Homework 1

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Problem 1. (Exercise 3.8) Let $a, b \in \mathbb{R}$. Prove that if $a \leq c$ for every $c > b$, then $a \leq b$.

Solution)

Proof by contrapositive: We will use the contrapositive of the given statement to prove it.
Given statement P: if $a \leq c$ for every $c > b$, then $a \leq b$.
Contrapositive of P: if $a > b$, then $a > c$ for some $c > b \cdots (1)$

Given $a > b$ from (1), $a - b > 0$ and $\frac{a-b}{2} > 0$.
Let c $=$ b $+$ $\frac{a-b}{2}$.
Since $\frac{a-b}{2} > 0$, c $=$ b $+$ $\frac{a-b}{2} > b$.
Then,

$$\begin{aligned} a - c &= a - \left(b + \frac{a-b}{2}\right) \\ &= a - b - \frac{a-b}{2} \\ &= \frac{a-b}{2} > 0. \end{aligned}$$

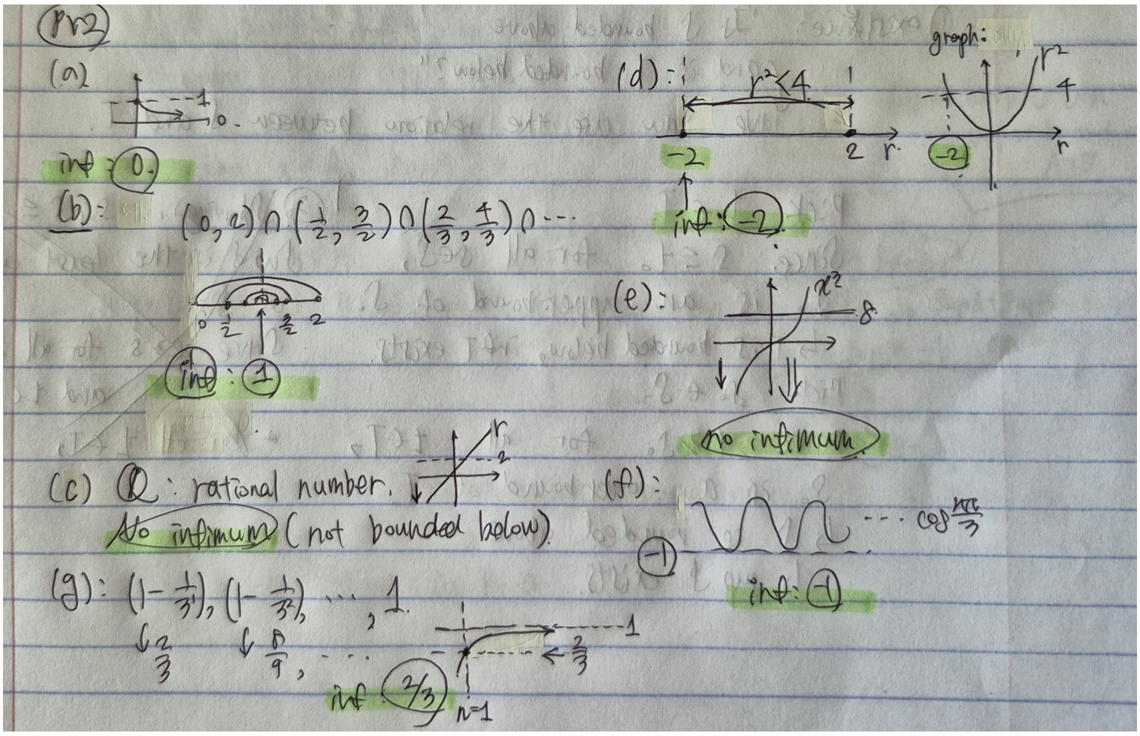
That is, $a > c$.
We proved the contrapositive of the given statement P. Thus, P is true.

Problem 2. (Parts of Exercise 4.4) For each set below either find the infimum (in $\mathbb{R}$) or write that the set is not bounded below:

1. $\left\{\frac{1}{n} : n \in \mathbb{Z}_{>0}\right\}$;
2. $\bigcap_{n=1}^{\infty} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$ (an intersection of infinitely many closed intervals);
3. $\{r \in \mathbb{Q} : r < 2\}$;
4. $\{r \in \mathbb{Q} : r^2 < 4\}$;
5. $\{x \in \mathbb{R} : x^3 < 8\}$;
6. $\left\{\cos\left(\frac{n\pi}{3}\right) : n \in \mathbb{Z}_{>0}\right\}$;
7. $\left\{1 - \frac{1}{3^n} : n \in \mathbb{Z}_{>0}\right\}$.

No need to explain your answers in this problem.

1. 0
2. 1
3. No infimum in $\mathbb{R}$.
4. -2
5. No infimum in $\mathbb{R}$.
6. -1
7. $\frac{2}{3}$



Problem 3. (Exercise 4.6) Let $S \subset \mathbb{R}$ be a non-empty subset.

(a) Prove $\inf S \leq \sup S$.

(b) What can you say about S if $\inf S = \sup S$?

Solution)

1. For all $x \in S$, inf $S \leq x$.
   For all $x \in S$, sup $S \geq x$.
   Combining these two, we obtain:
   For all $x \in S$, inf $S \leq x \leq$ sup S.

   Transitivity: If $x \geq y$ and $y \geq x$ then also $x \geq z$.
   Due to the transitivity, inf $S \leq$ sup S.

2. When inf $S =$ sup S,
   then for all $x \in S$, inf $S = x =$ sup S,
   which means that x is the only element in S.
   That is, S is a singleton set.

Problem 4. (Exercise 4.8, parts (a), (b)) Let $S, T \subset \mathbb{R}$ be two non-empty subsets such that $s \leq t$ for all $s \in S$, $t \in T$. Show that $\sup S \leq \inf T$ (this includes showing that both $\sup S$ and $\inf T$ exist!).

Solution)

1. Part 1: showing that both $\sup S$ and $\inf T$ exist

   We have to show the existence of $\sup S$ and $\inf T$.
   That is, we must show that S is bounded above and T is bounded below.

   To show this, we can use the given relation between S and T ($s \leq t$ for all $s \in S$, $t \in T$).

   Pick $t_1 \in T$. Since $s \leq t_1$ for all $s \in S$, t_1 is an upper bound of S.
   That is, T is bounded above and $\inf$ T exists.

   Pick $s_1 \in S$. Since $t \geq s_1$ for all $t \in T$, s_1 is a lower bound of T.
   That is, S is bounded below and $\inf$ T exists.

2. Part 2: Showing that $\sup S \leq \inf T$

   For any $s \in S$, $t \in T$, $t \geq s$.
   Then, for any $t \in T$, $t \geq$ sup S, which means sup S is the lower bound of T.

   Since inf T is the greatest lower bound of T, inf $T \geq$ sup S.

Problem 5. (Exercise 4.14) Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A+B$ be the set of all sums $a+b$ where $a \in A$ and $b \in B$.

(a) Prove $\sup(A+B) = \sup A + \sup B$;

(b) Prove $\inf(A+B) = \inf A + \inf B$;

Solution)

1. Since A and B are bounded sets of $\mathbb{R}$, $\sup A$ and $\sup B$ exist.

   That is, For any $a \in A$, $\sup A \geq a$
   and for any $b \in B$, $\sup B \geq b$.

   Combining these two, we know that
   for any $a \in A$ and $b \in B$, $\sup A + \sup B \geq a + b$.
   That is, $\sup A + \sup B$ is an upper bound of $A + B$.

   Also, for all $a \in A$ and $b \in B$,
   $(a+b) \leq \sup(A+B) \cdots (1)$.

   Since $\sup(A+B)$ is the least upper bound of $A+B$,
   $\sup(A+B) \leq \sup A + \sup B \cdots (2)$

   By (1),
   for all $a \in A$ and $b \in B$, $a \leq \sup(A+B) - b$.
   That is, for all $b \in B$, $\sup(A+B) - b$ is an upper bound of A.

   Since $\sup A$ is the least upper bound of A, for all $b \in B$, $\sup A \leq \sup(A+B) - b$.
   That is, $b \leq \sup(A+B) - \sup A$ for all $b \in B$.
   Then, $\sup(A+B) - \sup A$ is an upper bound of B.

   Since $\sup B$ is the least upper bound of B,
   $\sup B \leq \sup(A+B) - \sup A$. Then,
   $\sup B + \sup A \leq \sup(A+B) \cdots (3)$

   Combining (2) and (3),
   $\sup(A+B) = \sup A + \sup B$.

2. Since A and B are bounded sets of $\mathbb{R}$, $\inf A$ and $\inf B$ exist.

   That is, For any $a \in A$, $\inf A \leq a$
   and for any $b \in B$, $\inf B \leq b$.

   Combining these two, we know that
   for any $a \in A$ and $b \in B$, $\inf A + \inf B \leq a + b$.
   That is, $\inf A + \inf B$ is a lower bound of $A + B$.

   Also, for all $a \in A$ and $b \in B$,
   $(a+b) \geq \inf(A+B) \cdots (4)$.

   Since $\inf(A+B)$ is the greatest upper bound of $A+B$,
   $\inf(A+B) \geq \inf A + \inf B \cdots (5)$

   By (4),
   for all $a \in A$ and $b \in B$, $a \geq \inf(A+B) - b$.
   That is, for all $b \in B$, $\inf(A+B) - b$ is a lower bound of A.

   Since $\inf A$ is the greatest lower bound of A, for all $b \in B$, $\inf A \geq \inf(A+B) - b$.
   That is, $b \geq \inf(A+B) - \inf A$ for all $b \in B$.
   Then, $\inf(A+B) - \inf A$ is a lower bound of B.

Since $\inf B$ is the greatest lower bound of B,
$\inf B \geq \inf(A+B) - \inf A$. Then,
$\inf B + \inf A \geq \inf(A+B) \cdots (6)$

Combining (5) and (6),
$\inf(A+B) = \inf A + \inf B$.