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Boosting Exponential Gradient Strategy for Online Portfolio Selection: An Aggregating Experts' Advice Method

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Abstract

Online portfolio selection is one of the fundamental problems in the field of computational finance. Although existing online portfolio strategies have been shown to achieve good performance, we always have to set the values for different parameters of online portfolio strategies, where the optimal values can only be known in hindsight. To tackle the limits of existing strategies, we present a new online portfolio strategy based on the online learning character of *Weak Aggregating Algorithm* (WAA). Firstly, we consider a number of *Exponential Gradient* (EG(η)) strategies of different values of parameter η as experts, and then determine the next portfolio by using the WAA to aggregate the experts' advice. Furthermore, we theoretically prove that our strategy asymptotically achieves the same increasing rate as the best EG(η) expert. We prove our strategy, as EG(η) strategies, is universal. We present numerical analysis by using actual stock data from the American and Chinese markets, and the results show that it has good performance.

Keywords Online portfolio selection · Universal portfolio · Online expert advice · Weak aggregating algorithm

1 Introduction

Portfolio theory focuses on how to optimize the allocation of wealth across a set of assets. Markowitz (1952) proposed the *Mean-Variance* (MV) model for portfolio selection and opened a new era of quantitative analysis in the field of portfolio selection. For the MV model, the return is measured by the mean value and the risk is quantified by the variance of the portfolio return, respectively. The MV model aims to maximize the return for a given level of risk or minimize the risk for a given level of return. Based

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on Markowitz's work, different extensions of the MV model have been proposed (Zhou and Li 2000; Li and Ng 2000; Cui et al. 2012; Li et al. 2012; Cong and Oosterlee 2016; Kim et al. 2016).

The above MV model assumes the asset prices follow certain probability distributions, and the parameters are known. However, it is difficult to find suitable probability distributions to describe asset prices; even if found, it is difficult to accurately estimate the parameters. Thus, the MV model is not practical enough. To overcome the shortcomings of MV model, Cover (1991) proposed *Universal Portfolio* (UP) strategy that does not make any statistical assumptions about the future asset price. This kind of dynamic investment strategy with no assumption of statistical information is an online adjustment strategy, which is called online portfolio strategy. Learned from online learning, artificial intelligence and signal processing, online portfolio theory has constructed many well-performing strategies and obtained rich research results (Levina and Shafer 2008; Kozat and Singer 2009; Györf et al. 2012).

Different experts hold different views on the future performance of assets, so different experts will provide different advice for the same set of assets. In recent years, the online sequence decision algorithms by aggregating experts' advice have been further developed, which provide a new idea for the online portfolio research with comprehensive consideration of a series of experts' advice (Levina et al. 2010; Zhang and Yang 2016; Lin et al. 2017). Based on this idea, Zhang and Yang (2017) considered the *Constant Rebalanced Portfolios* (CRPs) as expert strategies, and presented a novel online portfolio strategy WAAC by using the WAA to aggregate experts' advice. However, the expert strategies are static, which adopt the same portfolio for each period. Financial market is an endlessly changing system, so considering static experts' advice doesn't seem realistic enough. In this paper, dynamic online experts are considered during constructing an online portfolio strategy. Specifically, we consider a class of Exponential Gradient strategies (EG(η)) of different values of parameter η as experts, apply the WAA to aggregate their advice, and propose an online portfolio strategy. More importantly, it is theoretically proved that the proposed strategy is a universal portfolio which exhibits the same asymptotic growth rate in the final cumulative logarithmic wealth as BCRP. The results of numerical analysis further verify that the proposed strategy has good competitive performance in different financial markets.

The rest of this paper is organized as follows. Section 2 formally formulates the online portfolio selection problem. Section 3 reviews the benchmarks, analyzes related works and describes the WAA. Section 4 proposes an online portfolio strategy and demonstrates its competitive performance. Section 5 presents and analyzes experiment results of our strategy. Section 6 concludes this paper.

2 Problem Setting

In this section, we define some related notations and formulate the online portfolio selection problem. We assume that an investor allocates his/her capital among m assets for n trading periods. To describe the changes of financial market, let $\mathbf{x}_t=(x_{t,1}, x_{t,2}, \ldots, x_{t,m})^\top \in \mathbf{R}_+^m$ denote the price relative vector on the t-th period, where $x_{t,i}$ represents the ratio of closing price to opening price of asset i on the t-th period ($1 \le t \le n$). We denote a price relative vector sequence by $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, and abbreviate as $\mathbf{x}^n$.

At the beginning of the t-th period, the investor allocates his/her capital among m assets corresponding to a portfolio vector $\mathbf{b}_t$=$(b_{t,1}, b_{t,2}, \ldots, b_{t,m}) \in \mathbf{R}_+^m$, where $b_{t,i}$ represents the proportion of wealth assigned to asset i on the t-th period ($1 \le t \le n$). Generally, we follow the typical assumption that the portfolio is self-financed and no margin/short is allowed. The cumulative return obtained at the end of the t-th period will be completely reinvested at the beginning of the $(t+1)$-th period and no additional wealth can be taken into the portfolio. The set of all possible portfolio vectors is denoted by Δ_m, where $\Delta_m = \{\mathbf{b} = (b_1, b_2, \ldots, b_m) : b_i \ge 0, \sum_{i=1}^m b_i = 1\}$. We denote a portfolio strategy for n trading periods by $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, and abbreviate as $\mathbf{b}^n$. At the very start, we usually assume the portfolio is uniformly allocated, that is, $\mathbf{b}_1 = (1/m, 1/m, \ldots, 1/m)$. On the t-th period, if the portfolio $\mathbf{b}_t$ is adopted, when the price relative vector $\mathbf{x}_t$ occurs at the end of the t-th period, the wealth increases by a factor of $\mathbf{b}_t \cdot \mathbf{x}_t = \sum_{i=1}^m b_{t,i} x_{t,i}$. After n trading periods, the final cumulative wealth of a portfolio strategy $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is achieved by a price relative vector sequence $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$. The final cumulative wealth S_n increases by the factor of

$$S_n = S_0 \prod_{t=1}^{n} \mathbf{b}_t \cdot \mathbf{x}_t = S_0 \prod_{t=1}^{n} \sum_{i=1}^{m} b_{t,i} x_{t,i}, \tag{1}$$

where S_0 denotes the initial wealth. Usually, S_0 is set to 1.

Online portfolio selection problem is a sequential decision-making problem. The investor is an online decision maker who updates the portfolio at the beginning of each period. We summarize the above procedure and formulate the whole online portfolio selection process to construct the portfolio strategy in Algorithm 1.

Algorithm 1 Online portfolio selection framework

Input: $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$: Historical daily price relative vector sequence.
Output: S_n : Final cumulative wealth.
1: Initialize $S_0 = 1, \mathbf{b}_1 = (1/m, 1/m, \ldots, 1/m)$.
2: **for** $t = 1, 2, \ldots, n$ **do**
3: Receive the price relative vector: $\mathbf{x}_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,m})^\top$.
4: Update the cumulative return until the t-th period: $S_t = S_{t-1} \times (\mathbf{b}_t \cdot \mathbf{x}_t)$.
5: Calculate the next portfolio: $\mathbf{b}_{t+1}$.
6: **end for**

3 Related Works

3.1 Benchmarks

Buy-and-hold (BAH) strategy is the most common benchmark strategy that allows the investor to make an initial portfolio among m assets and then leave all his/her holdings untouched during the subsequent trading periods. In particular, the BAH strategy with uniform initial portfolio, i.e., $\mathbf{b}_1 = (1/m, 1/m, \ldots, 1/m)$, is referred to uniform BAH strategy, which is usually adopted as market strategy to produce market index. According to the nature of the BAH strategy, the best BAH strategy is that puts all initial wealth on the single asset with best performance. Actually, the best BAH strategy can only be determined in hindsight, which makes this strategy impractical.

Another sophisticated benchmark strategy is the *Constant Rebalanced Portfolio* (CRP) strategy, which adopts the same portfolio $\mathbf{b} \in \Delta_m$ for each period. The prices of the m assets change differently during each period, which will cause the portfolio to change simultaneously. Therefore, a CRP strategy might require a mass of trading to rebalance the portfolio before the next period. The final cumulative wealth achieved by a CRP strategy is $S_n(CRP(\mathbf{b})) = \prod_{t=1}^{n} \mathbf{b} \cdot \mathbf{x}_t$. A special CRP is the best CRP (BCRP) strategy, which achieves the maximum final cumulative wealth. We denote this portfolio by $\mathbf{b}^*$, that is,

$$\mathbf{b}^* = \arg\max_{\mathbf{b} \in \Delta_m} S_n(CRP(\mathbf{b})), \tag{2}$$

and the corresponding final cumulative wealth is

$$S_n^* = \max_{\mathbf{b} \in \Delta_m} S_n(\mathbf{b}). \tag{3}$$

From Eq. (2), we know that BCRP, which can only be calculated with all price information, is a hindsight strategy. It is usually identified as a benchmark to judge the performance of online portfolio strategies because it possesses a series of good properties. Researchers always dedicate themselves to constructing an online portfolio strategy which can exhibit the same asymptotic growth rate as BCRP. Let us denote $LS^*(\mathbf{x}^n) = \ln S_n(\mathbf{b}^*)/n$ and $LS(\mathbf{b}^n, \mathbf{x}^n) = \sum_{t=1}^{n} \ln(\mathbf{b}_t \cdot \mathbf{x}_t)/n$ as the final cumulative average logarithmic return of BCRP and that of an online portfolio strategy, respectively. If an online portfolio strategy A satisfies

$$\lim_{n \to \infty} [LS^*(\mathbf{x}^n) - LS(\mathbf{b}^n, \mathbf{x}^n)] \leq 0, \tag{4}$$

the strategy A is called universal portfolio.

3.2 Existing Works

Based on the framework established by Cover (1991), many scholars put forward various online portfolio strategies. Singer (1997) constructed a *Bayesian* online portfolio strategy that tracks changes in the market and improved the return of the strategy. Helmbold et al. (1998) proposed *Exponential Gradient* (EG(η)), which is to track the best performer of the previous period as closely as possible while keeping the portfolio as close to the previous period as possible. Gaivoronski and Stella (2000) constructed the *Successive Constant Rebalanced Portfolio* (SCRP) and proved that the strategy is a universal portfolio. Making linear combination between the last portfolio and the SCRP strategy, *Weighted Successive Constant Rebalanced Portfolio* (WSCRP) is obtained. Empirical analysis shows that the method can significantly improve the performance of the strategy. Then, they generalized the above results and proposed an adaptive portfolio strategy (Gaivoronski and Stella 2003). Agarwal et al. (2006) put forward the *Online Newton Step* (ONS) strategy by using the Newton method from optimization problems. Zhang et al. (2012) improved the EG strategy based on the linear learning function and constructed a class of universal portfolio strategies. O'Sullivan and Edelman (2015) proposed the *Adaptive Universal Portfolio* (AUP), which retains much of the qualitative nature of Cover's UP strategy while enhancing early performance, and doesn't need to take a long time to achieve significant growth. Hazan and Kale (2015) combined the *Worst-Case* model with the *Geometric Brownian* model to create a universal portfolio by digging out the information in the Brownian motion hypothesis, whose circle of regret has been greatly improved. Using the WAA to aggregate the experts strategies, Zhang and Yang (2017) proposed WAAS and WAAC strategy, and further proved that the latter is universal portfolio.

Transaction costs is an important factor in the real-world trading. Blum and Kalai (1999) studied an online portfolio strategy with fixed-ratio transaction costs and proposed a solution that can be rapidly carried out in real-world application. Based on a new prediction method, Albeverio et al. (2001) constructed two online portfolio strategies with transaction costs, and further proved that the proposed strategies have considerable long-term returns. Li et al. (2017) proposed *Transaction Cost Optimization* (TCO) strategy, which may effectively handle reasonable transaction costs and improve existing strategies in the case of non-zero transaction costs. Yang et al. (2018) designed an online portfolio *Mean Reversion with Transaction Costs* (MRTC) to exploit the multi-period mean reversion property of financial market with transaction costs.

Side information can help to improve the return of the asset combinations, so many scholars introduced it into the construction of online portfolio strategy. Cover and Ordentlich (1996) first introduced the side information into the online portfolio model. Fagiuoli et al. (2007) proposed an online portfolio strategy with side information based on SCRP strategy.

While many scholars studied the momentum effect, that is, with high historical yield, future yields will continue to maintain a high level, the reversion of asset prices is also a common financial phenomenon. Based on this phenomenon, many scholars designed a series of online portfolio strategies. Borodin et al. (2004) put forward the Anticor strategy by calculating the correlation between the logarithmic returns of two pairs of stocks in two adjacent historical windows and using the market law of mean reversion to adjust the portfolio. Although the Anticor strategy was not proved to be a universal portfolio strategy, the empirical results show that the performance of the strategy surpasses not only UP and EG, but also BCRP. Li et al. (2012) constructed the PAMR strategy by using online passive aggressive learning technique from machine learning. Passively, the strategy possesses the current under-performing strategies; aggressively, adjusts the current better-performing strategies to effectively exploit the mean reversion characteristics of the financial market. Li et al. (2013) proposed *Confidence Weighted Mean Reversion* (CWMR) strategy by confidence-weighted online learning algorithm. Based on the mean reversion, Li et al. (2015) predicted the asset prices by using *Simple Moving Average* and *Exponential Moving Average*, and constructed two kinds of *Online Moving Average Revision* (OLMAR) strategies. Huang et al. (2016) used the L_1-median estimator to estimate the price relative vector and constructed an online portfolio strategy by using the passive aggressive learning algorithm from online learning theory. Lin et al. (2017) exploited mean reversion principle from a meta learning perspective and proposed a boosting method for price relative prediction.

3.3 Weak Aggregating Algorithm

The Weak Aggregating Algorithm (WAA), developed by Kalnishkan and Vyugin (2008), is a method of online sequence decisions with experts' advice. It makes decisions by aggregating experts' advice and aims to develop algorithms that catch up with a benchmark set of "experts", who can be free agents or decision strategies. Given a set of experts who provide advice each period, the decision maker employs the WAA to aggregate these advice in a certain way.

For the WAA, there are three kinds of decision-makers, i.e., the expert decision maker, the online decision maker, and the actual decision maker. The main purpose of WAA is to improve the competitive performance of the online decision maker. We use an index θ to uniquely identify each expert, which could be a simple scalar, a vector, or a more complex mathematical object. The collection of indices forms a set Θ. We assume that Θ is measurable respect to some suitable σ-algebra, and thus it is possible to define a probability measure over Θ. The σ-algebra depends on the specific setting. For our strategy, the set of expert indices Θ is measurable respect to the Borel σ-algebra on $\mathbf{R}$. The prediction set of the online decision maker and the expert decision maker are denoted by Γ, and the set of actual results are denoted by Ω. Let g be the return function. On the t-th period, given the online decision-maker's decision $r_t \in \Gamma$, the actual result $w_t \in \Omega$, his/her return is $g_t = \pi(r_t,\ w_t)$ and cumulative return is $G_t = G_{t-1} + \pi(r_t,\ w_t)$; given an expert θ's advice r_t^θ, his/her return is $g_t^\theta = \pi(r_t^\theta,\ w_t)$ and cumulative return is $G_t^\theta = G_{t-1}^\theta + \pi(r_t^\theta,\ w_t)$. On the basis of initializing each expert's weight, WAA uses the exponential weighted average method to update each expert's weight according to the return function. Let $q(d\theta)$ be a prior probability measure of Θ and each expert's weight is

$$p_t\,(d\theta) = \frac{\beta_t^{G_{t-1}^\theta} q\,(d\theta)}{\int_\Theta \beta_t^{G_{t-1}^\theta} q\,(d\theta)}, \tag{5}$$

where $\beta_t = e^{1/\sqrt{t}}$, and $1/\sqrt{t}$ is the learning rate, which can reduce the difference among expert's weights. According to Kalnishkan and Vyugin (2008), the pseudocode of the WAA is presented in Algorithm 2.

Levina et al. (2010) proved theoretical guarantee on the performance of the WAA for the case of the bounded return function. The result is presented in Lemma 1.

Algorithm 2 Weak aggregating algorithm (WAA)

1: Initialize: $G_0^\theta = 0,\ \theta \in \Theta$;
2: **for** $t = 1, 2, \ldots, n$ **do**
3: Calculate the experts' weight: $p_t(d\theta) = \beta_t^{G_{t-1}^\theta} q(d\theta) / \int_\Theta \beta_t^{G_{t-1}^\theta} q(d\theta)$.
4: Receive the experts' advice: $r_t^\theta \in \Gamma,\ \theta \in \Theta$.
5: Aggregate the experts' advice to make the decision: $r_t = \int_\Theta r_t^\theta p_t(d\theta)$.
6: Get the actual result: $w_t \in \Omega$.
7: Update the experts' return: $g_t^\theta = \pi(r_t^\theta,\ w_t),\ \theta \in \Theta$.
8: Update the experts' cumulative return: $G_t^\theta = G_{t-1}^\theta + \pi(r_t^\theta,\ w_t),\ \theta \in \Theta$.
9: **end for**

Lemma 1 *Let $-L \leq g \leq 0$. The WAA guarantees that, for all n,*

$$G_n \geq \left(\ln \int_\Theta e^{G_n^\theta / \sqrt{n}} q(d\theta) - L^2 \right) \sqrt{n}. \tag{6}$$

In this paper, we mainly apply the WAA to construct an online portfolio strategy and use Lemma 1 to theoretically analyze its competitive performance.

4 WAEG Strategy

4.1 Formulation

Helmbold et al. (1998) proposed the *Exponential Gradient* (EG(η)) by using the relative entropy as a distance function for motivating updates. The strategy calculates its next portfolio vector $\mathbf{b}_{t+1}$ as a function of $\mathbf{b}_t$ and the just-observed price relatives $\mathbf{x}_t$. It maximizes the logarithmic return

$$F(\mathbf{b}_{t+1}, \eta) = \max[\eta \ln(\mathbf{b}_{t+1} \cdot \mathbf{x}_t) - d(\mathbf{b}_{t+1}, \mathbf{b}_t)], \tag{7}$$

where $d(\mathbf{b}_{t+1}, \mathbf{b}_t) = \sum_{i=1}^m b_{t+1,i} \ln(b_{t+1,i}/b_{t,i})$ and $\eta > 0$. The portfolio can be updated by enforcing the additional constraint $\sum_{i=1}^m b_{t+1,i} = 1$, that is,

$$b_{t+1,i} = \frac{b_{t,i} \exp(\frac{\eta x_{t,i}}{\mathbf{b}_t \cdot \mathbf{x}_t})}{\sum_{i=1}^m b_{t,i} \exp(\frac{\eta x_{t,i}}{\mathbf{b}_t \cdot \mathbf{x}_t})}. \tag{8}$$

Helmbold et al. (1998) proved that EG(η) is a universal portfolio when the parameter η is fixed to a value associated with n trading periods, i.e., $\eta = (8(\ln m)^3/(m^2 n^3))^{1/4}$. We are excited to find that the above result is also true even if the value of parameter η changes a little. Thus, we derive a class of universal portfolios through EG(η).

Further, when using an online strategy for investment decision-making, we have to determine the values of different parameters in advance. In fact, when updating a new portfolio in the real financial market, it is difficult to find out an appropriate parameter ahead of time, and we can only estimate the range of parameter. In this paper, we consider the EG(η) corresponding to different values of parameter η as different experts, use the WAA to aggregate different experts' advice and determine the portfolio of the next period.

Firstly, we consider there are k expert strategies EG(η) of different values of parmeter η. The index set for experts is $\mathbf{E}_\eta = \{\mathrm{EG}(\eta_1), \mathrm{EG}(\eta_2), \ldots, \mathrm{EG}(\eta_k)\}$. We generate the experts by sampling the parameters uniformly from the range $\eta \sim U(\eta_{\min}, \eta_{\max})$. We generate k experts from EG($\eta = \eta_{\min}, \eta_{\min} + \varepsilon, \ldots, \eta_{\max}$), where the absolute value of the difference between any two adjacent parameters is $\varepsilon = (\eta_{\max} - \eta_{\min})/(k-1)$. In the initial stage, the online decision maker's cumulative return is $G_0 = 0$ and each expert's cumulative return is $G_0(\eta_j) = 0$, $j = 1, 2, \ldots, k$. On the t-th period, the online decision maker's return is $g_t = \ln(\mathbf{b}_t \cdot \mathbf{x}_t)$ and his/her cumulative return is $G_t = G_{t-1} + g_t$; the j-th expert's return is $g_t(\eta_j) = \ln(\mathbf{b}_t(\eta_j) \cdot \mathbf{x}_t)$ and his/her cumulative return is $G_t(\eta_j) = G_{t-1}(\eta_j) + g_t(\eta_j)$, where $\mathbf{b}_t(\eta_j)$ represents the portfolio of the j-th expert on the t-th period.

At the beginning of the $(t+1)$-th period, calculate the weight $p_{t+1}(\eta_j)$ based on each expert's historical cumulative return $G_t(\eta_j)$, that is,

$$p_{t+1}(\eta_j) = \frac{\beta_{t+1}^{G_t(\eta_j)}}{\sum_{j=1}^k \beta_{t+1}^{G_t(\eta_j)}}. \tag{9}$$

Calculate each expert's portfolio on the $(t+1)$-th period

$$b_{t+1,i}(\eta_j) = \frac{b_{t,i}(\eta_j)\exp\left(\frac{\eta_j \cdot x_{t,i}}{\mathbf{b}_t(\eta_j)\cdot \mathbf{x}_t}\right)}{\sum_{i=1}^m b_{t,i}(\eta_j)\exp\left(\frac{\eta_j \cdot x_{t,i}}{\mathbf{b}_t(\eta_j)\cdot \mathbf{x}_t}\right)}. \tag{10}$$

The portfolio at the beginning of the $(t+1)$-th period is obtained by aggregating all experts' advice, that is,

$$\mathbf{b}_{t+1} = \sum_{j=1}^k \mathbf{b}_{t+1}(\eta_j) p_{t+1}(\eta_j) = \frac{\sum_{j=1}^k \mathbf{b}_{t+1}(\eta_j)\exp\left(\frac{\sum_{\tau=1}^t \ln(\mathbf{b}_\tau(\eta_j)\cdot \mathbf{x}_\tau)}{\sqrt{t+1}}\right)}{\sum_{j=1}^k \exp\left(\frac{\sum_{\tau=1}^t \ln(\mathbf{b}_\tau(\eta_j)\cdot \mathbf{x}_\tau)}{\sqrt{t+1}}\right)}. \tag{11}$$

Without loss of generality, the portfolio vector on the first period is always set by $\mathbf{b}_1 = (1/m, 1/m, \ldots, 1/m)$. We refer this online strategy as *Weak Aggregating Exponential Gradient* (WAEG). Its pseudocode is presented in Algorithm 3.

Algorithm 3 WAEG strategy

Input: $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$: Historical daily price relative vector sequence.
Output: S_n : Final cumulative wealth.

Initialize: $S_0 = 1,\ \mathbf{b}_1 = (1/m, 1/m, \ldots, 1/m)\,, \mathbf{b}_1(\eta_j) = (1/m, 1/m, \ldots, 1/m)$, $G_0(\eta_j) = 0,\ \ j = 1, 2, \ldots, k$.
for $t = 1, 2, \ldots, n$ **do**
Receive the price relative vector: $\mathbf{x}_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,m})^\top$.
Calculate the cumulative return: $S_t = S_{t-1} \times (\mathbf{b}_t \cdot \mathbf{x}_t)$.
Calculate each expert's weight: $p_{t+1}(\eta_j) = \beta_{t+1}^{G_t(\eta_j)} / \sum_{j=1}^{k} \beta_{t+1}^{G_t(\eta_j)}$, where $G_t(\eta_j) = G_{t-1}(\eta_j) + \ln(\mathbf{b}_t(\eta_j) \cdot \mathbf{x}_t)$.
Calculate each expert's portfolio:

$$b_{t+1,i}(\eta_j) = \frac{b_{t,i}(\eta_j) \exp(\frac{\eta_j \cdot x_{t,i}}{\mathbf{b}_t(\eta_j) \cdot \mathbf{x}_t})}{\sum_{i=1}^{m} b_{t,i}(\eta_j) \exp(\frac{\eta_j \cdot x_{t,i}}{\mathbf{b}_t(\eta_j) \cdot \mathbf{x}_t})},\ \ j = 1, 2, \ldots, k.$$

Update the portfolio for the $(t+1)$-th period:

$$\mathbf{b}_{t+1} = \sum_{j=1}^{k} \mathbf{b}_{t+1}(\eta_j) p_{t+1}(\eta_j).$$

end for

4.2 Competitive Analysis

This section analyzes the competitive performance of WAEG. When definite assumptions on the price relative hold, we prove that WAEG is a universal portfolio, that is, it asymptotically approaches the same average exponential growth rate as BCRP.

Since $x_{t,i}$ represents price relative, we have that $x_{t,i} \geq 0$ for all i and t. In our competitive analysis, we first obtain Theorem 1 with the assumption $x_{t,i} \geq l > 0$ for all i and t. The assumed lower bound l on $x_{t,i}$ used in Theorem 1 can be viewed as a lower bound on the ratio of the worst to best price relatives for trading period t. And then, we relax the assumption of Theorem 1, and obtain Theorem 2 with the assumption $x_{t,i} \geq 0$ for all i and t.

Furthermore, we assume that $\max_i x_{t,i} = 1$ for all t, which will not make any difference between the logarithmic return achieved by WAEG and BCRP when we multiply the price relative $\mathbf{x}_t$ by a constant. In other words, the price relative can be normalized to satisfy $x_{t,i} \leq 1$. Therefore, the logarithmic wealth is also called normalized one since the wealth is calculated after price relative being normalized.

Theorem 1 *Let* $\mathbf{b}_t(\eta_j) \in \Delta_m$ *be the portfolio vector of the* j*-th expert strategy* $EG(\eta_j)$ *on the* t*-th period. Under the assumptions that* $x_{t,i} \geq r > 0$ *for all* i, t *and* $\max_i x_{t,i} = 1$ *for all* t*, the portfolio vector sequence* $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ *generated by WAEG satisfies*

$$\sum_{t=1}^{n} \ln(\mathbf{b}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{n} \ln(\mathbf{b}_t(\eta_j) \cdot \mathbf{x}_t) - (\ln k + (\ln l)^2)\sqrt{n}, \ \ j = 1, 2, \ldots, k. \quad (12)$$

Proof Based on the assumption $l \leq x_{t,i} \leq 1$, we have

$$\ln l \leq g = \ln(\mathbf{b}_t \cdot \mathbf{x}_t) \leq \ln 1 = 0.$$

By setting $L = -\ln l$ and using Lemma 1, we have

$$\begin{aligned} G_n &\geq \left(\ln \sum_{j=1}^{k} \left(\frac{1}{k} e^{\frac{G_n(\eta_j)}{\sqrt{n}}} \right) - L^2 \right) \sqrt{n} \\ &\geq \left(\ln \left(\frac{1}{k} e^{\frac{G_n(\eta_j)}{\sqrt{n}}} \right) - (-\ln l)^2 \right) \sqrt{n} \\ &= G_n(\eta_j) - (\ln k + (\ln l)^2)\sqrt{n}, \end{aligned}$$

where $G_n = \sum_{t=1}^{n} \ln(\mathbf{b}_t \cdot \mathbf{x}_t)$ and $G_n(\eta_j) = \sum_{t=1}^{n} \ln(\mathbf{b}_t(\eta_j) \cdot \mathbf{x}_t)$. □

When n is large enough, the inequality (12) shows that the final cumulative logarithmic wealth of WAEG is guaranteed to converge to that of the best expert strategy. However, the price relative of the lower bound l needs to be obtained in advance. Now we consider the case that there is no positive lower bound l, i.e., $0 \leq x_{t,i} \leq 1$. Firstly, we should deal with the expert strategy EG(η) which needs to use price relative $x_{t,i}$ to update portfolio vector. When the lower bound of $x_{t,i}$ is unknown, let

$$\tilde{\mathbf{x}}_t = (1 - \alpha/m)\mathbf{x}_t + (\alpha/m)\mathbf{1}, \ \alpha \in [0, 1],$$

where $\mathbf{1}$ is the all 1's vector. We obtain the updated portfolio vector $\mathbf{b}_{t+1}(\eta_j)$ by using $\tilde{\mathbf{x}}_t$ rather than $\mathbf{x}_t$, that is,

$$b_{t+1,i}(\eta_j) = \frac{b_{t,i}(\eta_j) \exp(\frac{\eta_j \cdot \tilde{x}_{t,i}}{\mathbf{b}_t(\eta_j) \cdot \tilde{\mathbf{x}}_t})}{\sum_{i=1}^{m} b_{t,i}(\eta_j) \exp(\frac{\eta_j \cdot \tilde{x}_{t,i}}{\mathbf{b}_t(\eta_j) \cdot \tilde{\mathbf{x}}_t})}, \ \ j = 1, 2, \ldots, k.$$

Meanwhile, we modify the portfolio vector slightly, that is, $\tilde{\mathbf{b}}_{t+1}(\eta_j) = (1 - \alpha)\mathbf{b}_{t+1}(\eta_j) + (\alpha/m)\mathbf{1}$. We refer this strategy as $\widetilde{\mathrm{EG}}(\eta)$. When considering $\widetilde{\mathrm{EG}}(\eta)$ strategies corresponding to different values of parameter η as different experts, the portfolio vector achieved by WAEG is slightly modified, that is,

$$\mathbf{b}_{t+1} = \frac{\sum_{j=1}^{k} \tilde{\mathbf{b}}_{t+1}(\eta_j) \exp\left(\frac{\sum_{\tau=1}^{t} \ln(\tilde{\mathbf{b}}_\tau(\eta_j) \cdot \tilde{\mathbf{x}}_\tau)}{\sqrt{t+1}} \right)}{\sum_{j=1}^{k} \exp\left(\frac{\sum_{\tau=1}^{t} \ln(\tilde{\mathbf{b}}_\tau(\eta_j) \cdot \tilde{\mathbf{x}}_\tau)}{\sqrt{t+1}} \right)}.$$

Further, the portfolio vector is also slightly modified, that is,

$$\tilde{\mathbf{b}}_{t+1} = (1-\alpha)\mathbf{b}_{t+1} + (\alpha/m)\mathbf{1}.$$

We call this modified strategy $\widetilde{\text{WAEG}}$.

Theorem 2 *Let $\tilde{\mathbf{b}}_t(\eta_j) \in \Delta_m$ be the portfolio vector of the j-th expert strategy $\widetilde{EG(\eta_j)}$ on the t-th period, and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a price relative sequence with $x_{t,i} > 0$ for all i, t and $\max_i x_{t,i} = 1$ for all t. For $\alpha \in (0, 1/2]$, the final cumulative logarithmic wealth achieved by the strategy $\widetilde{\text{WAEG}}$ is bounded as follows:*

$$\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - \left(\ln k + \left(\ln \frac{\alpha}{m}\right)^2\right)\sqrt{n} - 2\alpha n. \tag{13}$$

Proof According to $\max_i x_{t,i} = 1$, we have

$$\frac{\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t}{\mathbf{b}_t \cdot \tilde{\mathbf{x}}_t} = \frac{(1-\alpha)\mathbf{b}_t \cdot \mathbf{x}_t + (\alpha/m)\mathbf{1} \cdot \mathbf{x}_t}{(1-(\alpha/m))\mathbf{b}_t \cdot \mathbf{x}_t + (\alpha/m)\mathbf{b}_t \cdot \mathbf{1}} \geq \frac{(1-\alpha)\mathbf{b}_t \cdot \mathbf{x}_t + (\alpha/m)}{(1-(\alpha/m))\mathbf{b}_t \cdot \mathbf{x}_t + (\alpha/m)}. \tag{14}$$

As the function $f(x) = \frac{(1-\alpha)x+\alpha/m}{(1-\alpha/m)x+\alpha/m}$ decreases monotonically and $\mathbf{b}_t \cdot \mathbf{x}_t \leq 1$, we get $f(\mathbf{b}_t \cdot \mathbf{x}_t) \geq f(1)$, that is,

$$\frac{(1-\alpha)\mathbf{b}_t \cdot \mathbf{x}_t + (\alpha/m)}{(1-(\alpha/m))\mathbf{b}_t \cdot \mathbf{x}_t + (\alpha/m)} \geq 1 - \alpha + \alpha/m,$$

or equivalently,

$$\ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \ln(\mathbf{b}_t \cdot \tilde{\mathbf{x}}_t) + \ln(1-\alpha+\alpha/m).$$

And then, we use inequality $\ln(1-\alpha+\alpha/m) \geq \ln(1-\alpha) \geq -2\alpha, \alpha \in (0, 1/2]$ to obtain $\ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \ln(\mathbf{b}_t \cdot \tilde{\mathbf{x}}_t) - 2\alpha$. According to Theorem 1, it holds that

$$\sum_{t=1}^{n} \ln(\mathbf{b}_t \cdot \tilde{\mathbf{x}}_t) \geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \tilde{\mathbf{x}}_t) - (\ln k + (\ln l)^2)\sqrt{n},$$

so we obtain

$$\begin{aligned}\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) &\geq \sum_{t=1}^{n} \ln(\mathbf{b}_t \cdot \tilde{\mathbf{x}}_t) - 2\alpha n \\ &\geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \tilde{\mathbf{x}}_t) - \left(\ln k + \left(\ln \frac{\alpha}{m}\right)^2\right)\sqrt{n} - 2\alpha n.\end{aligned}$$

According to $\max_i x_{t,i} = 1$, we obtain

$$\tilde{\mathbf{b}}_t(\eta_j) \cdot \tilde{\mathbf{x}}_t = (1 - \alpha/m)\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t + (\alpha/m) \geq \tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t.$$

Thus, we obtain

$$\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - \left(\ln k + \left(\ln \frac{\alpha}{m}\right)^2\right)\sqrt{n} - 2\alpha n.$$

□

Theorem 3 *When $m \geq 2$ and $\alpha = (m^2/n)^{1/4}$, the final cumulative logarithmic wealth of $\widetilde{\mathrm{WAEG}}$ satisfies*

$$\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - 2\left(2\max\{m, k\}^8\right)^{\frac{1}{4}} n^{\frac{3}{4}}. \tag{15}$$

Proof When $m \geq k$, from (13), we have

$$\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - \left(\ln k + \frac{m}{\alpha}\right)\sqrt{n} - 2\alpha n.$$

When $\alpha = (m^2/n)^{1/4}$, we have

$$\begin{aligned}\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) &\geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - n^{\frac{1}{2}} \ln k - 3m^{\frac{1}{2}} n^{\frac{3}{4}} \\ &\geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - n^{\frac{1}{2}} \ln m - (2m^8)^{\frac{1}{4}} n^{\frac{3}{4}}.\end{aligned}$$

In addition, we have $\ln m/n^{1/4} \leq (2m^8)^{1/4}$, and obtain

$$\begin{aligned}\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) &\geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - n^{\frac{3}{4}} \ln m/n^{\frac{1}{4}} - (2m^8)^{\frac{1}{4}} n^{\frac{3}{4}} \\ &\geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - n^{\frac{3}{4}} (2m^8)^{\frac{1}{4}} - (2m^8)^{\frac{1}{4}} n^{\frac{3}{4}} \\ &\geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - 2(2m^8)^{\frac{1}{4}} n^{\frac{3}{4}}.\end{aligned}$$

Similarly, when $k > m$, we get

$$\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - 2(2k^8)^{\frac{1}{4}} n^{\frac{3}{4}}.$$

Thus, when $m \geq 2$, we obtain

$$\sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t(\eta_j) \cdot \mathbf{x}_t) - 2\Big(2\max\{m,k\}^8\Big)^{\frac{1}{4}} n^{\frac{3}{4}}.$$

□

In Theorem 3, the setting of α still needs to obtain n in advance. In this case, the stage technique can prove that the growth rate achieved by $\widetilde{\text{WAEG}}$ converges to that of the best expert strategy. Meanwhile, it is also easy to prove that $\widetilde{\text{WAEG}}$ is a universal portfolio. For large enough n ($n > 2\max\{m,k\}^8$), the specific steps are as follows. The first and second $2\max\{m,k\}^8$ periods are numbered by stage 0 and 1, respectively. And the following $2^i\max\{m,k\}^8 (i > 1)$ periods are numbered by stage i. At the beginning of each stage, the portfolio vector is reinitialized to the uniform portfolio vector and the setting of α is the same as that of Theorem 3, using $2^i\max\{m,k\}^8$ in stage $i > 1$ as the value for n. In each stage i, $\widetilde{\text{WAEG}}$ is carried out to obtain the staged portfolio vectors. We refer this strategy as staged $\widetilde{\text{WAEG}}$.

Theorem 4 *The staged* $\widetilde{\text{WAEG}}$ *is a universal portfolio.*

Proof We denote the last stage number by b. According to the staged $\widetilde{\text{WAEG}}$ mentioned above, we have inequalities

$$n > 2\max\{m,k\}^8 + 2\max\{m,k\}^8 + 2^2\max\{m,k\}^8 + \cdots + 2^{b-1}\max\{m,k\}^8$$

and

$$\begin{aligned} n \leq\ & 2\max\{m,k\}^8 + 2\max\{m,k\}^8 + 2^2\max\{m,k\}^8 + \cdots \\ & + 2^{b-1}\max\{m,k\}^8 + 2^b\max\{m,k\}^8. \end{aligned}$$

Thus, we have $b = [\log_2(n/2\max\{m,k\}^8)]$.

According to Theorem 3, for each stage, we have

$$\begin{aligned} \sum_{\tau=t_i}^{t_{i+1}-1} \ln(\tilde{\mathbf{b}}_\tau \cdot \mathbf{x}_\tau) \geq & \sum_{\tau=t_i}^{t_{i+1}-1} \ln(\tilde{\mathbf{b}}_\tau(\eta_j) \cdot \mathbf{x}_\tau) \\ & - 2\Big(2\max\{m,k\}^8\Big)^{\frac{1}{4}} \Big(\max\{2,2^i\}\max\{m,k\}^8\Big)^{\frac{3}{4}}. \end{aligned} \tag{16}$$

Helmbold et al. (1998) proved the staged $\widetilde{\text{EG}(\eta)}$ is universal, which can be shown for each stage, that is,

$$\sum_{\tau=t_i}^{t_{i+1}-1} \ln(\tilde{\mathbf{b}}_\tau(\eta_j) \cdot \mathbf{x}_\tau) \geq \sum_{\tau=t_i}^{t_{i+1}-1} \ln(\mathbf{b} \cdot \mathbf{x}_\tau) - 2(2m^2\ln m)^{\frac{1}{4}} \Big(\max\{2,2^i\}\max\{m,k\}^8\Big)^{\frac{3}{4}}. \tag{17}$$

Combining (16) with (17), we have

$$\sum_{\tau=t_i}^{t_{i+1}-1} \ln(\tilde{\mathbf{b}}_\tau \cdot \mathbf{x}_\tau) \geq \sum_{\tau=t_i}^{t_{i+1}-1} \ln(\mathbf{b} \cdot \mathbf{x}_\tau) - 4\left(2 \max\{m, k\}^8\right)^{\frac{1}{4}} \left(\max\{2, 2^i\} \max\{m, k\}^8\right)^{\frac{3}{4}}.$$

Thus, we obtain

$$\begin{aligned}\sum_{t=1}^{n} \ln(\mathbf{b} \cdot \mathbf{x}_t) - \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) &\leq \sum_{i=0}^{b} 4\left(2 \max\{m, k\}^8\right)^{\frac{1}{4}} \left(\max\{2, 2^i\} \max\{m, k\}^8\right)^{\frac{3}{4}} \\ &\leq 12 \max\{m, k\}^8 \left(1 + \left(\frac{n}{2 \max\{m, k\}^8}\right)^{\frac{3}{4}}\right).\end{aligned}$$

Now, setting $\mathbf{b}$ to the best constant rebalanced portfolio $\mathbf{b}^*$, we obtain

$$\lim_{n \to \infty} \frac{\sum_{t=1}^{n} \ln(\mathbf{b}^* \cdot \mathbf{x}_t) - \sum_{t=1}^{n} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t)}{n} \leq 0.$$

Then, the proof of the universality of the staged $\widetilde{\text{WAEG}}$ is completed. □

5 Numerical Experiments

In this section, we conduct numerical experiments on the proposed strategy WAEG and several existing classic strategies, and then compare their performance.

5.1 Experiment Data

In the following experiments, we use the real-world datasets, i.e., NYSE and S50I, which were collected from major financial markets. All of them are easy to obtain from public domains (such as Yahoo Finance, Google Finance, and RESSET Database). The dataset NYSE collected from New York Stock Exchange contains 5651 daily price relatives from Jul. 3rd 1962 to Dec. 31st 1984. The dataset S50I collected by us from Shares of the Shanghai 50 Index consists of 3034 daily price relatives ranging from Jan. 2nd 2004 to Jun. 30th 2016.

We construct twelve combinations of stocks from the datasets of NYSE and S50I, where six combinations contain two stocks and the rest six combinations contain four stocks. The brief names of stock combinations are listed in Table 1. The former three combinations in Table 1 were firstly used by Cover (1991) and followed by Helmbold et al. (1998). To make a distinct comparison with existing works, we

Table 1 Names of stock combinations

Names of combinations	Names of stocks
Comb. 1	Iroquois and Kin_ark
Comb. 2	Comm_meals and Kin_ark
Comb. 3	Comm_meals and Mei_corp
Comb. 4	Ford and Kin_ ark and Comm_meals and Iroquois
Comb. 5	Exxon and GM and Amer_brands and Kin_ ark
Comb. 6	Comm_meals and Schlum and MMM and Iroquois
Comb. 7	600406 and 600196
Comb. 8	600196 and 600690
Comb. 9	600150 and 600519
Comb. 10	600111 and 600519 and 600050 and 600518
Comb. 11	600104 and 600519 and 600332 and 600406
Comb. 12	600332 and 600111 and 600372 and 600703

continue to use these combinations to verify the competitive performance of our strategy.

5.2 Comparison Strategies

We conduct experiments to compare the proposed strategy WAEG with a number of benchmarks and existing online strategies. The parameters of these strategies are set according to their original studies as listed below.

1. Market: uniform BAH strategy;
2. Best: the best asset which gains the maximum cumulative wealth in a combination of assets;
3. BCRP: *Best Constant Rebalanced Portfolio* strategy in hindsight;
4. UP: Cover's *Universal Portfolio* according to Cover (1991);
5. EG: *Exponential Gradient* strategy with $\eta = 0.05$ according to Helmbold et al. (1998);
6. SCRP: *Successive Constant Rebalanced Portfolio* by Gaivoronski and Stella (2000).

Before the following experiments, we need to determine the number of expert strategies for our strategy reasonably. We know that the EG (η) strategies corresponding to different values of parameter η are considered as different experts. We set $\eta_{\min} = 0.01$ and $\eta_{\max} = 0.2$ in the range of parameter $\eta \sim U(\eta_{\min}, \eta_{\max})$ and $\varepsilon = 0.01$. We generate 20 expert strategies from $\mathrm{EG}(\eta = \eta_{\min}, \eta_{\min} + \varepsilon, \ldots, \eta_{\max} = 0.01, 0.02, \ldots, 0.2)$ in all experiments except sensitivity analysis.

5.3 Result Analysis

We compare the performance of online strategies by final cumulative wealth and the results are illustrated in Tables 2 and 3. From the results in Table 2, we find that

Table 2 Final cumulative wealth of different strategies on the six combinations in the American market

Strategies	Comb. 1	Comb. 2	Comb. 3	Comb. 4	Comb. 5	Comb. 6
Market	6.52	28.07	37.47	10.70	10.28	52.02
Best	8.92	52.02	52.02	22.91	16.10	27.51
BCRP	73.70	144.01	102.96	149.31	53.82	99.99
UP	37.14	75.95	67.41	46.59	13.39	41.93
EG	64.43	110.96	94.28	89.45	36.87	53.89
SCRP	16.54	25.97	28.44	7.23	7.34	12.11
WAEG	**67.90**	**114.95**	**95.29**	**95.16**	**38.75**	**55.04**

Remark: The top final cumulative wealth in each column, except that results of strategies in hindsight, are highlighted in bold

Table 3 Final cumulative wealth of different strategies on the six combinations in the Chinese market

Strategies	Comb. 7	Comb. 8	Comb. 9	Comb. 10	Comb. 11	Comb.12
Market	11.56	8.12	23.42	6.69	19.81	15.32
Best	13.83	9.31	39.07	13.82	39.07	39.07
BCRP	15.99	10.55	39.07	17.83	45.82	39.63
UP	13.86	9.41	24.72	7.82	16.06	14.61
EG	15.27	10.20	25.40	10.27	19.99	16.83
SCRP	8.04	4.05	15.64	7.80	23.37	22.60
WAEG	**15.40**	**10.30**	**25.55**	**10.42**	**20.07**	**17.11**

Remark: The top final cumulative wealth in each column, except that results of strategies in hindsight, are highlighted in bold

WAEG performs better than other online strategies on all combinations in the American market. Moreover, the results also show that the final cumulative wealth of WAEG are as good as that of the benchmark strategy BCRP on some combinations, such as Comb. 3. The above analysis indicates that our approach by using WAA to aggregate online experts' advice improves the performance of our strategy. From Table 3, we get the similar results to Table 2. The above analysis shows the strategy WAEG adapts to both the American and Chinese markets.

In addition to the comparison of the final cumulative wealth, we are also interested in observing how the final cumulative wealth achieved by online strategies change over different trading periods. We present the daily cumulative returns of BCRP, UP and WAEG on Combs. 1, 4, 7, 10 in Fig. 1. From the results, we see that the proposed strategy WAEG consistently surpasses UP over almost the whole trading period and performs as well as BCRP, which again validates the effectiveness of the proposed strategy.

From the tables and figures, it is clear that the final cumulative wealth achieved by WAEG is consistently higher than that of existing online strategies, such as UP, EG(η) and SCRP. More interesting, WAEG has generalized EG(η) by using the WAA and it performs better than EG(η). Some results show that WAEG performs almost as well as BCRP constructed with the perfected knowledge of

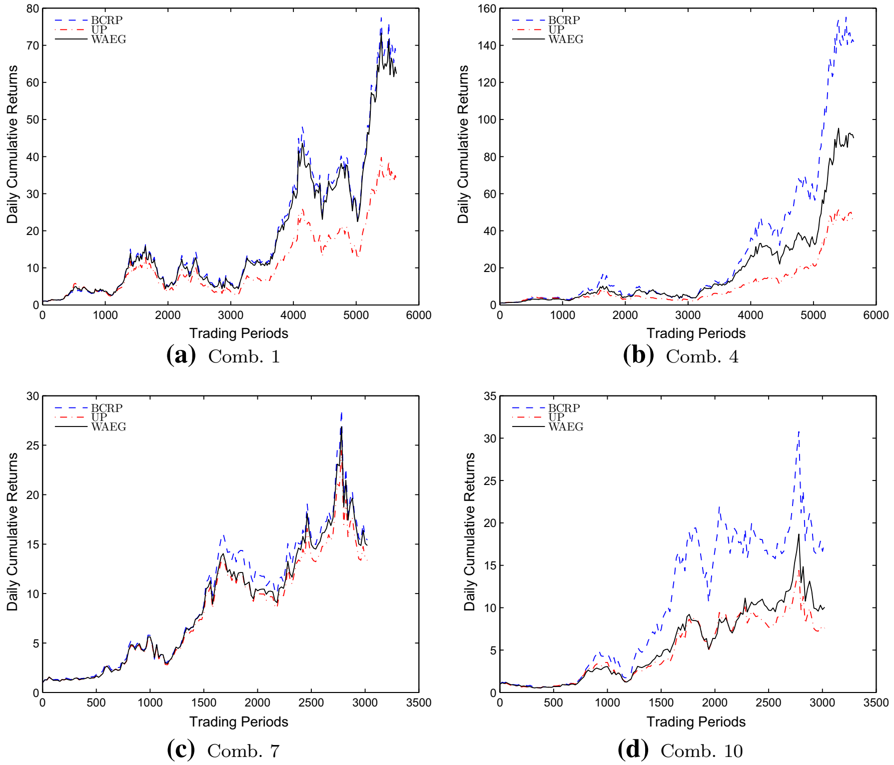

Fig. 1 Daily cumulative returns of BCRP, UP and WAEG

future. Thus, WAEG exhibits competitive performance compared with existing online strategies.

5.4 Sensitivity Analysis

Note that we need to determine the number and range of experts for our proposed strategy, which we can get by determining the number and range of different values of parameter η. Now we conduct the following experiments to analyze them.

5.4.1 Final Cumulative Wealths with Varying Numbers of Experts

Firstly, we examine the performance of WAEG with varying numbers of experts k from 2 to 20. We get different numbers of experts by determining a fixed range of values of parameter $\eta \sim U(\eta_{\min}, \eta_{\max})$ with $\eta_{\min} = 0.01$, $\eta_{\max} = 0.2$, and the numbers of values of parameter $k = \{2, 4, \ldots, 20\}$, which can be achieved by changing the value of ε. To facilitate the comparison, we also list the final cumulative wealths of BCRP and UP. From Fig. 2, the final cumulative wealth of WAEG hardly changes as the number of experts increases in most cases. It indicates that the performance of WAEG

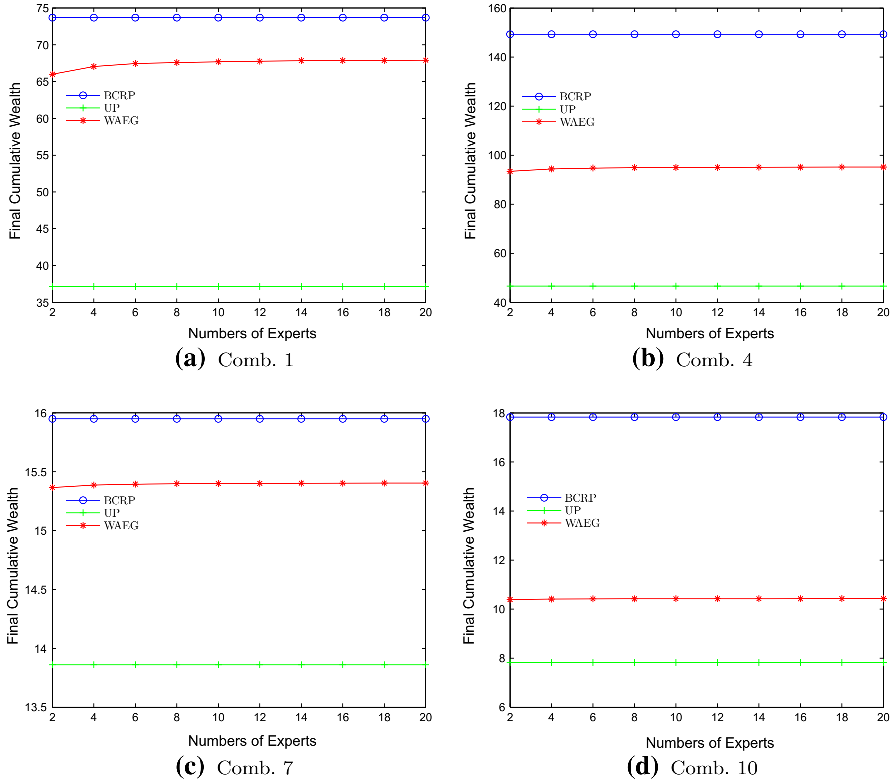

Fig. 2 Final cumulative wealth with varying numbers of experts (see online version for colours)

is not sensitive to the number of experts in a fixed range, so we can easily choose appropriate number of experts.

5.4.2 Final Cumulative Wealths with Varying Ranges of Experts

In addition, we evaluate the performance of WAEG with varying ranges of experts. We get different ranges of experts by determining different ranges of values of parameter $\eta \sim U(\eta_{\min}, \eta_{\max})$ with $\eta_{\min} = 0.01$, $\eta_{\max} = \{0.1, 0.2, \ldots, 0.8\}$, and the number of values of parameter $k = (\eta_{\max} - \eta_{\min})/\varepsilon + 1$, where $\varepsilon = 0.01$. For the convenience of comparison, we also list the final cumulative wealths of BCRP and UP. From Fig. 3, the final cumulative wealth of WAEG slowly drops as the range of experts expands in most cases. It illustrates that expanding the range of experts will have a negative impact on the performance of WAEG.

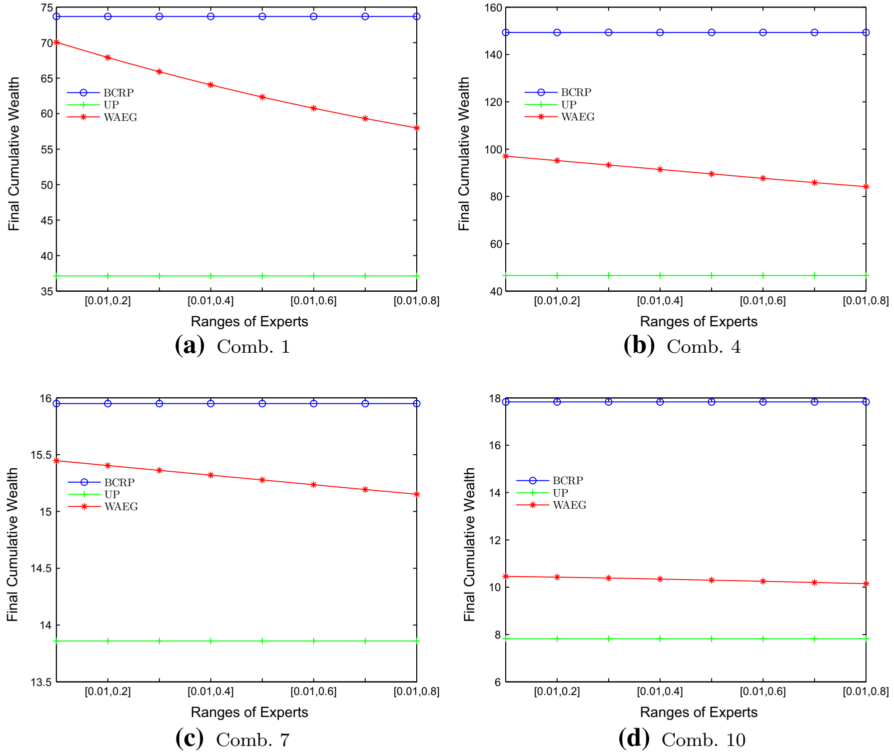

Fig. 3 Final cumulative wealth with varying ranges of experts (see online version for colours)

6 Conclusion

This paper investigates the online portfolio selection problem and presents a novel strategy WAEG that aggregates online experts' advice by using the WAA. We first consider a class of Exponential Gradient strategies (EG(η)) as experts, and use the WAA to aggregate their advice. More importantly, we proved that WAEG is universal. Experiments show that WAEG performs well in different financial markets. We think that using the WAA to aggregate different online strategies, such as UP, EG, SCRP and so on, is also a worthwhile study and we will consider this as future works. In addition, we ignore the transaction costs, which is an important factor of the real market. Therefore, how to integrate the transaction costs in the model is also interesting.

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