Mixed 0-1 Programming by Lift-and-Project in a Branch-and-Cut Framework

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We investigate the computational issues that need to be addressed when incorporating general cutting planes for mixed 0-1 programs into a branch-and-cut framework. The cuts we use are of the lift-and-project variety. Some of the issues addressed have a theoretical answer, but others are of an experimental nature and are settled by comparing alternatives on a set of test problems. The resulting code is a robust solver for mixed 0-1 programs. We compare it with several existing codes. On a wide range of test problems it performs as well as, or better than, some of the best currently available mixed integer programming codes.
(*Integer Programming; Cutting Planes; Branch-and-Bound; Branch-and-Cut*)

1. Introduction

For large scale integer programs, a straightforward use of branch-and-bound is often computationally prohibitive. In 1983, Crowder et al. (1983) demonstrated that such large scale problems can sometimes be solved to optimality in reasonable time by strengthening the integer programming formulation with automatic problem preprocessing and cut generation before applying branch-and-bound. Their work involved pure 0-1 programs. In 1987, Van Roy and Wolsey (1987) carried out a similar scheme for mixed 0-1 programs. The successes reported in these two papers had a considerable impact on recent practice in integer programming. In both cases, however, each cut strengthens only one (or a few) constraint(s) while ignoring the rest of the formulation. As a result, the combinatorial structure of the underlying problem may not be exploited. To remedy this limitation, the popular line of research in recent years has been to consider classes of problems with special structure and to devise combinatorial cut generation procedures for each case. This approach has led to a number of impressive successes which are well documented in the mathematical programming literature. However, its limitation is that each class of combinatorial problems requires a special purpose algorithm. A different approach, which was originated by Gomory (1960), is an automatic generation of cuts based on the full formulation. With this scheme, the structure of the integer program may be exploited even when this structure is not apparent to the user. The Gomory cuts fell out of favor due to poor computational experience reported in the sixties, but nowadays this disenchantment is probably excessive in view of the greatly improved linear programming codes that are available. Recently, lift-and-project cuts (Balas et al. 1993a, b) have been shown to be an effective way of strengthening mixed 0-1 programming formulations: not only are they numerically more stable than the classical mixed integer cuts of Gomory but, for every Gomory cut, there is a lift-and-project cut that dominates it. These cuts are related to earlier work of Balas (1979), Ursic (1984), Sherali and Adams (1990), and Lovász and Schrijver (1991). We use them extensively in the work presented in this paper.

Another significant recent development in computational integer programming is the branch-and-cut approach introduced by Padberg and Rinaldi (1991) in the context of the traveling salesman problem (see also Hoffman and Padberg (1991) for an application of branch-and-cut to pure 0-1 programs). In branch-and-cut, the automatic cut generation is performed not only

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prior to starting branch-and-bound but also at each node of the enumeration tree. For this approach to be successful, it is important that the cuts generated at a node of the enumeration tree be valid at all the nodes. So far, this requirement has limited the types of cuts used in branch-and-cut to combinatorial cuts. In this paper, we show how to use lift-and-project cuts within a branch-and-cut algorithm. Some of the issues that need to be addressed have a theoretical answer but others are of an experimental nature and can only be settled by comparing alternatives on a set of test problems. This combination of general cutting planes with branch-and-bound results in a versatile procedure for solving mixed integer programs. Our procedure does not require information about problem-specific structure and appears upon preliminary testing to be robust over a wide range of mixed 0-1 programs. We tested it over a standard set of real-world publicly available benchmark instances. In some cases, it dominates not only other general purpose codes, but also algorithms that use the specific structure of the problem.

Consider the mixed 0-1 program (MIP)

$$\text{Min} \quad cx$$

$$\text{subject to}$$

$$Mx \geq d,$$

$$x \geq 0,$$

$$x_i \in \{0, 1\}, \quad i = 1, \ldots, p,$$

where the first p variables are 0-1 constrained and the remaining variables x_i, $i = p + 1, \ldots, n$, are continuous.

At a generic step of the branch-and-cut algorithm, the original linear programming relaxation, $Mx \geq d$, $x \geq 0$, $x_i \leq 1$, $i = 1, \ldots, p$, is enriched by additional valid inequalities for (MIP), and some of the 0-1 constrained variables are fixed either at their upper or at their lower bound. We denote by $\mathcal{C}$ the current family of valid inequalities for (MIP), and we assume that the linear system $Ax \geq b$ defining $\mathcal{C}$ contains at least all the inequalities in $Mx \geq d$, $x \geq 0$, $x_i \leq 1$, $i = 1, \ldots, p$. We denote by $F_0, F_1 \subseteq \{1, \ldots, p\}$ the sets of variables that have been fixed at 0 and 1, respectively.

Let

$$K(\mathcal{C}, F_0, F_1) = \{x: Ax \geq b, x_i = 0 \quad \text{for } i \in F_0$$
$$x_i = 1 \quad \text{for } i \in F_1\},$$

and let $\text{LP}(\mathcal{C}, F_0, F_1)$ denote the linear program

$$\text{Min} \quad cx$$

$$x \in K(\mathcal{C}, F_0, F_1),$$

which is assumed to be feasible, with a finite minimum. The active nodes of the enumeration tree are represented by a list $\mathcal{S}$ of ordered pairs (F_0, F_1). Let UB stand for the current upper bound, i.e., the value of the best known solution to (MIP).

Branch-and-Cut Procedure
(Input c, M, d, p)

1. *Initialization*. Set $\mathcal{S} = \{(F_0 = \emptyset, F_1 = \emptyset)\}$, let $\mathcal{C}$ consist of the linear programming relaxation of (MIP) and $UB = \infty$.

2. *Node Selection*. If $\mathcal{S} = \emptyset$, stop. Otherwise choose an ordered pair $(F_0, F_1) \in \mathcal{S}$ and remove it from $\mathcal{S}$.

3. *Lower Bounding Step*. Solve the linear program $\text{LP}(\mathcal{C}, F_0, F_1)$. If the problem is infeasible go to Step 2, otherwise let $\bar{x}$ denote its optimal solution. If $c\bar{x} \geq UB$ go to Step 2. If $\bar{x}_j \in \{0, 1\}$, $j = 1, \ldots, p$, let $x^* = \bar{x}$, $UB = c\bar{x}$ and go to Step 2.

4. *Branching versus Cutting Decision*. Should cutting planes be generated? If yes, go to Step 5, else go to Step 6.

5. *Cut Generation*. Generate cutting planes $\alpha x \geq \beta$ valid for (MIP) but violated by $\bar{x}$. Add the cuts to $\mathcal{C}$ and go to Step 3.

6. *Branching Step*. Pick an index $j \in \{1, \ldots, p\}$ such that $0 < \bar{x}_j < 1$. Generate the subproblems corresponding to $(F_0 \cup \{j\}, F_1)$ and $(F_0, F_1 \cup \{j\})$, calculate their lower bounds, and add them to $\mathcal{S}$. Go to Step 2.

When the algorithm stops, if $UB < \infty$, then x^* is an optimal solution to (MIP), otherwise (MIP) is infeasible. In addition to the steps of the basic branch-and-cut procedure stated above, several steps can be performed periodically to improve efficiency: use of heuristics, preprocessing, variable fixing, etc., but this paper is not concerned with such refinements.

In this paper we investigate the use of general cutting planes in Step 5 of the branch-and-cut procedure. The cuts we use are lift-and-project cuts and some of the major issues that need to be addressed are the following.

(1) For computational reasons, it is important that the cuts generated at any node of the branch-and-cut tree be valid at all the other nodes.

(2) Not surprisingly, the degree to which general cutting planes help in solving (MIP) varies a lot depending on the instance. A key issue is a measure of cut quality. Such a measure is used in this paper for comparing alternative versions of the lift-and-project cuts and, in the branch-and-cut algorithm, for deciding what is the right amount of cutting relative to branching.

(3) A difficult question is the branching versus cutting decision that needs to be made in Step 4 of the algorithm.

(4) When the decision to generate cuts is made in Step 4, it is often efficient to generate more than one cut. For a given amount of time devoted to generating cuts, is it better to generate cuts sequentially, updating $\mathcal{C}$ and $\bar{x}$ before generating a new cut, or is it better to generate the cuts in a batch without updating $\bar{x}$?

A follow-up paper will discuss other issues, not dealing directly with lift-and-project cuts, such as the incorporation of Gomory cuts and of some special purpose cuts, general integer variables, heuristics and preprocessing, all of which are part of a code currently under development, called MIPO (this stands for MIP Optimizer).

Section 2 discusses the basic aspects of generating lift-and-project cuts. In particular, it is shown how the cuts generated at any node of the branch-and-cut tree can be made valid at all the other nodes, and how to strengthen the cuts. Section 3 focuses on computational considerations, such as cut quality versus computing time. The issue of whether it is better to generate cuts in small batches or in large batches is also addressed in this section. Section 4 deals with the key decision of branching versus cutting at a node of the enumeration tree. Other decisions such as node and branching variable selection in the enumeration tree are investigated in §5. Finally, in §6 we compare our optimization procedure with other existing algorithms, such as CPLEXMIP, OSL, and MINTO.

2. Cut Generation

In this section we address the problem of generating cutting planes in Step 5 of the branch-and-cut procedure. The use of combinatorial cutting planes, namely, cuts which arise from an underlying combinatorial structure of the problem is well documented in the literature. In contrast, the literature contains no serious attempt to incorporate general cutting planes within branch-and-cut, namely cuts obtained by imposing integrality conditions on one or more of the 0-1 constrained variables. Such general cutting planes include among others, Gomory cuts (Gomory 1960), intersection cuts (Balas 1971), disjunctive cuts (Balas 1979), and lift-and-project cuts (Balas et al., 1993a). Here we focus on lift-and-project cuts, including several improvements and variations. A major goal of this paper is to compare cuts in terms of some quality measure and of the time necessary to generate them. We start this section by addressing different measures of cut quality. Another key issue addressed in this section is the guarantee that cuts generated at any node of the branch-and-cut tree be valid for all the nodes.

2.1. Measure of Cut Quality

An important problem arising in a computational study of cutting planes is to define meaningful measures for comparing them. It is quite common among integer programmers to compare cutting planes based on the improvement obtained in the objective function once the inequality is added to the formulation. The main drawback of this measure is that it does not take into account improvements in the formulation that do not have an immediate effect on the objective function value. For example, some hard mixed 0-1 programs may not have any integrality gap but nevertheless, the cutting planes may be very useful in driving the current solution to integrality. In other mixed 0-1 programs where there is an integrality gap, the difficulty may reside in proving that there is no feasible solution whose objective value equals that of a linear programming relaxation. In this case, the objective function may start improving only after several iterations, even with strong cuts. In general, the behaviour of the cutting planes may vary so much from problem to problem that a meaningful reduction of the integrality gap in one problem may not necessarily be meaningful for another. Geometric measures compare cutting planes $\alpha x \geq \beta$ by establishing a notion of "depth" relative to the solution $\bar{x}$ that they are designed to cut off. For a full dimensional polyhedron, an appealing geometric measure of cut quality is the Euclidean distance between $\bar{x}$ and the hyperplane $\alpha x = \beta$, namely the distance between $\bar{x}$ and its orthogonal projection on this hyperplane. It turns out to be a reliable guide (in the use that we make of it later) even when the underlying integer polyhedron is not full

dimensional. Thus, it will be our preferred measure when we report computational results in §3. Another useful measure in our work is the amount $\beta - \alpha\bar{x}$ by which the cut is violated by $\bar{x}$. Note that this measure assumes that the cut $\alpha x \geq \beta$ has been normalized. We consider mainly two normalizations: For cuts with nonzero right-hand side, we can use the *β-normalization*: $|\beta| = 1$; in general, we can use the *α-normalization*: $\Sigma_i |\alpha_i| = 1$.

2.2. Lift-and-Project Cuts

The cutting planes that we are using for solving (MIP) are called *lift-and-project* cuts, because they can be obtained by a procedure that lifts the constraint set of the LP relaxation into a higher dimensional space by introducing new variables, adds to it some equations implied by the 0-1 conditions, and, finally, projects it back onto the original space by eliminating the new variables (see Balas et al. 1993a for details). The outcome of this process of lifting-strengthening-projecting is a constraint set tighter than the original one but still satisfied by every feasible solution to (MIP).

In particular, such lift-and-project cuts can be derived by imposing the 0-1 condition on a single 0-1 constrained variable x_j. These cuts are valid inequalities for

$$P_j(K) = \text{conv}(K \cap \{x \in \Re^n : x_j \in \{0, 1\}\}),$$

where $K = K(\mathcal{C}, F_0, F_1)$.

A theorem of Balas (1979) can be used to characterize all valid inequalities for $P_j(K)$, as follows. Let $F = \{1, \ldots, n\} \setminus (F_0 \cup F_1)$ denote the set of free variables at node (F_0, F_1). We will assume, without loss of generality, that $F_1 = \emptyset$, since if $F_1 \neq \emptyset$ then all the variables x_k in F_1 can be replaced by $1 - x_k$ (which amounts to replacing the column A_k and the right-hand side b with $-A_k$ and $b - A_k$, respectively). Then $\Sigma_{i \in F} \alpha_i x_i \geq \beta$ (which we denote by $\alpha^F x^F \geq \beta$) is a valid inequality for $P_j(K)$, if and only if there exist vectors u^F, v^F and scalars u_0, v_0 satisfying:

$$\begin{array}{llllll} \alpha^F & -u^F A^F & -u_0 e_j^F & & \geq & 0 \\ \alpha^F & & & -v^F A^F \quad -v_0 e_j^F & \geq & 0 \\ & u^F b^F & & & = & \beta \\ & & & v^F b^F \quad +v_0 & = & \beta \\ & & & u^F, v^F & \geq & 0, \end{array} \qquad (1)$$

where e_j^F is the jth unit vector in $\Re^{|F|}$, A^F is the matrix obtained from A by removing the columns A_i for $i \in F_0$ and the rows corresponding to the inequalities $x_i \geq 0$ for $i = 1, \ldots, n$, and $-x_i \geq -1$ for $i \in \{1, \ldots, p\} \cap F_0$, and b^F is obtained from b by removing the components corresponding to the rows removed from A.

Let $\bar{x}$ be the solution obtained when solving LP$(\mathcal{C}, F_0, F_1)$. In this section, we assume that $\bar{x}$ is not feasible to (MIP), and we let $j \leq p$ be the index of a 0-1 variable such that $0 < \bar{x}_j < 1$. In order to find a valid inequality for $P_j(K)$ that cuts off $\bar{x}$, we proposed in Balas et al. (1993a) to solve the following linear program:

$$\max \quad \beta - \alpha^F \bar{x}^F$$

$$\text{subject to} \quad (1) \text{ and } (\alpha^F, \beta) \in S,$$

where S is a normalization set such as the α-normalization or β-normalization defined earlier. We called its optimal solution (α^F, β), the *deepest cut*. Notice that such a cut is only valid for the current node and its descendants, since the variables x_i, for $i \in F_0$, have been eliminated from the formulation. In the next section we discuss how to make this cut valid for the whole branch-and-cut tree. For this reason we introduce a more general cut generating linear program.

For any $R \subseteq \{1, \ldots, n\}$, let e_j^R be the jth unit vector in $\Re^{|R|}$, let A^R be the matrix obtained from A by removing the columns A_i for $i \notin R$ and the rows corresponding to the inequalities $x_i \geq 0$ for $i = 1, \ldots, n$, and $-x_i \geq -1$ for $i \in \{1, \ldots, p\} \setminus R$, and let b^R be obtained from b by removing the components corresponding to the rows removed from A. Denote by LP(R) the linear program

$$\begin{array}{lllllll} \max & \beta - \alpha^R \bar{x}^R & & & & & \\ \text{subject to} & & & & & & \\ & \alpha^R & -u^R A^R & & -u_0 e_j^R & \geq & 0 \\ & \alpha^R & & -v^R A^R & -v_0 e_j^R & \geq & 0 \\ & & u^R b^R & & & = & \beta \\ & & & v^R b^R & + v_0 & = & \beta \\ & & & & u^R, v^R & \geq & 0 \\ & & & & (\alpha^R, \beta) & \in & S. \end{array}$$

Notice that in any optimal solution to LP(R), we have $\alpha_i^R = \max\{\alpha_i^1, \alpha_i^2\}$ for all $i \in R$, where

$$\alpha^1 = u^R A^R + u_0 e_j^R, \quad \alpha^2 = v^R A^R + v_0 e_j^R.$$

With the β-normalization, we must decide whether to set $\beta = 1$ or $\beta = -1$. In either case, α^R can be eliminated (after introducing slack variables in LP(R)), reducing the size of the linear program to be solved. The drawback of this normalization is that, in some cases, the linear program may be unbounded. The α-normalization is always applicable and can be implemented by choosing $S := \{(\alpha^R, \beta)\colon \sum_{i\in R} |\alpha_i^R| = 1\}$. The absolute value constraint used here can be linearized by introducing new variables α_i^+, α_i^-, writing

$$S := \left\{(\alpha^R, \beta)\colon \alpha^R = \alpha^+ - \alpha^-;\ \alpha^+, \alpha^- \geq 0;\ \sum_{i\in R} (\alpha_i^+ + \alpha_i^-) = 1\right\},$$

and eliminating α^R.

The variables α^R are eliminated from the linear program LP(R) to reduce its size but are easy to reconstruct from the other variables, namely $\alpha_i^R = \max\{\alpha_i^1, \alpha_i^2\}$ in case of the β-normalization, $\alpha^R = \alpha^+ - \alpha^-$ in case of the α-normalization.

2.3. Cut Lifting

In general, a cut generated at a node (F_0, F_1) of the enumeration tree is valid only when the variables in $F_0 \cup F_1$ remain at their fixed values. Such a cut may not be valid for (MIP) and therefore it cannot be used in other parts of the tree. From a computational point of view, it is extremely important that the cuts that are generated be made valid throughout the enumeration tree: it not only reduces the need for extensive bookkeeping, but such "shared" cuts may improve the bounds at many nodes of the tree. A cut which is valid at node (F_0, F_1) is made valid for (MIP) by computing appropriate coefficients for the variables x_j such that $j \in F_0 \cup F_1$. This operation is called *lifting* the cut. In this section, we show how to lift lift-and-project cuts (intersection cuts, being a special case, can be lifted as well).

We briefly summarize the results of Balas et al. (1993a). Let $\bar{x}$ be the optimum solution obtained when solving LP($\mathcal{C}, F_0, F_1$). We will work with a subvector $\bar{x}^R$ of $\bar{x}$ which contains all the fractional components $\bar{x}_i$ for $i \leq p$, all the positive components $\bar{x}_i$ for $i \geq p + 1$, and possibly others. We will compute a lift-and-project cut in the space of variables x^R and then lift the inequality into the original space. W.l.o.g. we can assume that if $i \notin R$ then $\bar{x}_i = 0$, since for those i such that $\bar{x}_i = 1$ the variable x_i can be complemented (by replacing A_i and b with $-A_i$ and $b - A_i$, respectively). Let $j \in \{1, \ldots, p\}$ be an index such that $0 < \bar{x}_j < 1$ and consider the inequality $\alpha^R x^R \geq \beta$ generated by the linear program LP(R). The following theorem shows how the inequality $\alpha^R x^R \geq \beta$ can be extended into a valid cut $\alpha x \geq \beta$ in the original space.

THEOREM 2.1. *Let (α^R, β) be an optimal solution to LP(R) where $R \supseteq \{i \in \{1, \ldots, p\} : 0 < \bar{x}_i < 1\} \cup \{i \in \{p + 1, \ldots, n\} : \bar{x}_i > 0\}$. W.l.o.g. assume $\{i \in \{1, \ldots, p\} : \bar{x}_i = 1\} = \emptyset$. Then the inequality $\alpha x \geq \beta$ defined below is valid for (MIP) and cuts off $\bar{x}$.*

$$\alpha_i = \begin{cases} \alpha_i^R & \text{if } i \in R, \\ \max\{\alpha_i^1, \alpha_i^2\} & \text{if } i \notin R, \end{cases}$$

where $\alpha_i^1 = u^R A_i^R$, $\alpha_i^2 = v^R A_i^R$ and A_i^R denotes the subvector of A_i with the same row set as A^R.

PROOF. Since $i \notin R$ implies $\bar{x}_i = 0$, it is clear that the inequality $\alpha x \geq \beta$ cuts off $\bar{x}$, so we have to show only that $\alpha x \geq \beta$ is valid for (MIP). That is, we must show that $(\alpha, \beta) = (\alpha^N, \beta)$ is a feasible solution of LP(N), where $N = \{1, \ldots, n\}$. The linear programs LP(R) and LP(N) differ in the following ways.

LP(N) contains more variables and constraints, namely the constraints

$$\alpha_i^N - u^N A_i^N \geq 0 \quad \text{for } i \in N\backslash R,$$
$$\alpha_i^N - v^N A_i^N \geq 0 \quad \text{for } i \in N\backslash R,$$

and the variables u_i and v_i corresponding to the constraints $-x_i \geq -1, i \in N\backslash R$ that had been removed from $A^N x \geq b^N$.

A feasible solution to LP(N) can be obtained from a feasible solution to LP(R) by setting the extra variables u_i and v_i equal to zero, and setting $\alpha_i, i \in N\backslash R$ as stated in the theorem. Then the above extra constraints are satisfied. □

An important property shown in Balas et al. (1993a) is that, for the β-normalization, the cut $\alpha^R x \geq \beta$ is "optimally" lifted by Theorem 2.1, in the sense that the resulting cut $\alpha x \geq \beta$ is identical to the one that would be obtained by solving LP(N). The more complicated statement (Theorem 3.2 in Balas et al. 1993a) arises because Balas et al. (1993a) use an equality version of LP(R). For the α-normalization, however, there is usually a difference between the cuts obtained from LP(R) by lifting and those obtained by solving LP(N).

The lifting procedure of Theorem 2.1 is computationally inexpensive, since each lifted coefficient can be computed in time proportional to the number of rows of A^R. This theorem is crucial to the success of a branch-and-cut algorithm based on lift-and-project cutting planes. First, by choosing R such that $R \cap (F_0 \cup F_1) = \emptyset$, Theorem 2.1 provides an easy and efficient way of making the cuts that are generated at the individual nodes of the branch-and-cut tree valid for the whole tree. Second, it implies that the size of the linear programs that need to be solved for generating cutting planes can be substantially reduced, rendering the approach computationally more efficient. In fact, a good choice for R is to take it as small as possible, that is $R = \{i \in \{1, \ldots, p\} : 0 < \bar{x}_i < 1\} \cup \{i \in \{p+1, \ldots, n\} : \bar{x}_i > 0\}$.

2.4. Cut Strengthening

The lift-and-project cut $\alpha x \geq \beta$ of Theorem 2.1 is derived from the 0-1 condition on x_j. The cut $\alpha x \geq \beta$ can be strengthened to $\gamma x \geq \beta$ by using the integrality condition on variables other than x_j, as shown by Balas and Jeroslow (1980) (see also §7 of Balas 1979).

THEOREM 2.2. *The inequality $\gamma x \geq \beta$ is valid for (MIP), where*

$$\gamma_k = \min\{\alpha_k^1 + u_0\lceil \bar{m}_k \rceil, \alpha_k^2 - v_0\lfloor \bar{m}_k \rfloor\} \quad \text{for } k = 1, \ldots, p,$$

$$\gamma_k = \max\{\alpha_k^1, \alpha_k^2\} \quad \text{for } k = p+1, \ldots, n,$$

with α^1 and α^2 as defined in §2.2 and

$$\bar{m}_k = \frac{\alpha_k^2 - \alpha_k^1}{u_0 + v_0}, \quad k = 1, \ldots, p. \tag{2}$$

PROOF. (This version of the proof is due to L. Wolsey.) We show that the cut is valid by proving that it arises from a valid disjunction for the 0-1 solutions of MIP. Suppose that in order to derive a cutting plane we use the valid disjunction

$$-x_j + mx \geq 0 \vee x_j - mx \geq 1,$$

instead of $-x_j \geq 0 \vee x_j \geq 1$, where m is a vector of all integer components such that $m_k = 0$, $k = p+1, \ldots, n$. Then the coefficients of the cut obtained by imposing the new disjunction are defined as:

$$\gamma_k = \min\{\alpha_k^1 + u_0 m_k, \alpha_k^2 - v_0 m_k\} \quad \text{for } k = 1, \ldots, p,$$

$$\gamma_k = \max\{\alpha_k^1, \alpha_k^2\} \quad \text{for } k = p+1, \ldots, n.$$

We can now choose m_k, $k = 1, \ldots, p$, as the integers that make γ_k, $k = 1, \ldots, p$, as small as possible. It is easy to see that these values are obtained by first finding the value of m_k that makes the two terms in the brackets equal, which is $\bar{m}_k$ as in (2), and then taking $m_k = \lfloor \bar{m}_k \rfloor$ or $m_k = \lceil \bar{m}_k \rceil$, whichever gives the minimum value for γ_k. This yields

$$\gamma_k = \min\{\alpha_k^1 + u_0\lceil \bar{m}_k \rceil, \alpha_k^2 - v_0\lfloor \bar{m}_k \rfloor\} \quad \text{for } k = 1, \ldots, p,$$

$$\gamma_k = \max\{\alpha_k^1, \alpha_k^2\} \quad \text{for } k = p+1, \ldots, n,$$

and the theorem follows. □

It is important to note that the cut strengthening is done once the multipliers u^R, v^R, u_0, and v_0 are defined by the solution to LP(R), i.e., the disjunction is changed only a posteriori.

3. Computational Testing of the Cuts

This section is devoted to testing several cutting plane strategies. Given a solution $\bar{x}$ of LP($\mathcal{C}$, F_0, F_1) which is not feasible to (MIP), we call a *round* of cuts a set of cutting planes generated for some or all $j \in \{1, \ldots, p\}$ such that $0 < \bar{x}_j < 1$. One of the key issues investigated in this section is whether it is better to generate cuts in large or small rounds. Before we deal with this question, we first settle three other computational issues: lifted versus nonlifted cuts, strengthened versus nonstrengthened cuts, and whether LP(R) is the best linear program we can use to generate the cuts. In each case, the issue boils down to the quality of cuts versus the time required to generate them. Each experiment, except for the last, was conducted as follows: we ran 10 rounds of cuts (or fewer whenever an optimal solution of (MIP) was found) and, in each round, we generated a cutting plane for each fractional component of the current solution. The cut comparisons were performed at the root node, i.e., $F_0 = F_1 = \emptyset$. The cuts generated in a round were sorted according to the amount by which they were violated by $\bar{x}$ (most violated first), and a cut was added to $\mathcal{C}$ if the cosine of the angle between its normal vector and each previously added cut was at most $\theta < 1$, where θ is a parameter. Here we took $\theta = 0.999$ (we did not explore smaller values of θ since our main purpose was to discard duplicate cuts generated in the same round, if any). The linear programs encountered during the procedure were solved using the CPLEX 2.1 callable library. The times reported in this paper refer to seconds

Table 1 Problem Characteristics

Problem Name	Constraints	0-1 Variables	Continuous Variables	Value of LP Optimum	Value of IP Optimum
BM23	20	27	0	20.57	34
CTN2	150	120	120	169.79	239.21
EGOUT	98	55	86	149.589	568.101
FXCH3	161	141	141	152.01	197.98
MISC05	300	74	62	2,930.9	2,984.5
MODGLOB	291	98	324	20,430,947	20,740,508
MOD008	6	319	0	290.93	307
P0033	15	33	0	2,520.57	3,089
P0201	133	201	0	6,875.00	7,615
P0282	241	282	0	176,867.50	258,411
P0291	252	291	0	1,705.13	5,223.749
P2756	755	2,756	0	2,688.75	3,124
SCPC2S	385	468	0	209.85	216
SET1AL	492	240	472	11,145.62	15,869.7
TSP43	143	1,117	0	5,611	5,620
VPM1	234	168	210	15.4167	20

on an HP720 Apollo desktop workstation with 64 megabytes of memory.

For the purpose of testing computationally both our cuts (in this section) and our branch-and-cut algorithm (in §6), we used a wide variety of mixed 0-1 programs arising from applications. Since mixed 0-1 programs can have very different structures, any algorithm for this class of problems must be tested on a wide variety of instances. Many of our test problems are taken from MIPLIB, a publicly available library of real-world mixed integer programs compiled by Bixby et al. (1992). The instances Pxxxx are pure 0-1 problems. They were first described in Crowder et al. (1983). SCPC2S is a set-covering problem generated by Beasley (1990) and preprocessed by Nobili and Sassano (1993). BM23 is a small but relatively difficult 0-1 problem due to Bouvier and Messoumian. The problems EGOUT, MODGLOB, SET1AL, and VPM1 are mixed 0-1 programs with fixed-charge network flow structure, described in Van Roy and Wolsey (1987). CTN2 and FXCH3 are fixed-charge network flow problems described in Balas et al. (1993a). MOD008 originated at IBM France and MISC05 comes from circuit design. These problems range from difficult to not-so difficult. We also found it interesting to test our algorithm on some pure 0-1 problems arising from combinatorial optimization problems, where special purpose algorithms have been developed and tested. TSP43 is a 43-city asymmetric traveling salesman problem arising from a scheduling problem at DuPont, and was provided to us by D. Miller and J. Pekny. The formulation used here contains the degree constraints and a collection of subtour elimination constraints that, together, provide a tight lower bound (see Balas et al. 1993a for a more detailed discussion of this instance). Table 1 contains the characteristics of the problems used in our test bed.

The first issue addressed in our experiment is that of cut lifting. For the α-normalization, there is usually a difference between the cuts obtained from $R = \{1, \ldots, n\}$ ("not lifted") and those obtained from $R = \{i \in \{1, \ldots, p\} : 0 < \bar{x}_i < 1\} \cup \{i \in \{p+1, \ldots, n\} : \bar{x}_i > 0\}$ followed by the lifting step of Theorem 2.1 ("lifted"). However, the following computational experiment demonstrates that the difference in cut quality between the two choices of R does not justify the enormous increase in time needed to generate the cutting planes in the full space of variables. The results are summarized in Table 2. For both choices of R we report the total number of cuts generated during ten rounds of cutting, the average depth of the cuts (using the geometric measure introduced in §2.1, namely the Euclidean distance between $\bar{x}$ and the cut hyperplane $\alpha x = \beta$) and the percentage of the integrality gap closed. The time gained by lifting the cuts increases very significantly with the

Table 2 Cuts, Lifted Versus Not Lifted

	Lifted				Not Lifted			
Problem	Number of Cuts	Average Distance	Gap Closed	CPU Time	Number of Cuts	Average Distance	Gap Closed	CPU Time
BM23	103	0.05	39.2%	8	107	0.05	34.8%	23
CTN2	334	0.11	95.8%	240	340	0.12	95.0%	1,346
EGOUT	77	0.71	100%	15	84	0.67	100%	39
FXCH3	291	0.14	88.8%	205	304	0.13	89.9%	1,999
MISC05	244	0.03	12.4%	284	250	0.02	21.7%	1,228
MODGLOB	412	0.31	96.6%	14,498	341	0.50	96.4%	56,618
MOD008	106	0.01	43.0%	10	88	0.01	32.2%	882
P0033	123	0.05	72.9%	7	126	0.05	69.1%	20
P0201	654	0.03	59.8%	1,714	792	0.04	72.2%	24,040
P0282	340	0.06	94.1%	125	282	0.07	96.3%	745
P0291	122	0.09	98.8%	8	138	0.06	97.7%	280
P2756	498	0.23	90.3%	163	388	0.3	97.2%	31,383
SCPC2S	1,132	0.02	60.5%	15,171	351	0.03	43.1%	16,169*
SET1AL	257	0.71	99.9%	113	246	0.74	99.9%	961
TSP43	709	0.01	62.2%	2,782	505	0.01	63.8%	15,000*
VPM1	300	0.05	72.3%	211	203	0.08	84.5%	655

* Space limit or time limit exceeded.

number of variables: for the four largest instances in our test bed (in terms of the number of variables), the non-lifted cuts required CPU time about 90 times greater than lifted cuts in one case, 200 times greater in another case, and had to be terminated in the last two cases due to excessive time or space used. As a consequence of this experiment, our code always solves LP(R) in the space $R = \{i \in \{1, \ldots, p\} : 0 < \bar{x}_i < 1\} \cup \{i \in \{p+1, \ldots, n\} : \bar{x}_i > 0\}$.

The next experiment compares the effect of using "strengthened" versus "unstrengthened" cuts. In both cases, the cuts were lifted using Theorem 2.1. The strengthening step, when applied, was performed after lifting. As in the previous experiment, we report the number of cuts generated in ten rounds, the average depth of the cuts, the percentage of the integrality gap closed and the CPU time. It can be seen from Table 3 that the strengthening step improves the quality of lift-and-project cuts in most cases, while the time needed to perform it is relatively small. Therefore MIPO performs cut strengthening whenever possible. It may seem surprising that, in two or three cases, the effect of strengthening was detrimental in terms of the gap closed or average depth of cuts. But remember that our results are reported after the completion of ten rounds of cuts. After one round, strengthening can only improve the gap closed and the average depth of cuts, but in the following rounds, the fractional points to be cut off are likely to be different.

Next, we turn to the use of cuts derived from weaker relaxations. The number of variables in LP(R) is twice the number of rows of A^R when using the β-normalization and twice the number of rows plus columns of A^R when using the α-normalization. So, clearly the effort involved in generating lift-and-project cuts is substantial. This leads to the study of cuts obtained when a relaxation K' of $K(\mathcal{C}, F_0, F_1)$ is used to obtain lift-and-project cuts. The first relaxation K' that we consider only contains the constraints whose slacks are nonbasic in the solution to LP($\mathcal{C}, F_0, F_1$). The cuts obtained from this relaxation are exactly the intersection cuts associated with the convex set $0 \le x_j \le 1$. They are reported in Table 4 under the heading "not reoptimized." The second relaxation reported in Table 4 contains the constraints whose slacks are nonbasic at $\bar{x}$ together with the inequalities $x_i \ge 0, i \in R$, $x_i \le 1$ for $i \in \{1, \ldots, p\} \cap R$. We call these cuts "semi-reoptimized." Finally the cuts generated from $K(\mathcal{C}, F_0, F_1)$, as discussed in Theorem 2.1,

Table 3 Cuts, Strengthened Versus Not Strengthened

	Strengthened				Not Strengthened			
Problem	Number of Cuts	Average Distance	Gap Closed	CPU Time	Number of Cuts	Average Distance	Gap Closed	CPU Time
BM23	103	0.05	39.2%	8	124	0.03	34.3%	13
CTN2	334	0.11	95.8%	240	331	0.11	95.5%	225
EGOUT	77	0.71	100%	15	87	0.62	100%	19
FXCH3	291	0.14	88.8%	205	284	0.16	82.1%	176
MISC05	244	0.03	12.4%	284	237	0.11	11.7%	243
MODGLOB	412	0.31	96.6%	14,498	402	0.29	96.2%	12,030
MOD008	106	0.01	43.0%	10	145	0.003	20.9%	24
P0033	123	0.05	72.9%	7	82	0.05	70.3%	2
P0201	654	0.03	59.8%	1,714	686	0.02	55.2%	2,024
P0282	340	0.06	94.1%	125	305	0.05	95.2%	59
P0291	122	0.09	98.8%	8	128	0.07	95.9%	8
P2756	498	0.23	90.3%	163	472	0.24	97.4%	128
SCPC2S	1,132	0.02	60.5%	15,171	1,199	0.02	59.5%	16,559
SET1AL	257	0.71	99.9%	113	257	0.71	99.9%	111
TSP43	709	0.01	62.2%	2,782	547	0.01	34.4%	641
VPM1	300	0.05	72.3%	211	290	0.05	62.3%	198

are called "fully reoptimized." The following observations can be made from Table 4. The intersection cuts (under the heading "not reoptimized") are substantially worse than the semi-reoptimized or fully reoptimized lift-and-project cuts. Quite surprisingly, the semi-reoptimized cuts are not always dominated by the fully

Table 4 Comparison of Three Types of Cuts

	Not Reoptimized				Semireoptimized				Fully Reoptimized			
Problem	# of Cuts	Avg Dist	Gap Closed	CPU Time	# of Cuts	Avg Dist	Gap Closed	CPU Time	# of Cuts	Avg Dist	Gap Closed	CUP Time
BM23	58	0.02	27.9%	4	109	0.04	36.4%	5	103	0.05	39.2%	8
CTN2	329	0.08	73.6%	92	325	0.12	93.4%	141	334	0.11	95.8%	240
EGOUT	133	0.41	99.0%	20	93	0.84	100%	12	77	0.71	100%	15
FXCH3	244	0.09	44.3%	40	294	0.12	76.7%	134	291	0.14	88.8%	205
MISC05	170	0.02	7.7%	30	197	0.04	17.2%	58	244	0.03	12.4%	284
MODGL.	443	0.17	67.0%	2,254	477	0.28	87.8%	1,412	412	0.31	96.6%	14,498
MOD008	39	.002	33.5%	6	115	0.01	52.1%	8	106	0.01	43.0%	10
P0033	51	0.04	68.9%	4	136	0.05	74.6%	7	123	0.05	72.9%	7
P0201	513	0.02	24.6%	471	548	0.03	63.5%	408	654	0.03	59.8%	1,714
P0282	206	0.05	23.4%	15	346	0.05	94.7%	71	340	0.06	94.1%	125
P0291	97	0.06	90.5%	7	122	0.08	98.5%	5	122	0.09	98.8%	8
P2756	184	0.20	3.3%	42	448	0.26	97.6%	80	498	0.23	90.3%	163
SCPC2S	556	0.01	12.2%	534	1,465	0.01	47.5%	10,551	1,132	0.02	60.5%	15,171
SET1AL	278	0.66	99.9%	68	261	0.70	99.9%	99	257	0.71	99.9%	113
TSP43	423	0.01	55.6%	308	380	0.02	55.6%	257	709	0.01	62.2%	2,782
VPM1	345	0.02	23.6%	89	301	0.05	72.7%	95	300	0.05	72.3%	211

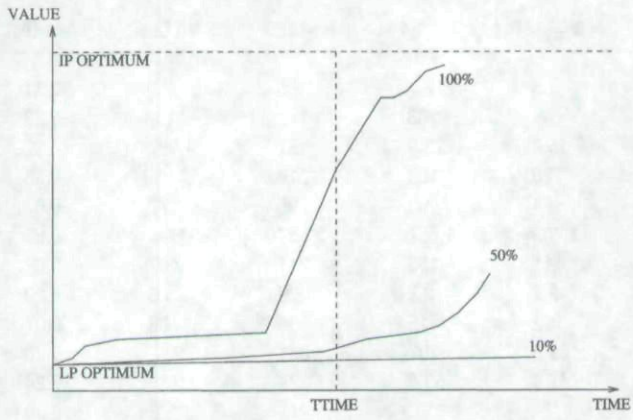
Table 5 Large Versus Small Rounds of Cuts

	100%			50%			10%		
Problem	# of Cuts	Avg Dist	Gap Closed	# of Cuts	Avg Dist	Gap Closed	# of Cuts	Avg Dist	Gap Closed
BM23	44	0.07	31.8%	41	0.06	32.8%	33	0.05	24.6%
CTN2	138	0.22	84.4%	120	0.20	77.3%	70	0.16	46.8%
EGOUT	93	0.61	99.7%	77	0.58	73.2%	44	0.47	36.8%
FXCH3	117	0.19	63.0%	91	0.21	48.9%	67	0.14	31.8%
MISCO5	85	0.04	4.3%	62	0.03	3.7%	48	0.03	5.7%
MODGL.	223	0.44	79.8%	227	0.30	73.5%	271	0.11	24.7%
MOD008	38	0.01	34.4%	29	0.01	22.8%	19	0.01	11.2%
P0033	54	0.09	67.9%	60	0.10	47.2%	52	0.09	25.8%
P0201	181	0.06	32.2%	184	0.05	55.4%	85	0.03	44.7%
P0282	157	0.09	88.9%	158	0.08	94.1%	85	0.10	92.3%
P0291	57	0.16	89.0%	49	0.13	98.0%	27	0.14	69.1%
P2756	326	0.35	62.1%	295	0.26	4.6%	122	0.03	0.3%
SCPC2S	663	0.02	37.6%	470	0.02	38.2%	309	0.02	40.0%
SET1AL	233	0.78	99.9%	207	0.76	91.5%	111	0.73	50.9%
TSP43	95	0.03	19.4%	46	0.03	16.1%	26	0.02	26.7%
VPM1	126	0.07	34.8%	119	0.07	30.2%	67	0.06	13.8%

reoptimized cuts (both in terms of gap closed and average distance cut off). In fact, quite to the contrary, in almost half the cases, the semi-reoptimized cuts are stronger than the fully reoptimized cuts. We offer the following explanation for this unexpected behavior. Even though a fully reoptimized cut is typically deeper than a semi-reoptimized cut, it also tends to be "more parallel" to the objective function. So, as a family, fully reoptimized cuts may not improve the polyhedron as much as semi-reoptimized cuts. In subsequent rounds, having cuts that improve the polyhedron in diverse directions is important for the generation of new cuts. So, as a family, the semi-reoptimized cuts are often more effective. To illustrate this point, consider problem P0201. For each cut that we generated, we computed the cosine of the angle formed by its normal direction and the objective direction, and we averaged these cosines over all cuts generated in a round. In the first round, the average cosine was 0.439 for the semi-reoptimized cuts and 0.565 for the fully reoptimized cuts. So indeed, for this instance, the fully reoptimized cuts were "more parallel" to the objective direction. As a consequence MIPO uses the semi-reoptimized lift-and-project cuts.

We now turn to the important issue of deciding how many cuts to generate at a given node. The time needed for generating a given number of cuts is usually greater with many rounds of few cuts than with fewer rounds of many cuts, because of the need for reoptimization after the generation of every round. Hence we designed our computational experiment as follows. First we ran five rounds of cuts, where in every round we generated a cut for every fractional component $\bar{x}_j$, $j = 1, \ldots, p$, of the current LP relaxation solution. Let TTIME denote the total time taken to generate the cuts in these five rounds. We then ran for TTIME an experiment where in every round we generated cuts for 50% of the fractional components and for 10%. The fractional components closest to $\frac{1}{2}$ are those selected to generate the cuts. In Table 5 we report for each alternative the number of cuts generated, the percentage of the gap closed, and the average distance cut off by the cuts. Although none of the three approaches is uniformly dominant, the first approach closes the largest gap in 9 out of the 16 instances. In some cases, the difference between the three runs is enormous. This is the case for P2756 (see Table 5). It is useful to study this example in greater detail and to look at the evolution of the gap closed over time

Figure 1 Objective Function Value as a Function of Time



for each of the three runs. In Figure 1, the objective function value is plotted as a function of time. For several rounds, the cuts do not improve much the objective function value, but they eventually become very effective. The point in time where the objective value starts improving significantly occurs sooner when the cuts are generated in large rounds. Although P2756 is an extreme case, several other instances exhibit a similar, less pronounced, behaviour. For other problems, such as SCPC2S, generating cuts for 100% of the fractional variables is inferior after five rounds (the choice we made for TTIME in Table 5), but becomes superior for a larger number of rounds.

Based on this set of experiments, our current implementation generates rounds of semi-reoptimized cuts for all the 0-1 variables which are fractional in the current solution $\bar{x}$ or some upper bound MAXROUND, whichever is smaller. In our implementation, we use MAXROUND = 40 for instances with up to 400 0-1 variables and MAXROUND = 80 for problems with more 0-1 variables.

A computational issue related to the solution of the linear programs LP(R) is worth mentioning here. One of the most common problems found during the solution of these linear programs was the presence of unbounded solutions, even when the α-normalization was used. This is to be expected when the β-normalization is used, as discussed in Balas et al. (1993a). In the case of the α-normalization, the cut generation problem LP(R) will be unbounded if the polyhedra obtained by fixing the fractional variable to 0 and 1 are empty. Another possible circumstance when this may occur is when the linear program LP(R) using the α-normalization is actually not unbounded, but the LP solver (CPLEX 2.1) finds it unbounded because of numerical difficulties. These may originate either in degeneracy, or in the fact that the reduction step eliminates variables close to, but not actually at their upper or lower bounds. For example, it is typically the case that variables which are at values close to zero, but not exactly zero (say 10^{-5}) are not included in the set R. The appearance of an unbounded solution is handled in practice by taking as the optimal solution to LP(R) the last basic feasible solution given by the LP solver. Our experience is that the cuts obtained in this manner are still quite deep, probably because in most cases the last basic feasible solution is indeed the optimal one. If the cut obtained in this fashion is not violated by the current solution, it is discarded.

4. Branching Versus Cutting Decision

One of the key decisions to be taken in branch-and-cut procedures is in Step 4: the branching versus cutting decision. This issue has received little attention in the literature. This may be due to the fact that, in most branch-and-cut algorithms that use combinatorial cutting planes, the branching versus cutting decision is easier due to two factors: first, restricted classes of inequalities are considered as cutting planes, and second, the heuristics used to find violated inequalities from these restricted classes might only manage to generate a small fraction of them, if any. With cutting plane procedures guaranteed to find a cut, the branching versus cutting decision has to be confronted. Indeed, pure cutting is possible, but often at a prohibitive computational cost. At the other extreme, pure branch-and-bound may also fail, whereas an appropriate balance between branching and cutting may result in small computing times.

We have shown in the previous section that when we decide to generate a cut by lift-and-project, it is usually computationally efficient to generate a full round of cuts. So, we generate cuts in rounds of cardinality equal to the number of fractional 0-1 variables, or the upper bound MAXROUND, whichever is smaller. In this section, we address the following questions: Should cuts be generated at all the nodes of the enumeration tree?

Table 6 Computing Times for Various Skip Factors k

Problem	$k = 1$	$k = 2$	$k = 4$	$k = 8$	$k = 16$	$k = 32$	$k = 64$	$k = 128$	No Cut	Avg Dist
BM23	16.8	9.5	4.9	3.6	2.8	2.8	2.7	2.5	2.3	0.11
CTN2	220	223	153	118	132	95	183	179	***	0.38
EGOUT	9.1	8.3	8.6	8.9	11.2	15.0	23.9	37	4,927	0.92
FXCH3	287	195	137	101	109	120	169	216	***	0.36
MISC05	864	364	169	118	102	94	93	95	77	0.07
MODGL.	3,584	***	***	2,370	***	1,703	1,657	3,164	***	0.85
MOD008	244	191	171	142	96	111	155	211	407	0.02
P0033	5.3	3.3	3.6	4.1	1.8	2.1	2.8	3.9	35.6	0.29
P0201	1,674	926	432	177	163	112	115	96	85	0.07
P0282	72.6	65.7	39.8	24.8	40.3	59.0	107	351	***	0.20
P0291	4.9	4.5	9.2	8.0	17.4	47.0	53.8	128	***	0.24
P2756	944	1,894	2,211	12,618	***	***	***	***	***	0.38
SCPC2S	1,296	705	239	247	91	76	67	66	38	0.03
SET1AL	134	125	131	165	222	294	542	899	***	0.81
TSP43	954	429	238	196	164	533	354	598	208	0.03
VPM1	4,621	8,010	15,078	3,523	2,502	3,320	4,763	4,096	***	0.07

*** Space limit or time limit exceeded.

If cuts are not generated at all the nodes, how frequently should they be generated?

We call a *fixed strategy* one that generates one round of cuts every k nodes of the enumeration tree, for some fixed integer k (called the *skip factor*). These fixed strategies are used as a benchmark for comparison. We then consider a problem dependent *automatic strategy* in the sense that the skip factor is chosen automatically by the algorithm, depending on the instance to be solved. More sophisticated strategies would be *adaptive strategies* where the skip factor may vary throughout the enumeration tree: when the cuts in $\mathcal{C}$ are considered effective according to the measure, rounds of cuts are generated more frequently; and when the effectiveness measure is small, cuts are generated with less frequency. We first investigate fixed strategies.

Figure 2 Normalized Computing Time as a Function of the Skip Factor

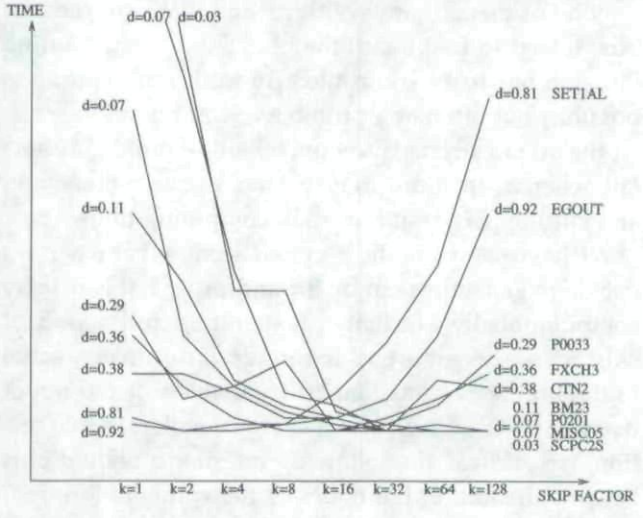


Table 6 compares computing times for our test problems with various cut frequencies. We report the CPU times for skip factors $k = 1, 2, 4, 8, 16, 32, 64, 128$ and when no cuts at all are generated, i.e., our code is run as a pure branch-and-bound code. The last column gives the average depth of the cuts generated at the root node. In order to reduce the variations in computing time due to factors other than k, we used "best bound" as the node selection strategy, and we set the initial upper bound UB equal to the optimum value of (MIP). An immediate observation from Table 6, comparing columns "$k = 128$" and "No cut," is that many problems that could not be solved by pure branch-and-bound became manageable when cuts were added at the root node and then sporadically throughout the enumeration tree. More importantly, Table 6 indicates a correlation between the fixed strategy that yields the best time and the average depth of the cuts generated at the root node: the deeper the cuts at the root node, the more frequently we should cut in the branch-and-cut algo-

rithm. To illustrate this point, Figure 2 plots the computing time as a function of the skip factor, for nine of the sixteen instances in Table 6. Each curve represents a problem and all computing times have been normalized so that, on each curve, the smallest time equals one. The curves are labeled by the average distance d cut off by the cuts at the root node (refer to Table 6). Figure 2 shows that, by using a fixed strategy of $k = 8$ or $k = 16$, eight of the nine instances are solved within a factor of two of the best computing time over all fixed strategies. Furthermore, a pattern clearly emerges from Figure 2, with the instances having largest d solved fastest using the smallest skipping factors. Unfortunately, the correlation is far from being perfect: the reader can use the data from Table 6 to plot the seven missing curves. This should not surprise anyone familiar with the diversity of instances of MIP and the difficulty of devising a robust MIP solver. Nevertheless, there is a clear (negative) correlation between the best skip factor k and the measure d of cut quality.

When the lift-and-project cuts are very effective, it might be worthwhile not only to use a skip factor of $k = 1$ but even to generate several rounds of cuts in each node of the enumeration tree. Indeed, for P2756, we obtain substantially better results this way. However, with rare exceptions, we found that it is more efficient to generate a single round of cuts per node (see Balas et al. 1993b Table 4). In our current implementation, we never generate more than one round of cuts per node.

The times in Table 6 were obtained using semi-reoptimized cuts. The argument used in §3 in favor of the semi-reoptimized cuts was based mainly on cut quality. Here we reconsider this decision based on time: the intersection cuts are extremely fast to generate using the basis inverse, so we re-ran the experiments of Table 6 using intersection cuts instead. Even with almost no computing time spent generating the cuts, the overall computing time was greater for 10 of the 16 instances. In fact, four of the instances could not even be solved within the time limit. So we conclude that the additional time spent generating the semi-reoptimized cuts is well worth it.

Next we turn to the design of an automatic strategy. We devised our automatic strategy by relating the value of the best skip factor from Table 6 to our measure of cut quality and some other problem dependent parameters. First, we generate one round of cuts at the root node and compute the average distance cut off by these cuts. This parameter, which we denote by d, plays a key role in setting the skip factor k for the automatic strategy. We have observed that d is a reasonably good indicator of the average distance cut off by the cuts generated overall throughout the branch-and-cut tree (d is typically two to three times greater, as the cuts have a general tendency to become shallower when the process advances). The next observation is that, in the best run from Table 6, the ratio number of cuts/number of nodes for the full branch-and-cut tree is a function not only of the skip factor k but also of the average number of cuts generated in a round, and it is this ratio that we feel should be proportional to d. Since k needs to be set at the root node, we use the number of cuts generated at the root node as a proxi for the average size of a round. Denote this parameter by f. So f is the minimum of MAXROUND and the number of fractional variables at the root node. Finally, it appears that problem size also affects the optimal skip factor: given d and f, the skip factor should be made slightly smaller as the number of 0-1 variables increases. Based on these observations, we chose

$$k = \min\left\{\text{KMAX}, \left\lceil \frac{f}{cd \log_{10} p} \right\rceil\right\},$$

where KMAX and c are constants. In our current implementation we set KMAX = 32 and $c = 5$. The reason for never using a skip factor greater than KMAX = 32 is robustness. When using a skip factor equal to 32, the fraction of time spent generating cuts is small but, in some cases, these cuts make the difference between being able to solve the problem or not, even when the cut quality as measured by d is poor.

As to adaptive strategies, we experimented with different choices, but we could not find a strategy that obtains overall better results than the automatic strategy. Some simple improvements on the automatic strategy should work, such as recomputing the skip factor k after a new round of cuts has been generated with parameters d and f redefined dynamically. We tried more ambitious strategies that favor generating cuts at the higher nodes of the tree (those closest to the root), but these efforts were disappointing. For example, we tried an adaptive strategy that generates a round of cuts at a node with a probability which is inversely proportional to d and the level of the node in the enumeration tree.

5. Enumeration Strategies and Constraint Management

In all the experiments we used the "best bound" rule for node selection. This is done by listing the nodes in the queue in increasing order of objective function value and always picking the first one. It is important to note, however, that these values were computed when the node was generated and they do not reflect improvements obtainable from more recently generated cutting planes. This implies that in general the ordering of the queue is just an approximation to the best bound criterion.

If the current node cannot be discarded, a variable is selected for branching. In order to select the branching variable we use the following simple criterion proposed by Padberg and Rinaldi in 1991: we first determine the largest value of $\bar{x}_j \leq 0.5$ and the smallest value $\bar{x}_j \geq 0.5$. Let f_0 and f_1 be the respective values (if f_0 or f_1 is undefined, set it equal to 0.5). We define the set of candidate variables for branching as

$$I = \left\{ i \in \{1, \ldots, p\} : \frac{f_0}{2} \leq \bar{x}_i \leq \frac{1 + f_1}{2} \right\}.$$

Then we select a variable in I with the largest (in absolute value) objective function coefficient.

A successful implementation of branch-and-cut requires paying attention to the management of the cuts that have been generated in the course of the algorithm. If all the generated cuts were kept active in the problem formulation the linear programs LP($\mathcal{C}$, F_0, F_1) would become too large and sometimes difficult to solve. This is the main reason why the generated cuts are kept in two separate lists, the *active list* and the *pool*, as in Padberg and Rinaldi (1991). In the active list we keep the constraints which are currently active in the formulation, while the pool contains all other generated cuts. In the initial step both lists are empty. After a first set of cuts is generated, they are all added to the list of active cuts, and periodically, cuts that become inactive are transferred to the pool and deleted from the active list. On the other hand, once a node (F_0, F_1) is retrieved from the queue, the pool is searched for cuts violated by the solution $\bar{x}$ retrieved at this node. These violated inequalities are added to the active list and removed from the pool. The resulting LP is then solved, and the procedure is iterated until no violated cuts remain in the pool. If the number of constraints in the pool becomes larger than an upper limit, inequalities that are inactive at every node in the queue are removed from the pool, in order to make space for new inequalities. Based on our experimentation, the upper limit was set to 500.

6. Computational Experience

We compare our code with CPLEX's branch-and-bound code (CPLEXMIP 2.1), with MINTO 1.4 (using CPLEX 2.1 as the LP solver) and with OSL (subroutine EKKMPRE). Default options were used for all the codes, including ours.

6.1. Expanded Test-Bed

In addition to the test problems given in Table 1, we used some difficult problems from the literature to test our branch-and-cut code. These additional instances were obtained from the following sources. The CFAT and SAN instances are maximum stable set problems from the 1993 DIMACS computational challenge (the CFAT problem is a fault diagnosis problem originating with Berman and Pelc and the SAN problem is from Laura Sanchis). The GENOVA instance is a set covering problem obtained from Antonio Sassano. AIR04 and AIR05 are set partitioning problems originating from crew scheduling at American Airlines. L152LAV, LSEU and MOD010 originated at IBM Yorktown Heights. RGN and SET1AL are fixed charge problems obtained from Laurence Wolsey. STEIN45 is the Steiner triple formulation as given in MIPLIB. The three remaining instances (CTN, MISC and P0548) are from classes already discussed in connection with Table 1. The problem characteristics of all instances in our test set can be found in Table 7. Most of these instances were obtained from MIPLIB. Note, however, that not all the instances in MIPLIB are part of our test bed. Instances were excluded mainly for being too easy for branch-and-bound (AIR01, AIR02, AIR03, AIR06, CRACPB1, DIAMOND, ENIGMA, FIXNET3, KHB05250, LP4L, MISC01, MISC02, MISC03, MISC04, MISC06, MOD013, P0040, PIPEX, SAMPLE2, SENTOY, STEIN9, STEIN15, STEIN27). A couple of problems were excluded because the resulting linear programming relaxations were too large (MOD011, RENTACAR). In fact, RENTACAR is an easy instance if the problem is preprocessed with a linear programming preprocessor.

Table 7 Problem Characteristics

Problem Name	Constraints	0–1 Variables	Continuous Variables	LP Value (not preprocessed)	LP Value (preprocessed)	Value of IP Optimum
AIR04	823	8,904	0	55,535.436	55,535.436	56,137
AIR05	426	7,195	0	25,877.609	25,877.609	26,374
BM23	20	27	0	20.57	20.57	34
CFAT200-1	1,919	200	0	−14.517	−14.517	−12
CTN2	150	120	120	169.79	207.73	239.21
CTN3	182	142	142	313.80	388.31	432.28
EGOUT	98	55	86	149.589	511.618	568.101
FXCH3	161	141	141	152.01	152.01	197.98
GENOVA6	98	904	0	10,213.88	10,213.88	10,267
L152LAV	97	1,989	0	4,656.36	4,656.36	4,722
LSEU	28	89	0	834.68	947.96	1,120
MISC05	300	74	62	2,930.9	2,930.9	2,984.5
MISC07	212	259	1	1,415	1,415	2,810
MOD008	6	319	0	290.93	290.93	307
MOD010	146	2,655	0	6,532.08	6,532.08	6,548
MODGLOB	291	98	324	20,430,947	20,430,947	20,740,508
P0033	15	33	0	2,520.57	2,828.33	3,089
P0201	133	201	0	6,875.00	7,125.00	7,615
P0282	241	282	0	176,867.50	176,867.5	258,411
P0291	252	291	0	1,705.13	1,749.88	5,223.749
P0548	176	548	0	315.29	3,142.88	8,691
P2756	755	2,756	0	2,688.75	2,701.14	3,124
RGN	24	100	80	48.8	48.8	82.2
SAN200-0.9-3	1,126	200	0	−49.9676	−49.9676	−44
SCPC2S	385	468	0	209.85	209.85	216
SET1AL	492	240	472	11,145.62	11,651.62	15,869.7
STEIN45	331	45	0	22	22	30
TSP43	143	1,117	0	5,611	5,611	5,620
VPM1	234	168	210	15.4167	16.4333	20

Since preprocessing may play a big role in some cases (a case in point is P0548: this is why we excluded it from the experiments of §§3 and 4), all instances are preprocessed using the MINTO preprocessor. These preprocessed instances are fed to the four codes compared in Table 8. The reason for this choice is to focus the comparison on the algorithms themselves. All computational experiments were first run with the β-normalization. In the case where no violated cuts were found for either $\beta = 1$ or $\beta = -1$, the same problem was rerun with the α-normalization. In the end the β-normalization was used successfully in 20 of the 29 instances.

6.2. Comparison Between MIP Solvers

In this section, we compare a preliminary version of our code, called MIPO, with several existing codes. OSL was run on a RISC 530, whereas the three other codes were run on an Apollo HP720. All times are reported in seconds. The conversion factor between a RISC 530 and an Apollo HP720 is roughly one to one.

Our purpose here is to show that MIPO is robust. Only the semi-reoptimized lift-and-project cuts are used throughout the runs of MIPO. For most instances, not having a good estimate of *UB* is not a big handicap. However, for some of the very large problems, such as the AIRxx problems, it does make a difference. For this reason, we implemented rudimentary heuristics: at each node where cuts are generated, MIPO uses the "diving heuristic" described in Hoffman and Padberg (1993). The subproblem is solved and all variables at 0 or 1 are fixed at these values. One more 0, 1 variable is then fixed to both 0 and 1 using the branching rule described in

Table 8 Comparison of OSL, CPLEXMIP, MINTO, and MIPO

	OSL		CPLEXMIP2.1		MINTO			MIPO		
Problem	# of Nodes	CPU Time	# of Nodes	CPU Time	# of Nodes	# of Cuts	CPU Time	# of Nodes	# of Cuts	CPU Time
AIR04	***	***	***	***	***	***	***	308	394	7,662
AIR05	***	***	***	***	***	***	***	886	1,123	18,734
BM23	35	11	571	2	155	922	17	254	72	4
CFAT200-1	4	2,726	30	962	***	***	***	76	80	848
CTN2	***	***	41,071	1,300	197	165	23	188	305	86
CTN3	***	***	***	***	4,005	302	650	496	625	243
EGOUT	1	1.7	84	0.6	3	19	0.6	16	14	1.3
FXCH3	***	***	***	***	565	241	86	880	587	237
GENOVA6	3,787	2,375	18,826	2,006	7,533	0	1,281	3,970	2,111	1,508
L152LAV	1,842	7,413	27,255	5,591	***	***	***	4,716	3,457	4,772
LSEU	87	641	24,995	674	161	153	13	1,040	112	24
MISC05	454	220	948	37	505	100	79	788	74	88
MISC07	***	***	40,713	3,832	***	***	***	7,766	2,354	2,634
MOD008	53	107	15,203	263	537	479	293	1,376	215	50
MOD010	18	212	577	146	47	6	970	18	30	58
MODGLOB	***	***	***	***	263	360	56	960	3,346	7,413
P0033	8	2.6	1,058	2.5	15	34	0.8	138	23	1.6
P0201	164	282	2,476	63	1,233	116	160	998	292	145
P0282	210	341	***	***	3,367	438	1,007	218	186	36
P0291	8	10	***	***	31	101	8.5	60	185	17
P0548	0	12	***	***	***	***	***	6,422	2,433	11,036
P2756	37	305	***	***	1,188	875	1,867	724	595	1,933
RGN	2,836	284	5,213	49	3,607	211	255	846	134	55
SAN200-0.9-3	***	***	2,149	4,074	***	***	***	310	201	995
SCPC2S	202	277	673	123	833	0	600	62	137	93
SET1AL	***	***	***	***	27	400	10	68	201	161
STEIN45	***	***	90,922	3,877	***	***	***	55,824	16,248	9,010
TSP43	***	***	***	***	***	***	***	746	568	324
VPM1	***	***	***	***	183	44	8.4	2,934	1,438	1,848

*** Space limit or time limit exceeded.

§5.2, and only the most promising branch is followed. These two steps are repeated sequentially until either an infeasible problem occurs or an integer solution is found. Also, at each node of the enumeration tree, we use two rounding heuristics. In one we round to the closest integer and in the other we round up. By incorporating these heuristics, the computing times deteriorate somewhat for several instances, but overall the resulting code is more robust.

Next we summarize some of the major observations that can be made from Table 8.

In terms of the number of nodes in the enumeration tree, the performance of the four codes is aggregated in Table 9 where, for each code, we give the number of times it came in first (smallest number of nodes) and the number of times it came in second. As seen from the table, OSL achieves the smallest number of nodes in 15 of the 29 instances. The literature often stresses the need

Table 9 Ranking by Number of Search Tree Nodes

OSL		CPLEXMIP2.1		MINTO		MIPO	
first	second	first	second	first	second	first	second
15	2	0	4	4	10	11	10

to reduce the number of enumerated nodes as much as possible. This seems to be the strategy followed by OSL, but unfortunately, OSL does not report the number of cuts it generates or the cut generating strategy it uses. The strategy followed by MIPO pays more attention to the trade-off between node enumeration and cut generation. There are situations where enumerating nodes is more efficient than generating cuts.

As far as computing time is concerned, Table 10 summarizes the performance of the four codes. While MINTO and MIPO achieved shortest computing times on a similar number of instances (ten and twelve respectively), MIPO was in second place fourteen times against three times for MINTO.

Overall, the most robust code was MIPO, which solved all 29 instances and was best or second best in computing time in 26 of the 29 instances. Of course, there were instances that MIPO could not solve, such as P6000 (a problem from Hoffman and Padberg 1991), but the other codes failed as well, so we did not include such instances in the tables since there is nothing to report for any of the codes.

MINTO did particularly well on the fixed charge problems (CTN2, EGOUT, FXCH3, MOD008, SET1AL, VPM1). Indeed, MINTO generates special purpose cuts for this class of problems.

MIPO performed particularly well on the very large instances, that is, those with more than a thousand variables or constraints (AIR04, AIR05, CFAT200-1, L152LAV, MOD010, SAN200-0.9-3, TSP43). Surprisingly, MIPO even managed to outperform MINTO on some instances, like CTN3, where MINTO exploits the problem-specific structure.

In some cases, our general purpose code was competitive with special purpose codes. For example, Hoffman and Padberg (1993) give the following results for AIR04 and AIR05 on a RISC 550: AIR04 required 14441 seconds, 90 nodes, and 3787 cuts whereas AIR05 required 139337 seconds, 494 nodes, and 15449 cuts. In both cases we were able to solve these instances with a few more nodes and much fewer cuts. Another example is TSP43, which can be converted into a symmetric TSP and solved using a special purpose code. The resulting computing time is of a similar order of magnitude as with MIPO (Jünger 1993).

Table 10 Ranking by CPU Time

OSL		CPLEXMIP2.1		MINTO		MIPO	
first	second	first	second	first	second	first	second
2	3	6	6	10	3	12	14

7. Conclusions

In this paper, we investigated the use of lift-and-project cutting planes in a branch-and-cut procedure. The major conclusions are:

(1) A useful measure of cut quality can be obtained by simply computing the Euclidean distance between the hyperplane defining the cut and the point it cuts off.

(2) It is better to generate cuts in large rounds rather than one at a time or in small rounds.

(3) Contrary to a widespread belief, it is efficient to incorporate general cuts within branch-and-cut.

(4) The branching versus cutting decision is one of the most important issues in the implementation of an efficient general purpose branch-and-cut solver. Small enumeration trees do not always correspond to smaller computing times: it is more important to have the right balance between cutting and branching. This decision seems difficult to automate satisfactorily for all instances. We found that a simple strategy, based on the quality of the cuts at the root node, performs well in most instances.

(5) Our branch-and-cut code is an efficient solver for mixed 0-1 programs. On several instances of pure and mixed 0-1 programs, it performs as well as, or better than some of the best currently available mixed integer programming codes. The main advantage of our code is that it is able to generate cutting planes independently of the structure of the problem, thus yielding a more robust MIP solver. In some cases it is even more efficient than state-of-the-art algorithms that make use of the problem-specific structure.[1]

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