Complexity of Columns variant

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1 Introduction

Over the past years, there has been a lot of interest in the complexity of the so called "casual games".

This trend started in 2002 with a paper about the complexity of the game Tetris [1]

Also, recently there have been papers studying the complexity of famous match three games bejeweled [2] and candy crush [3]

Yet, there is no information about the complexity of the game Columns, the originator of the match three kind of games.

We give partial results in this direction by proving that a restricted variant of the game is np-hard

2 Description

Columns was a game introducided in 1990 to compete with the popular Tetris. It was the originator of the match three kind of games.

In the game, we are presented with pieces that are formed by three jewels and which we can rotate inside the piece, we cant rotate the pieces, only translate them and change the position of the jewels inside the piece.

The objective is to avoid the pieces to reach the top of the board. To do that, we must eliminate jewels by matching three or more jewels of the same kind in horizontal, vertical or diagonal orientation.

There are six types of jewels in columns we will refer to them by their colours: red (r), orange (o), green (g), sapphire(s), "brown"(b) and yellow(y).

In the variant of the game we are presenting now, we can only translate the pieces, not change the position of the jewels inside them . Also, in this variant of the game, we can see all the remaining pieces of the game, is an offline variant

3 Description of the reduction

We are going to prove the np hardness of the problem of clearing a particular board in columns by reducing the problem 3-partition to this problem

4 The 3-PARTITION problem

The 3-Partition problem can be defined as follows:

Given a multiset A of positive numbers a_1,...,a_3s and a positive value B such that $B/4 < a_1 < B/2$ for all $1 \leq i \leq 3s$ such that $\sum_{i=1}^{3s} a_i = sB$. Can A be divided into s disjoint subsets A_1,...,A_s such that $\sum_{a_i \in A_j} = B$ for all $1 \leq j \leq s$

Theorem 1. *(Garey and johnson [4]) 3-PARTITION is np-complete in the strong sense.*

5 Reduction

We start with an empty board, this board will have a width of $8 \times s + 5$.

This board is divided in sections, each section represents a subset.

In the three middle rows of the section there are planted pieces, and a space of five between this bucket and the one on the left and the one on the right.

Every bucket is composed of a base of height $B \times 6 + 24$ where B is the number that each subset should add to.

This base is formed by blocks that are composed by orange, brown and yellow pieces for each number in $B + 4$, represented in unary.Now we show a figure representing a block of the bottom part:

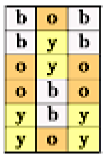


Figure 1: Example of a block that forms part of the base, these blocks are stacked on top of each other for every digit of $B + 4$ represented in unary

The purpose of the base is simply to make sure that the middle part can only be cleared by the top

After the base, there is the middle part.

The middle part is on top of the base part and is formed by the following block:

Figure 2: The middle part

This part of the bucket has a height of thirteen

Finally, in top of the middle part, we have the following blocks, that form part of the top part.

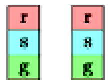


Figure 3: Example of a block that belongs to the top part

These blocks are repeated and stacked on top of another for each digit of the number $B + 4$, written in unary, for a total height of this part of the bucket of $B \times 3 + 4 \times 3$

We have just described the board, below, there is an example of the board for $S = 18$, $B = 6$ and $s = 3$.

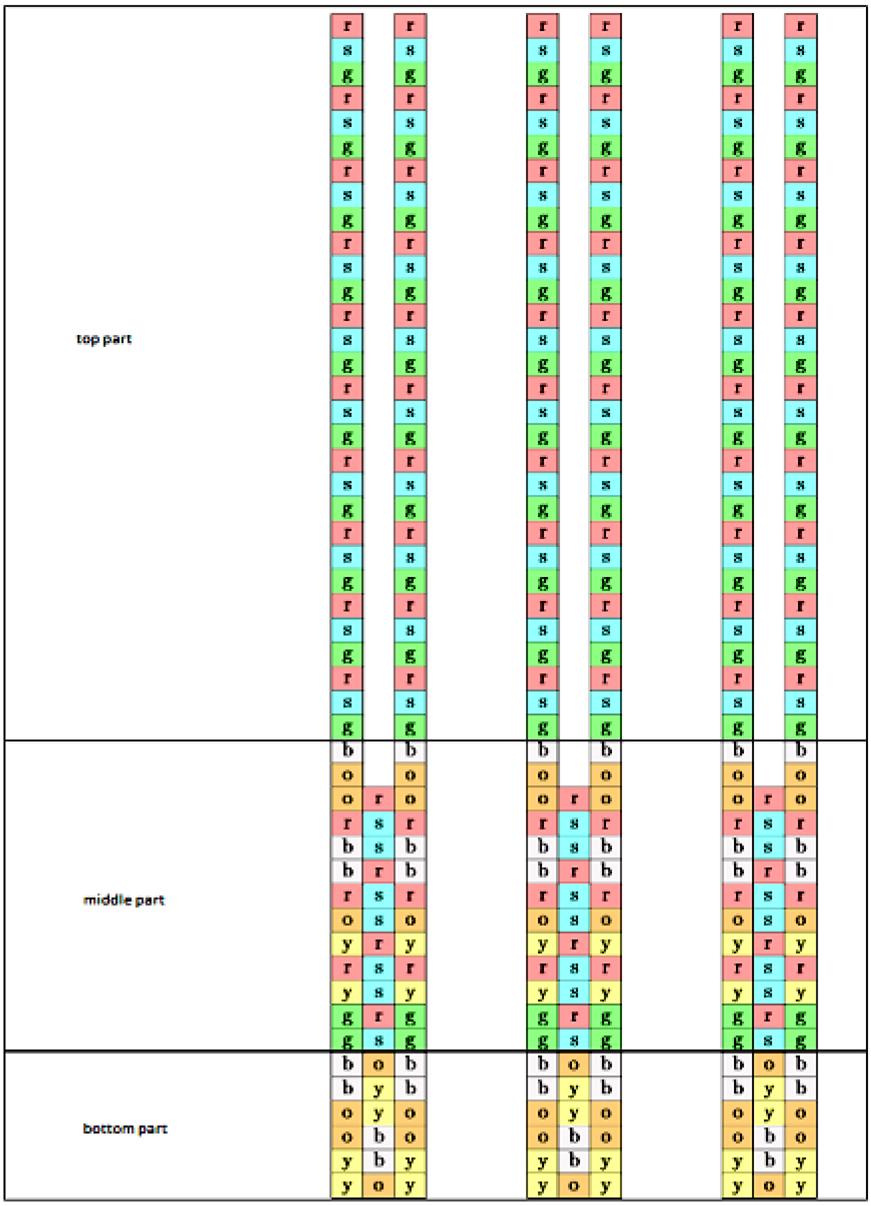


Figure 4: Example of an initial board,the bottom part is truncated

The board of the figure 4 corresponds to the multiset $\{3, 4, 2, 1, 1, 1, 2, 2, 2\}$.
A possible solution to this problem is $\{3, 2, 1\}$,$\{4, 1, 1\}$,$\{2, 2, 2\}$

Now, we are going to describe the pieces that we will use for the reduction.

First, we receive an orange piece and a brown piece, with these pieces we must decide in which bucket (subset) we want to put the first number

This is the orange piece:

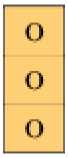


Figure 5: All orange piece

Is formed by three orange jewels

And this is the brown piece:

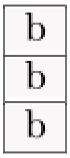


Figure 6: All brown piece

We put these pieces in a bucket to open it in the following way:

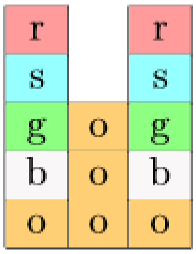


Figure 7: Clearing orange pieces

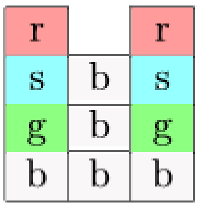


Figure 8: Clearing brown pieces

This clears the orange and brown pieces,opening the bucket, and allows us to put other pieces in that bucket

Then we receive a_1 pieces for the first number, we must use these pieces in the bucket that we have just opened

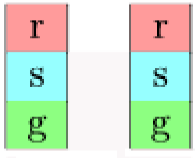


Figure 9: The bucket is open

These pieces are formed by a red jewel a sapphire jewel and a green jewel

To do that, we simply put the pieces in the bucket that we have just opened as seen in the figure:

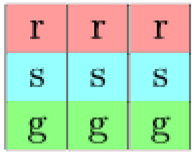


Figure 10: Clearing a block of the top bucket

We repeat this process for each digit of a_1 written in unary. In our example we will repeat this process four times, one for each digit of four written in unary and the left bucket will have its height reduced by twelve blocks:

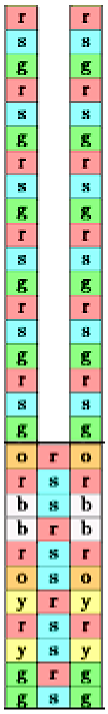


Figure 11: After clearing four blocks of the top part, the left column has lost height

After placing the number inside the bucket we receive a piece formed by a sapphire jewel a sapphire jewel and a green jewel:

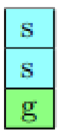


Figure 12: We receive this piece at the end of every number

With this piece we must clear part of the pieces of the middle part of the bucket we have opened, and close the bucket at the same time.

To do that, we simply put the piece in that bucket:

r	s	r
s	s	s
g	g	g
o	r	o
r	s	r
b	s	b

We clear the bottom two lines and now two red pieces and a sapphire piece fall down:

r		r
s		s
g		g
r	s	r
o	r	o
r	s	r
b	s	b

The red pieces at the diagonal are cleared:

r		r
s		s
g		g
	s	
o		o
	s	
b	s	b

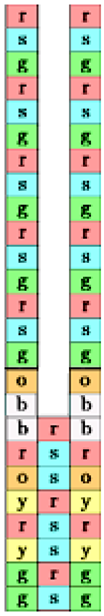
The sapphire pieces fall down:

r		r
s		s
g	s	g
o	s	o
b	s	b

Clearing a part of the middle section and closing the bucket:

r		r
s		s
g		g
o		o
b		b

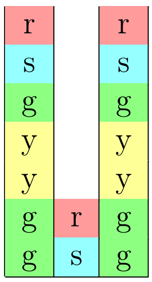
Here we see the state of the left column after this:



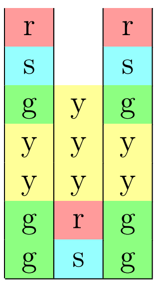
As can be seen in the picture, a part of the middle has been cleared, and five blocks from the top have been cleared

We repeat this process for each element that belongs to A, placing the pieces in the corresponding buckets. At the end of this sequence we receive s yellow pieces, one for each subset, with these pieces we must clear the remaining pieces of the top and the middle part of the buckets, as seen in the following pictures.

This is what remains to be cleared of each column at the end of the sequence of numbers.



To do that, we put the all yellow piece, and we put it in the middle of the bucket as seen in the picture:



The yellow pieces are cleared, making the pieces in the top fall down:

r		r
s		s
g		g
g	r	g
g	s	g

The green pieces are cleared, making the rest of the pieces fall down:

r	r	r
s	s	s

With this the top and middle parts are cleared

The only remaining blocks are those that correspond to the bottom of the bucket

To clear them we follow these steps:

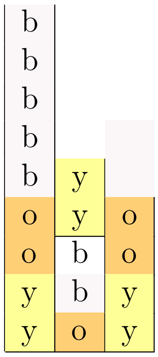
For the first block of a column, we receive an all orange, piece formed by three orange jewels. We receive this piece only for the first block of every column. We must put the piece in the first block of the column we want to clear in the following way:

	o	
	o	
	o	
b	o	b
b	y	b
o	y	o
o	b	o
y	b	y
y	o	y

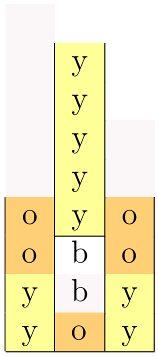
Then for every block of a particular column we follow this sequence:

First we receive two brown pieces, formed by three brown jewels, we can put the pieces like this:

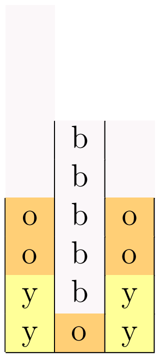
		b
		b
		b
b		b
b	y	b
o	y	o
o	b	o
y	b	y
y	o	y



Now we receive a yellow piece formed by three yellow pieces, me put this piece in top of the yellow ones:



We receive another brown piece, with this piece we finish with the remaining brown pieces of the block:



Now we put an all orange piece to end with the remaining orange pieces of the block, this also clears the orange piece on the top of the next block, if there is any block remaining on the column:

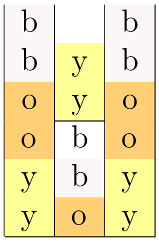
o	o	o
o	o	o
y	o	y
y	o	y
b	o	b
b	y	b
o	y	o
o	b	o
y	b	y
y	o	y

Finally we clear the remaining yellow pieces on the block, to do that, we receive two yellow pieces.

These pieces are formed by three yellow jewels. With these pieces we can clear the yellow jewels in the following way:

y		
y		
y		
y		y
y		y
b		b
b	y	b
o	y	o
o	b	o
y	b	y
y	o	y

		y
		y
		y
		y
		y
b		b
b	y	b
o	y	o
o	b	o
y	b	y
y	o	y



And with this we have finished clearing the block. We repeat the sequence of movements for every column, to clear the bottom part.

Is possible that the player can find a better way to clear the bottom part using these pieces, but that doesnt change the correctness of the reduction, because the player will never we able to clear pieces in the top part using pieces of the bottom part.

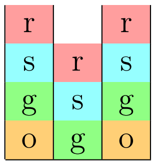
6 Correctness of the reduction

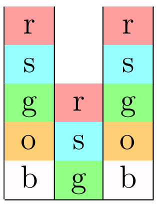
In this section we are going to prove that a multiset that cannot be satisfied, maps to an instance of columns in which the board cannot be cleared

We will do this through a series of lemmas.

Lemma 1. *If we put a number piece in a closed bucket, we will close that bucket forever*

Proof. If we put a number piece in a closed bucket, that bucket will have a piece that cannot be cleared blocking the bucket:



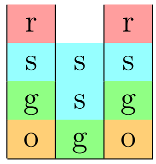


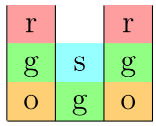
And we will not be able to clear this bucket later on

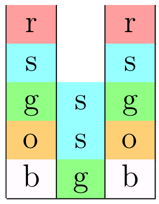
□

Lemma 2. *If we put an ending piece in a closed bucket, we will close that bucket forever*

Proof. If we put an ending piece in a closed bucket, that bucket will have a piece that cannot be cleared blocking the bucket:







And we will not be able to clear this bucket later on

□

Lemma 3. *We only can clear blocks of the bottom part by placing vertical pieces on top of them*

Proof. We cannot clear pieces of the bottom part by placing pieces adjacent to the block because there are no two contiguous pieces of the same kind horizontally or diagonally in that block □

Lemma 4. *If we try to clear a block of the top part in other way than the ways specified here, we will not be able to clear the board*

Proof. Notice that we have just enough pieces to clear the top part of the board. If we try to clear a block from the top by putting adjacent pieces to the left and the right, we will need four pieces instead of one (two to clear the piece on the left, and two to clear the piece on the right) □

Lemma 5. *If we try to maintain two buckets open at the same time, we will not be able to clear the board*

Proof. After we open a bucket, we receive a series of pieces for a number, we should place these pieces in the bucket that we have just opened as we have seen in lemmas one and two.

When we finish this series of pieces, we receive a piece to close the bucket, in order to maintain this bucket open, we must discard this piece. This piece is necessary to clear a part of that bucket.

As we have just enough pieces to clear the top part, we will not be able to clear the board □

Lemma 6. *We need at least three "pieces from the top" to clear the middle part of a bucket*

Proof. When we say "top pieces", we are refering to pieces that form the top part, pieces that belong to a sequence of numbers (pieces formed by a green jewel at the bottom, a sapphire jewel in the middle and a red jewel on top) and "ending pieces" (The pieces that we use to close a bucket. These pieces are formed by a green jewel at the bottom, a sapphire jewel in the middle and a sapphire jewel on top)

The middle part of the bucket has red and sapphire jewels on it that we must clear to in order to clear the middle part.

To do that, we need pieces that contain red or sapphire jewels, the only pieces that contain these jewels are "top pieces" and "ending pieces".

To clear the red or sapphire jewels, we must put in contact red or sapphire jewels.

We cant put in contact red or sapphire jewels directly because there are green jewels blocking the pieces,and as we cant cycle the positon of the pieces, we need to clear these green pieces.

To do that we need to put in contact three or more green jewels to clear them, so we need at least three "pieces from the top" □

Theorem 2. *Clearing the board in this "columns" variant is NP-hard*

References

[1] Ron Breukelaar, Erik D. Demaine, Susan Hohenberger, Hendrik Jan Hoogeboom, Walter A. Kosters, and David Liben-Nowell. *"Tetris is Hard, Even to Approximate"*. International Journal of Computational Geometry and Applications, volume 14, number 1–2, 2004, pages 41–68.

[2] Luciano Gualà, Stefano Leucci, Emanuele Natale. *"Bejeweled, Candy Crush and other Match-Three Games are (NP-)Hard"*. arXiv preprint arXiv:1403.5830.

[3] Toby Walsh. *"Candy crush is np-hard"*. arXiv preprint arXiv:1403.1911.

[4] Garey, Michael R.; Johnson, David S. *"Computers and Intractability: A Guide to the Theory of NP-Completeness"*. W. H. Freeman, ISBN 0-7167-1045-5.