Time Series

A brief Introduction

Dr. Umberto Michelucci

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Lucerne University of
Applied Sciences and Arts

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Introduction

Definitions

A time series is a collection of observations x_t made sequentially in time.
A discrete-time time series is a collection of observations x_t in which the set T_0 of times at which observations are made is a discrete set.
A continous-time time series is a collection of observations x_t are made continously over some time interval.

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Time

With *time* we can mean many things:

- Seconds, Minutes, Hours, Years, ...
- (Spatial interpretation) first, second, third on a road, words in a sentence, etc.

The important point is to have an *ordered variable* (like time) where there is a meaning of direction in its values. So the concept of *past*, *presence* and *future* has meaning.

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Visualisation

Plotting a time series is important. A plot can reveal:

- Trend: upward or downward pattern.
- Periodicity: repetition of behaviour in a regular pattern.
- Seasonality: preiodic behaviour with a known period (monthly, every two months, every weekend, day and night, etc.
- Heteroskedasticity: changing variance.
- Dependence: positive (successive observations are similar) or negative (successive observations are dissimilar).

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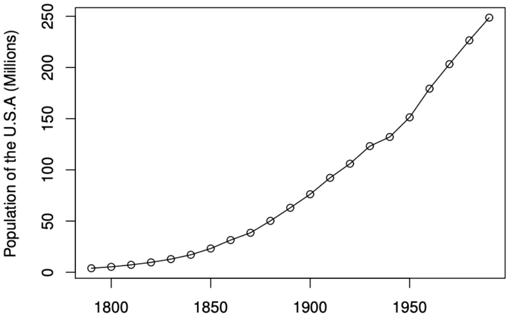
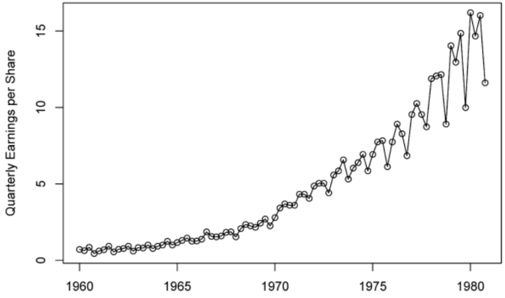
Examples

USA Population at ten years intervals from 1790 to 1990.



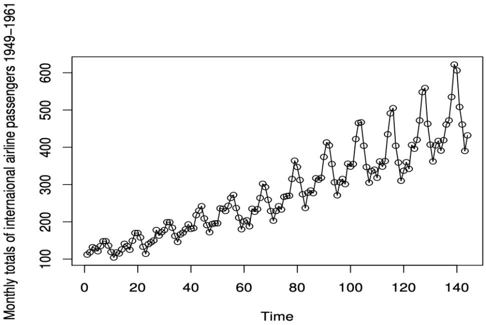
Examples

Johnson & Johnson Quarterly Earnings. Note the gradually increasing underlying trend PLUS regular variations superimposed on the trend that seems to repeat regularly.



Examples

Airline passengers fro 1949-1961.



Examples

Monthly real exchange rates between U.S. and Canada. No obvious seasonality or trend. Hard to make a long range prediction. Positive Dependence: successive observations tend to lie on the same side of the mean.

Remark

It is difficult to distinguish between trend, seasonality or dependence. There is no unique decomponosition of a series into the various parts.

If we had several “realisations” we could extract the trend by averaging, for example. But we never have them.

Goals

What do we want to achieve with a time series analysis?

- Provide a model of the data. To understand a phenomena for example.
- Predict future data.
- Produce a compact description of the data.

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Modeling

Time Series Modeling Assumptions

We assume that the data is a random variable. However in Time Series:

- the variables are usually **not** independent (affected by trend and/or seasonality).
- Variance may change significantly (the variables are not iid, indipendently and identically distributed).
- Variables are not identically distributed (different distributions in different periods for example).

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Modeling

The first goal in time series modeling is to reduce the analysis to a simpler case: eliminate trend, seasonality and heteroskedasticity. Then we can use many techniques from statistics that assume that the variables are iid.

Modeling

Let's call our variable x_t. A time series will have a certain "realisation", for example $(C_1, C_2, ..., C_n)$. Meaning that our variable (for example the stock price) will have assume the value C_1 at time $t_!$, C_2 at time t_2 and so on.

To model a time series we will need to model the probability of the variable x_t assume certain values. For example

$$P(x_1 \leq C_1, x_2 \leq C_2, ..., x_n \leq C_n)$$

Note that if we know $P()$ we can predict any realisation or future values.

Models with zero-mean

Modeling

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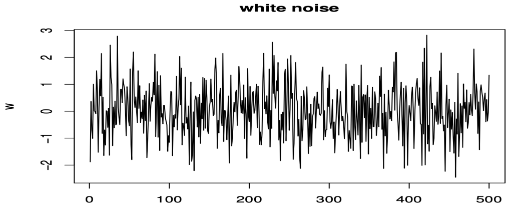
White noise

For example the simplest model for a time series: no trend, no seasonal components where the observations are iid with zero mean (white noise).

- In this csae we can write

$$P(x_1 \leq C_1, x_2 \leq C_2, ..., x_n \leq C_n) = P(x_1 \leq C_1)...P(x_n \leq C_n)$$



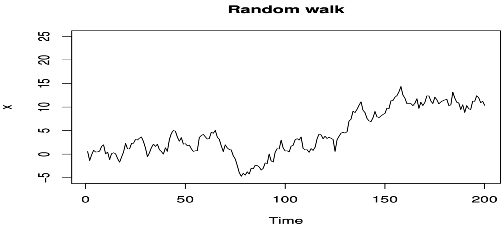
Random Walk

Random Walk

The random walk $\{S_t\}, t = 0, 1, ..., n$ is obtained by cumulatively summing iid random variables:

$$S_t = x_1 + x_2 + ... + x_t, \quad t = 0, 1, ..., n$$

where x_t is random iid noise.



Models with Trend

Trend Types

A trend T_i can be of two types:

- Additive, such that $x_i = T_1 + y_i$
- Multiplicative, such that $x_i = T_i y_i$

Note that

- An additive trend corresponds to a changing mean.
- A multiplicative trend correspnds to a changing mean **and** variance.

We typically deal with additive trends.

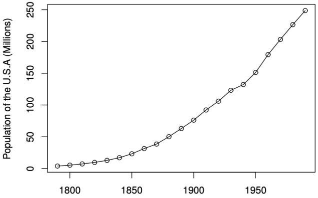
Model with Trend I/II

Model with Trend

In this case a zero-mean model is not working (clearly). But we can separate the contributions to the model in two parts, of which one is zero-mean.

$$x_t = m_t + y_t$$

where m_t is the trend, and y_t has zero mean.



Model with Trend II/II

m_t can be predicted, for example, using a least squares regression method for example.

Models with Seasonality

Model with Seasonality

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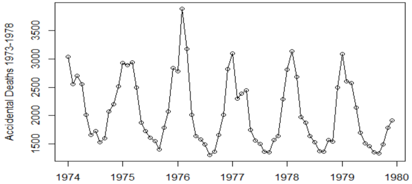
Models with Seasonality

In this case a zero-mean model is not working (clearly). But we can separte the contributions to the model in two parts, of which one is zero-mean.

$$x_t = S_t + y_t$$

where S_t contains the seasonality components, and y_t has zero mean. How to extract S_t?



Modelling Time Series

Modelling Time Series Steps

1. Plot the series and examine the main characteristics (trend, seasonality, etc.)
2. Remove the trend and seasonal components to get "stationary residuals/models"
3. Choose a model to fit the residuals
4. Forecasting will be given by forecasting the residuals over the trend and seasonality.

Stationarity and trends

If the sample mean

$$\sum_{i=0}^{n} \frac{x_i}{n}$$

is constant the model (or system) is said to be **stationary** (no trend present).

The first step in a time series analysis is typically to make it stationary (remove or detect trends).

Trend Estimation

Moving averages and spectral smoothing are an essential nonparametric methods for trend estimation.

1. Smoothing with a finite moving average filter.
2. Exponential Smoothing
3. Smoothing by eliminating the high-frequency components (Fourier Series).
4. Polynomial fitting (Regression).

Exponential Smoothing

Exponential Smoothing

The averaged signal w_t is obtained with

$$w_t = \alpha x_t + (1-\alpha) w_{t-1}, \quad t = 2, ..., n, \quad \alpha < 1$$

and $w_1 = x_1$. By using the recursive relationship one can show that

$$w_t = \alpha x_t + \alpha(1-\alpha) x_{t-1} + \alpha(1-\alpha) x_{t-2} + ... + (1-\alpha)^t x_1$$

So points close in time (at t) are weighted with α, while points far in time, for example at 1,2, etc. are weighted with powers of $(1-\alpha)$, that are smaller and smaller since $1-\alpha < 1$.

Smoothing Moving Average

Moving Average

The averaged signal w_t is obtained with

$$w_t = \frac{\sum_{i=-q}^{q} x_{t-i}}{2q+1}$$

It is useful to think about w_t as being obtained from x_t by application of a linear operator (or filter) with weights $1/(2q+1)$. This is a low-pass filter in the sesne that it removes the rapidly fluctuating compnents.

Weigthed Moving Average

Weigthed Moving Average

The averaged signal w_t is obtained with

$$w_t = \sum_{i=-q}^{q} k(i) x_{t-i}$$

Where $k(i)$ is a kernel (or filter) that can be tuned to obtain different results.

Differencing

Differencing

The smoothed signal w_t is obtained with

$$w_t = x_t - x_{t-1}$$

Decomposition

Decomposition

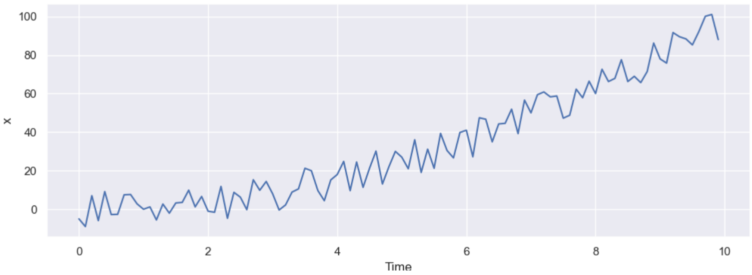
One of the main component of a time-series analysis is **Decomposition**: separating the different components in the series (trends, seasonal components, etc.).

Decomposition Example

We consider the model

$$x_t = t^2 + y_t, \quad t = 1, 2, ..., n$$

where y_t is white noise with a a mean of zero, and values that can go from 0 to 10.



Decomposition

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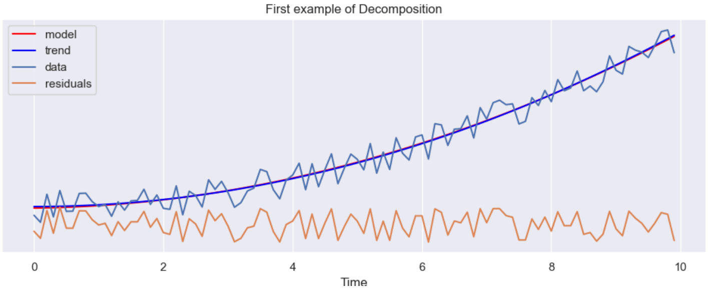
With a simple least square fit to a quadratic polynomial we can extract the trend and separate it from a zero-mean white noise.



Air Passenger Data

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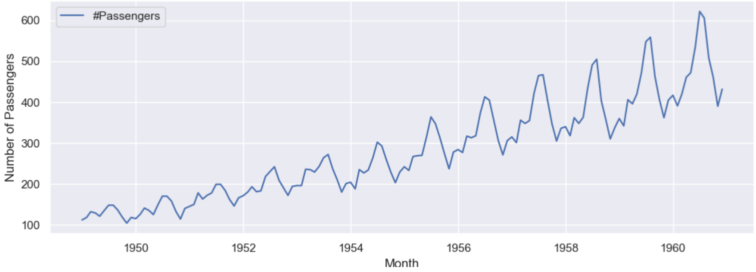
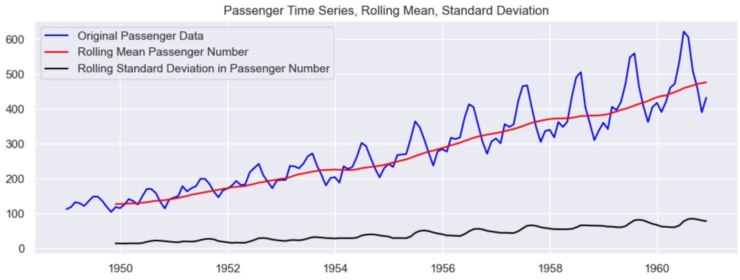
Data can be downloaded from `https://www.kaggle.com/datasets/rakannimer/air-passengers`. We have here several components: trend, seasonal components, white noise.



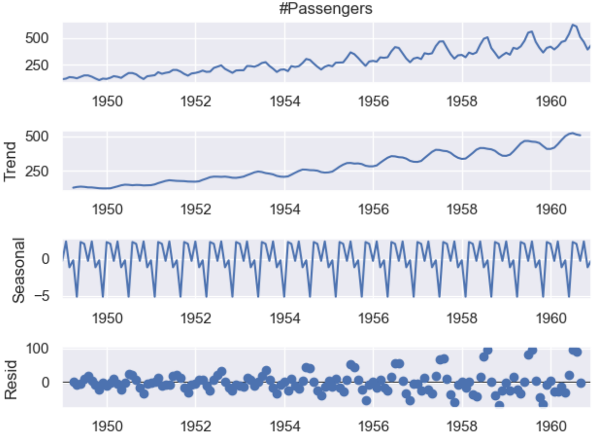
Trend Extraction (moving average)

One way to extract a trend and predict how it will develop is by using a moving average, that can isolate the trend from the seasonality and white noise (if the average window is large enough).



Example of Decomposition (automatic)


```
from statsmodels.tsa.seasonal import seasonal_decompose
```



Autocorrelation Function

Autocorrelation Function

One define the **autocorrelation function** $\gamma_x(r,s)$ as

$$\gamma_x(r,s) = \mathrm{Cov}(x_r, x_s) = \mathbb{E}((x_r - \mu_x(r))(x_s - \mu_x(s)))$$

where

$$\mu_x(h) = \mathbb{E}(x_h)$$

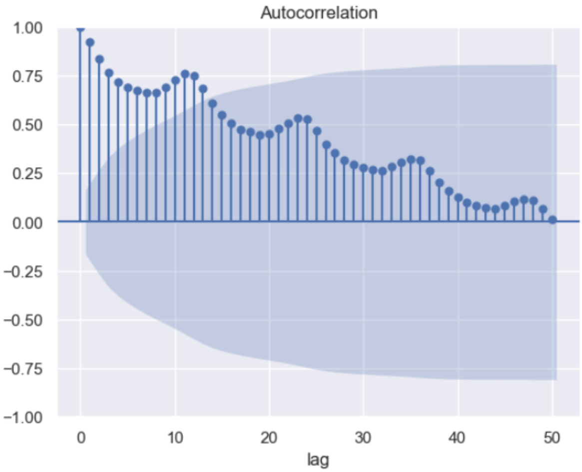
the distance in time $h = s - r$ is sometime called *lag*.

Sample Autocorrelation Function

The autocorrelation function will reflect and gives us indication about a time series. A few examples are:

1. White noise: zero
2. Trend: slow decay
3. Periodic: periodic
4. Moving Agerage(q): Zero for $|h| > q$

Sample Autocorrelation Function (passenger dataset)



Most common Time-Series Analysis models

- Box-Jenkins ARIMA models: These univariate models are used to better understand a single time-dependent variable, such as temperature over time, and to predict future data points of variables. These models work on the assumption that the data is stationary. Analysts have to account for and remove as many differences and seasonalities in past data points as they can.
- Box-Jenkins Multivariate Models: Multivariate models are used to analyze more than one time-dependent variable, such as temperature and humidity, over time.
- Holt-Winters Method: The Holt-Winters method is an exponential smoothing technique. It is designed to predict outcomes, provided that the data points include seasonality.

Example of ARIMA model