50 Years of Integer Programming 1958-2008

From the Early Years to the State-of-the-Art

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1st Edition. 2009. Buch. xx, 804 S.
ISBN 978 3 540 68274 5
Format (B x L): 15,5 x 23,5 cm
Gewicht: 1345 g

Chapter 12
Fifty-Plus Years of Combinatorial Integer Programming

William Cook

Abstract Throughout the history of integer programming, the field has been guided by research into solution approaches to combinatorial problems. We discuss some of the highlights and defining moments of this area.

12.1 Combinatorial integer programming

Integer-programming models arise naturally in optimization problems over combinatorial structures, most notably in problems on graphs and general set systems. The translation from combinatorics to the language of integer programming is often straightforward, but the new rendering typically suggests direct lines of attack via linear programming.

As an example, consider the stable-set problem in graphs. Given a graph $G = (V,E)$ with vertices V and edges E, a *stable set* of G is a subset $S \subseteq V$ such that no two vertices in S are joined by an edge. The *stable-set problem* is to find a maximum-cardinality stable set. To formulate this as an integer-programming (IP) problem, consider a vector of variables $x = (x_v : v \in V)$ and identify a set $U \subseteq V$ with its characteristic vector $\overline{x}$, defined as $\overline{x_v} = 1$ if $v \in U$ and $\overline{x_v} = 0$ otherwise. For $e \in E$ write $e = (u,v)$, where u and v are the ends of the edge. The stable-set problem is equivalent to the IP model

$$\begin{gathered} \max \sum (x_v : v \in V) \qquad (12.1) \\ x_u + x_v \leqslant 1,\ \forall\, e = (u,v) \in E, \\ x_v \geqslant 0,\ \forall\, v \in V, \\ x_v \text{ integer},\ \forall\, v \in V. \end{gathered}$$

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M. Jünger et al. (eds.), *50 Years of Integer Programming 1958-2008*,
DOI 10.1007/978-3-540-68279-0_12, © Springer-Verlag Berlin Heidelberg 2010

To express this model in matrix notation, let A denote the edge-vertex incidence matrix of G, that is, A has rows indexed by E, columns indexed by V, and for each $e \in E$ and $v \in V$, entry $A_{ev} = 1$ if v is an end of e and $A_{ev} = 0$ otherwise. Letting $\mathbf{0}$ and $\mathbf{1}$ denote the vectors of all zeros and all ones, respectively, problem (12.1) can be written as

$$\max(\mathbf{1}^T x : Ax \leqslant \mathbf{1},\ x \geqslant \mathbf{0},\ x \text{ integer}). \tag{12.2}$$

In a similar fashion, the *vertex-cover problem* can be modeled as

$$\min(\mathbf{1}^T x : Ax \geqslant \mathbf{1},\ x \geqslant \mathbf{0},\ x \text{ integer}). \tag{12.3}$$

This later problem asks for a minimum-cardinality set $C \subseteq V$ such that every edge in E has at least one of its ends in C.

By dropping the integrality constraints on the variables, we obtain linear programming (LP) relaxations for the IP models. From these relaxations we get the LP dual

$$\min(y^T \mathbf{1} : y^T A \geqslant \mathbf{1}^T,\ y \geqslant \mathbf{0}) \tag{12.4}$$

for the stable-set problem and the LP dual

$$\max(y^T \mathbf{1} : y^T A \leqslant \mathbf{1}^T,\ y \geqslant \mathbf{0}) \tag{12.5}$$

for the vertex-cover problem. Solutions to (12.4) and (12.5) give upper bounds and lower bounds for the two combinatorial problems, respectively, through the weak LP-duality theorem. When these bounds are unsatisfactory in a given application, IP techniques can be employed to improve the relaxations and reduce the gap between the cardinality of a stable set or vertex cover and the value of the corresponding dual LP. The primary tool for obtaining such an improvement is to add, to the LP relaxation, inequalities that are satisfied by all integer solutions, but violated by an optimal solution to the LP problem. This is known as the *cutting-plane method* and it was first proposed in the context of a combinatorial problem, as we discuss in the next section.

The use of cutting planes is a practical step to improve a given model, but the LP-duality framework can also be a powerful tool in proving theorems in combinatorics. The idea is to formulate a relaxation such that the LP optimum can always be attained by an integer-valued vector. Such a relaxation gives a characterization of optimal solutions to the combinatorial problem. If it can be shown that the dual LP also always has integer solutions, then strong LP duality provides a form of combinatorial min-max theorem. Such statements are among the most beautiful results in combinatorics.

The pursuit of LP-based proofs of min-max theorems often involves showing that a polyhedron P, defined as the solution set to a system of linear inequalities $Ax \leqslant b$, has the property that each of its vertices is integer valued. In this case P is said to be an *integer polyhedron*. If for each integer objective vector w the dual LP $\min(y^T b : y^T A = w^T, y \geqslant \mathbf{0})$ has an integer optimal solution, then the system $Ax \leqslant b$ is called *totally dual integral* (TDI). Such systems are often the final goal of research efforts, since they translate to min-max results for the weighted version of the combinatorial problem, where *w* provides the weights and the integer dual solutions correspond to combinatorial structures.

The study of integer polyhedra and totally dual integral systems for combinatorial problems is known as *polyhedral combinatorics*. The area we are calling *combinatorial integer programming* includes both the application of IP techniques to solve particular instances of possibly difficult problems, as well as the development of general methods and examples in polyhedral combinatorics.

In this paper we give a brief history of combinatorial IP. It is not our intent to be comprehensive in any form, but rather to touch on some of the highlights in the development and growth of the field, with particular emphasis on breakthroughs in combinatorial methods that have led to successful techniques for general integer programming. For a marvelously complete account of the history of the wider field of combinatorial optimization, the reader is directed to the work of Schrijver [135]. We refer the reader also to the reprints of classical papers contained in this volume, and to the fascinating historical perspectives offered by the authors of these papers in their newly written introductions.

12.2 The TSP in the 1950s

The birth of combinatorial integer programming occurred in the spring of 1954, pre-dating the start of general integer-programming research by several years. The event was described as follows in the popular journal *Newsweek*, July 26, 1954.

> Finding the shortest route for a traveling salesman—starting from a given city, visiting each of a series of other cities, and then returning to his original point of departure—is more than an after-dinner teaser. For years it has baffled not only goods- and salesman-routing businessmen but mathematicians as well. If a drummer visits 50 cities, for example, he has 10^{62} (62 zeros) possible itineraries. No electronic computer in existence could sort out such a large number of routes and find the shortest.
>
> Three Rand Corp. mathematicians, using Rand McNally road-map distances between the District of Columbia and major cities in each of the 48 states, have finally produced a solution. By an ingenious application of linear programming—a mathematical tool recently used to solve production-scheduling problems—it took only a few weeks for the California experts to calculate "by hand" the shortest route to cover the 49 cities: 12,345 miles.

The California experts were George Dantzig, Ray Fulkerson, and Selmer Johnson, part of an exceptionally strong and influential center for the new field of mathematical programming, housed at the RAND Corporation in Santa Monica. Dantzig et al. took up the computational challenge of the traveling salesman problem (TSP), solving a 49-city instance with hand-only computations. Along the way they set the stage for the study of integer programming.

The traveling salesman problem

Before going into the details of the RAND team's work, it is appropriate to consider the research environment where conditions were set for their breakthrough study. The starting point of the discussion is the TSP itself and how the problem came to the prominent role it has played in the history of integer programming. Concerning this issue, Dantzig et al. [29] write the following.

> The origin of this problem is somewhat obscure. It appears to have been discussed informally among mathematicians at mathematics meetings for many years. Surprisingly little in the way of results has appeared in the mathematical literature. It may be that the minimal-distance tour problem was stimulated by the so-called Hamiltonian game which is concerned with finding the number of tours possible over a specified network. The latter problem is cited by some as the origin of group theory and has some connections with the famous Four-Color Conjecture. Merrill Flood (Columbia University) should be credited with stimulating interest in the traveling-salesman problem in many quarters. As early as 1937, he tried to obtain near optimal solutions in reference to routing of school buses. Both Flood and A. W. Tucker (Princeton University) recall that they heard the problem first in a seminar talk by Hassler Whitney at Princeton in 1934 (although Whitney, recently queried, does not seem to recall the problem).

This brief summary of TSP history is expanded in Hoffman and Wolfe [81] and Schrijver [135, 136]. In these works it is noted that Karl Menger described a geometric variant of the TSP in a record of a mathematics colloquium held in Vienna on February 5, 1930 [113]. Schrijver [135, 136] also points out that Menger and Whitney met at Harvard University in 1930–31, during a semester-long visit by Menger. This Menger-Whitney interaction supports the idea of a possible connection between Menger's Vienna colloquium and Whitney's Princeton seminar.

It remains a question whether Whitney did in fact discuss the TSP at Princeton. There unfortunately is not an accessible record at Princeton University covering the seminars delivered in the Department of Mathematics in the 1930s. The Pusey Library at Harvard University does, however, contain an archive of 3.9 cubic feet of Whitney's papers, and within the collection there is a set of handwritten notes that appear to be preparation for a seminar by Whitney, written sometime in the years shortly after 1930. The notes give an introduction to graph theory, including the following paragraph.

> A similar, but much more difficult problem is the following. Can we trace a simple closed curve in a graph through each vertex exactly once? This corresponds to the following problem. Given a set of countries, is it possible to travel through them in such a way that at the end of the trip we have visited each country exactly once?

This is an unusual example for the *Hamilton-circuit problem*, and clearly not a far step away from the TSP. The geographic aspect also matches well with Flood's recollection of the "'48-states problem' of Hassler Whitney" in a 1984 interview [45] with Albert Tucker.

There is not a record of TSP research, under the TSP name, in the late 1930s and into the 1940s, but by the end the 1940s it had become a known challenge in Princeton and RAND, supported by the interest of Merrill Flood. On the Princeton side, Harold Kuhn writes the following in a recent email letter [95].

> The traveling salesman problem was known by name around Fine Hall by 1949. For instance, it was one of a number of problems for which the RAND corporation offered a money prize. I believe that the list was posted on a bulletin board in Fine Hall in the academic year 1948–49.

At the RAND Corporation, Julia Robinson published a research report in 1949 that appears to be the first reference to the TSP by name [132]. Interestingly, Robinson formulates the problem as finding “the shortest route for a salesman starting from Washington, visiting all the state capitals and then returning to Washington.”

Heller and Kuhn

Robinson’s work considers an LP approach to the TSP, treating a variation of the *assignment problem*. This problem asks for an optimal assignment of n workers to n tasks, where the quality of assigning worker i to task j is specified by a weight w_{ij}. The goal is to maximize the total weight of the n assignments. A feasible solution to the problem can be viewed as a graph having vertices labeled 1 up to n, with the assignment of worker i to task j indicated by an edge directed from i to j. The assignment solution gives a collection of disjoint directed circuits meeting every vertex. The connection to the TSP is clear: a TSP solution is the special case when the assignment yields a single circuit containing all vertices.

The assignment problem is a member of a more general class called *transportation problems*. Efforts to solve instances from this class played a prominent role in the early history of linear programming. Julia Robinson’s research in this area is mentioned in the following quote from Dantzig et al. [29].

> The relations between the traveling-salesman problem and the transportation problem appear to have been first explored by M. Flood, J. Robinson, T. C. Koopmans, M. Beckmann, and latter by I. Heller and H. Kuhn.

Robinson’s paper begins an LP line of attack, but it is the work of Isidor Heller and Harold Kuhn that appears to have had the most influence on the computational study of Dantzig et al. Both Heller [72] and Kuhn [91] began investigations of linear descriptions of the convex hull of TSP tours, considering tours as characteristic vectors of their edge sets. Their approach aims at a direct LP formulation of the TSP.

In notes from a George Dantzig Memorial Lecture delivered in 2008 [94], Kuhn writes the following concerning his TSP study.

> My efforts centered around the formulation of the traveling salesman problem as a linear program that has as feasible solutions the convex hull of the “tours”. A tour is defined as a 0-1 matrix that presents a permutation that is a single cycle. For example, with 5 cities, there are 24 tours that are the extreme points of an 11 dimensional feasible set. In the summer of 1953, I found that this convex polyhedron has 390 faces, a very discouraging fact.
>
> I had a number of contacts with George throughout the summer discussing this and other problems. And I know that George attended my lecture at the end of the summer (as did Selmer Johnson, Ray Fulkerson, and Alan Hoffman). We were both keenly aware of the fact that, although the complete set of faces (or constraints) in the linear programming formulation of the Traveling Salesman Problem was enormous, if you could find an optimal

solution to a relaxed problem with a subset of the faces that is a tour, then you had solved the underlying Traveling Salesman Problem.

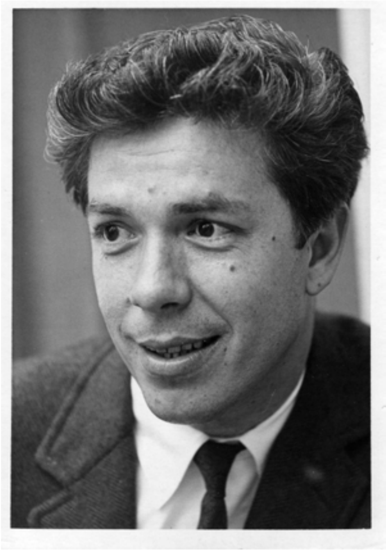

Fig. 12.1 Harold Kuhn, 1961. Photograph courtesy of Harold Kuhn.

Kuhn presented his TSP work at the Sixth Symposium in Applied Mathematics, held in Santa Monica on August 26–28, 1953. His lecture is titled “The traveling salesman problem” in the conference program [2], and Dantzig, Fulkerson, and Johnson are listed as participants of the meeting. Kuhn [95] notes that his Santa Monica lecture included the following points.

1. A statement of the TSP as an LP and a clear statement that if you solved over a subset of the constraints and obtained a tour then the tour was optimal.
2. Results of a shooting experiment on the five-city TSP polytope to explore the distribution of its faces. (This experiment is described in [55] and in [93].)
3. The fact that in the five-city polytope any two vertices are contained in a face of dimension one, that is, the polytope is neighborly, and that the six-city polytope does not have this property.
4. An account of constructions of classes of faces whose number grows exponentially with the number of cities.

Concerning this last point, Kuhn writes that the number of faces “was so large that it discouraged me from pursuing this direction of research.”

Kuhn’s work was inspired in part by a study of the five-city TSP polytope by Heller, which was also carried out in 1953. Heller presented his research in a lecture “On the problem of shortest path between points” [72] at the Summer Meeting of the

American Mathematical Society, held in Kingston, Ontario, August 31 to September 5, 1953. Dantzig again participated in the meeting, while Fulkerson and Johnson are not listed as participants [3].

The studies of Heller and Kuhn conclude with the fact that the natural LP model of the TSP necessarily contains far too many inequalities for any solver to handle directly. Undeterred, Dantzig et al. saw this as an opportunity to demonstrate the versatility of the simplex algorithm.

The cutting-plane method

The approach adopted by the RAND team is laid out in a preliminary version [28] of their paper. In the following quote, C_1 denotes the solution set of the LP relaxation, T_n denotes the convex hull of all tours through n cities, and d_{ij} is the cost of travel between city i and city j.

> What we do is this: Pick a tour x which looks good, and consider it as an extreme point of C_1; use the simplex algorithm to move to an adjacent extreme point e in C_1 which gives a smaller value of the functional; either e is a tour, in which case start again with this new tour, or there exists a hyperplane separating e from the convex of tours; in the latter case cut down C_1 by one such hyperplane that passes through x, obtaining a new convex C_2 with x as an extreme point. Starting with x again, repeat the process until a tour $\hat{x}$ and a convex $C_m \supset T_n$ are obtained over which $\hat{x}$ gives a minimum of $\sum d_{ij}x_{ij}$.

The process clearly applies to problems beyond the TSP, and it is known today as the *primal cutting-plane method*; see, for example, Letchford and Lodi [102].

The published version of the paper excludes a general description of their method, relying on a sequence of five-city and six-city examples to convey the idea. There was a four-month gap between the release of the preliminary report and the submission of their paper, and the authors appear to have changed their minds as to how best to describe the methodology. Among other things, the preliminary report also contains a discussion of the convex hull of tours, similar in style to the modern treatment of the TSP polytope. Regarding this, Fulkerson writes in a September 2, 1954, letter to *Operations Research* editor George Shortly: “In an effort to keep the version submitted for publication elementary, we avoid going into these matters in any detail.” It is a pity this choice was made, but it is not a surprising decision given the nature of operations research literature at the time.

The LP relaxation adopted by Dantzig et al. has a variable x_{ij} for each unordered pair of cities (i, j). It is convenient to describe this model in terms of a complete graph $G = (V,E)$, denoting variable x_{ij} as x_e, where e is the edge having ends i and j. The initial relaxation consists of the *degree equations*

$$\sum(x_e : v \text{ is an end of } e) = 2 \quad \text{for all cities } v \tag{12.6}$$

together with the restriction $x_e \geqslant 0$ for all $e \in E$. A ready supply of potential cutting planes is derived from the observation that every proper subset of k cities can contain at most $k-1$ edges in a tour. The corresponding *subtour constraints* are

$$\sum(x_e: \text{ both ends of } e \text{ are in } S) \leqslant |S|-1 \quad \text{for all } S \subseteq V, S \neq V. \tag{12.7}$$

These inequalities are called "loop conditions" in [29] and they are the first line-of-defense in the RAND computations.

The published descriptions of the small TSP examples in [29] focus on the integration of the simplex algorithm and the cutting-plane method, suggesting how Dantzig et al. were able to handle the 49-city LP relaxation with hand-only calculations. For this large TSP, using LP duality, they present a succinct proof that their method produced an optimal tour. The final LP relaxation contains a set of 23 subtour inequalities (of which 16 have the form $x_e \leqslant 1$ for edges $e \in E$, that is, the set S has only two cities) and two additional inequalities. The second of these two non-subtour cutting planes points to Irving Glicksberg as an unsung hero in the TSP effort, cited in the footnote: "We are indebted to I. Glicksberg of Rand for pointing out relations of this kind to us."

Fig. 12.2 Irving Glicksberg, 1978. Photograph copyright Mathematisches Forschungsintitut Oberwolfach.

It should be noted that the Dantzig et al. study considers the symmetric version of the TSP, where the travel cost between city i and city j is the same as the cost between city j and city i. This differs from the Heller and Kuhn studies, where the directed version of the problem is considered. This point generated some discussion among TSP researchers. In his September 2, 1954, letter to George Shortley, Fulkerson writes the following.

> The assumption $d_{ij} = d_{ji}$ certainly seems to be of some importance, although we are not sure that it is crucial. (Dr. I. Heller, who has done considerable research on the problem, feels that the symmetry assumption, which permits representing the convex of tours in a different space, is of the utmost importance.) It is true, as the referee says, that the loop conditions and combinatorial analysis can be used for directed tours as well, and some work should be done along these lines. (The fact that the analogues of the loop conditions are faces of the convex of directed tours has been known for a couple of years.) However, if one has a symmetric problem, much is gained by using undirected tours. This is probably due to two

> things: (1) The simplex algorithm of linear programming becomes especially easy, and (2) there is some reason to believe that the convex of undirected tours may have significantly fewer faces than the directed tours.

This point is also brought up in a letter from Fulkerson to Heller, dated March 11, 1954.

> I read your abstracts "On the problem of shortest path between points" in the November issue of the Bulletin of the American Mathematical Society with much interest. If it is not too much trouble, I would greatly appreciate it if you would send me more details of your results.
>
> Recently, G. Dantzig, S. Johnson, and I have been working on computational aspects of the problem via linear programming techniques even though we don't know, of course, all the faces of the convex C_n of tours for general n. The methods we have been using seem hopeful, however; in particular, an optimal tour has been found by hand computation for a large scale problem using 48 cities, rather quickly. We have found it convenient in translating Dantzig's simplex algorithm in terms of the map of points, to identify tours which differ only in direction of traversal. For example, C_5 can be characterized by a system of 25 hyperplanes in 10 dimensional space. We don't know very much about C_n in general, but thought we might learn more from reading your papers, if they are available.

Similar requests for polyhedral results were sent from Dantzig to Kuhn (March 11, 1954) and from Dantzig to Tucker (March 25, 1954). It is clear that Dantzig et al. were actively seeking more information on the facial structure of the TSP polytope, to better equip their cutting-plane method. This is a topic that was taken up in force two decades later, as we describe in Section 12.6.

Reduced-cost fixing and branch-and-bound algorithms

In the 1954 reports and in a follow-up 1959 paper [30], Dantzig et al. assert the effectiveness of a method for transforming the TSP into a problem on a sparse graph. This transformation is known as *reduced-cost fixing* and it was described for the first time in this TSP work. The idea is the following. When a minimization LP problem with nonnegative variables is solved by the simplex method, the objective function is rewritten in the form $z = z_o + \sum(\overline{c}_j x_j : j = 1,\ldots,m)$ such that z_o is a constant, $\overline{c}_j \geqslant 0$ for all j, and a solution vector x^* is found such that $x_j^* = 0$ for all j such that $\overline{c}_j$ is positive. If the variables are required to take on integer values, then any x_j such that $z_o + \overline{c}_j$ is greater than the cost of a known feasible integer solution can be set to the value 0, and eliminated from the problem. This process is used in modern integer-programming solvers, reducing the problem space in a preliminary step to an enumeration phase.

The RAND team's implementation of this idea is more subtle, since they do not actually solve the LP relaxation in their primal cutting-plane method, carrying out only single pivots of the simplex algorithm. Nevertheless, they show that reduced-cost fixing can be accomplished, taking advantage of the fact that the variables in the TSP relaxation are bounded between 0 and 1. The explicit variable bounds allow one to obtain a bound on the objective value even in the case when some of the $\overline{c}_j$ values are negative, and again variables can be eliminated. In the following comment on this process from Dantzig et al. [29], E denotes a value such that variables x_j with $\overline{c}_j > E$ can be eliminated.

> During the early stages of the computation, E may be quite large and very few links can be dropped by this rule; however, in the latter stages often so many links are eliminated that one can list all possible tours that use the remaining admissible links.

A general method for carrying out this enumeration of tours is not given, but in [30] an example is used to describe a possible scheme, relying on forbidding subtours and vertices of degree three when growing a tour in the sparse edge set.

The enumeration aspect of the Dantzig et al. work has not been followed up to any large degree in modern computational studies of the TSP, but it was pursued in the late 1950s in various combinatorial approaches by Frederick Bock [14], G. A. Croes [24], and others. These studies, in turn, contributed to the development of the *branch-and-bound* algorithm, where the set of solutions is split into two or more subsets (the branching step), a lower-bounding method is applied separately to each of the subsets (the bounding step), and the process is applied repeatedly to the resulting subproblems (growing a search tree).

The first full-version of the branch-and-bound method may be the TSP algorithm described in the 1958 Ph.D. thesis of Willard Eastman [32]. In Eastman's algorithm, the lower bound is provided by the solution of a variant of the assignment problem. In his branching step, a subtour having k edges in the assignment solution is chosen, and k subproblems are created by setting to 0, one at a time, each of the variables corresponding to the edges in the subtour. Eastman carried out his method on a ten-city TSP; an illustration of part of his search tree is given in Figure 12.3.

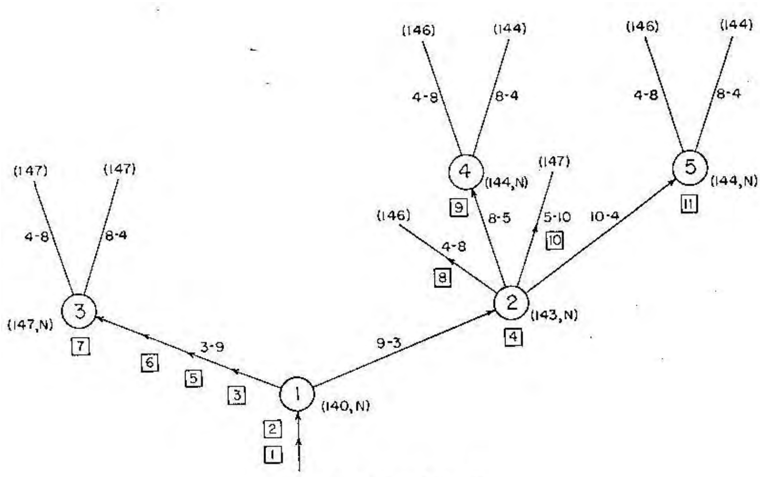

Fig. 12.3 Branch-and-bound search tree from W.L. Eastman's 1958 Ph.D. thesis.

The TSP-driven branch-and-bound research had a great impact on the practical solution of general integer-programming instances, starting with the important 1960 paper of Land and Doig [98]. We close this section by noting that the name "branch-and-bound" first appeared several years after Land and Doig, when the method was again applied to the TSP in the 1963 work of Little, Murty, Sweeney, and Karel [104].

12.3 Proving theorems with linear-programming duality

In the preface to a collection of his papers [114], Alan Hoffman thanks Harold Kuhn and David Gale: "in fond recollection of the early '50s, when we taught each other to use the ostensibly practical subject of linear programming to prove aesthetic combinatorial theorems that were ostentatiously useless." The work of these gentlemen and their colleagues set basic research directions that formed a roadmap for the early development of polyhedral combinatorics. The topics considered include the assignment problem by Kuhn [92], systems of distinct representatives by Hoffman and Kuhn [78], bipartite matching by Hoffman [74], network flows by Lester Ford and Ray Fulkerson [46] and David Gale [51], and partially ordered sets by Dantzig and Hoffman [31]. This work was carried out in an incredibly active span of years in the mid-1950s. A nice overview of the activity can be seen in the volume *Linear Inequalities and Related Systems*, edited in 1956 by Kuhn and Tucker [96]. Leafing through the pages of the book, it is striking how many household names appear among the authors. The volume also contains a bibliography of 289 books and papers covering research on systems of linear inequalities, with the majority written after 1950.

An important general concept that came out of this body of work is the notion of totally unimodularity, introduced by Hoffman and Joseph Kruskal [77]. A matrix is called *totally unimodular* if each of its subdeteriminants is 0, 1, or -1. The well-known Hoffman-Kruskal result states that an integral matrix A is totally unimodular if and only if for each integral vector b the set $\{x : Ax \leqslant b, x \geqslant \mathbf{0}\}$ is an integer polyhedron.

The following illustration of this concept is adopted from Hoffman's short survey paper "Linear programming" in *Applied Mechanics Reviews*, 1956 [75]. Consider a *bipartite* graph $G = (V,E)$. By definition, V can be written as the disjoint union of sets U and W such that each $e \in E$ has one end in U and one end in W. An inductive proof shows that the edge-vertex incidence matrix A of such a graph is totally unimodular. It follows that both sides of the LP-duality equation

$$\max(\mathbf{1}^T x : Ax \leqslant \mathbf{1},\ x \geqslant \mathbf{0}) = \min(y^T \mathbf{1} : y^T A \geqslant \mathbf{1}^T,\ y \geqslant \mathbf{0})$$

are attained by integer solutions, $\bar{x}$ and $\bar{y}$, assuming that the optima exist. Note that $\bar{x}$ is the incidence vector of a stable set of G, while $\bar{y}$ is the incidence vector of a set of edges F such that each vertex in V meets at least one edge in F, that is,

Fig. 12.4 Esther and Alan Hoffman, Washington D.C., 1951. Photograph courtesy of Alan Hoffman.

Fig. 12.5 Alan Hoffman, 2000. Photograph by Sue Clites.

F is an *edge cover*. Also, the optimality condition is satisfied as long as G has no isolated vertices. We conclude that for such a graph, the maximum cardinality of a stable set is equal to the minimum cardinality of an edge cover. Perhaps not "ostentatiously useless", but a pretty min-max result nonetheless. Hoffman [114] writes the following concerning his joint work with Kruskal.

> In this paper the concept (not the name) of total unimodularity was shown to be a neat explanation (via Cramer's rule) of the fact that some linear programming problems have all their vertices integral. I do not think this paper would have been accepted for publication if we had not fancied it up with a supçon of generalization: the main idea is too obvious and folklorish. And we also thought that we introduced a new class of matrices with the "unimodular property", but Jack Edmonds later found that our new class wasn't really new after all. It is nevertheless true that totally unimodular matrices (as Berge christened them), and unimodular matrices generally, are key to understanding how linear programming duality underlies a wide variety of extremal combinatorial analysis.

Indeed, total unimodularity provides a unifying theme for combinatorial min-max theorems, and it remains a fundamental tool in polyhedral combinatorics.

12.4 Cutting-plane computation

Returning to the cutting-plane method, the publication of the 1954 TSP paper did not begin an immediate revolution in the practical solution of integer-programming problems. Within the RAND Corporation, however, the cutting-plane strategy was explored as a computational tool in the years following the Dantzig-Fulkerson-Johnson success.

Markowitz and Manne

An important contribution in this effort is documented in the paper "On the solution of discrete programming problems" by Harry Markowitz and Alan Manne [111], first published as a RAND research paper in 1956 [110]. Markowitz and Manne formulate a general mixed-integer-programming model and describe, in abstract terms, how it can be solved with a variant of the cutting-plane method. They introduce their procedure as follows [111].

> We do not present an automatic algorithm. We present, rather, a general approach susceptible to a number of variations depending on the problem and the judgment of the user. The approach is of little or no purely mathematical interest. Its only recommendation consists of a few empirical observations: When applied to very small discrete problems (with a few thousand a priori possibilities) it has produced and confirmed the answer almost immediately. Its application to two moderate-size problems is described subsequently. There is a danger, of course, in generalizing from so few observations. They provide encouragement, rather than proof.
>
> Our procedure for handling discrete problems was suggested by that employed in the solution of the 'traveling-salesman' problem by Dantzig, Fulkerson, and Johnson.

Fig. 12.6 Harry Markowitz, 2000. Photograph by Sue Clites.

Despite these modest words, the Markowitz-Manne approach is an interesting variation of the method used in the TSP work. The new ideas are to (1) allow cuts that possibly remove integer solutions that are known to have objective value no better than a previously computed solution, (2) use linear constraints to partition the feasible region, allowing the cutting-plane method to be applied independently to each of the subregions, and (3) allow the simplex algorithm to compute optimal LP solutions to the relaxations, rather than carrying out single simplex pivots, thus obtaining bounds to measure the quality of previously computed solutions. The second point is a clear precursor to the modern branch-and-cut version of the branch-and-bound algorithm, and the third point is the adoption of the now common "dual" cutting-plane method.

Markowitz and Manne begin their presentation with a description of the primal cutting-plane approach of Dantzig et al. and then lay out a general step of the partition+relaxation strategy. Details of a possible implementation of the abstract ideas are provided through two examples, one in production scheduling and one in air transportation. The step-by-step elaboration of the technique on these problems provides great insight into the practical application of LP arguments in integer programming.

The following simple remark concludes the Markowitz-Manne paper [111].

> The solutions to the two examples presented here, along with those to the traveling-salesman problem, suggest that the human being with simple aids can frequently produce solutions with near-optimum payoffs.

It is interesting to see these famous researchers (Markowitz was awarded a Noble Prize in 1990) getting their hands dirty with detailed integer-programming calculations. One must imagine that the members of the RAND Corporation were a driven group of problem solvers, using the focus of real computations to guide their research.

Dantzig in 1957

Foremost among these IP problem solvers is undoubtedly George Dantzig. He returns to the cutting-plane method in a 1957 paper [27], summarizing some of his work following the TSP study. Here Dantzig also describes the "dual" cutting-plane approach, considering cuts that remove fractional optimal LP solutions.

> The linear programming approach consists in putting such additional linear-inequality constraints on the system that the fractional extreme points of C where the total value of z is maximized will be excluded, while the set of extreme points of the convex hull C^* of admissible solutions will be unaltered. The procedure would be straightforward except that the rules for generating the *complete set* of additional constraints is not known. For practical problems, however, rules for generating a partial set of constraints is often sufficient to yield the required solution.

This methodology is applied to an example of the *knapsack problem*, that is, an IP model where the feasible region is determined by a single inequality constraint and all variables are restricted to take on 0 or 1 values. In the knapsack metaphor, the coefficients in the inequality are the weights of the objects and the right-hand-side is the knapsack's capacity. To run his cutting-plane approach, Dantzig considers what are now known as *cover inequalities*, expressing that the sum of k variables can be no more than $k-1$ if the weight of the corresponding k objects exceeds the capacity of the knapsack. In the following quote from [27], condition "(14)" refers to such a cutting plane, form "(11)" refers to the objective function, and condition "(9)" refers to the single knapsack constraint.

> Form (11) is maximized under conditions (9) and $0 \leqslant x_j \leqslant 1$, but with the constraint (14) added. Again a new fractional extreme point may turn up for the new convex C, and it will be necessary again to seek a condition that will exclude it. For the most part the conditions added will be other partial sums of the x_j similar to (14). However, at times more subtle relations will be required until an extreme point is obtained that is admissible.
>
> Since the discovery of these more subtle relations is more an art than a science, the reader may dismiss the whole approach as worthless. However, experiments with many problems by the author and others indicate that very often a practical problem can be solved using only such obvious supplementary conditions as (14).

The use of the phrase "many problems" suggests that the cutting-plane method was indeed in use, at least at the RAND Corporation.

Gomory's IP algorithm

A common thread in the discussions of Dantzig, Fulkerson, and Johnson [29], Markowitz and Manne [111], and Dantzig [27] is the need for creativity in the discovery of inequalities to add as cutting planes, with appeals to Irving Glicksberg, to a "human being", and to "more of an art than a science", respectively. Such creativity would limit the automation of the procedure on the class of digital computers that was becoming available. This subject was addressed by Princeton researcher Ralph Gomory, with the publication of his stunning four-page paper [54] in 1958.

Fig. 12.7 George Dantzig, Ralph Gomory, and Ellis Johnson, 1982. Photograph by Sue Clites.

Fig. 12.8 Ralph Gomory as a TSP tour. Image by Robert Bosch, December 2007.

> It is the purpose of this note to outline a finite algorithm for obtaining integer solutions to linear programs. The algorithm has been programmed successfully on an E101 computer and used to run off the integer solution to small (seven or less variables) linear programs completely automatically.
>
> The algorithm closely resembles the procedures already used by Dantzig, Fulkerson, and Johnson, and Markowitz and Manne to obtain solutions to discrete variable programming problems. Their procedure is essentially this. Given the linear program, first maximize the objective function using the simplex method, then examine the solution. If the solution is not in integers, ingenuity is used to formulate a new constraint that can be shown to be satisfied by the still unknown integer solution but not by the noninteger solution already attained. This additional constraint is added to the original ones, the solution already attained becomes nonfeasible, and a new maximum satisfying the new constraint is sought. This

process is repeated until an integer maximum is obtained, or until some argument shows that a nearby integer point is optimal. What has been needed to transform this procedure into an algorithm is a systematic method for generating the new constraints. A proof that the method will give the integer solution in a finite number of steps is also important. This note will describe an automatic method of generating new constraints. The proof of the finiteness of the process will be given separately.

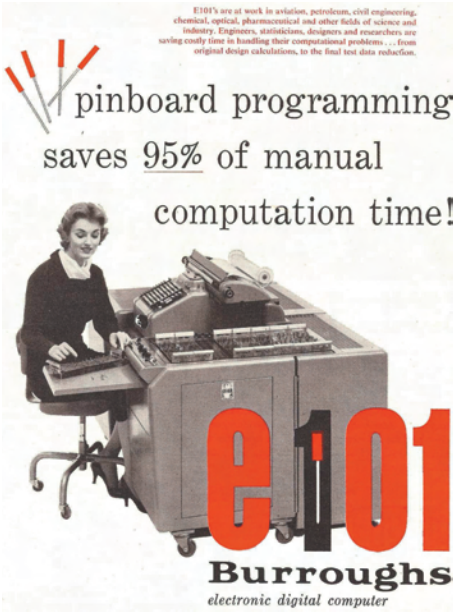

Fig. 12.9 Magazine advertisement for the Burroughs E101 computer, 1959.

The great importance of Gomory's algorithm is covered in detail in other parts of this volume; we focus here only on its connections to developments in combinatorial integer programming.

In one direction, the connection to combinatorial IP is not as strong as one might guess. Indeed, in his 1991 paper "Early integer programming" [56], Gomory describes how he became aware of the existing cutting-plane research only after the main details of his procedure had been worked out. Nonetheless, the history of success with the cutting-plane method in combinatorial integer programming likely played a major role in the acceptance of Gomory's algorithm as a viable technique for solving general IP problems.

In the other direction, an initial wave of combinatorial projects took the direct approach of formulating IP models and turning Gomory loose on small instances. Representative papers at this kind are those by Lambert [97] and Miller, Tucker, and Zemlin [116], where IP formulations of the TSP are presented together with reports of solutions to instances having five cities and four cities, respectively. Rapid growth in the size of the IP formulations in these studies limits the applicability of the methodology.

An interesting hybrid approach was explored several years later by Glenn Martin, described in an unpublished manuscript from 1966 [112]. Martin considers the TSP, but he begins with a simple LP relaxation consisting of the degree equations and a subtour constraint for the ends of the cheapest edge incident to each city. He applies Gomory's algorithm to obtain an integer optimal solution x^* to the relaxation. If x^* is a tour, then it is an optimal solution to the TSP; otherwise he adds, by hand, subtour inequalities violated by x^* and applies Gomory again. Using three rounds of the procedure, Martin repeated the Dantzig-Fulkerson-Johnson feat of solving the 49-city USA instance. This effective approach is considered in further studies by Takis Miliotis [115] in the mid-1970s.

12.5 Jack Edmonds, polynomial-time algorithms, and polyhedral combinatorics

The work of Gomory centers on the automation of the cutting-plane procedure, making it suitable for implementation on a digital computer. In 1960, a branch-and-bound alternative was proposed by Ailsa Land and Alison Doig [98], working at the London School of Economics. In the following quote, these two authors comment on their IP algorithm.

> Until recently there was no general automatic routine for solving such problems, as opposed to procedures for proving the optimality of conjectured solutions, and the work here is intended to fill the gap.

Fig. 12.10 Ailsa Land, Banff, 1977. Photograph courtesy of Ailsa Land.

Fig. 12.11 Alison Doig, *The Sun*, October 21, 1965. Courtesy of Alison (Doig) Harcourt.

Variations of their approach became the dominant practical method for the solution of IP instances. Concerning this, Ailsa Land and Susan Powell [99] make the following remark in a 2007 paper.

> While branch and bound began to be built into computer codes, the cutting plane approach was obviously more elegant, and we spent a great deal of time experimenting with it. ... Work was done, but it was not published because as a method to solve problems branch and bound resoundingly won.

They go on to write: “It is gratifying that the combination, ‘branch and cut’, is now often successful in dealing with real problems.”

The importance of the automatic nature of the Gomory and Land-Doig algorithms cannot be disputed, but a critical theoretical question remained. The algorithms were shown to be finite, but this in itself is not a substantial issue for the problem class. Consider, for example, the TSP, where it is obvious that one can simply list all possible tours. That this approach is not an acceptable solution is suggested already by Karl Menger [113], in his initial description of the problem.

> This problem can naturally be solved using a finite number of trials. Rules which reduce the number of trials below the number of permutations of the given point set are not known.

What is sought is an algorithm that is efficient, not just finite.

A search for a better-than-finite algorithm for the assignment problem was a focus of early mathematical-programming research in the United States, starting with a 1951 lecture of John von Neumann at Princeton. Two years later, an efficient solution method was famously developed by Harold Kuhn [92], armed with a copy of Jenő Egerváry’s 1931 paper [44] and a large Hungarian-English dictionary.

Kuhn's [93] personal account of the events leading up to this work is delightful to read, as is Schrijver's [135] thorough description of the history of assignment-problem computation and algorithms.

An analysis of Kuhn's Hungarian algorithm appeared in a 1957 paper of James Munkres [117], showing that it can be implemented to run in time polynomial in n, the number of vertices. This notion of polynomial time did not immediately become a standard means for evaluating algorithms, however. In particular, the criterion was not used in the discussions of the finite algorithms for integer programming.

Jack Edmonds took charge of this issue, several years later, dramatically bringing the notion of polynomial-time algorithms and good characterizations into the hearts and minds of the research community. His efforts of persuasion began at a workshop in the summer of 1961, held at the RAND Corporation. Edmonds, working at the National Bureau of Standards in Washington, D.C., joined a group of young researchers invited to take part in the workshop together with leading figures in the field, including Dantzig, Fulkerson, Hoffman and others. His RAND lecture, and a 1963 research paper [33, 35], concerned the problem of finding optimal matchings in a general graph. Edmonds [35] writes the following.

> I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph.
>
> There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether *or not* there exists an algorithm whose difficulty increases only algebraically with the size of the graph.

Not only did this paper of Edmonds establish the basis for complexity theory, the technique he employed opened up the world of polyhedral combinatorics beyond unimodularity. The linear constraints present in the natural formulation of the matching problem do not define an integer polyhedron. Edmonds nonetheless provides a simple description of a full set of inequalities defining the convex hull of the integer points in the relaxation.

A paper of Gomory [55] has a fascinating section covering the discussion that took place after Gomory's TSP lecture at the IBM Scientific Computing Symposium on Combinatorial Problems, March 16–18, 1964. This record includes the following remarks of Edmonds, in response to a comment of Harold Kuhn.

> The algorithm I had in mind is one I introduced in a paper submitted to the Canadian Journal of Mathematics. This algorithm depends crucially on what amounts to knowing all the bounding inequalities of the associated convex polyhedron—and, as I said, there are many of them. The point is that the inequalities are known by an easily verifiable characterization rather than by an exhausting listing—so their number is not important.
>
> This sort of thing should be expected for a class of extremum problems with combinatorially special structure. For the traveling salesman problem, the vertices of the associated polyhedron have a simple characterization despite their number—so might the bounding inequalities have a simple characterization despite their number. At least we should hope they have, because finding a really good traveling salesman algorithm is undoubtedly equivalent to finding such a characterization.

The thesis of Edmonds was clear: the existence of polynomial-time algorithms goes hand-in-hand with polyhedral characterizations.

Fig. 12.12 Jack Edmonds, 2009. Photograph by Marc Uetz.

The application of Edmonds' thesis to matching problems begins, for a graph $G = (V,E)$, with the simple inequalities

$$\begin{aligned} \sum(x_e : e \text{ meets vertex } v) \leqslant 1 \quad &\text{for all vertices } v \in V, \\ x_e \geqslant 0 \quad &\text{for all edges } e \in E. \end{aligned} \tag{12.8}$$

The polyhedron P defined by this system has as vertices each incidence vector of a matching of G, but it may have non-integer vertices as well. Consider, for example, three edges f, g, and h that form a triangle in G. Setting $\overline{x}_f = \overline{x}_g = \overline{x}_h = 1/2$ and $\overline{x}_e = 0$ for all other edges e gives a vertex $\overline{x}$ of P. Such half-integer vertices can be cut off from P by the addition of inequalities

$$\sum(x_e : e \text{ has both ends in } S) \leqslant (|S| - 1)/2$$

for each set $S \subseteq V$ of odd cardinality. Edmonds calls these constraints *blossom inequalities*. His remarkable theorem is that adding these inequalities to (12.8) gives a defining system for the convex hull of matchings. Edmonds' proof is via a polynomial-time algorithm that constructs a matching and a corresponding dual solution that together satisfy the LP-duality equation.

The Chvátal closure

The method of Edmonds considers the full set of blossom inequalities in a single stroke, rather than introducing them one at a time in a cutting-plane implementation. An exquisite theory considering waves of inequalities was developed by Vašek Chvátal [17], summarized by the famous slogan

$$\boxed{\text{combinatorics} = \text{number theory} + \text{linear programming}}$$

from his paper "Edmonds polytopes and a hierarchy of combinatorial problems", published in 1973. The waves considered by Chvátal are the following. Given a

Fig. 12.13 Vašek Chvátal. Photograph by Adrian Bondy.

polyhedron P and an inequality $c^T x \leqslant \delta$, with c integral, satisfied by each of its members, each integer vector in P also satisfies $c^T x \leqslant \lfloor \delta \rfloor$, where $\lfloor \delta \rfloor$ denotes δ rounded down to the nearest integer. Let P' denote the members of P that satisfy all such inequalities. Chvátal called P' the "elementary closure" of P; nowadays it is referred to as the *Chvátal closure*. The main result of [17] is that for any bounded polyhedron P, a finite number of applications of the closure operation results in the convex hull of the integer points in P. Thus combinatorial theorems can be proved by repeatedly rounding down inequalities obtained as linear combinations of previously derived inequalities. This theory can be interpreted either in terms of *cutting-plane proofs* [20, 21] or geometrically as the *Chvátal rank* of polyhedra [133]; it provides an important connection between the polyhedral methods of Edmonds and the IP algorithm of Gomory.

Polyhedral combinatorics in the 1970s

Edmonds himself followed the matching breakthrough with a series of results, applying his polyhedral methods to spanning trees [39], branchings [37], matroid intersection [40], submodular functions [38], and, together with Ellis Johnson, the Chinese postman problem [36, 42, 43]. His leadership and amazing research moved polyhedral combinatorics into high gear. Highlights of the maturing field in the 1970s include the following projects.

Fig. 12.14 Bernhard Korte and László Lovász, 1982. Photograph courtesy of the Research Institute for Discrete Mathematics, University of Bonn.

- Ray Fulkerson [48, 49] develops his theory of blocking and anti-blocking polyhedra.
- László Lovász [105, 106] proves the weak perfect-graph conjecture.
- Egon Balas and Manfred Padberg [4, 5] study set-covering problems.
- Manfred Padberg [122], George Nemhauser and Leslie Trotter [118, 119], Vašek Chvátal [19], and Laurence Wolsey [143] study the stable-set polytope.
- Jack Edmonds and Rick Giles [41] and Alan Hoffman [76] show that total dual integrality implies primal integrality, that is, if $Ax \leqslant b$ is TDI and b is integer, then $P = \{x : Ax \leqslant b\}$ is an integer polyhedron. Further properties of TDI systems are investigated by Rick Giles and William Pulleyblank [52] and Alexander Schrijver [134].
- William Pulleyblank and Jack Edmonds [130, 129] describe the facet-defining inequalities of the matching polytope.
- Paul Seymour [138] provides a deep characterization of a certain combinatorial class of integer polyhedra and TDI systems, receiving a Fulkerson Prize in 1979.

- Jack Edmonds and Rick Giles [41] propose a general LP framework that includes a min-max theorem for directed cuts in graphs proved by Cláudio Lucchesi and Daniel Younger [109]. Other LP-based min-max frameworks are described by András Frank [47] and Alan Hoffman and Donald Schwartz [80]
- William Cunningham and Alfred Marsh, III [26] show that the blossom system for matchings is TDI. Further studies of the blossom system are made by Alan Hoffman and Rosa Oppenheim [79] and Alexander Schrijver and Paul Seymour [137].
- Paul Seymour's [139] decomposition theorem for regular matroids yields a polynomial-time algorithm to test if a matrix is totally unimodular.

This period was a golden era for polyhedral combinatorics, attracting great talent to the field and establishing a standard of quality and elegance.

Fig. 12.15 James Ho, Ellis Johnson, George Nemhauser, Jack Edmonds, and George Dantzig, 1985. Photograph courtesy of George Nemhauser.

12.6 Progress in the solution of the TSP

On the computational side, the TSP continued to lead the way in studies of combinatorial IP methods. TSP research in the 1960s and 1970s was dominated first by the work of Michael Held and Richard Karp, and later by the return of the cutting-plane method.

Dynamic programming

The straightforward enumeration algorithm for the TSP, listing all tours and choosing the cheapest, solves an n-city instance in time proportional to $n!$. Analysis of the cutting-plane method has not improved this result—the number of cuts needed in a worst-case example cannot be easily estimated. A breakthrough occurred in 1962, however, using an alternative algorithmic technique called *dynamic programming*. This method was shown by Held and Karp [69] to solve any instance of the TSP in time proportional to $n^2 2^n$.

The general approach was layed out in Richard Bellman's book *Dynamic Programming*, published by Princeton University Press in 1957 [7]. Bellman was another prominent member of the mathematical-programming group at the RAND Corporation, where he introduced the dynamic-programming technique in a 1953 technical report [6]. The subject goes well beyond its application to IP problems, encompassing general models in multistage decision making.

At roughly the same time as the Held and Karp study, dynamic programming for the TSP was also proposed by Bellman [8, 9] and R. H. Gonzales [57]. The idea used in all three projects is to build a TSP tour by computing optimal paths visiting subsets of cities, gradually increasing the number of cities covered in each path. The $n^2 2^n$ worst-case bound for the method is significantly better than $n!$, although it is still far from a practical result for instances of the size tackled by Dantzig et al. with cutting planes.

The Held and Karp paper [69] includes a computational study, giving very fast solutions for examples having 13 or fewer cities, and good approximate solutions for larger problems. An IBM press release from January 2, 1964, describes the availability of the TSP code as follows.

> IBM mathematicians (left to right) Michael Held, Richard Shareshian and Richard M. Karp review the manual describing a new computer program which provides business and industry with a practical scientific method for handling a wide variety of complex scheduling tasks. The program, available to users of the IBM 7090 and 7094 data processing systems, consists of a set of 4,500 instructions which tell the computer what to do with data fed into it. It grew out of the trio's efforts to find solutions for a classic mathematical problem—the "Traveling Salesman" problem—which has long defied solution by man, or by the fastest computers he uses.

The accompanying photograph of Held, Karp, and Shareshian is shown in Figure 12.16.

An effective branch-and-bound algorithm

Held and Karp's $n^2 2^n$ result is to this day the best known bound for general TSP algorithms. The dynamic-programming approach does not, however, extend to a practical method for large-scale instances, and the 49-city USA solution remained a computational record throughout the 1960s. At the end of the decade Held and Karp struck again, this time with a branch-and-bound method that succeeded in pushing

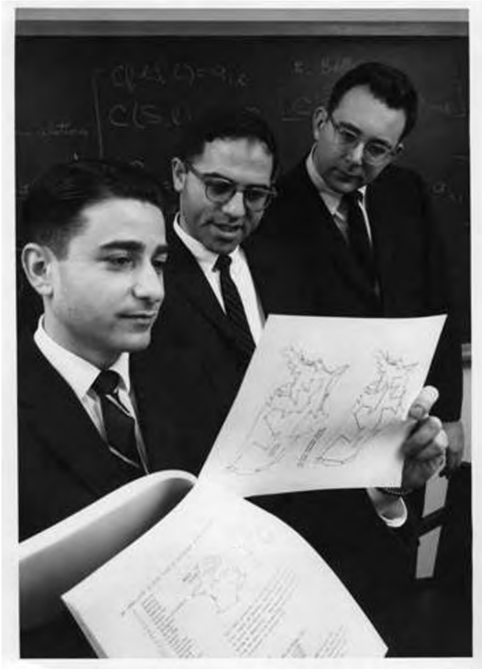

Fig. 12.16 Michael Held, Richard Shareshian, and Richard Karp, 1964. Photograph courtesy of IBM Corporate Archives.

the limits of TSP computation. Richard Karp made the following remarks on this joint work in his 1985 Turing Award Lecture [86].

> After a long series of unsuccessful experiments, Held and I stumbled upon a powerful method of computing lower bounds. This bounding technique allowed us to prune the search severely, so that we were able to solve problems with as many as 65 cities. I don't think any of my theoretical results have provided as great a thrill as the sight of the numbers pouring out of the computer on the night Held and I first tested our bounding method.

The Held-Karp bounding technique relies on an iterative approach for obtaining a good approximation to the value of the LP relaxation consisting of the degree equations and all subtour constraints, avoiding the simplex method and the cutting-plane approach. Each step of the bounding algorithm computes an optimal spanning tree for a problem obtained by deleting a single city; the value of the tree plus the cost of the cheapest two edges meeting the deleted city is a lower bound on the cost of any tour. The edge costs are adjusted after each step, according to the shape of the resulting tree. The form of the cost adjustment is to add or subtract, for each city v, a fixed value δ_v from the cost of each edge meeting v. This adjustment does not alter the TSP, but it can change the spanning tree and the implied lower bound. The spanning-tree polyhedron result of Edmonds [39] connects the subtour constraints with the tree computations, and the edge-cost adjustment accounts for dual variables on the degree equations, in a technique known as Lagrangian relaxation.

The tenacity of Held and Karp set a new standard in TSP computation. Using a computer implementation written together with Linda Ibrahim, their branch-and-bound algorithm solved a set of instances having up to 64 cities [71].

Implementing the cutting-plane method

Despite the great early success of the cutting-plane method, the approach was not really pursued as a TSP tool after the solution of the 49-city problem. Except for the publication of a RAND report by John Robacker in 1955, describing a set of tests on nine-city TSP instances, no further computations were reported with the Dantzig-Fulkerson-Johnson technique in the decade following their published result. This point is brought up in a 1964 lecture by Gomory [55].

> I do not see why this particular approach stopped where it did. It should be possible to use the same approach today, but it an algorithmic manner. We no longer have to be artistic about generating the separating hyperplanes or cuts, since this is now done automatically in integer programming. It seems likely that one can get over the difficulties of maintaining the basis as well. So it should be possible to do the whole thing now systematically. This is an approach one might not expect to work, but we already know that it does.

Saman Hong, supervised by Mandell Bellmore, responded to this call in his Ph.D. work at The Johns Hopkins University. His thesis *A Linear Programming Approach for the Traveling Salesman Problem* appeared in 1972, and reports the first fully automatic version of the cutting-plane method for the TSP.

Fig. 12.17 Saman Hong, 1971. Photograph courtesy of Saman Hong.

Hong's work uses subtour constraints together with a version of Edmonds' blossom inequalities, embedded in a combined branch-and-bound and cutting-plane approach, now called *branch-and-cut*. His computational results are modest, solving instances with up to 25 cities, but he opened the way for a renewed attack with TSP cuts.

Following Hong, the team of Martin Grötschel and Manfred Padberg took up the study of TSP cutting planes, combining to push all aspects of the technology. The pair started their effort in 1974, working at Bernhard Korte's Institut für Ökonometrie und Operations Research at the University of Bonn. They focused on the study of structural properties of the TSP polytope, including an important proof that a generalization of the *comb inequalities*, introduced by Vašek Chvátal [18], are facet defining.

Their work set the stage for a big push in TSP computation, beginning with a study by Grötschel using an instance consisting of 120 cities from West Germany. Grötschel [59] describes his method as follows.

> After every LP-run we represented the optimal solution graphically by hand on a map. In the beginning a plotter was used, but as the number of different fractional components of the solutions increased there were not enough symbols to distinguish them and the plottings became too cluttered. Using the graphical representation of the optimal solution we looked for subtour elimination constraints and comb inequalities to cut off the present solution and added them to the present constraint system. Drawing and searching took from 30 man-minutes in the beginning up to 3 man-hours after the last runs.

After thirteen rounds of the procedure, an optimal solution to the 120-city TSP was found. This work was carried out in 1975, and it is first described in Grötschel's 1977 Ph.D. thesis [58].

Manfred Padberg commented on this successful computation in his 2007 paper [123], and notes that a study together with Hong was begun hot on the heels of Grötschel's project.

> To my pleasant surprise, Martin included numerical experiments in his dissertation; he solved a 120-city traveling salesman problem to optimality which was a world record. In early 1975 I met Saman Hong, a Korean 1972 Johns Hopkins' Ph.D., in New York. We started a project to solve symmetric TSPs, by implementing an exact arithmetic, primal simplex algorithm using Don Goldfarb's "steepest edge" criteria for pivot column selection with automatic constraint generation.

The joint work of Padberg and Hong [124] was a computational success, automating the primal cutting-plane algorithm, solving instances with up to 75 cities, and computing good lower bounds on other instances. The largest example treated in the study is a 318-city instance considered earlier by Shen Lin and Brian Kernighan [103]. This instance arose in an application to guide a drilling machine through a set of holes; it is treated as a *Hamiltonian-path* problem, asking for a single path joining specified starting and ending points and covering all other cities.

Padberg was not satisfied with the good approximations for the large instances, and several years after the completion of his study with Hong he continued his pursuit for a solution to the Lin-Kernighan example. Padberg [123] comments on this effort in the following passage.

> Some day in early 1979 I approached Harlan Crowder of IBM Research with a proposal to push the exact solvability of TSPs up a notch, just like he had done earlier with Mike

Held and Phil Wolfe. He was all for it and we sat down to discuss what had to be done. It seemed feasible and so we did it. It took maybe a couple of months to string it all together. Harlan had other duties as well and I was back teaching at NYU. One evening we had it all together and submitted a computer run for the 318-city symmetric TSP. We figured it would take hours to solve and went to the "Side Door", a restaurant not far from IBM Research, to have dinner. On the way back we discussed all kinds of "bells and whistles" we might want to add to the program in case of a failure. When we got to IBM Research and checked the Computer Room for output it was there. The program proclaimed optimality of the solution it had found in under 6 minutes of computation time!

Fig. 12.18 Ellis Johnson, Tito Ciriani, Manfred Padberg, Mario Lucertini, Harlan Crowder (sitting), 1982. Photograph courtesy of Manfred Padberg.

The Crowder-Padberg study [25] concluded with the solution to the 318-city instance and a large collection of smaller examples. This was a triumph of the cutting-plane method and the long-term research efforts of Grötschel and Padberg.

12.7 Widening the field of application in the 1980s

In the twenty-five years following Dantzig-Fulkerson-Johnson, the few computational studies with the cutting-plane method focused on efforts at solving instances of the TSP. The landscape changed dramatically, however, in the 1980s. By the end of the decade the technique was in wide use, covering numerous fundamental models as well as applied problems from business and industry.

The transition away from the confines of the TSP was led by a focused effort of Martin Grötschel, Michael Jünger, and Gerhard Reinelt, with their study of the linear-ordering problem [61, 62, 63, 84, 131]. Over a period of four years, the team managed to duplicate the scope of research that had evolved for the TSP over the previous two and half decades. In a single project, Grötschel et al. formulated an

LP relaxation, carried out a polyhedral study, developed a class of potential cutting planes, created efficient separation algorithms to produce cuts, implemented a branch-and-cut framework, gathered problem data, and carried out a large-scale computational test. This work set a standard for future studies and demonstrated that the algorithmic success with the TSP was not an isolated event.

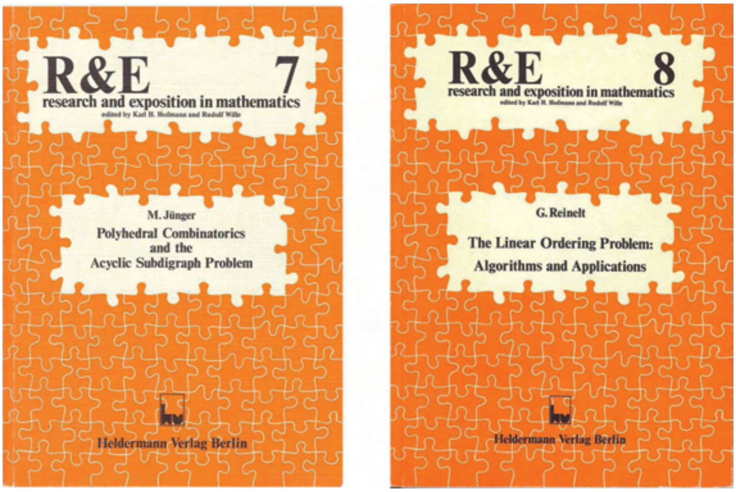

Fig. 12.19 Ph.D. theses of Michael Jünger and Gerhard Reinelt, 1985.

Fig. 12.20 Michael Jünger, Martin Grötschel, Jeff Edmonds, Yoshiko Wakabayashi, Mario Sakumoto, and Gerhard Reinelt, providing input to order a selection of beers, 1984. Photograph courtesy of Gerhard Reinelt.

The linear-ordering problem

Fresh from theoretical and computational work with the TSP, Grötschel began a study with Ph.D. students Jünger and Reinelt, aiming to apply the TSP lessons to a new problem area. The team began their work in 1981 at the University of Bonn, moving in 1983 to the University of Augsburg. The *linear-ordering problem* they consider is defined as follows. Let $D = (V,E)$ be a complete directed graph with weights $(w_e : e \in E)$ on the edges. The problem is to find an ordering of the vertices $v_1, v_2, \ldots, v_n$ that maximizes the sum of the weights of the edges that are directed consistent with the ordering, that is, from a vertex lower in the order to a vertex higher in the order. The ordering can be thought of as a ranking of the vertices, with the weight w_e, for directed edge $e = (u,v)$, giving the payoff for ranking u before v. The problem was formulated by Bernhard Korte and Walter Oberhofer [89, 90] in the late 1960s; applications are described in Jan Karel Lenstra's 1977 thesis [101]. The Grötschel et al. study provides a complete package of tools for solving real-world instances arising in a variety of settings.

In the years following the linear-ordering project, cutting-plane research entered a period of rapid growth, with studies covering a wide range of models. A 1994 survey paper by Jünger, Reinelt, and Stephan Thienel [85] lists twenty-one cutting-plane projects carried out by various research teams.

The breadth of this cutting-plane work was aided by the development of high quality of LP solvers such as Robert Bixby's CPLEX code, making it much easier to experiment with combinatorial IP techniques. Bixby [13] describes this point as follows.

> What was needed was a numerically robust code that was also flexible enough to be embedded in these integer programming applications. It had to be a code that made it easy to handle the kinds of operations that arose in a context in which it was natural to begin with a model instantiated in one form followed by a sequence of problem modifications (such as row and column additions and deletions and variable fixings) interspersed with resolves. These needs were among the fundamental motivations behind the development of the callable-library version of the CPLEX code.

The growth and availability of such flexible LP codes went hand-in-hand with the expansion of the cutting-plane method.

Advancing the TSP

During these expansion years, the TSP was certainly not left behind. Indeed, two large-scale computational projects were initiated by Padberg and Giovanni Rinaldi [126, 127, 128] and by Grötschel and Olaf Holland [82, 60]. The important goal of these studies was to assess whether the performance of the cutting-plane method could be substantially enhanced by digging deeper into the polyhedral structure of problem classes and by considering more sophisticated computational tools available in branch-and-cut implementations. The spectacular success of the two studies demonstrates that this is indeed the case. Among the computational achievements were the solution of a 666-city world TSP instance by Grötschel and Holland and the solution of a 2,392-city circuit-board drilling problem by Padberg and Rinaldi. The term “branch-and-cut” was first used in the Padberg-Rinaldi study, and their work introduced important components that continue to be used in modern implementations of the solution scheme.

Fig. 12.21 Giovanni Rinaldi, Michele Conforti, Monique Laurent, M.R. Rao, and Manfred Padberg, 1985. Photograph courtesy of Monique Laurent.

12.8 Optimization $\equiv$ Separation

The most widely circulated news event in the history of mathematical programming occurred in the summer of 1979, when Leonid Khachiyan [88] published a polynomial-time algorithm for solving linear-programming problems. The story was covered on the front page of the *New York Times* and in newspapers around the world. Khachiyan’s work made use of the ellipsoid method for convex programming developed by David Yudin and Arkadi Nemirovski [144], and the papers [88] and [144] were jointly awarded a Fulkerson Prize in 1982.

In the introduction to the current paper, we took care to state that LP duality is used to obtain bounds on combinatorial problems, rather than assuming one could actually solve a given relaxation. With Khachiyan’s result this is no longer an issue in theoretical work, since the ellipsoid method can deliver the required optimal LP value. The real power of the result goes well beyond this, however, as discovered by Martin Grötschel, László Lovász, and Alexander Schrijver [64], Richard Karp and Christos Papadimitriou [87], and Manfred Padberg and M. R. Rao [125]. Each of these teams showed that subject to some modest technical conditions, the ellipsoid

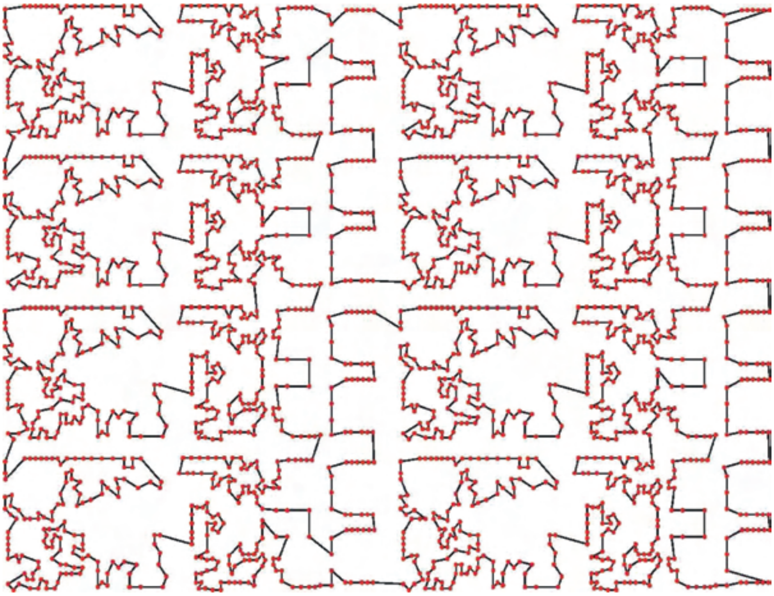

Fig. 12.22 Padberg and Rinaldi's 2,392-city TSP.

method can be used to prove that the problem of optimizing a linear function over a rational polyhedron P is polynomial-time equivalent to the separation problem for P, that is, given a vector $\bar{x}$, either show that $\bar{x} \in P$ or find a linear inequality $c^T x \leqslant \beta$ that is satisfied by all points in P but violated by $\bar{x}$. A particularly sharp version of this equivalence is derived in the Grötschel et al. paper [64], and it too was awarded a Fulkerson Prize in 1982.

The optimization $\equiv$ separation result has wide-reaching applications in combinatorial integer programming, giving a precise algorithmic realization of Jack Edmonds' appeal for linking polynomial-time algorithms and polyhedral descriptions. Recall the quote of Edmonds concerning a characterization of the TSP polytope: "finding a really good traveling salesman algorithm is undoubtedly equivalent to finding such a characterization." As with many other aspects of combinatorial optimization, Edmonds' insight was right on the money. The ellipsoid method tells us that what is needed to solve the TSP is a polyhedral characterization yielding a polynomial-time separation algorithm. The study of separation algorithms for classes of combinatorial problems, and for particular classes of inequalities, is now a standard part of the field, in both practical and theoretical research.

The optimization $\equiv$ separation paradigm is the central theme of a beautiful monograph *Geometric Algorithms and Combinatorial Optimization* by Grötschel et al. [66], published in 1988. This work intertwines combinatorics, polyhedra, and the geometry of numbers, to produce polynomial-time algorithms and deep insights into a host of combinatorial optimization problems. The monograph is on a very short list of must-read books for any student of integer programming.

Fig. 12.23 Alexander Schrijver, László Lovász, and Martin Grötschel, Amsterdam, 1991. Photograph courtesy of Martin Grötschel.

The discovery and elaboration of optimization $\equiv$ separation is a crowning achievement in combinatorial integer programming and it might well be viewed as marking the end of the initial development phase of the field.

12.9 State of the art

The applied and theoretical work of the 1980s brought to fruition the early visions of Dantzig-Fulkerson-Johnson, Edmonds, Gomory, Hoffman, Kuhn, and others. The accomplishments of that decade set the stage for the now mature field of combinatorial integer programming, where deep theoretical questions and ever more complex practical computations drive the growth of the discipline.

At this point in the narrative we cannot hope to do justice to the wide range of activities being carried out by the research community, and we refer the reader to state-of-the-art surveys included in this volume. We limit our discussion to several highlights that are representative of the overall advancement of the field.

Balanced matrices and perfect graphs

Claude Berge, a pioneer in both graph theory and optimization, was the catalyst for two major results in the late 1990s and early 2000s. Both studies concern the integrality of polyhedra and are thus central to the theme of combinatorial integer programming.

The first of the two results is the 1999 publication of an algorithm for decomposing and recognizing balanced matrices by Michele Conforti, Gérard Cornuéjols, and M. R. Rao [22]. Berge [11, 12] introduced this class of $0/1$ matrices in the early 1970s, generalizing the notion of a bipartite graph. Matrix A is called *balanced* if it does not contain a square submatrix of odd order having exactly two ones in every row and exactly two ones in every column. If A is balanced, then both $Ax \leqslant \mathbf{1}, x \geqslant \mathbf{0}$ and $Ax \geqslant \mathbf{1}, x \geqslant \mathbf{0}$ are totally dual integral systems [50]. The definition of the class tells us which matrices are not balanced. The achievement of Conforti et al. was to answer the other natural question, showing which matrices are in fact balanced. Their study was awarded a Fulkerson Prize in 2000.

Fig. 12.24 W. Cunningham, A. Schrijver, M. Laurent, B. Gamble, F. B. Shepherd, D. Williamson, A. Hoffman, C. De Simone, D. Shmoys, J. Geelen, J. Kleinberg, S. Fekete, M. Goemans, M. Conforti, G. Cornuéjols, and A. Gerards. Bellairs Research Institute, March 1995. Photograph courtesy of David Williamson.

The second major result was the proof of Berge's [10] strong perfect-graph conjecture by Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas [16]. Perfection is defined in terms of the stable sets and cliques of a graph. A clique in $G = (V,E)$ is a set $C \subseteq V$ such that each pair of vertices in C is joined by an edge in E; the clique-covering number of G is the minimum number of cliques covering all vertices, that is, each $v \in V$ is a member of a least one of the cliques. If G has a stable set of cardinality k, then the clique-covering number of G must be at least k.

Graph G if called *perfect* if for every induced subgraph H of G the clique-covering number of H is equal to the cardinality of its largest stable set. It follows that if we let A denote the clique-vertex incidence matrix of G, then G is perfect if and only if for each $0/1$ vector $w = (w_v : v \in V)$ the optimal values in the LP duality equation

$$\max(w^T x : Ax \leqslant \mathbf{1}, x \geqslant \mathbf{0}) = \min(y^T \mathbf{1} : y^T A \geqslant w^T, y \geqslant \mathbf{0})$$

are attained by $0/1$ vectors. This is just a reinterpretation of the definition, but it hints at the result that G is perfect if and only if $\{x : Ax \leqslant \mathbf{1}, x \geqslant \mathbf{0}\}$ is an integer polyhedron [49, 105, 19]. This close connection with integer programming is one of the drivers of the great interest in the study of perfection.

The strong perfect-graph conjecture states that a graph is perfect if and only if neither it or its complement contains as an induced subgraph an odd circuit having at least five edges. The simplicity of this possible characterization drew considerable attention in the forty years after Berge proposed the problem in 1960, leading to a great body of work in the literature. This research culminated in the Chudnovsky et al. proof, announced in May 2002, just one month before Berge passed away. Their published version runs 179 pages in the *Annals of Mathematics* and it is one of the great achievements in graph theory and polyhedral combinatorics. An interesting account of the steps and missteps along the way to the final proof can be found in Seymour [140].

Fig. 12.25 Robin Thomas, Paul Seymour, Neil Robertson and Maria Chudnovsky signing copies of their proof of the strong perfect graph conjecture, November 1, 2002. Photograph courtesy of the American Institute of Mathematics.

With the conjecture now a theorem, the characterization was used to obtain a polynomial-time recognition algorithm for perfect graphs [15]. Thus three important classes of matrices, totally unimodular, balanced, and perfect, all are recognizable in polynomial time.

This area of polyhedral combinatorics is well-developed, but many interesting open questions remain. A nice treatment can be found in the book of Cornuéjols [23], including offers of cash rewards of $5,000 for each of eighteen conjectures. Of this potential $90,000, the teams Chudnovsky-Seymour and Chudnovsky-Robertson-Seymour-Thomas have thus far collected $25,000, leaving plenty of money on the table for future work.

Semidefinite programming

One of the important applications of the ellipsoid method in the Grötschel et al. [64, 65, 66] study is the development of a polynomial-time algorithm to find a maximum-weight stable set in a perfect graph. Building on earlier work of Lovász [107] concerning the Shannon capacity of graphs, Grötschel et al. consider a convex relaxation of the stable-set problem involving a matrix of variables that must be symmetric and positive semidefinite. The optimization $\equiv$ separation theory provides a polynomial-time algorithm to optimize over this non-polyhedral set, and in the case of perfect graphs the relaxation coincides with the stable-set polytope.

The semidefinite-relaxation idea was further developed by Lovász and Schrijver [108] in a hierarchy of relaxations for $0/1$ integer-programming problems. This framework was shown to have particularly interesting consequences for the study of stable-set polytopes beyond the class of perfect graphs.

The Lovász-Schrijver study, in turn, generated interest in the model of semidefinite programming (SDP). Here linear programming is extended by replacing the standard vector of variables by a symmetric and positive semidefinite matrix. The interest in SDP models was heightened by two additional developments in the early 1990s. First, Farid Alizadeh [1] and Yurii Nesterov and Arkadi Nemirovski [120, 121] showed that LP interior-point methods could be extended to semidefinite programming, providing an efficient practical tool for solving these models. A nice description of this work can be found in Todd [141]. Second, Michel Goemans and David Williamson [53] utilized SDP relaxations in their breakthrough result on the max-cut problem in graphs, yielding a strong new approximation algorithm; their result was awarded a Fulkerson Prize in 2000.

With these applications and algorithms in place, the past decade has seen the SDP area grow by leaps and bounds. Henry Wolkowicz [142] lists 1,060 references in an online SDP bibliography; IP-related SDP work is covered in surveys by Christoph Helmberg [73] and Monique Laurent and Franz Rendl [100]. SDP methods are now an exciting tool in the study of approximation algorithms and in the study of lower bounds for combinatorial optimization problems.

Schrijver's Meisterwerk

A milestone in combinatorial integer programming was reached in 2003 with the publication of Alexander Schrijver's three-volume monograph *Combinatorial Optimization: Polyhedra and Efficiency* [135]. The breadth and depth of coverage in the monograph is breathtaking, as are Schrijver's historical treatments. The volumes total 1,881 pages and include over 4,000 references. The importance of this work can hardly be overstated. Schrijver's beautiful scholarly writing has defined the field, giving combinatorial optimization an equal footing with much older, more established areas of applied mathematics. His monograph received the 2004 Lanchester Prize.

The following quote from Schrijver's preface emphasizes the role of polyhedral methods in the broad study of combinatorial optimization.

> Pioneered by the work of Jack Edmonds, polyhedral combinatorics has proved to be a most powerful, coherent, and unifying tool throughout combinatorial optimization. Not only has it led to efficient (that is, polynomial-time) algorithms, but also, conversely, efficient algorithms often imply polyhedral characterizations and related min-max theorems. It makes the two sides closely intertwined.

This connection will undoubtedly continue, advancing both combinatorial optimization and general integer programming.

Fig. 12.26 Alexander Schrijver, 2007. Photograph copyright Wim Klerkx, Hollandse Hoogte.

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