, labeling, wide diameters, coloring, hamiltonicity, and crossing numbers of generalized Petersen graphs. In [2], and [1], Mahmoodian et al. studied minimum vertex cover, and recently, in [6] vertex domination of generalized Petersen graphs has been studied.

As $\alpha(G)+\beta(G)=|V(G)|$, therefore consider the results of above papers, we will have:

i) $\alpha(P(n,1))=\begin{cases} n & n \quad is \quad even \\ n-1 & n \quad is \quad odd. \end{cases}$ ([1])

ii) For all n, $\alpha(P(n,2))=\lfloor\frac{4n}{5}\rfloor$. ([2])

iii) $\alpha(P(n,3))=\begin{cases} n & n \;\; is \;\; even \\ n-2 & n \;\; is \;\; odd. \end{cases}$ ([1])

iv) $\alpha(P(n,5))=\begin{cases} n & n \quad is \quad even \\ n-3 & n \quad is \quad odd. \end{cases}$ ([4])

v) If both n and k are odd, then $\alpha(P(n,k))\geq n-\frac{k+1}{2}$.
Also, if $k\mid n$, then $\alpha(P(n,k))=n-\frac{k+1}{2}$. ([1])

vi) $\alpha(P(n,k))=n$ if and only if n is even and k is odd. ([1])

vii) For all even k, we have

- If $k-1\mid n$ then $\alpha(P(n,k))\geq n-\frac{n}{k-1}$.
- If $k-1\nmid n$ then $\alpha(P(n,k))\geq n-\frac{n}{k-1}-2k$. ([1])

viii) For all odd n, we have $\alpha(P(n,k))\leq n-\frac{(n,k)+1}{2}$. ([1])

ix) For any integer $k\geq 1$, we have that $\alpha(P(3k,k))=\lceil\frac{5k-2}{2}\rceil$. ([4])

Notice that the problem of finding the size of a maximum independent set in the graph $P(n,k)$ is trivial for even n and odd k, since $P(n,k)$ is a bipartite graph, but we can remove some $k+1$ edges from $P(n,k)$ to obtain a bipartite graph. In contrary, for even k, $P(n,k)$ has a lot of odd cycles. In fact, the number of odd cycles in $P(n,k)$ is at least as large as O(n). This observation shows that for even k, the graph $P(n,k)$ is far from being a bipartite graph and as we see in continuation we need much more complicated arguments for finding lower and upper bounds for $\alpha(P(n,k))$, for even k, compare to the case that k is an odd number.

This paper is organized as follows. In Section 2, we introduce an upper bound for even $k>2$. In Section 3, we introduce some lower bounds when both n and k are even, and when n is odd and k is even. This lower bounds are much better than the previous bounds for general even k. Some exact values for $\alpha(P(n,k))$ are given in Section 4 by applying results presented in Sections 2 and 3. Finally, in Section 5 we proved conjecture using known lower bounds for some cases. We checked the conjecture with our Table for $n<78$, and it had no inconsistency.

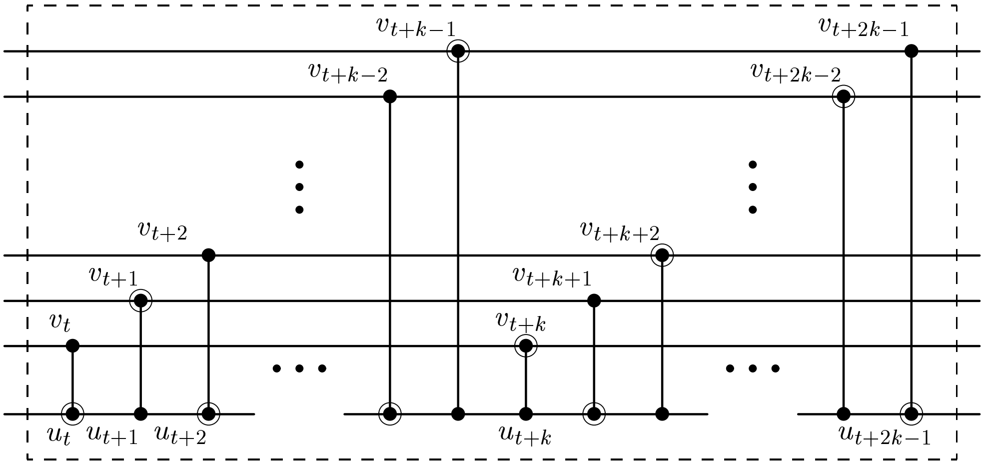

Figure 1: Type 1, in I_t segment.

2 Upper bound

In this section we present an upper bound for $\alpha(P(n,k))$ when k is even, $(k>2)$, and we will show that the given upper bound is equal to $\alpha(P(n,k))$ in some cases.

For $t=1,2,\ldots,n$, we call the set $\{u_t,u_{t+1},\cdots,u_{t+2k-1},v_t,v_{t+1},\cdots,v_{t+2k-1}\}$ a $2k$-segment and we denote it by I_t. Let $G[I_t]$ be the subgraph of $P(n,k)$ induced by I_t.

Let $\mathcal{S}$ be the set of all maximum independent sets of $P(n,k)$. For every $S\in\mathcal{S}$ we denote by $f(S)$ $2k$-segments I_t for which $|I_t\cap S|=2k$ $(1\leq t\leq n)$.
Define $\mathcal{S}_{\min}=\{S\in\mathcal{S}|\forall S'\in\mathcal{S},f(S)\leq f(S')\}$.
There is no need to say that $\mathcal{S}$ and $\mathcal{S}_{\min}$ are nonempty sets. Let $S_0\in\mathcal{S}_{\min}$.

Proposition 1 *For any $S\in\mathcal{S}$, $S\in\mathcal{S}_{\min}$ if and only if $f(S)=f(S_0)$.*

Definition 1 *For any $S\in\mathcal{S}$, we say I_t is of* Type 1 *with respect to S if $|I_t\cap S|=2k$, of* Type 2 *with respect to S if $|I_t\cap S|=2k-1$, and of* Type 3 *with respect to S if $|I_t\cap S|\leq 2k-2$.*
Since $G[I_t]$ has a perfect matching of spokes $\{u_tv_t,u_{t+1}v_{t+1},\cdots,u_{2k+t-1},v_{2k+t-1}\}$, hence $|I_t\cap S|\leq 2k$. So every I_t is in either of the above Types.
Let $T_i(S)=\{I_t|I_t$ is of Type i with respect to $S\}$, for $i=1,2,3$. Note that $f(S)=|T_1(S)|$.
For a given $I_t\in T_2(S)$, we say I_t is of Special type 2 *with respect to S, if $u_t\notin S$ and $\{u_t\}\cup(I_t\cap S)$ becomes an independent set for $G[I_t]$.*

Lemma 1 *If k is an even number then $\alpha(G[I_t])=2k$ and $G[I_t]$ has a unique α-set shown in Figure 1.*

Proof. $G[I_t]$ has a perfect matching $\{u_i\ v_i|\ t\leq i\leq t+2k-1\}$. So $\alpha(G[I_t])\leq\frac{|V(G[I_t])|}{2}=2k$. On the other hand, Figure 1 is an example of an independent set of $G[I_t]$ of size $2k$. So $\alpha(G[I_t])=2k$.
To show the uniqueness of α-set of $G[I_t]$, let S be an α-set of $(G[I_t])$. Since $\alpha(G[I_t])=2k$, $|S|=2k$ and S must contain precisely one vertex from each edge $\{u_i,v_i\}$ where $t\leq i\leq t+2k-1$. Notice that the set of u-vertices of $(G[I_t])$ induces a path of lengths $2k$. Therefore

$|S \cap \{u_i : t \le i \le t + 2k - 1\}| \le k$. The set of v-vertices of $(G[I_t])$ induces a matching of size k. This means that $|S \cap \{v_i : t \le i \le t + 2k - 1\}| \le k$. These two observations show that any α-set S of $G[I_t]$ has k vertices from u-vertices and k other vertices from v-vertices of $V(G[I_t])$. Moreover, every such S, contains precisely one vertex from each edge $\{u_i, v_i\}$ where $t \le i \le t + 2k - 1$ and $\{v_i, v_{i+k}\}$ where $t \le i \le t + k - 1$. Now, consider two cases:
Case 1: $v_t \in S$.
In this case, u_t and v_{t+k} are forced not to be in S. So u_{t+k} is forced to be in S. Then u_{t+k-1} and u_{t+k+1} are forced not to be in S and this forces v_{t+k-1} and v_{t+k+1} to be in S. Since v_{t+k+1} is in S, $v_{t+1} \notin S$. Therefore $u_{t+1} \in S$, so $u_{t+2} \notin S$ and thus $v_{t+2} \in S$. So, we showed that if $v_t \in S$ then $v_{t+2} \in S$ too. Now, if we repeat the same argument for v_{t+2} instead of v_t, we can deduce that $v_{t+4} \in S$ and by a simple induction, it turns out that $v_{t+2l} \in S$ for any $0 \le l \le \frac{k}{2} - 1$. Particularly, $v_{t+k-2} \in S$. Therefore $v_{t+2k-2} \notin S$. This shows that $u_{t+2k-2} \in S$. Hence $u_{t+2k-1} \notin S$ and $v_{t+2k-1} \in S$. So $v_{t+k-1} \notin S$. But we already showed that v_{t+k-1} is forced to be in S. This contradiction shows that there is no Type1 I_t for which $v_t \in S$.
Case 2: $u_t \in S$.
In this case, similar to the argument in Case 1, each vertex is either forced to be in S or it is forced not to be in S. So, there is a unique pattern for $S \cap I_t$ when $I_t \in T_1(S)$. Since the pattern shown in Figure 1 is an instance of an independent set of size $2k$ for $G[I_t]$, it is the unique pattern for such independent set. ■

Lemma 1 guarantees that there is a unique pattern for $I_t \cap S$, if I_t is of Special type 2 with respect to S.

Lemma 2 *For every $S \in \mathcal{S}$, if $I_t \in T_1(S)$ then $u_{2k+t}, v_{2k+t}, u_{t-1}$ and $v_{t-1} \notin S$. Also, if $I_t \in T_1(S)$ is of Special type 2 with respect to S then u_{2k+t} and $v_{2k+t} \notin S$.*

Proof. If $I_t \in T_1(S)$ then $|I_t \cap S| = 2k$. So by Lemma 1, there is a unique pattern for $I_t \cap S$. Based on this pattern, u_{2k+t-1} and $v_{k+t} \in S$, therefore u_{2k+t} and $v_{2k+t} \notin S$ since S is an independent set of vertices of $P(n, k)$. The similar argument shows that u_{t-1} and $v_{t-1} \notin S$. Similar argument proves the second part of the lemma. ■

Corollary 1 *If $I_t \in T_1(S)$ then $I_{t+1}, I_{t+2}, \cdots, I_{t+2k} \notin T_1(S)$.*

Proof. Notice that if $I_r \in T_1(S)$ then for any edge $u_i v_i \in E(G[I_r])$ either $u_i \in S$ or $v_i \in S$. Since $I_t \in T_1(S)$, Lemma 2 implies that u_{2k+t} and $v_{2k+t} \notin S$. On the other hand, $u_{2k+t} v_{2k+t} \in E(G[I_{t+i}])$ for $i = 1, 2, \cdots, 2k$. Thus $I_{t+1}, I_{t+2}, \cdots, I_{t+2k} \notin T_1(S)$. ■

Theorem 1 *$\alpha(P(n, k)) \le \lfloor \frac{(2k-1)n}{2k} \rfloor$ for any even number $k > 2$ and any integer $n > 2k$.*

Proof. Let $S_0 \in \mathcal{S}_{\min}$. We consider two cases.
case 1: $f(S_0) = 0$.
In this case, $T_1(S_0) = \emptyset$. So $|I_t \cap S_0| \le 2k - 1$ for any $1 \le t \le n$. If we add all of these n inequalities, we get:

$$\sum_{t=1}^{n} |I_t \cap S_0| \le (2k - 1)n. \tag{1}$$

On the other hand $\sum_{t=1}^{n} |I_t \cap S_0| = 2k|S_0|$. Since every element of S_0 is contained in precisely $2k$ of the sets I_t, thus:

$$2k|S_0| \le (2k - 1)n \Longrightarrow \alpha(P(n, k)) = |S_0| \le \frac{2k - 1}{2k} n.$$

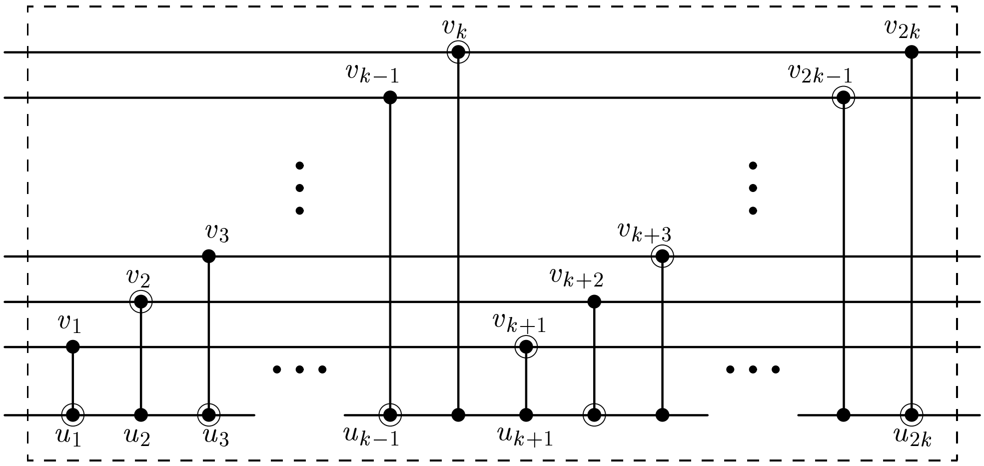

Figure 2: Type 1, in I_1 segment.

case 2: $f(S_0) > 0$.
In this case, $T_1(S_0) \neq \emptyset$. Similar to the inequality 1, we have:

$$2k|S_0| = \sum_{t=1}^{n} |I_t \cap S_0| \leq (2k-1)n + |T_1(S_0)| - |T_3(S_0)|.$$

So, to prove the theorem, it suffices to show that there exits $S_0 \in \mathcal{S}_{\min}$ such that $|T_1(S_0)| \leq |T_3(S_0)|$. $2k|S_0| \leq (2k-1)n \Rightarrow |S_0| \leq \frac{2k-1}{2k}n$.
If we can show that for any $I_r \in T_1(S_0)$, there exists an $I_{r'} \in T_3(S_0)$ so that $I_{r+1}, I_{r+2}, \cdots, I_{r'} \notin T_1(S_0)$, it turns out $|T_1(S_0)| \leq |T_3(S_0)|$.
On the contrary, suppose that there exists $I_t \in T_1(S_0)$ in such a way that in the sequence $I_{t+1}, I_{t+2}, \cdots$ before we see an element of $T_3(S_0)$, we see an element of $T_1(S_0)$. Without loss of generality, assume that $t = 1$. By Lemma 1, $(I_1 \cap S_0)$ is of the form depicted Figure 2.
Since $I_1 \in T_1(S_0)$, by Corollary 1, $I_2, I_3, \ldots, I_{2k+1} \notin T_1(S_0)$. If any of $I_2, I_3, \ldots, I_{2k+1} \in T_3(S_0)$, we are done. Otherwise, all of them are in $T_2(S_0)$. In particular, $I_{2k+1} \in T_2(S_0)$. As $I_{2k+1} \in T_2(S_0)$ and $u_{2k+1}, v_{2k+1} \notin S_0$, S_0 must have one vertex from each edge $u_i v_i$ where $2k+2 \leq i \leq 4k$. Since, $2k+2 \leq 2k+3 \leq 4k$, either u_{2k+3} or $v_{2k+3} \in S_0$. But notice that v_{2k+3} is adjacent to v_{k+3} which is in S_0, for $k > 2$. Thus $v_{2k+3} \notin S_0$ and u_{2k+3} must be in S_0. This means that $u_{2k+2} \notin S_0$. Now, define $S_1 := (S_0 - \{u_{2k}\}) \cup \{u_{2k+1}\}$. One can easily see that $S_1 \in \mathcal{S}$. On the other hand $I_1 \in T_1(S_0)$ but $I_1 \notin T_1(S_1)$. Based on the choice of S_0, $f(S_0) \leq f(S_1)$. Therefore, there must be an index $2 \leq r \leq n$ in such a way that $I_r \in T_1(S_1) \setminus T_1(S_0)$. Since S_0, S_1 agree on every element except u_{2k}, u_{2k+1}, the only candidate for r is $r = 2k+1$. So $I_{2k+1} \in T_1(S_1)$. Moreover, $f(S_0) = f(S_1)$ and by Proposition 1, $S_1 \in \mathcal{S}_{\min}$. Notice that if any of $I_{2k+2}, I_{2k+3}, \ldots, I_n$ are of Type i with respect to S_1, they are of Type i with respect to S_0, as well. So, in the sequence $I_{2k+2}, I_{2k+3}, \ldots, I_n$ any Type 3 element with respect to S_1 appears after an element of Type 1 with respect to S_1.
This means that the same argument can be applied to S_1 and it turns out that if we define $S_2 := (S_1 \cup \{u_{4k+1}\}) - \{u_{4k}\}$, then $S_2 \in \mathcal{S}_{\min}$ and so on. If we repeat this argument for $S_1, S_2, S_3, \ldots, S_m$ where $m = \lfloor \frac{n}{2k} \rfloor - 1$ and $S_i = (S_{i-1} \cup \{u_{2ik+1}\}) - \{u_{2ik}\}$, consecutively, we can observe that $S_i \in \mathcal{S}_{\min}$, $I_{2ki+1} \in T_1(S_i)$ for $i = 1, 2, \ldots, m$, and non of $I_2, \ldots, I_{2(m+1)k}$ are of Type 1 with respect to S_0. In the other words, if I_1 belongs to $T_1(S_0)$ and the next element of $T_1(S_0)$ appears before the first element of $T_3(S_0)$ in the sequence $I_2, I_3, I_4, \ldots,$ then $I_{2k+1}, I_{4k+1}, \ldots, I_{2km+1}$ are all of Special type 2 with respect to S_0.

Moreover, $I_2, I_3, \ldots, I_{2(m+1)k} \notin T_1(S_0)$. Also $v_{2k(m+1)+1}$ and $u_{2k(m+1)+1} \notin S_m$ and since S_m and S_0 agree on the set $\{u_{2k(m+1)+1}, u_{2k(m+1)+2}, \ldots, u_n, v_{2k(m+1)+1}, v_{2k(m+1)+2}, \ldots, v_n\}$ we have $u_{2k(m+1)+1}, v_{2k(m+1)+1} \notin S_0$ as well.
Now consider three cases:

- $2k(m+1)+1 \equiv 1 \pmod{n}$:
  Since I_{2km+1} is of Special type 2 with respect to S_0, by Corollary 1, we have $u_{2km+1+2k} = u_1 \notin S_0$. This is a contradiction as we assumed $I_1 \in T_1(S_0)$ and therefore $u_1 \in S_0$.
- $2k(m+1)+1 \not\equiv 0, 1 \pmod{n}$:
  Since non of $u_{2k(m+1)+1}, v_{2k(m+1)+1}, u_n, v_n$ are in S_0, $I_{2k(m+1)+1}$ is of Type 3 with respect to S_0 and non of $I_2, I_3, \ldots, I_{2k(m+1)}$ are of Type 1 with respect to S_0 which is a contradiction.
- $2k(m+1)+1 \equiv 0 \pmod{n}$:
  I_1 is of Type 1 with respect to S_0 and for every $1 \leq r \leq m$, I_{2rk+1} is of Special type 2 with respect to S_0. In particularly, I_{2km+1} is of Special type 2 with respect to S_0, and therefore $v_{2k(m+1)-k+3} \in S_0$. On the other hand, $v_2 \in S_0$ as $I_1 \in T_1(S_0)$, and since $n = 2(m+1)k+1$, v_2 is adjacent to $v_{2k(m+1)-k+3}$. This is a contradiction.

So in all the cases, we get a contradiction which means, after any Type 1 block I_r, a Type 3 block I_r will appear before we see another Type 1 block. This means that $|T_1(S_0)| \leq |T_3(S_0)|$ and the theorem follows, as we argued earlier. ∎

3 Lower bounds

In this section, we introduce some lower bounds for $\alpha(P(n,k))$ where k is even, $(k > 2)$. Recently, Fox et al. in [4] proved the following results.

Theorem A ([4])**.** *If n, k are integers with n odd and k even, then $\alpha(P(n,k)) \geq \frac{d-1}{2} + \frac{(2d+1)n}{4d}$, where $d = \gcd(n,k)$.*

Theorem B ([4])**.** *If n, k are even, then $\alpha(P(n,k)) \geq \frac{n}{2} + \frac{d}{2}\lfloor\frac{n}{2d}\rfloor$, where $d = \gcd(n,k)$.*

Here we explain a construction for an independent set in $P(n,k)$ for n,k even numbers. It turns out that for every even $n, k < 78$, our lower bound is exact, using a computer program for finding the maximum independence number in $P(n,k)$.

Theorem 2 *If n and $k > 2$ are even, then:*

$$\alpha(P(n,k)) \geq (2k-1)\lfloor\frac{n}{2k}\rfloor + \begin{cases} \frac{r}{2} & \text{if } r \leq k, \\ \frac{3r}{2} - k - 1 & \text{if } r > k. \end{cases}$$

where r is the remainder of n module $2k$.

Proof. We partition the vertices of $P(n,k)$ into $\lfloor\frac{n}{2k}\rfloor$ $2k$-segments and one r-segment. Since n,k are even numbers, r is also an even number and it is straight forward to see that if we choose a subset of the form shown in Figure 3, from each $2k$-segment, they form an independent set S_0 of size $(2k-1)\lfloor\frac{n}{2k}\rfloor$.

Then we try to extend this independent set by adding more vertices form the remaining r-segment. Without loss of generality, we may assume that the r-segment consist of the vertices $\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_r\}$. consider two cases:

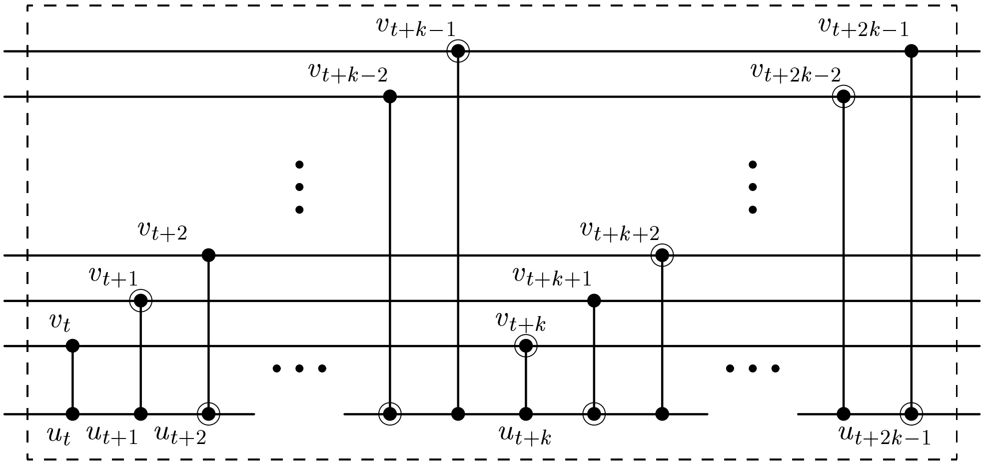

Figure 3: Special type 2 in I_t segment.

- $r \leq k$:
  In this case the set $S_0 \cup \{u_2, u_4, \ldots, u_{r-2}, u_r\}$ is an independent set of size $(2k-1)\lfloor \frac{n}{2k} \rfloor + \frac{r}{2}$.

- $r > k$:
  In this case the set $S_0 \cup \{u_3, u_5, \ldots, u_{r-k-3}, u_{r-k-1}\} \cup \{v_2, v_4, \ldots, v_{r-k-2}, v_{r-k}\} \cup \{u_{r-k+1}, u_{r-k+3}, \ldots, u_{k-3}, u_{k-1}\} \cup \{u_{k+2}, u_{k+4}, \ldots, u_{r-2}, u_r\} \cup \{v_{k+1}, v_{k+3}, \ldots, v_{r-3}, v_{r-1}\}$ is an independent set of size $(2k-1)\lfloor \frac{n}{2k} \rfloor + \frac{r-k-2}{2} + \frac{r-k}{2} + \frac{2k-r}{2} + \frac{r-k}{2} + \frac{r-k}{2} = (2k-1)\lfloor \frac{n}{2k} \rfloor + \frac{3r}{2} - k - 1$.

■

Corollary 2 *If $n, k > 2$ are even numbers then $\alpha(P(n,k)) \geq (2k-1)(\lfloor \frac{n}{2k} \rfloor)$.*

Notice that our lower bound is considerably better than the lower bound obtained by Fox et al. in [4].

In the next theorem we establish an upper bound for $\alpha(P(n,k))$, for odd n, and even k.

Theorem 3 *For odd n, and even k, $(k > 2)$, we have:*

$$\alpha(P(n,k)) \geq (2k-1)\lfloor \frac{n}{2k} \rfloor + \begin{cases} -\frac{k}{2} + 2 & \text{if } r = 1, \\ \frac{3r-k-1}{2} & \text{if } 1 < r < k, \\ \frac{k}{2} + \frac{r-1}{2} & \text{if } k < r < 2k. \end{cases}$$

where where r is the remainder of n module $2k$.

Proof. We construct an independent set for the graph $P(n,k)$. Similar to the proof Theorem 2, first we partition the vertices of the graph into $\lfloor \frac{n}{2k} \rfloor$, $2k$-segments and a remaining segment of size r. Without loss of generality, we can assume that the last $2k$-segment starts from the first spoke, and the remaining segment starts from the $(2k+1)$st spoke and finishes at the $(2k+r)$-th spoke. We also label the $2k$-segments with indices $1, 2, \dots, \lfloor \frac{n}{2k} \rfloor$.

From each of $2k$-segments $1, 2, \dots, \lfloor \frac{n}{2k} \rfloor - 1$, we choose $2k-1$ vertices as shown in Figure 3. We also choose the following vertices from the last $2k$-segment and the remaining r-segment:

- $r = 1$:
  $\{u_2, u_4, \dots, u_{k-2}, u_k\} \cup \{u_{k+3}, u_{k+5}, \dots, u_{2k-1}, u_{2k+1}\} \cup \{v_{k+1}\} \cup \{v_{k+2}, v_{k+4}, \dots, v_{2k-2}, v_{2k}\}$.
- $1 < r < k$:
  $\{u_3, u_5, \dots, u_{k-3}, u_{k-1}\} \cup \{u_{k+2}, u_{k+4}, \dots, u_{2k-2}, u_{2k}\} \cup \{v_2, v_4, \dots, v_{k-2}, v_k\} \cup \{v_{k+1}, v_{k+3}, \dots, v_{k+r-2}, v_{k+r}\} \cup \{u_{2k+3}, u_{2k+5}, \dots, u_{2k+r-2}, u_{2k+r}\} \cup \{v_{2k+2}, v_{2k+4}, \dots, v_{2k+r-3}, v_{2k+r-1}\}$.
- $k < r < 2k$:
  $\{u_3, u_5, \dots, u_{k-3}, u_{k-1}\} \cup \{u_{k+2}, u_{k+4}, \dots, u_{2k-2}, u_{2k}\} \cup \{v_2, v_4, \dots, v_{k-2}, v_k\} \cup \{v_{k+1}, v_{k+3}, \dots, v_{2k-3}, v_{2k-1}\} \cup \{u_{2k+3}, u_{2k+5}, \dots, u_{2k+r-2}, u_{2k+r}\} \cup \{v_{2k+2}, v_{2k+4}, \dots, v_{3k-2}, v_{3k}\}$.

One can easily check that in each case, the given set is an independent set of size specified in the theorem. ■

Corollary 3 *For even $k > 2$, and odd number n, $\alpha(P(n,k)) \geq (2k-1)(\lfloor \frac{n}{2k} \rfloor) - \frac{k}{2} + 1$.*

Notice that the lower bound given in Theorem 1 and the upper bound in Theorem 2, 3 are very close to each other for every fixed even $k > 2$. More precisely we have the following corollary:

Corollary 4 *If $k > 2$ is an even number then $\alpha(P(n,k)) = \frac{(2k-1)}{2k} n + O(k)$.*

4 Some exact values

In this section, we will find the exact value of $\alpha(P(n,k))$ for special set of n, k.

Proposition 2 *If $n > 8$, then:*

$$\alpha(P(n,4)) = \begin{cases} \frac{7n}{8} & \text{if } n \equiv 0 \pmod 8, \\ \frac{7}{8}(n-1) & \text{if } n \equiv 1 \pmod 8, \\ \frac{7}{8}(n-2)+1 & \text{if } n \equiv 2 \pmod 8, \\ \frac{7}{8}(n-3)+2 & \text{if } n \equiv 3 \pmod 8, \\ \frac{7}{8}(n-5)+4 & \text{if } n \equiv 5 \pmod 8. \end{cases}$$

$$\alpha(P(n,4)) \geq \begin{cases} \frac{7}{8}(n-4)+2 & \text{if } n \equiv 4 \pmod 8, \\ \frac{7}{8}(n-6)+4 & \text{if } n \equiv 6 \pmod 8, \\ \frac{7}{8}(n-7)+5 & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Proof. This result is straight consequence of Theorems 1,2,and 3. ■

Notice that for $k = 4$ and $n \equiv 4, 6$ or $7 \pmod 8$, the upper bound and lower bound differ by 1. In fact, for $n < 700$ the exact of $\alpha(P(n, k))$ is the same as our lower bound as we checked by computer.

Proposition 3 *If $k > 2$ is an even number and $n \equiv 0, 2, k-1$ or $k+1 \pmod{2k}$ then $\alpha(P(n, k)) = \lfloor \frac{(2k-1)n}{2k} \rfloor$.*

Proof. This assertion is trivial consequence of Thoremes 1,2,and 3. In fact the upper bound and lower bounds we have for $\alpha(P(n, k))$ are identical in this case. ■

5 conjecture

Conjecture 1 ([1]). *For all n, k we have $\alpha(P(n, k)) \geq \lfloor \frac{4n}{5} \rfloor$.*

Theorem 4 *Conjecture 1 is valid in the following cases:*

a) n is even and k is odd.

b) n, k are odd and $n > \frac{5(k-1)}{2}$.

c) k is even.

Proof. a) In this case $P(n, k)$ is a bipartite graph and $\alpha(P(n, k)) = n$.

b) $\alpha(P(n, k)) \geq n - \frac{k+1}{2}$ ([1]). For $n > \frac{5(k-1)}{2}$ this lower bound is greater than $\lfloor \frac{4n}{5} \rfloor$.

c) We consider three subcases;

- if $k = 2$ then $\alpha(P(n, k)) = \lfloor \frac{4n}{5} \rfloor$.

- If $k \geq 4$ and n is even number then by Theorem 2, we have $\alpha(P(n, k)) \geq \lfloor \frac{4n}{5} \rfloor$, for any $n \geq 77$. For $n \leq 77$ conjecture 1 holds based on the information provided in Table 1.
- If $k \geq 4$ and n is odd number then

■

6 A polynomial time algorithm

Here we present an algorithm for finding maximum independent set in $P(n, k)$ that is polynomial time for fixed k. In this section we denote Maximum Independent Set by MIS.

In $G = P(n, k)$, let A_1 and A_2 be two consecutive k-segment, such that $A = A_1 \cup A_2$, and B the remaining subgraph.

Definition 2 MIS_{A_1,B,A_2} *is a function whose for each independent subset X of A, $\mathrm{MIS}_{A_1,B,A_2}(X)$ is an independent subset of B with the following properties:*

1) $X \cup \mathrm{MIS}_{A_1,B,A_2}(X)$ *is an independent subset of* G.

2) $\mathrm{MIS}_{A_1,B,A_2}(X)$ *is maximum with respect to the property (1).*

Clearly there exists an independent set X in A, such that $X \cup \mathrm{MIS}_{A_1,B,A_2}(X)$ forms a maximum independent set in G. There are at most 3^{2k} ways to choose an independent set in A, three different ways for each spoke. Assume that we have stored values of $\mathrm{MIS}_{A_1,B,A_2}(X)$ in memory for all X. So having X, we can get $\mathrm{MIS}_{A_1,B,A_2}(X)$ in $O(1)$. since k is assumed to be a fixed number, The maximum independent set in G can be found by performing an exhaustive search of X in A in polynomial time. In the next section we propose an algorithm to compute and store $\mathrm{MIS}_{A_1,B,A_2}(X)$ for all X in polynomial time, $O(3^{2k})$.

6.1 Computing $\mathrm{MIS}_{A_1,B,A_2}(X)$

$\mathrm{MIS}_{A_1,B,A_2}(X)$ can be computed using the following dynamic programing algorithm:

Algorithm MIS_{A_1,B,A_2}:

Input: A_1, B and A_2 as three consecutive vertex disjoint segments of $G = P(n,k)$.

Output: MIS_{A_1,B,A_2}.

If B has less than $3k$ spokes:

Create and return MIS_{A_1,B,A_2} for all X using an exhaustive search.

Else:

Step1. Partition B's spokes into three segments B_1, B_2 and B_3. which B_1 and B_3 are k-segments and B_2 is the remainder segment.

Step 2. Recursively compute $\mathrm{MIS}_{B_1,B_2,B_3}$ and store the results in memory.

Step 3. For every independent set $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$:

Perform an exhaustive search in independent subsets of B_1 and B_3. Find $Y_1 \subseteq B_1$ and $Y_2 \subseteq B_2$ as independent sets with the following properties:

1) $X_1 \cup X_2 \cup Y_1 \cup Y_3 \cup \mathrm{MIS}_{B_1,B_2,B_3}(Y_1 \cup Y_3)$ is an independent set in G.

2) $|Y_1 \cup Y_3 \cup \mathrm{MIS}_{B_1,B_2,B_3}(Y_1 \cup Y_3)|$ is maximum with respect to the property (1).

$\mathrm{MIS}_{A_1,B,A_2}(X_1 \cup X_2) = Y_1 \cup Y_3 \cup \mathrm{MIS}_{B_1,B_2,B_3}(Y_1 \cup Y_3)$.

Step 4. Return the computed MIS_{A_1,B,A_2}.

The overall running time of algorithm for finding a maximum independent set of $P(n,k)$ would be of $O(\frac{3^{4k}n}{2k})$.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
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34	34	27	34	29	34	30	34	31	34	29	34	28	34	30	34	32																						
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37	36	29	35	32	34	32	33	33	32	32	32	35	30	33	34	33	30	29																				
38	38	30	38	32	38	34	38	33	38	35	38	31	38	33	38	34	38	36																				
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41	40	32	39	35	38	37	37	38	36	35	35	37	34	39	35	36	37	36	34	32																		
42	42	33	42	36	42	36	42	36	42	39	42	37	42	34	42	37	42	38	42	40																		
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44	44	35	44	37	44	38	44	39	44	40	44	40	44	36	44	38	44	39	44	41	44																	
45	44	36	43	39	42	40	41	40	40	40	39	39	41	42	37	42	40	39	40	39	37	36																
46	46	36	46	39	46	41	46	42	46	41	46	43	46	40	46	38	46	40	46	42	46	44																
47	46	37	45	40	44	41	43	41	42	43	41	40	42	43	39	45	41	42	44	43	42	39	37															
48	48	38	48	42	48	44	48	45	48	42	48	46	48	42	48	39	48	44	48	43	48	45	48															
49	48	39	47	42	46	43	45	43	44	46	44	42	42	44	42	47	40	45	45	42	43	42	40	39														
50	50	40	50	43	50	45	50	46	50	43	50	47	50	45	50	41	50	43	50	44	50	46	50	48														
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53	52	42	51	46	50	48	49	48	48	49	47	47	46	46	49	49	44	51	46	48	50	47	46	47	44	42												
54	54	43	54	46	54	47	54	48	54	48	54	49	54	51	54	48	54	44	54	49	54	48	54	50	54	52												
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56	56	44	56	49	56	49	56	49	56	51	56	50	56	54	56	50	56	46	56	49	56	49	56	51	56	53	56											
57	56	45	55	49	54	51	53	53	52	51	51	53	51	49	49	51	51	54	47	54	50	51	54	52	51	51	47	45										
58	58	46	58	50	58	52	58	51	58	54	58	51	58	55	58	53	58	50	58	48	58	53	58	52	58	54	58	56										
59	58	47	57	51	56	52	55	54	54	52	53	56	54	51	51	53	55	55	49	57	51	54	55	51	53	53	51	49	47									
60	60	48	60	51	60	55	60	54	60	57	60	52	60	56	60	56	60	55	60	49	60	54	60	53	60	55	60	57	60									
61	60	48	59	53	58	54	57	55	56	54	55	58	54	54	53	53	56	56	53	59	50	57	55	54	57	57	56	54	50	48								
62	62	49	62	53	62	56	62	57	62	58	62	54	62	57	62	59	62	55	62	51	62	54	62	55	62	56	62	58	62	60								
63	62	50	61	54	60	56	59	56	58	56	57	59	56	57	55	54	56	57	56	60	52	60	57	57	60	56	56	56	56	52	50							
64	64	51	64	56	64	57	64	60	64	59	64	57	64	58	64	62	64	58	64	56	64	53	64	60	64	57	64	59	64	61	64							
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76	76	60	76	65	76	68	76	69	76	70	76	71	76	69	76	68	76	72	76	72	76	68	76	66	76	63	76	70	76	67	76	69	76	71	76	73	76	
77	76	61	75	67	74	70	73	70	72	70	71	70	70	71	69	73	72	69	67	67	70	70	70	73	64	75	67	72	70	69	74	70	69	70	69	69	64	61

Table 1: Independence numbers for $\alpha(P(n,k))$, $n \leq 77$.

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