My Math Memoir

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§1.1 Problems

Problem 1 (∞ · MO 2019)**.** Let $\mathbb{Z}_{>1} = \{2, 3, \ldots\}$. Decide if there exists a function $f : \mathbb{Z}_{>1} \to \mathbb{Z}_{>1}$ which obeys the identity

$$f^{f(n)}(m) = m^n$$

for all positive integers m and n greater than 1.

§1.2 Solutions

Problem 1 (∞ · MO 2019)**.** Let $\mathbb{Z}_{>1} = \{2, 3, \ldots\}$. Decide if there exists a function $f : \mathbb{Z}_{>1} \to \mathbb{Z}_{>1}$ which obeys the identity

$$f^{f(n)}(m) = m^n$$

for all positive integers m and n greater than 1.
Solution. We claim no such function exists.
Let's suppose there exists such a function from now on.

> **Claim** — f is an exponential function.

Proof. It's evident that all perfect powers are contained in $\operatorname{im} f$.[1] So there exists $n_1, n_2 \in \mathbb{Z}_{>1}$ such that $f(n_1) = 8$ and $f(n_2) = f(n_1) + 1 = 9$. We then have,

$$m^{f(n_2)} = f(m^n) = f^{f(n_1)+1}(m) = f(m)^{f(n_1)}.$$

Thus the result follows. □

But exponential functions induces size issues i.e, $f^{f(n)}(m)$ will be much larger than m^n which is a contradiction and we are done. □

[1] Then somehow it came to me to use the fact that 8 and 9 are consecutive perfect powers.

Chapter II

Combinatorics

§1 Processes

§1.1 Problems

Problem 1 (IMO Shortlist C1, 2022). A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C so that, for any ± 1-sequence, there exists an integer k and indices $1 \leq t_1 < \ldots < t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq C.$$

Problem 2 (India P5, 2025). Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins he could have collected?

Problem 3 (IMO P1, 2022). The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to ...$
Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 4 (India P5, 2018). There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps. (Assume that, the teacher has an abundant supply of apples.)

§1.2 Solutions

Problem 1 (IMO Shortlist C1, 2022). A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C so that, for any ± 1-sequence, there exists an integer k and indices $1 \le t_1 < \ldots < t_k \le 2022$ so that $t_{i+1} - t_i \le 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \ge C.$$

Solution. We claim that the optimal value of C is 506.
Constructing a sequence that gives an upper bound of 506 on C is very natural. Consider the sequence,

$$1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1, 1, -1, -1, 1, -1.$$

There are 505 copies of $1, 1, -1, -1$ chunks with an anomaly at the end. Say we want to maximise the number of 1s in the list, then we must have at least 507 more 1s than -1s and this is impossible as appending a 1 in our list comes at the cost of appending a -1 unless we are yet to start.

¶ Establishing The Lower bound on C. We will show that no matter what $a_1, a_2, \ldots, a_{2022}$ are, we can always achieve

$$\left| \sum_{1 \le i \le k} a_{t_i} \right| \ge 506 \qquad \text{with } t_{i+1} - t_i \le 2,$$

for some k. Say there are at least 1011 ones in the sequence and we want to include all of it in the list i.e, $a_{t_1}, a_{t_2}, \ldots, a_{t_k}$. We do so according to the following algorithm,

- Begin from the smallest index that contains a 1 and append it to the list.
- If the next nearest 1 is at most two steps away, jump to that index and append it to the list.
- And if you don't find a 1, then always jump to the right by two steps until you find a 1 and of course appending the number you jumped on each time. Eventually if you encounter a 1 which is at most two steps away, follow the above procedure.

Aha! following these steps, the worst case is you would have collected at most 505 (a jump of length 2 is essentially avoiding a -1) number of -1s in the list and we are done. $\square$

Problem 2 (India P5, 2025). Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins he could have collected?
Solution. We claim Griphook can pick a maximum of 1998 coins and a minimum of 668 coins. We prove a more general proposition: that given any n-gon for $n \ge 3$, Griphook

can pick at most $n-2$ coins and the minimizing case is a bit tricky given by the following formula,

$$\text{Minimum number of coins Griphook must pick} = \begin{cases} \lceil n/3 \rceil + 1 & \text{if } n \equiv 2 \pmod 3, \\ \\ \lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

(this proposition is simply motivated by checking for small cases). Firstly, we shall deal with the maximizing case and we shall do that by constructing an algorithm for collecting $n-2$ coins and then arguing that it cannot get any better. For convenience we represent the number of coins on each vertex as a string with i^{th} number representing the number of coins on the i^{th} vertex. Consider picking the coins in the following manner, i.e, pick the next vertex each time starting from the second vertex,

$$1\underline{1}1111\ldots111 \to 02\underline{0}111\ldots111 \to 011\underline{0}11\ldots111 \to 0101\underline{0}1\ldots111 \to \cdots \to 010000\ldots010.$$

As you may see there will be exactly two coins left that are two vertices away from each other at the end. Suppose we could do better, clearly we cannot pick all the n coins, so we are left with the only possibility of picking $n-1$ coins and assume we could. By backtracking we have,

$$010000\ldots000 \leftarrow 101000\ldots000 \leftarrow 110100\ldots000 \leftarrow 111010\ldots000 \leftarrow \cdots \leftarrow 111111\ldots101.$$

Note that the above sequence of moves is the only possibility (ahh ignore the cyclic permutations). But, the configuration we get at the end is never achievable hence $n-2$ is the best upper bound.

Now we shall deal with the minimizing case. Divide the string into substrings each of length 3 and for Griphook to stop, it must be that each of them contain at least one 0 at the extremities. Notice, each move can generate only one substring with at least a single 0 at the extremity, hence the lower bound. When $n \equiv 2 \pmod 3$ there will always be a $\overline{202}$ string at the end and we are done. ☐

Problem 3 (IMO P1, 2022)**.** The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to \ldots$
Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.
Solution. We claim that (n, k) is such a pair if and only if it is contained in the following set defined as,

$$S = \left\{ (n, k) \in \mathbb{Z}^2_{\geq 1} \middle| n \leq k \leq n + \left\lceil \frac{n}{2} \right\rceil \right\}$$

Firstly note that for $k < n$ we can construct a counter sequence of coins which shows that the final state cannot be achieved;

$$\underbrace{A \dots A}_{n-1\ A\text{'s}} \underbrace{C \dots C}_{n-1\ C\text{'s}} AC.$$

Similarly we show that $k \leq n + \lceil n/2 \rceil$ by constructing the sequence which loops to itself every time it completes a cycle. Also, to keep things neat and clean denote the contiguous string of coins of the same metal M of size $\#$ as $M^{\#}$, consider the sequence;

$$A^{\lfloor \frac{n}{2} \rfloor} C^{\lceil \frac{n}{2} \rceil} A^{\lceil \frac{n}{2} \rceil} C^{\lfloor \frac{n}{2} \rfloor} \to C^{\lfloor \frac{n}{2} \rfloor} A^{\lfloor \frac{n}{2} \rfloor} C^{\lceil \frac{n}{2} \rceil} A^{\lceil \frac{n}{2} \rceil} \to \dots \to A^{\lfloor \frac{n}{2} \rfloor} C^{\lceil \frac{n}{2} \rceil} A^{\lceil \frac{n}{2} \rceil} C^{\lfloor \frac{n}{2} \rfloor} \to \cdots$$

Now, we are only left with the case when $n \leq k \leq n + \lceil n/2 \rceil$ and as per our claim this range of values of k works. Assume that for some k in this range, some sequences do not attain the favourable end state. And for this to happen, there has to be a stagnant point after which the size of any contiguous string of coins of the same metal does not increase at all if not it has to reach the end state.

We would like to characterize the sequences that have reached the stagnant point. Note that the right-most contiguous string of the same metal has to have at least $2n + 1 - k$ coins at each iteration, else, it will contradict the stagnant point. With this argument, we assert that size of every contiguous string of coins of the same metal has to be at least $2n + 1 - k$ and as $n \leq k \leq n + \lceil n/2 \rceil$ we may certainly state that the stagnant point is the end state itself and we are done. □

Problem 4 (India P5, 2018)**.** There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps. (Assume that, the teacher has an abundant supply of apples.)

Solution. Let $\mathcal{G}$ denote the set of girls g_i where $i = 1, 2, \dots, n$ and let $\alpha_j(g)$ denote the number of apples with g after j^{th} iteration for $j = 0, 1, \dots, n$.

Something cool pops up when we study the extremal object $\max\limits_{g \in \mathcal{G}} \alpha_j(g)$.

Lemma 1. $\max\limits_{g \in \mathcal{G}} \alpha_j(g) \geq \max\limits_{g \in \mathcal{G}} \alpha_{j+1}(g)$.

By the above result we have an upper bound for the total number of apples in the class i.e,

$$\sum_{g \in \mathcal{G}} \alpha_j(g) \leq |\mathcal{G}| \max_{g \in \mathcal{G}} \alpha_0(g).$$

But the total number of apples after each iteration seems to be increasing by 1 which forces this process to terminate after a finite number of iterations. □

§2 Constructions

§2.1 Problems

Problem 1 (India P3, 2020). Let S be a subset of $\{0, 1, 2, \ldots, 9\}$. Suppose there is a positive integer N such that for any integer $n > N$, one can find positive integers a, b so that $n = a + b$ and all the digits in the decimal representations of a, b (expressed without leading zeros) are in S. Find the smallest possible value of $|S|$.

Problem 2 (Sleepy Students – OTIS). There are n sleepy students working on a morning constest. The contest has six problems, and the score on each problem is a non negative integer less than or equal to 10. Given that no two students got the same score on two or more problems, what is the greatest possible value of n?

Problem 3 (RMM P1, 2015). Does there exist an infinite sequence of postivie integers $a_1, a_2, \ldots$ such that $\gcd(a_m, a_n) = 1$ if and only if $|m - n| = 1$?

Problem 4 (Junior Balkan Shortlist C2, 2021). Let n be a positive integer. We are given a $3n \times 3n$ board whose unit squares are colored in black and white in such way that starting with the top left square, every third diagonal is colored in black and the rest of the board is in white. In one move, one can take a 2×2 square and change the color of all its squares in such way that white squares become orange, orange ones become black and black ones become white. Find all n for which, using a finite number of moves, we can make all the squares which were initially black white, and all squares which were initially white black.

Problem 5 (India P4, 2025). Let $n \geq 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

$$\max\left(|a_1 + a_2|, |a_2 + a_3|, \ldots, |a_{n-1} + a_n|, |a_n + a_1|\right) \geq t_n \cdot \max(|a_1|, |a_2|, \ldots, |a_n|)$$

holds for all real numbers $a_1, a_2, \ldots, a_n$.

Problem 6 (Coluring Numbers Efficiently-Kithun). What is the least number required to colour the integers $1, 2, \ldots, 2^n - 1$ such that for any set of consecutive integers taken from the given set of integers, there will always be a colour colouring exactly one of them? That is, for all integers i, j such that $1 \leq i \leq j \leq 2^n - 1$, there will be a colour coloring exactly one integer from the set $i, i + 1, \ldots, j - 1, j$.

Problem 7 (Ahan Chakraborty-Unknown). Let $n \in \mathbb{N}$. Let $X = \{1, 2, 3, ..., n^2\}$.Let $A \subset X$ with $|A| = n$. Prove that $X \setminus A$ contains an arithmetic progression with n terms.

Problem 8 (IM0 P2, 2014). Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any its k^2 unit squares.

§2.2 Solutions

Problem 1 (India P3, 2020)**.** Let S be a subset of $\{0, 1, 2, \dots, 9\}$. Suppose there is a positive integer N such that for any integer $n > N$, one can find positive integers a, b so that $n = a + b$ and all the digits in the decimal representations of a, b (expressed without leading zeros) are in S. Find the smallest possible value of $|S|$.
Solution. We claim $\min |S| = 5$, which can be achieved by taking $S = \{0, 1, 2, 3, 7\}$.

¶ Why Does the Construction Work? Note that sum of pairs of elements (not necessarily distinct) of S generate the whole residue class modulo 10,

	0	1	2	3	7
0	**0**	**1**	**2**	3	**7**
1	·	2	**3**	4	**8**
2	·	·	**4**	**5**	**9**
3	·	·	·	**6**	0
7	·	·	·	·	4

The nice thing about the construction is that it makes sure there's no carrying-over stuff, so we may choose each digit of a and b independently. For example, say we want to generate the number 59078294316, we simple choose a, b according to the above table

$$\begin{array}{cccccccccccc} & 2 & 2 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 3 \\ + & 3 & 7 & 0 & 7 & 7 & 2 & 7 & 2 & 2 & 1 & 3 \\ \hline & \mathbf{5} & \mathbf{9} & \mathbf{0} & \mathbf{7} & \mathbf{8} & \mathbf{2} & \mathbf{9} & \mathbf{4} & \mathbf{3} & \mathbf{1} & \mathbf{6} \end{array}$$

Note. In fact we get $N = 1$ which is cool.

¶ Establishing the Lower Bound. We shall show that $|S| > 4$. Suppose there exists a single digit number d such that $s_1 + s_2 \not\equiv d \pmod{10}$ for each $(s_1, s_2) \in S \times S$ then we will not be able to generate

$$\underbrace{\overline{** \dots *d}}_{\text{take sufficiently large number}}$$

as a sum of a, b. Hence it must be that,

$$\{s_1 + s_2 \pmod{10} \mid s_1, s_2 \in S\} = \mathbb{Z}_{10} \quad \text{meaning,} \quad \binom{|S|}{2} + |S| \geq 10.$$

From the above mentioned bound it is quite clear that $|S| \geq 4$, so we are left to show that $|S| \neq 4$. Suppose $S = \{s_1, s_2, s_3, s_4\}$, notice $2s_i \pmod{10}$ generates four distinct even numbers among $\mathbb{Z}_{10}$. So either exactly one element of S is even or all are odd, the latter implies $s_i + s_j$ is even for $i \neq j$ meaning the number of even numbers generated exceeds $10/2$ which is ridiculous. Similarly, we argue that the former case is also not possible, therefore $|S| > 4$ and we are done. $\square$

Problem 2 (Sleepy Students – OTIS)**.** There are n sleepy students working on a morning constest. The contest has six problems, and the score on each problem is a non negative integer less than or equal to 10. Given that no two students got the same score on two or more problems, what is the greatest possible value of n?

Solution. The answer is 121.

By pigeonhole principle it is evident that n is atmost 121. And fortunately by the virtue of this artificial setup, there exists a construction for $n = 121$ – consider the scores of each student to be an arithmetic progression of course under modulo 11. There are 11 choices each for the intial term and the common difference hence making $11 \times 11 = 121$ of them.

¶ But how are we sure that no two of them have two or more coinciding terms? Suppose some two different arithmetric progressions namely, $\{a, a+d, \ldots, a+5d\}$ and $\{a', a'+d', \ldots, a'+5d'\}$ have two coinciding terms say $a+nd = a'+nd'$ and $a+md = a'+md'$ with $0 \le n \ne m \le 5$. As 11 is prime, by cancellation law we have that $n \equiv m$ which is a contradiction and we are done. ☐

> **Remark.** I had to look up the ARCH for hints regarding the construction.

Problem 3 (RMM P1, 2015)**.** Does there exist an infinite sequence of postivie integers $a_1, a_2, \ldots$ such that $\gcd(a_m, a_n) = 1$ if and only if $|m-n| = 1$?

Solution. The answer is yes.

Denote the n^{th} prime number as p_n. Consider the following construction –

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
p_1	•		•		•		•		•	
p_2		•		•		•		•		•
p_3	•			•		•		•		•
p_4		•			•		•		•	
p_5			•			•		•		•
p_6				•			•		•	
p_7					•			•		•
p_8						•			•	
p_9							•			•
p_{10}								•		

proof with little to no words[1]

Clearly each a_n is finite and satisfy the hypothesis. ☐

Problem 4 (Junior Balkan Shortlist C2, 2021)**.** Let n be a positive integer. We are given a $3n \times 3n$ board whose unit squares are colored in black and white in such way that starting with the top left square, every third diagonal is colored in black and the rest of the board is in white. In one move, one can take a 2×2 square and change the color of all its squares in such way that white squares become orange, orange ones become black and black ones become white. Find all n for which, using a finite number of moves, we can make all the squares which were initially black white, and all squares which were initially white black.

Solution. The desired end state can be acheived if and only if n is even.

[1]Here a '•' is at some position (i, j) if and only if $p_i | a_j$.

> **Proposition 1.** A square which was initially black should be subjected to $1 \pmod 3$ moves and $2 \pmod 3$ moves if the concerning square was white initially inorder to attain the end state.

Define score of a 2×2 square as the number of times it has been chosen throughout. Observe the top row closely, the rightmost 2×2 square must have a score which is $1 \pmod 3$, so this forces the next square to the left to have a score that is $1 \pmod 3$ and so on till we reach the left most square. Denote black squares as B, white square as W and orange squares as O. For example consider $n = 3$, the scores of squares in the top rows modulo 3 arranged in order must look like,

$$\begin{array}{c} \begin{matrix} \_ & \color{red}{1} & \color{red}{1} & \color{red}{0} & \color{red}{2} & \color{red}{0} & \color{red}{1} & \color{red}{1} & \color{red}{1} \end{matrix} \\ \begin{bmatrix} W & W & B & W & W & B & W & W & B \\ W & B & W & W & B & W & W & B & W \\ B & W & W & B & W & W & B & W & W \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B & W & W & B & W & W & B & W & W \end{bmatrix} \end{array}.$$

Case 1. When n is odd.
Considering the previous example, note the top-left most square cannot turn black at the end of the process no matter what. This is because the score of the only square it is a part of is $1 \pmod 3$. By induction we can easily show this holds for all odd n. Which means the first row itself can never reach the end state. Not to mention, one can easily show the case $n = 1$ does not work.

Case 2. When n is even.
It would suffice to show that end state can be acheived for 6×6 grid as the successive grids $12 \times 12, 18 \times 18$ and so can be partitioned into 6×6 grids that repeat themselves. The algorithm is, flip the colours of a rows one at a time starting from the top then traversing till the bottom.

$$\begin{bmatrix} W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \\ W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix} \to \begin{bmatrix} B & B & W & B & B & W \\ B & O & O & B & O & O \\ B & W & W & B & W & W \\ W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix} \to \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ B & B & O & B & B & O \\ W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix}$$

$$\to \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \\ O & W & W & O & W & W \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix} \to \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \\ B & B & W & B & B & W \\ O & O & W & O & O & W \\ B & W & W & B & W & W \end{bmatrix} \to \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \\ B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \end{bmatrix}.$$

Longing turned to having. □

Problem 5 (India P4, 2025)**.** Let $n \geq 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

$$\max\left(|a_1+a_2|, |a_2+a_3|, \ldots, |a_{n-1}+a_n|, |a_n+a_1|\right) \geq t_n \cdot \max(|a_1|, |a_2|, \ldots, |a_n|)$$

holds for all real numbers $a_1, a_2, \ldots, a_n$.

Solution. We claim $t_n = 0$ when n is even and $t = 2/n$ otherwise.

The case when n is even is trivial because of the following construction,

$$(a_1, a_2, \ldots, n) = (1, -1, \ldots, -1).$$

Which gives $t_n = 0$.

Now we deal with the case when n is odd. The central theme of the solution is to exploit the triangle inequality, by which we have

$$\begin{aligned} n \cdot \max_{1\leq i<n} \left\{|a_i + a_{i+1}|, |a_n + a_1|\right\} &\geq |(a_n + a_1)| + \sum_{1\leq i<n} |(a_i + a_{i+1})| \\ &\geq 2 \max_{1\leq i\leq n} |a_i| \end{aligned}$$

So we have a nice lower bound for t_n which is $t \geq 2/n$, and fortunate enough this is the upper bound as well. Consider the construction that make sures $\max_{1\leq i<n} \left\{|a_i + a_{i+1}|, |a_n + a_1|\right\} = 2$,

$$(a_1, a_2, \ldots, a_n) = (1, -3, 5, \ldots, (-1)^{\frac{n-1}{2}} n, \ldots, 5, -3, 1).$$

□

Problem 6 (Coluring Numbers Efficiently-Kithun)**.** What is the least number required to colour the integers $1, 2, \ldots, 2^n - 1$ such that for any set of consecutive integers taken from the given set of integers, there will always be a colour colouring exactly one of them? That is, for all integers i, j such that $1 \leq i \leq j \leq 2^n - 1$, there will be a colour coloring exactly one integer from the set $i, i+1, \ldots, j-1, j$.

Solution. Denote the minimum number of colours required to colour the integers $1, 2, \ldots, 2^n - 1$ such that for any set of consecutive integers taken from the given set of integers, there will always be a colour colouring exactly one of them as χ_n. For convenience call a colouring *minimally-valid* if the colouring is "valid" and uses only χ_n different colours. The key idea is to come up with good enough non-trivial bounds on χ_n. If we are lucky, the lower bound might turn out to be the same as that of the upper bound and we win.

Before jumping into the details of χ_n, let us try colouring the numbers $1, 2, \ldots, 2^4 - 1$ (base case being $n = 3$ is rather trivial giving us little to no insights), maybe we can come up with a nice construction that could be generalised easily.

1 2 3 4 5 6 7 | 8 9 10 11 12 13 14 15

Note that, the above colouring is valid and turns out it is minimally valid as well (proof simply follows by considering a colouring using three or less distinct colours and getting a contradiction). Cool, but how can we proceed for cases $n = 5, 6, 7$ and so on? For $n = 5$ just colour the numbers $1, 2, \ldots, 15$ as shown above and colour 16 with a new colour which is not used to colour the numbers before it. Colour the remaining numbers $17, 18, \ldots, 31$

in the same manner in which we coloured $1, 2, \ldots, 15$. For instance 17 is orange, 18 is blue and so on. This means $\chi_5 \leq 5$. Aha!! we can continue this process indefinitely for bigger n introducing exactly one new colour each time giving us the upper bound $\chi_n \leq n$. We now ask the question, if this upper bound can be refined further and fortunately the answer is a clear no.

> **Claim 1.** $\chi_n \geq n$.

Proof. Partition the given set of integers into three subsets as $A = \{1, 2, \ldots, 2^{n-1} - 1\}$, $B = \{2^{n-1}\}$ and $C = \{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n - 1\}$. Notice, the numbers in A must be coloured with at least χ_{n-1} distinct colours and similarly C must be coloured with at least χ_{n-1} distinct colours. Our goal isto show that $\chi_n = \chi_{n-1} + 1$. Suppose not, then we know that χ_n is at least χ_{n-1}. So the only case we have to deal with is when $\chi_n = \chi_{n-1}$.If you look closely, there must be a number among $1, 2, \ldots, 2^n - 1$ with a unique colour, if not, consider the whole set as $2^n - 1$ consecutive integers giving us a contradiction. In order to minimize χ_n, it must be that numbers in A and C must be coloured with the same set of colours. But note that every colour is used at least two times, which means the number in B that is 2^{n-1} must be coloured with a unique colour and that is a contradiction as we had initially assumed $\chi_n = \chi_{n-1}$. Hence the lower bound. $\square$

It is evident that $n \leq \chi_n \leq n$, hence giving us the required result that is $\chi_n = n$. $\square$

> **Remark.** A further question one may ask is, how many minimally-valid colourings are possible in total given a set of distinct colours $C_1, C_2, \ldots, C_{\chi_n}$?

Problem 7 (Ahan Chakraborty-Unknown)**.** Let $n \in \mathbb{N}$. Let $X = \{1, 2, 3, ..., n^2\}$.Let $A \subset X$ with $|A| = n$. Prove that $X \setminus A$ contains an arithmetic progression with n terms.
Solution. Suppose I could construct a subset A such that $X \setminus A$ does not contain an arithmetic progression with n elements. We shall device an algorithm for constructing it and then arrive at a conclusion that it is not possible. Arrange the elements on X on a $n \times n$ grid as follows,

$$\begin{array}{cccccc}
1 & 2 & 3 & \cdots & n-1 & n \\
n+1 & n+2 & n+3 & \cdots & 2n-1 & 2n \\
2n+1 & 2n+2 & 2n+3 & \cdots & 3n-1 & 3n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^2-2n+1 & n^2-2n+2 & n^2-2n+3 & \cdots & n^2-n-1 & n^2-n \\
n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2-1 & n^2
\end{array}$$

The nice thing about this arrangement is that, several n-element arithmetic progressions are very easy to spot i.e, just walk along the rows, columns, main diagonals. Notice that we must pick exactly one element from each row and column and place it in A, or else the row/column which doesn't share an element with A will be a counter example. If we pick 1, then we cannot pick any other element from the first row and also we cannot pick $n+1$, which means $\{2, 3, \ldots, n+1\} \subseteq X \setminus A$ and that does not sound good to us. Similarly we can argue that none of the elements in the first column can be picked except for the last one (with value $= n^2 - n + 1$). Continuing this argument we can see that the ideal construction is to pick all the diagonal elements (the main diagonal that contains $n^2 - n + 1$). Note that $\{n-1, 2n-2, 3n-3, \ldots, (n-1)^2\} \cup \{(n-1)^2 + n\} \subseteq X \setminus A$, hence contradicting the existence of such an A and we are done. $\square$

> **Remark.** This way of constructing the algorithm resembles placing n non-attacking rooks on a $n \times n$ chessboard, which seem like two totally unrelated topics.

Problem 8 (IM0 P2, 2014)**.** Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any its k^2 unit squares.

Solution. The answer is $\lfloor n - 1 \rfloor$.

Let $f(n)$ denote the greatest such k for a given n.

§3 Graph Theory

§3.1 Problems

Problem 1 (MTRP). In the planet of MTRPia, one alien named Bob wants to build roads across all the cities all over the planet. The alien government has imposed the condition that this construction must be carried out in such a way so that one can go from one city to any other city through the network of roads thus constructed. To have consistency in the whole process, Bob decides to have an even number of lords originating from each city. Prove that starting from an arbitrary city one can traverse the whole network of roads without ever traversing the same road twice.

§3.2 Solutions

Problem 1 (MTRP). In the planet of MTRPia, one alien named Bob wants to build roads across all the cities all over the planet. The alien government has imposed the condition that this construction must be carried out in such a way so that one can go from one city to any other city through the network of roads thus constructed. To have consistency in the whole process, Bob decides to have an even number of lords originating from each city. Prove that starting from an arbitrary city one can traverse the whole network of roads without ever traversing the same road twice.

Solution. The key question to ask is, if the contrary were true why would one fail traversing the whole network of roads without ever returning to the same road twice? For convenience we shall talk in terms of graph theory terminologies, where the cities are an analogue to vertices and roads to that of edges. Let the graph be $\mathcal{G}(V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ for some natural n. Consider the process,

- Pick any vertex say v_{a_1} and then pick any of its neighbour v_{a_2}.
- Delete the edge (v_{a_1}, v_{a_2}) from E.
- Now repeat the above steps but starting from v_{a_2}.

Claim — The above process will do the job.

Proof. The virtue of the defined process is that it preserves $\deg(v_i) \pmod 2$ for all $i = 1, 2 \ldots, n$ throughout. Which means it will only stop when it encounters a vertex v when $\deg v = 0$, so ignoring v we get a new graph with $n - 1$ vertices, from here it follows from induction with base case being a $3-$clique. And that my friend is easy to work with. □

So yeah Bob's idea is pretty cool. □

§4 Global Ideas and the Probabilistic Method

§4.1 Problem

Problem 1 (USAMO P4, 1985). There are n people at a party. Prove that there are two people such that, of the remaining $n-2$ people, there are at least $\left\lfloor \frac{n}{2} \right\rfloor - 1$ of them, each of whom either knows both or else knows neither of the two. Assume that knowing is a symmetric relation, and that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

§4.2 Solutions

Problem 1 (USAMO P4, 1985). There are n people at a party. Prove that there are two people such that, of the remaining $n-2$ people, there are at least $\left\lfloor \frac{n}{2} \right\rfloor - 1$ of them, each of whom either knows both or else knows neither of the two. Assume that knowing is a symmetric relation, and that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.
Solution. Pick pairs of two people uniformly at random and define a random variable X such that

$$X \overset{\text{def}}{=} \begin{cases} \text{\# People among the remaining } n-2 \text{ people who are acquainted to both or neither.} \end{cases}$$

Our goal is to show that $\mathbb{E}[X] \geq \left\lfloor \frac{n}{2} \right\rfloor - 1$, as it will automatically assure the existence of two such people. But computing $\mathbb{E}[X] = \sum_{v \in V} v\mathbb{P}[X = v]$ seems to be a pain in the as*, not a big deal as linearity of $\mathbb{E}[X]$ is a boon. We shall introduce auxiliary random variables X_i for $i = 1, 2, \ldots, n-2$ which will help in the computation, consider

$$X_i \overset{\text{def}}{=} \begin{cases} 0 & \text{if } i^{\text{th}} \text{ person among the remaining is acquainted to exactly one of them,} \\ 1 & \text{otherwise.} \end{cases}$$

Notice $\mathbb{P}[X_i = v] = 1/2$. As $X = \displaystyle\sum_{1 \leq i \leq n-2} X_i$ and using the boon we mentioned earlier we have the following,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{1 \leq i \leq n-2} X_i\right] = \sum_{1 \leq i \leq n-2} \mathbb{E}[X_i] = \sum_{v \in V} v\mathbb{P}[X_i = v] = \frac{n-2}{2} \geq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

And we are done. □

§5 Combinatorial Games

§5.1 Problems

Problem 1 (India P4, 2021)**.** A Magician and a Detective play a game. The Magician lays down cards numbered from 1 to 52 face-down on a table. On each move, the Detective can point to two cards and inquire if the numbers on them are consecutive. The Magician replies truthfully. After a finite number of moves, the Detective points to two cards. She wins if the numbers on these two cards are consecutive, and loses otherwise.
Prove that the Detective can guarantee a win if and only if she is allowed to ask at least 50 questions.

Problem 2 (IMO P4, 2018)**.** A *site* is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.
Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.
Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

§5.2 Solutions

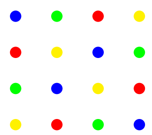
Problem 1 (India P4, 2021)**.** A Magician and a Detective play a game. The Magician lays down cards numbered from 1 to 52 face-down on a table. On each move, the Detective can point to two cards and inquire if the numbers on them are consecutive. The Magician replies truthfully. After a finite number of moves, the Detective points to two cards. She wins if the numbers on these two cards are consecutive, and loses otherwise.
Prove that the Detective can guarantee a win if and only if she is allowed to ask at least 50 questions.
Solution. Consider a complete graph K_{52} where each vertex C_i denotes the i^{th} card. So each edge represents a potential win. Delete an edge C_iC_j if the Detective has already inquired about the card C_i and C_j in a move.

❡ Strategy for The Detective. It is easy to that she can guarantee a win with 50 or less questions as she may fix a card say C_1 throughout and inquire about every other card.

❡ Strategy for The Magician. Suppose the Detective can guarantee a win with 49 questions then we shall show that she has a non zero probability of losing. Let us say she has a pre-planned strategy of picking pairs of cards from the positions $(a_1, b_1), (a_2, b_2), \ldots, (a_{49}, b_{49})$ in that order with $1 \leq a_i, b_i \leq 52$ and $a_i \neq b_i$. But notice if there is a Hamiltonian path even after deleting 49 edges of K_{52}, the Magician could have simply placed the cards $C_1, C_2, \ldots, C_{52}$ in such a way, that these cards lie along the Hamiltonian path in that order. Which means none of the pairs of cards picked from the positions $(a_1, b_1), (a_2, b_2), \ldots, (a_{49}, b_{49})$ contain cards with consecutive numberings and hence a non zero probability of the Detective losing. And yes, if you delete any 49 edges from the graph a Hamiltonian path always exists due to **Ore's theorem**. Therefore we may conclude, 50 questions is the least number of questions required to guarantee a win. □

Problem 2 (IMO P4, 2018). A *site* is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.
Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.
Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.
Solution. The answer is 100.

❡ Strategy for Ben. The key idea partition the 20×20 set of lattice points into 25 pieces of 4×4 lattice points which we'll call **units** as depicted below. And then colour each of the sites in one of four colours in this manner.



Suppose Amy places her stone on a site, then Ben can always place his stone in the same 4×4 unit and same coloured site in which Amy had placed her stone just before in such way that Amy can place no more stones in the same coloured site in that unit. And this strategy of Ben forces Amy to not place more than 4 stones in each unit. Hence the upper bound of 100.

❡ Strategy for Amy. Our goal is to show that Amy can place at least 100 stones regardless of the wildest of strategies Ben designs. We simply colour the lattice points in checkered pattern with black and white. Then Amy can always choose to place her stones in white spots (non-attacking knights). Ben can best ruin this strategy by placing all his stones in white sites. But even then Amy would placed 100 of her stones.

And we are done. □

§6 Miscellaneous

§6.1 Problems

Problem 1 (India P1, 2021)**.** Suppose $r \geq 2$ is an integer, and let $m_1, n_1, m_2, n_2, \dots, m_r, n_r$ be $2r$ integers such that

$$|m_i n_j - m_j n_i| = 1$$

for any two integers i and j satisfying $1 \leq i < j \leq r$. Determine the maximum possible value of r.

§6.2 Solutions

Problem 1 (India P1, 2021)**.** Suppose $r \geq 2$ is an integer, and let $m_1, n_1, m_2, n_2, \dots, m_r, n_r$ be $2r$ integers such that

$$|m_i n_j - m_j n_i| = 1$$

for any two integers i and j satisfying $1 \leq i < j \leq r$. Determine the maximum possible value of r.

Solution. We claim the maximum possible value of r is 3.
What does the subscripts of $m_i n_j$ and $m_j n_i$ remind us of - matrix elements that are reflections of each other in the main diagonal. So let us arrange the numbers as follows for better intuition,

$$\begin{matrix} m_1n_1 & m_1n_2 & m_1n_3 & m_1n_4 & \cdots & m_1n_r \\ m_2n_1 & m_2n_2 & m_2n_3 & m_2n_4 & \cdots & m_2n_r \\ m_3n_1 & m_3n_2 & m_3n_3 & m_3n_4 & \cdots & m_3n_r \\ m_4n_1 & m_4n_2 & m_4n_3 & m_4n_4 & \cdots & m_3n_r \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_rn_1 & m_rn_2 & m_rn_3 & m_rn_4 & \cdots & m_rn_r \end{matrix}$$

Notice that entries which are reflections of each other (the subscripts) about the main diagonals are of different parity. Assume n_1 is even which means all the entries in the first row are odd (except the first entry of course). Say m_2 is even then we know both m_3 and n_2 are odd (as m_2n_3 and m_3n_2 are reflections of each other) also m_4 is odd. Here comes the catch if you look closely, both m_4n_3 and m_3n_4 are odd which is a contradiction. Hence $r \leq 3$ and for $r = 3$ a number of constructions are already given. $\square$

Chapter III

Geometry

It's very hard to sort geometry problems as each problem involves ideas from several different kinds. So it's very likely that I will not sort out these problems as I have for other disciplines.

§1 Assorted Geometry

§1.1 Problems

Problem 1 (EGMO Day 2 P6, 2023)**.** Let ABC be a triangle with circumcircle Ω. Let S_b and S_c respectively denote the midpoints of the arcs AC and AB that do not contain the third vertex. Let N_a denote the midpoint of arc BAC (the arc BC including A). Let I be the incenter of ABC. Let ω_b be the circle that is tangent to AB and internally tangent to Ω at S_b, and let ω_c be the circle that is tangent to AC and internally tangent to Ω at S_c. Show that the line IN_a, and the lines through the intersections of ω_b and ω_c, meet on Ω.

Problem 2 (IMO P5, 2012)**.** Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.

Problem 3 (Balkan P1, 2018)**.** A quadrilateral $ABCD$ is inscribed in a circle k where $AB > CD$,and AB is not paralel to CD.Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E.If EM bisects the angle CED prove that AB is diameter of k.

Problem 4 (Int'l Festival of Young Mathematicians 2019)**.** The perpendicular bisector of AB of an acute ΔABC intersects BC and the continuation of AC in points P and Q respectively. M and N are the middle points of side AB and segment PQ respectively. If the lines AB and CN intersect in point D, prove that ΔABC and ΔDCM have a common orthocenter.

Problem 5 (USAMO P6, 2006)**.** Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF,

TCF, and TDE pass through a common point.

Problem 6 (EGMO P4, 2019)**.** Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Problem 7 (EGMO P4, 2025)**.** Let ABC be an acute triangle with incentre I and $AB \neq AC$. Let lines BI and CI intersect the circumcircle of ABC at $P \neq B$ and $Q \neq C$, respectively. Consider points R and S such that $AQRB$ and $ACSP$ are parallelograms (with $AQ \parallel RB, AB \parallel QR, AC \parallel SP$, and $AP \parallel CS$). Let T be the point of intersection of lines RB and SC. Prove that points R, S, T, and I are concyclic.

Problem 8 (IMO Shortlist G3, 2006)**.** Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Problem 9 (Turkey TST 1998)**.** In a triangle ABC, the circle through C touching AB at A and the circle through B touching AC at A have different radii and meet again at D. Let E be the point on the ray AB such that $AB = BE$. The circle through A, D, E intersect the ray CA again at F . Prove that $AF = AC$.

Problem 10 (USA TST Day 3-P7, 2008)**.** Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

Problem 11 (India TST Day1-P1, 2019)**.** In an acute angled triangle ABC with $AB < AC$, let I denote the incenter and M the midpoint of side BC. The line through A perpendicular to AI intersects the tangent from M to the incircle (different from line BC) at a point P¿ Show that AI is tangent to the circumcircle of triangle MIP.

Problem 12 (USAJMO P5, 2011)**.** Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that

(i) lines PB and PD are tangent to ω,

(ii) P, A, C are collinear, and

(iii) $DE \parallel AC$.

Prove that BE bisects AC.

Problem 13 (IMO Shortlist G8, 2004)**.** Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

Problem 14 (ELMO Shortlist G1, 2023)**.** Let $ABCDE$ be a cyclic pentagon. Let P be a variable point on the interior of segment AB such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at Q. Let R be the circumcenter of $\triangle DPQ$. Show that

as P varies, R lies on a fixed line.
Note. The original problem had the condition that $ABCDE$ is regular but in this solution we will see that it is not necessary as cyclicity is sufficient.

Problem 15 (Crux Mathematicorum 5044)**.** Given a circle Γ with center O and a chord AB, let X be the midpoint of the larger arc AB, and C be an arbitrary point of that arc. Define K to be the point where the bisector of $\angle ACB$ intersects the tangent to Γ at B, while M is the intersection of AC and BX. Prove that the line MK contains the midpoint of AB.

§1.2 Solutions

Problem 1 (EGMO Day 2 P6, 2023)**.** Let ABC be a triangle with circumcircle Ω. Let S_b and S_c respectively denote the midpoints of the arcs AC and AB that do not contain the third vertex. Let N_a denote the midpoint of arc BAC (the arc BC including A). Let I be the incenter of ABC. Let ω_b be the circle that is tangent to AB and internally tangent to Ω at S_b, and let ω_c be the circle that is tangent to AC and internally tangent to Ω at S_c. Show that the line IN_a, and the lines through the intersections of ω_b and ω_c, meet on Ω.

Solution. Let S_a be the midpoint of arc BC not containing A (it is also the antipode of N_a in Ω). Define $X \neq N_a$ to be the meeting point of IN_a and Ω.

The problem simply asks us to show that X lies on the radical axis of ω_b and ω_c.

> **Claim** — Point A lies on the radical axis of ω_b and ω_c.

Proof. It suffices to show that $\text{Pow}_{\omega_b} A = \text{Pow}_{\omega_c} A$.

Shooting star lemma tells us that ω_b is tangent to AB at a point say M such that it passes through $\overline{S_b S_c}$ and by symmetry, one can argue the same for ω_c as well (define a point N on $\overline{AC}$ such that AN is tangent to ω_c).By Incentre-Excentre lemma, it follows that $\odot(S_b, |\overline{S_b A}|)$ and $\odot(S_c, |\overline{S_c A}|)$ pass through C and B respectively. Now, as AI is the radical axis of $\odot(S_b, |\overline{S_b A}|)$ and $\odot(S_c, |\overline{S_c A}|)$, it follows that $\overline{AI} \perp \overline{S_b S_c}$. We finish off by sending line BC to line $S_B S_c$ by perspectivity through A. As $AN_a \parallel S_b S_c$, it follows that,

$$-1 = (BC; AI \cap BC, AN_a \cap BC) \stackrel{A}{=} (MN; AI \cap S_b S_c, \infty_{S_b S_c}).$$

So $|AM| = |AN|$ as desired. □

Define K to be the radical centre of circles Ω, ω_b and ω_c (also K passes through the radical axis of ω_b and ω_c). And note, KS_b and KS_c are tangent to ω_b and ω_c. So it suffices to show, XA is the $X-$ symmedian of $\triangle XS_bS_c$.

> **Claim** — The cyclic quadrilateral AS_cXS_b is harmonic i.e,
> $$-1 = (AX; S_c S_b)_\Omega.$$

Proof. Recall that N_a is the antipode of S_a in Ω, meaning N_aBS_aC is harmonic. Consider sending Ω to itself by perspectivity through I to get,

$$-1 = (N_a S_a; BC)_\Omega \stackrel{I}{=} (XA; S_b S_c)_\Omega,$$

as desired. □

This indeed solves the problem and we are done. □

> **Remark.** First I thought of performing a Bary-bash with reference triangle ABC and then showing $\text{Pow}_{\omega_b} A = \text{Pow}_{\omega_c} A$ as PoP has a nice form in Barycentric coordinates. But the main problem is, determining the equations of ω_b and ω_c (maybe that's why you didn't spot a single Bary-bash solution in this thread). I even tried complex bashing it, but here the main problem is, how you translate the PoP characterization. Does this problem really justify it's placement? Not really as it's mostly based on well known configs and requires little to no revelatory observations.

Problem 2 (IMO P5, 2012). Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.

Solution. Naturally we construct ω_A, ω_B centred at A and B with radii $|AC|$ and $|BC|$ respectively. We note that CD is the radical axis of ω_A and ω_B. Let the two circles meet again at point say $E \neq C$. Moreover, these two circles are orthogonal.

We now make a slightly unnatural move that is to extend AX to meet ω_B again at $P \neq K$. Similarly let BX meet ω_A again at $Q \neq L$. Consider the following key claim.

> **Claim 1.** The cyclic quadrilateral $CKEP$ is harmonic. In other words,
>
> $$-1 = (CE; KP)_{\omega_B} \overset{C}{=} (AX; KP).$$

Proof. Simply follows from the fact that AC and AE are tangent ω_B at C and E respectively, meaning, A is the pole of line CE with regards to ω_B. □

Notice as X lies on the radical ω_A and ω_B, it follows,

$$\text{Pow}_{\omega_A}(X) = \text{Pow}_{\omega_B}(X) \quad \text{which in turn implies, } LPQK \text{ is cyclic.}$$

Aha, we get that $LPQK$ is harmonic and furthermore, AL is tangent to $(BPQK)$ at L and by symmetry one can argue that BK is tangent to $(BPQK)$ at K as well. And we are done. □

> **Remark.** Here is the motivation behind extending AX and BX to meet ω_B and ω_A at P and Q respecitively: I had initially thought of performing a bary bash (it is more than good that I abandoned it), inorder to compute K, you'll have to first rule out the the case of P as it lies on the same cevian AX by consider a quadratic, then Vieta blah blah blah...

Problem 3 (Balkan P1, 2018). A quadrilateral $ABCD$ is inscribed in a circle k where $AB > CD$,and AB is not paralel to CD.Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E.If EM bisects the angle CED prove that AB is diameter of k.

Solution 1. We present a rather natural Bary Bash solution.

Notice, how showing M is the incentre of $\triangle ECD$ suffices. This naturally suggests us to set $\triangle ECD$ as the reference. Let $E = (1 : 0 : 0)$, $C = (0 : 0 : 1)$ and $D = (0 : 0 : 1)$. Consider the labelling, $a = CD$, $b = DE$ and $c = EC$. As EM is the angle bisector of $\angle CED$, we have $M = (t_M : b : c)$ for some $t_M \in \mathbb{R}$. And as AB is the external angle bisector of $\angle ECD$, it follows that,

$$A = (t_A : -b : c) \quad \text{and} \quad B = (t_B : -b : c)$$

for some $t_A, t_B \in \mathbb{R}$. Using the fact that points C, M and A lie on the same line, we have

$$\det \begin{bmatrix} 0 & 1 & 0 \\ t_M & b & c \\ t_A & -b & c \end{bmatrix} = -t_M c + ct_A = 0.$$

So, $t_A = t_M$ and similarly using the fact that D, M and B lie on the same line it follows $t_B = -t_M$. Now, let's consider the equation of k i.e,

$$k : -a^2yz - b^2zx - c^2xy + (x + y + z)ux = 0 \quad \text{for some } u \in \mathbb{R}.$$

As A lies on k, it follows that,

$$k(A) = a^2bc - b^2ct_M + c^2t_Mb + (t_M - b + c)ut_M = 0 \quad \implies \quad u = \frac{-a^2bc + b^2ct_M - c^2t_Mb}{t_M^2 - t_Mb + ct_M}.$$

The above can be treated as a quadratic in t_M i.e,

$$ut_M^2 + t_M \underbrace{(-ub + cu - b^2c + c^2b)}_{=0} + a^2bc = 0.$$

As $B = (-t_M : -b : c)$ lies on k, it must be that even $-t_M$ satisfies the above equation. So by Vieta's formula, the coefficient of t_M is 0. Which then yields us,

$$u = -bc.$$

So the quadratic simply boils down to the following,

$$-bct_M^2 + a^2bc = 0 \quad \implies \quad t_M = \pm a.$$

But as M lies in the interior of $\triangle ECD$, $t_M > 0$ meaning $t_M = a$ and hence $M = (a : b : c)$ which is precisely the coordinates of the incentre of $\triangle ECD$ and we are done. $\square$

Solution 2. This setup immediately induces complete quadrilateral $ABCD$, so naturally we define $N = AD \cap BC$ and $P = AB \cap CD$ (not to mention neither N nor P are points at infinity). Consider the following key result.

> **Claim** — Points E, M and N lie on the same line.

Proof. Define $Q = EM \cap DC$ and define the phantom point $Q' = NM \cap PC$. We wish to show that Q and Q' coincide. Consider the perspectivity through N sending line PB to line PC, then by **Ceva-Menelaus** we have,

$$(A, B; NM \cap PB, P) \overset{N}{=} (D, C; Q', P) = -1.$$

But we know that $(D, C; Q, P) = -1$, so Q' coincides with Q. $\square$

Now, by **Brokard's theorem**, it follows that the circumcentre O of $ABCD$ is the orthocentre of $\triangle MNP$ or equivalently, M is the orthocentre of $\triangle ONP$ which means $OP \perp MN$. But by claim 1, we know that $AB \perp MN$ meaning O must lie on $\overline{AB}$ and we are done. □

Problem 4 (Int'l Festival of Young Mathematicians 2019)**.** The perpendicular bisector of AB of an acute ΔABC intersects BC and the continuation of AC in points P and Q respectively. M and N are the middle points of side AB and segment PQ respectively. If the lines AB and CN intersect in point D, prove that ΔABC and ΔDCM have a common orthocenter.
Solution. Let $\triangle XYZ$ be the orthic triangle of $\triangle ABC$ with X, Y and Z lying on $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. Define H to be the orthocentre of $\triangle ABC$. Notice that it suffices to show $MH \perp CD$.

> **Claim 1.** Points X, Y and D lie on the same line.

Proof. Consider the perspectivity through C sending line PQ to line AB by which we have,

$$(PQ; N\infty_{PQ}) \stackrel{C}{=} (BA; DZ) = -1.$$

And on the other hand by Ceva-Menelaus we have that $(BA; XY \cap ABZ) = -1$. Thus, $XY \cap AB = D$ and the result follows.

□

Now to finish off, we invoke Brokard's theorem on the complete cyclic quadrilateral $ABXY$. As M is the circumcentre of $(ABXY)$ it follows that $MH \perp CD$ as desired and we are done. □

Problem 5 (USAMO P6, 2006)**.** Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.
Solution. Let M be the Miquel point of $ABFE$. It suffices to show that M is also the Miquel point of $EFCD$.
We know that there exists a spiral similarity centred at M such that $A \mapsto E$ and $E \mapsto F$. As E and F divide $\overline{AD}$ and $\overline{BC}$ respectively in the same ratio, it follows that M is the spiral centre with $E \mapsto D$ and $F \mapsto C$(recall that spiral similarity is just a composition of homothety and rotation). Aha! by uniqueness of spiral centre, M is indeed the Miquel point of $EFCD$ and we are done. □

Problem 6 (EGMO P4, 2019)**.** Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.
Solution. Let the second tangent to the incircle through P other than PB touch it at point T. It suffices to show that $Q = PT \cap AC$ which is equivalent to showing that $\angle BPT = \angle BPQ$.

Claim 1. $BCQP$ is cyclic.

Proof. Begin by noticing that AI is the radical axis of (BIP) and (CIQ). It follows that

$$\text{Pow}_{(BIP)}A = \text{Pow}_{(CIQ)}A \quad \text{which implies } BCQP \text{ is cyclic.}$$

□

Now, consider the following angle chase.

$$\begin{aligned}
\angle BPT &= 2\angle BPI && (\angle IPT = \angle BPI)\\
&= 2(180^\circ - \angle AIB) && (AI \text{ is tangent to } (BIP) \text{ at } I)\\
&= 2\left(\frac{\angle BAC}{2} + \frac{\angle CBA}{2}\right) && (\text{angle sum in } \triangle AIB)\\
&= \angle BAC + \angle CBA\\
&= 180^\circ - \angle ACB.
\end{aligned}$$

So, the points P, B, C and $PT \cap AC$ are concyclic but we have already seen that Q lies on (PBC) which means $Q = PT \cap AC$ and we are done. □

Problem 7 (EGMO P4, 2025)**.** Let ABC be an acute triangle with incentre I and $AB \neq AC$. Let lines BI and CI intersect the circumcircle of ABC at $P \neq B$ and $Q \neq C$, respectively. Consider points R and S such that $AQRB$ and $ACSP$ are parallelograms (with $AQ \parallel RB, AB \parallel QR, AC \parallel SP$, and $AP \parallel CS$). Let T be the point of intersection of lines RB and SC. Prove that points R, S, T, and I are concyclic.

Solution. We shall first show that T lies on (BIC) and then proceed to show, there exists a spiral similarity centred at I with $B \mapsto R$ and $C \mapsto S$. Then by uniqueness of spiral centre, we would end up showing that R, S, T, and I are concyclic.

Claim 1. Points B, C, T and I are concyclic.

Proof. It suffices to show $\measuredangle BTC = \measuredangle BIC$. Consider the following angle chase.

$$\begin{aligned}
\measuredangle ACT &= 180^\circ - \angle SCA && (T, C \text{ and } S \text{ are collinear})\\
&= \angle APS && (ACSP \text{ is a parallelogram})\\
&= (180^\circ - \angle CBA) + \angle CPS && (\angle APS = \angle APC + \angle CPS)\\
&= (180^\circ - \angle CBA) + \frac{\angle CBA}{2}\\
&= 180^\circ - \frac{\angle CBA}{2}.
\end{aligned}$$

So, $\measuredangle ACT = \frac{\angle CBA}{2}$ and by symmetry, we end up with $\measuredangle TBA = \frac{\angle ACB}{2}$. Now, as $\measuredangle ACT + \angle TCB = \angle ACB$ it follows that $\angle TCB = \frac{\angle CBA}{2}$. Again by symmetry we get $\angle CBT = \frac{\angle ACB}{2}$ and thereby the result follows. □

Claim 2. $\triangle IBR \overset{+}{\sim} \triangle ICS$.

Proof. We begin by showing that $\frac{IB}{BR} = \frac{IC}{CS}$. Consider the following length chase.

$$\begin{aligned} \frac{IB}{BR} &= \frac{IB}{QA} && (BR = QA \text{ as } AQRB \text{ is a parallelogram}) \\ &= \frac{IB}{QB} && (Q \text{ is the midpoint of arc } BQA) \\ &= \frac{IC}{CS}. && (\triangle IBQ \sim \triangle TCP) \end{aligned}$$

We are just left to show that $\measuredangle ICS = \measuredangle IBR$ which follows from a simple angle chase.

$$\begin{aligned} \measuredangle ICS &= 180^\circ - \measuredangle TCI && (\text{points } T,\ C \text{ and } S \text{ are collinear}) \\ &= 180^\circ - \measuredangle TBI && (BCTI \text{ is cyclic}) \\ &= \measuredangle IBR. && (\text{points } T,\ B \text{ and } R \text{ are collinear}) \end{aligned}$$

Hence the result. $\square$

It is now evident that I is the spiral centre sending B to C and R to S which means I is also the spiral centre sending B to R and C to S and we are done. $\square$

Problem 8 (IMO Shortlist G3, 2006)**.** Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.
Solution. As (ABC) and (ADE) are tangent to $\overline{CD}$ at C and D respectively, it suffices to show that AP is the radical axis of these two circles as it would instantly imply that $\text{Pow}_{(ABC)}(AP \cap CD) = \text{Pow}_{(ADE)}(AP \cap CD)$.

Claim 1. P lies on (ABC) and (ADE).

Proof. Observe that A is the spiral centre sending B to D and C to E. And as $P = \overline{BD} \cap \overline{CE}$, it follows that P lies on (ABC) and (ADE) by uniqueness of spiral centre. $\square$

By the above claim, it is clear that AP is the radical axis and we are done. $\square$

Problem 9 (Turkey TST 1998)**.** In a triangle ABC, the circle through C touching AB at A and the circle through B touching AC at A have different radii and meet again at D. Let E be the point on the ray AB such that $AB = BE$. The circle through A, D, E intersect the ray CA again at F . Prove that $AF = AC$.
Solution 1. We invoke Barycentric Coordinates. Let ABC be the reference triangle with $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$ and $C = (0 : 0 : 1)$. Also we follow the usual convention $a = BC$, $b = CA$ and $c = AB$. Define F' as a point on the ray CA with $F'A = AC$ i.e, $F' = (2 : 0 : -1)$. Our goal is to show that F' satisfies the Barycentric equation of (ADE). Note that D is simply the $A-$ dumpty point, so using the fact that it is also the midpoint of $A-$symmedian chord of (ABC) we get $D = (2S_A : b^2 : c^2)$. Lastly $E = (-1 : 2 : 0)$ indeed. As (ADE) passes through A, the equation is given by

$$\Omega : -a^2yz - b^2zx - c^2xy + (x + y + z)(vy + wz) = 0 \quad \text{for some } v, w \in \mathbb{R}.$$

Evaluating Ω at point E,we get,

$$\Omega(E) = 2c^2 + 2v = 0 \quad \text{which implies } v = -c^2.$$

Now, similarly evaluating Ω at D, we get,

$$\begin{aligned}\Omega(D) &= -a^2b^2c^2 - 4S_Ab^2c^2 + (2b^2 + 2c^2 - a^2)(-c^2b^2 + wc^2)\\ &= -a^2b^2 - 4S_Ab^2 + (2b^2 + 2c^2 - a^2)(-b^2 + w)\\ &= 0 \quad \text{meaning, } w = \frac{4S_Ab^2 + a^2b^2}{2b^2 + 2c^2 - a^2} + b^2.\end{aligned}$$

Let's evaluate Ω at F' and we have,

$$\begin{aligned}\Omega(F') &= 2b^2 + \left(\frac{-4S_Ab^2 - a^2b^2}{2b^2 + 2c^2 - a^2} - b^2\right)\\ &= b^2 - \underbrace{\frac{4S_Ab^2 - a^2b^2}{2b^2 + 2c^2 - a^2}}_{=b^2}\\ &= 0.\end{aligned}$$

Therefore F' coincides with F and we are done. □

Solution 2. Define F' to be the point on ray CA with $F'A = AC$ and our goal is to show that F' coincides with F. It is evident that D is the spiral centre sending B to A and A to C. Call the spiral similarity σ. The key result is the following.

Proposition 1. Let X be a point on line AB and Y be the point on line AC with $X \overset{\sigma}{\mapsto} Y$. Then it follows that,

$$\frac{AX}{XB} = \frac{CY}{YA}.$$

Proof is redundant.

From the above result, we can say that $E \overset{\sigma}{\mapsto} F'$ or in other words $\triangle EDA \sim \triangle F'DC$. On the other hand, F lies on both (ADE) and (FDC) which implies D is the spiral centre with $E \mapsto F$ and $A \mapsto C$. Aha, this means that both spiral similarities are identical and hence F' coincides with F and we are done. □

Problem 10 (USA TST Day 3-P7, 2008)**.** Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

Solution. We shall use barycentric coordinates. Let ABC be the reference triangle with $A = (1:0:0)$, $B = (0:1:0)$ and $C = (0:0:1)$. Parametrize P as $(0:u:v)$ for $u, v \in \mathbb{R}$ with $u + v = a$. Then we blindly proceed to compute the coordinates of Q, R using Thale's theorem by which we end with,

$$Q = (u:0:v) \quad \text{and} \quad R = (v:u:0).$$

As we now have the coordinates of A, Q and R, we can simply find the equation of (AQR) and we shall refer it to as Γ. The equation is given by,

$$\Gamma : -a^2yz - b^2zx - c^2xy + (x+y+z)(v'y + w'z) = 0 \quad \text{for some } v', w' \in \mathbb{R}.$$

Putting the coordinates of Q and R in the above equation tells us what these v' and w' are.

$$\Gamma(Q) = -b^2uv + a(w'u) = 0 \quad \text{so, } w' = \frac{b^2u}{a}.$$
$$\Gamma(R) = -c^2vu + a(v'v) = 0 \quad \text{so, } v' = \frac{b^2u}{a}.$$

So the refined form of Γ is given by,

$$\Gamma(x:y:z) = -a^2yz - b^2zx - c^2xy + (x+y+z)\left(\frac{b^2u}{a}x + \frac{c^2v}{a}y\right) = 0.$$

As AX is isogonal to AG, we can parametrize X as $(t : b^2 : c^2)$ for some $t \in \mathbb{R}$ (using the fact that symmedian point $K = (a^2 : b^2 : c^2)$ lies on AX). We then proceed to show that t is fixed as we vary the point P.

$$\begin{aligned}
\Gamma(X) &= -a^2b^2c^2 - b^2c^2t - c^2b^2t + (t + b^2 + c^2)\left(\frac{c^2b^2}{a}v + \frac{b^2c^2}{a}u\right) \\
&= -a^2b^2c^2 - 2tb^2c^2 + t\frac{b^2c^2}{a}(u+v) + (b^2+c^2)(u+v)\frac{b^2c^2}{a} \\
&= -t - a^2 + b^2 + c^2 \\
&= 0.
\end{aligned}$$

Hence $t = 2S_A$ meaning Γ always passes through $X = (2S_A : b^2 : c^2)$ and we are done. □

> **Remark.** X is simply the $A-$dumpty point i.e, the midpoint of $A-$symmedian chord of (ABC).

Problem 11 (India TST Day1-P1, 2019). In an acute angled triangle ABC with $AB < AC$, let I denote the incenter and M the midpoint of side BC. The line through A perpendicular to AI intersects the tangent from M to the incircle (different from line BC) at a point P¿ Show that AI is tangent to the circumcircle of triangle MIP.

Solution. It suffices to show that $\measuredangle IMP = \measuredangle AIP$. Define D as the point of contact of the incircle and line BC and let D be the antipode of D on the incircle. Let E be the reflection of D in M and say line MP is tangent to the incircle at point X. The key result is the following.

> **Claim 1.** The points A, D', X and E all lie on the same line.

Proof. We only need to show that X lies on AD' as it is well known that A, D' and E are collinear. Consider a homothety $\mathcal{H}_D(2)$. It is evident that

$$I \overset{\mathcal{H}}{\longmapsto} D' \quad \text{and} \quad M \overset{\mathcal{H}}{\longmapsto} E.$$

As X is the reflection of D in the line MI, it follows that $\mathcal{H}$ sends the foot of D on MI to X and hence the claim. $\square$

By the virtue of $\mathcal{H}$, it follows that $\overline{IM} \parallel \overline{AE}$. We then have,

$$\begin{aligned} \measuredangle AIP &= \measuredangle AXP && A, I, X, P \text{ are concyclic,} \\ &= \measuredangle EXM && \text{vertically opposite angles,} \\ &= \measuredangle IMX && \text{alternate interior angles,} \\ &= \measuredangle IMP && M, X, P \text{ are collinear.} \end{aligned}$$

Which is exactly what we wished to show and we are done. $\square$

Problem 12 (USAJMO P5, 2011). Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that

(i) lines PB and PD are tangent to ω,

(ii) P, A, C are collinear, and

(iii) $DE \parallel AC$.

Prove that BE bisects AC.

Solution. As $ABCD$ is harmonic it suffices to show that BD and BE are isogonal conjugates in $\triangle BCD$ i.e, $\angle DBC = \angle ABE$ (this is because BD is the B-symmedian of $\triangle BCA$). Clearly it follows that,

$$\angle DBC = \angle DEC = \angle ACE = \angle ABE.$$

Hence BE bisects AC. $\square$

Problem 13 (IMO Shortlist G8, 2004). Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

Solution. Define $H = CD \cap EF$ and $G = AB \cap CD$. As $(DC; HG) = -1$ and M being the midpoint of $\overline{CD}$ it follows that

$$GH \cdot GM = GD \cdot GC = GA \cdot GB.$$

This implies H lies on (ABM). Now by considering the perspectivity sending line AB to (ABM) through H, we have,

$$-1 = (AB; G, EF \cap AB) \stackrel{H}{=} (AB; M, EF \cap (ABM)).$$

But we also know that $(AB; MN) = -1$ meaning $EF \cap (ABM) = N$ and we are done. $\square$

Problem 14 (ELMO Shortlist G1, 2023). Let $ABCDE$ be a cyclic pentagon. Let P be a variable point on the interior of segment AB such that $PA \neq PB$. The circumcircles of $\triangle PAE$ and $\triangle PBC$ meet again at Q. Let R be the circumcenter of $\triangle DPQ$. Show that

as P varies, R lies on a fixed line.
Note. The original problem had the condition that $ABCDE$ is regular but in this solution we will see that it is not necessary as cyclicity is sufficient.
Solution. Note that $X := SD \cap (ABCDE)$ lies on (PDQ) as $AE \cap BC$ is the radical centre of (PDQ), $(ABCDE)$ and $(PBCQ)$. So R must always lie on the perpendicular bisector of XD, and we are done. □

Problem 15 (Crux Mathematicorum 5044)**.** Given a circle Γ with center O and a chord AB, let X be the midpoint of the larger arc AB, and C be an arbitrary point of that arc. Define K to be the point where the bisector of $\angle ACB$ intersects the tangent to Γ at B, while M is the intersection of AC and BX. Prove that the line MK contains the midpoint of AB.
Solution. We present a **Projective solution**. Define P as the midpoint of arc AB of Γ not containing X and define N to be the midpoint of line segment AB. Call the tangent to Γ at B as ℓ. We wish to prove that M, N and K are collinear. The key idea is to define two projective maps $\varphi_1, \varphi_2 : \widehat{AXB} \to \ell$ with

1. $\varphi_1 : C \mapsto CP \mapsto CP \cap \ell \overset{\text{def}}{=} K_1$,
2. $\varphi_2 : C \mapsto AC \cap BX = M \mapsto MN \cap \ell \overset{\text{def}}{=} K_2$,

then showing these two maps are identical, finishes the problem as it would imply $K_1 \equiv K_2 \equiv K$ hence M, N and K are collinear.

We begin by showing φ_1, φ_2 are projective. The map with $C \mapsto CP$ preserves cross ratios thus is projective and similarly the map with $CP \mapsto K_1$ is projective as well. Now, as composition of two projective maps is also projective, it follows that φ_1 is projective. Following similar lines, one can conclude that φ_2 is projective as well. The following result is the reason for emphasizing on the 'projective' nature of these maps.

Lemma 1. Let $f, g : C_1 \to C_2$ be two projective maps that coincide at at least three distinct points then it follows that f and g are identical. Where C_1, C_2 could be conic sections, pencil of lines or a single line.

Proof. Let A_1, A_2, A_3 be three distinct points on C_1 such that f, g coincide on all three points. Then, as for any $B \in C_2 \setminus \{A_1, A_2, A_3\}$ there exists a unique point X on C_2 such that

$$(f(A_1), f(A_2); f(A_3), X) = (g(A_1), g(A_2); g(A_3), X) = (A_1, A_2; A_3, B).$$

Thus, it follows that $f(B) = g(B) = X$ as desired. □

It follows from *lemma 1* that it suffices to check for three distinct cases for C:

- Consider $C \equiv B$. Quite evidently $C \xmapsto{\varphi_1} B$ and $C \xmapsto{\varphi_2} B$.
- Consider $C \equiv X$. As $X \equiv M$ it follows that $C \xmapsto{\varphi_1} XP \cap \ell$ and $C \xmapsto{\varphi_2} XP \cap \ell$.
- Lastly we consider a slightly trickier case i.e. $C \equiv A$. We begin by noting that in the limiting case, the line CA is simply the tangent to Γ at A. Let Y be the meeting

point of the tangent to Γ at A and ℓ. It is easy to see that $APBX$ is a harmonic quadrilateral and hence

$$(X, P; A, B) = (X, P; N, Y) = -1.$$

As $\angle XBP = 90^\circ$, it follows that BP is the angle bisector of $\angle ABY$ and by symmetry P is the incenter of $\triangle ABY$. Define $K = AP \cap BY$ and let $M' = NK \cap AY$. It is well known that $(A, Y; BP \cap AY, M') = -1$ and as we have already seen that BX is the external angle bisector of $\angle ABY$, it follows that $(A, Y; BP \cap AY, M) = -1$. And this implies $M' \equiv M$, so φ_1, φ_2 coincide once again for the third time.

So φ_1, φ_2 are indeed identical and we are done. $\square$

> **Remark.** Note that the restricted location of C in the larger arc AB is not required as the result follows otherwise. The method we used is an advanced projective technique which mainly shows up in the world of high school olympiads and is often referred to as 'Method of Moving Points' or 'MMP' for short.

Chapter IV

Number Theory

§1 Divisibility

§1.1 Problems

Problem 1 (IMO Shortlist N2, 2022). Find all positive integers $n > 2$ such that

$$n! \mid \prod_{p<q\leq n, p,q \text{ primes}} (p+q).$$

Problem 2 (India P4, 2024). Let p be an odd prime and a, b, c be integers so that the integers

$$a^{2023} + b^{2023}, \quad b^{2024} + c^{2024}, \quad a^{2025} + c^{2025}$$

are divisible by p.Prove that p divides each of a, b, c.

Problem 3 (Simon Marais A2, 2021). Define the sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, and

$$a_{n+1} = (n + 1 - \gcd(a_n, n)) \times a_n$$

for all integers $n \geq 1$. Prove that $\frac{a_{n+1}}{a_n} = n$ if and only if n is prime or $n = 1$. Here $\gcd(s, t)$ denotes the greatest common divisor of s and t.

Problem 4 (CRT-Evan Chen's Handout). Let n be a positive integer. Determine, in terms of n, the number of $x \in \{1, 2, \ldots, n\}$ for which $x^2 \equiv x \pmod{n}$.

§1.2 Solutions

Problem 1 (IMO Shortlist N2, 2022)**.** Find all positive integers $n > 2$ such that

$$n! \mid \prod_{p<q\leq n, p,q \text{ primes}} (p+q).$$

Solution. We claim that $n = 7$ is the only solution, easy to check why it works.

Now we will show that no other value works. Clearly any $n < 7$ and $7 < n \leq 11$ does not work and let us assume some $n > 11$ works. Let $p_{\max}$ be the largest prime less than n. By the given given condition, it is evident that distinct prime $p < q \leq n$ exist that satisfy,

$$p + q \equiv 0 \pmod{p_{\max}}.$$

But, note as $p < q \leq p_{\max}$ it must be that $p + q = p_{\max}$ and by parity constraint, it is necessary that $p = 2$. Similarly there exist primes $p' < q' \leq n$ such that $p' + q' \equiv 0 \pmod{p_{\max} - 2}$. By comparing sizes, it is clear that either

$$p' + q' = p_{\max} - 2 \quad \text{or} \quad p' + q' = 2p_{\max} - 4.$$

We shall deal with the latter case first. Both p' and q' cannot be less than $p_{\max}$ else we have a contradiction, so this forces $p_{\max} - 4$ to also be a prime. Recall that there does not exist a prime p such that both $p + 2$ and $p + 4$ are primes unless $p = 3$ (as a hint consider modulo 3). So this case is ruled out and the former case is obvious as p must be 2 and $p_{\max} - 4$ is a prime which is a contradiction. □

> **Remark.** The problem does not justify its positioning in N2 by any means.

Problem 2 (India P4, 2024)**.** Let p be an odd prime and a, b, c be integers so that the integers

$$a^{2023} + b^{2023}, \quad b^{2024} + c^{2024}, \quad a^{2025} + c^{2025}$$

are divisible by p.Prove that p divides each of a, b, c.

Solution. Showing that any one of a, b, c is divisible by p would suffice. Assume all of them are invertible on $\mathbb{Z}_p$. As $b^{2024} \equiv -ba^{2023}$ we may say,

$$c^{2024} + b^{2024} \equiv c^{2024} - ba^{2023} \quad \text{and} \quad c^{2025} \equiv cba^{2023}.$$

We then have $bca^{2023} + a^{2025} \equiv 0$. Since a is invertible we are left with the key result,

$$a^2 \equiv -bc.$$

Using the above fact, $a(-bc)^{1012} \equiv -c^{2025}$ which deduces to $ab^{1012} \equiv -c^{1013}$ and as $a^{2025} \equiv cb^{2024}$ we can easily show that $b^{1012} \equiv ac^{1011}$. Now we are left with,

$$ab^{1012} \equiv a^2c^{1011} \equiv -c^{1013} \quad \Longrightarrow \quad a^2 \equiv -c^2.$$

But we had already seen $a^2 \equiv -bc$ and is absurd, contradicting our assumption, hence none of them are invertible as desired. □

Problem 3 (Simon Marais A2, 2021). Define the sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, and

$$a_{n+1} = (n + 1 - \gcd(a_n, n)) \times a_n$$

for all integers $n \geq 1$. Prove that $\frac{a_{n+1}}{a_n} = n$ if and only if n is prime or $n = 1$. Here $\gcd(s, t)$ denotes the greatest common divisor of s and t.
Solution. Let us write down few terms just to build some intuition,

a_n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
n	1	1	2	$2 \cdot 3$	$2 \cdot 3^2$	$2 \cdot 3^2 \cdot 5$	$2 \cdot 3^2 \cdot 5$	$2 \cdot 3^2 \cdot 5 \cdot 7$	$2 \cdot 3^2 \cdot 5 \cdot 7^2$

Notice that every prime less than n divides a_n for $n \leq 12$ at least and more precisely no prime bigger than $N - 1$ divides a_N as well. We ask the question, if this holds for any n.

Claim 1. A prime p divides a_n if and only if $p < n$.

Proof. This simply follows from an inductive argument (base case holds-refer the table). Suppose this property holds for some natural N then we have,

$$a_{N+1} = \underbrace{(N + 1 - \gcd(N, a_N))}_{\text{less than } N+1} \times a_N$$

By hypothesis it is true that every prime less than N divides a_N and are the only ones. As $N + 1 - \gcd(N, a_N) < N + 1$, our claim holds! $\square$

To finish off we just need to observe that $\gcd(n, a_n) = 1$ if and only if n is a prime which is obvious from our claim and we are done. $\square$

Problem 4 (CRT-Evan Chen's Handout). Let n be a positive integer. Determine, in terms of n, the number of $x \in \{1, 2, \ldots, n\}$ for which $x^2 \equiv x \pmod{n}$.
Solution. The answer is 2^k, where k corresponds to the number of distinct prime divisors of n.
Say $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. We then have,

$$\begin{aligned} x^2 &\equiv x \pmod{p_1^{e_1}} \\ x^2 &\equiv x \pmod{p_2^{e_2}} \\ &\vdots \\ x^2 &\equiv x \pmod{p_k^{e_k}}. \end{aligned}$$

As $\gcd(x, (x - 1)) = 1$, each time we either have $p_i^{e_i} | x$ or $p_i^{e_i} | x - 1$ (something like a binary tree). So, there are 2^k possible constructions for x and hence the answer. $\square$

§2 Number Theory Constructions

§2.1 Problems

Problem 1 (USAMO P1, 2017). Prove that there exists infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b | a^b + b^a$.

Problem 2 (USAJMO P3, 2024). Let $a(n)$ be the sequence defined by $a(1) = 2$ and $a(n+1) = (a(n))^{n+1} - 1$ for each integer $n \geq 1$. Suppose that $p > 2$ is a prime and k is a positive integer. Prove that some term of the sequence $a(n)$ is divisible by p^k.

Problem 3 (India P3, 2020). Let S be a subset of $\{0, 1, 2, \ldots, 9\}$. Suppose there is a positive integer N such that for any integer $n > N$, one can find positive integers a, b so that $n = a + b$ and all the digits in the decimal representations of a, b (expressed without leading zeros) are in S. Find the smallest possible value of $|S|$.

Problem 4 (IMO P5, 1989). Prove that for every positive integer n, theree exists n consecutive positive integers such that noe of them is a power of a prime.

Problem 5 (USAMO P4, 2011). Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

§2.2 Solutions

Problem 1 (USAMO P1, 2017). Prove that there exists infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b | a^b + b^a$.
Solution 1. We claim $(a, b) = (4k^2 - 1, 4k^2 + 1)$ is a valid construction for any odd natural k.
Evidently, $\gcd(4k^2 - 1, 4k^2 + 1)$. So, we are only left to show

$$8k^2 | (4k^2 - 1)^{4k^2+1} + (4k^2 + 1)^{4k^2-1}.$$

As $\gcd(8, k^2) = 1$, we can separately work in mod k^2 and mod 8. We then have,

$$\begin{aligned}(4k^2 - 1)^{4k^2+1} + (4k^2 + 1)^{4k^-1} &\equiv (-1)^{\text{odd}} + (1)^{\text{odd}} \pmod{k^2} \\ &\equiv 0.\end{aligned}$$

Note that $k^2 \equiv 1 \pmod 8$, then,

$$\begin{aligned}(4k^2 - 1)^{4k^2+1} + (4k^2 + 1)^{4k^2-1} &\equiv 3^5 + 5^3 \pmod 8 \\ &\equiv 0.\end{aligned}$$

Indeed, the proposed construction is valid and we are done. □

Solution 2. (Courtesy: `@v_Enhance`) My construction: let $d \equiv 1 \pmod 4$, $d > 1$. Let $x = \frac{d^d + 2^d}{d+2}$. Then set

$$a = \frac{x + d}{2}, \qquad b = \frac{x - d}{2}.$$

To see this works, first check that b is odd and a is even. Let $d = a - b$ be odd. Then:

$$\begin{aligned} a+b \mid a^b + b^a &\iff (-b)^b + b^a \equiv 0 \pmod{a+b} \\ &\iff b^{a-b} \equiv 1 \pmod{a+b} \\ &\iff b^d \equiv 1 \pmod{d+2b} \\ &\iff (-2)^d \equiv d^d \pmod{d+2b} \\ &\iff d+2b \mid d^d + 2^d. \end{aligned}$$

So it would be enough that

$$d + 2b = \frac{d^d + 2^d}{d+2} \implies b = \frac{1}{2}\left(\frac{d^d+2^d}{d+2} - d\right)$$

which is what we constructed. Also, since $\gcd(x, d) = 1$ it follows $\gcd(a, b) = \gcd(d, b) = 1$. □

> **Remark.** Bruhh after reading, Evan's comment that any consecutive odd pair works, I feel so stupid.

Problem 2 (USAJMO P3, 2024)**.** Let $a(n)$ be the sequence defined by $a(1) = 2$ and $a(n+1) = (a(n))^{n+1} - 1$ for each integer $n \geq 1$. Suppose that $p > 2$ is a prime and k is a positive integer. Prove that some term of the sequence $a(n)$ is divisible by p^k.
Solution. The structure of the recurrence somewhat reminds us of the famous Euler's Totient theorem, turns out there is a little more to that. Consider the terms with indices which are one less than multiples of $\varphi(p^k)$. If any of them say $a(m\varphi(p^k) - 1)$ is invertible over $\mathbb{Z}_{p^k}$ for some natural m, then we are done as

$$a(m\varphi(p^k)) \equiv 0 \pmod{p^k}.$$

Suppose there isn't one then it is certain that

$$a(m(p-1)-1) = (a(m(p-1)-2))^{m(p-1)-1} - 1 \equiv 0 \pmod p \qquad \text{for each } m,$$

From the above one might agree, $\gcd(p, a(m(p-1)-2)) = 1$ and $a(m(p-1)-2) \equiv 1 \pmod p$. Wait how does that help?! Lifting the Exponent lemma (LTE for short) to the rescue my friend! Recollect that given a prime $p|a-b$ with $\gcd(a,p) = \gcd(b,p) = 1$ we have

$$\nu_p(a^n - b^n) = \nu_p(a-b) + \nu_p(n),$$

where $\nu_p()$ denotes the $p-$adic valuation. We then have,

$$\nu_p((a(m(p-1)-2))^{m(p-1)-1} - 1) = \nu_p(a(m(p-1)-2) - 1) + \underbrace{\nu_p(m(p-1)-1)}_{\text{can take arbitrarily large values}}.$$

> **Claim 1.** The value of $\nu_p(m(p-1)-1)$ can be made arbitrarily large with a right choice for m.

Proof. Easily follows from an elementary result. As $\gcd(p-1, p^k) = 1$, Bézout's lemma (or more precisely a corollary) assures the existence of two positive integers x, y such that $x(p-1) - 1 = yp^k$. And $\nu_p(x(p-1)-1) \geq k$, hence the claim. $\square$

By claim 1 we have a valid construction for m with $p^k | a(m(p-1)-1)$ and we are done. $\square$

Problem 3 (India P3, 2020)**.** Let S be a subset of $\{0, 1, 2, \ldots, 9\}$. Suppose there is a positive integer N such that for any integer $n > N$, one can find positive integers a, b so that $n = a + b$ and all the digits in the decimal representations of a, b (expressed without leading zeros) are in S. Find the smallest possible value of $|S|$.
Solution. We claim $\min |S| = 5$, which can be achieved by taking $S = \{0, 1, 2, 3, 7\}$.

❡ Why Does the Construction Work? Note that sum of pairs of elements (not necessarily distinct) of S generate the whole residue class modulo 10,

	0	1	2	3	7
0	**0**	**1**	**2**	3	**7**
1	·	2	**3**	4	**8**
2	·	·	**4**	**5**	**9**
3	·	·	·	**6**	0
7	·	·	·	·	4

The nice thing about the construction is that it makes sure there's no carrying-over stuff, so we may choose each digit of a and b independently. For example, say we want to generate the number 59078294316, we simple choose a, b according to the above table

$$\begin{array}{cccccccccccc} & 2 & 2 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 3 \\ + & 3 & 7 & 0 & 7 & 7 & 2 & 7 & 2 & 2 & 1 & 3 \\ \hline & \mathbf{5} & \mathbf{9} & \mathbf{0} & \mathbf{7} & \mathbf{8} & \mathbf{2} & \mathbf{9} & \mathbf{4} & \mathbf{3} & \mathbf{1} & \mathbf{6} \end{array}$$

Note. In fact we get $N = 1$ which is cool.

❡ Establishing the Lower Bound. We shall show that $|S| > 4$. Suppose there exists a single digit number d such that $s_1 + s_2 \not\equiv d \pmod{10}$ for each $(s_1, s_2) \in S \times S$ then we will not be able to generate

$$\underbrace{\overline{** \ldots *d}}_{\text{take sufficiently large number}}$$

as a sum of a, b. Hence it must be that,

$$\{s_1 + s_2 \pmod{10} \mid s_1, s_2 \in S\} = \mathbb{Z}_{10} \quad \text{meaning,} \quad \binom{|S|}{2} + |S| \geq 10.$$

From the above mentioned bound it is quite clear that $|S| \geq 4$, so we are left to show that $|S| \neq 4$. Suppose $S = \{s_1, s_2, s_3, s_4\}$, notice $2s_i \pmod{10}$ generates four distinct even numbers among $\mathbb{Z}_{10}$. So either exactly one element of S is even or all are odd, the latter implies $s_i + s_j$ is even for $i \neq j$ meaning the number of even numbers generated exceeds $10/2$ which is ridiculous. Similarly, we argue that the former case is also not possible, therefore $|S| > 4$ and we are done. $\square$

Problem 4 (IMO P5, 1989)**.** Prove that for every positive integer n, theree exists n consecutive positive integers such that noe of them is a power of a prime.
Solution. Consider $2n$ parwise distinct primes p_i. Consider the system of equations such that,

$$\begin{aligned} a &\equiv -1 \pmod{p_1 p_2} \\ a &\equiv -2 \pmod{p_3 p_4} \\ &\vdots \\ a &\equiv -n \pmod{p_{2n-1} p_{2n}}. \end{aligned}$$

We are done is we are able to find a solution for the above system of equations. Voilà!! by **Chinese Remainder theorem** such an a exists. □

Problem 5 (USAMO P4, 2011)**.** Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.
Solution.

§3 Diophantine Equations

§3.1 Problems

Problem 1 (Junior Balkan Olympiad P3, 2022). Find all quadruples of positive integers (p, q, a, b), where p and q are prime numbers and $a > 1$, such that

$$p^a = 1 + 5q^b.$$

§4 Solutions

Problem 1 (Junior Balkan Olympiad P3, 2022)**.** Find all quadruples of positive integers (p, q, a, b), where p and q are prime numbers and $a > 1$, such that

$$p^a = 1 + 5q^b.$$

Solution. We claim $(p, q, a, b) = (2, 3, 4, 1), (3, 2, 4, 4)$ are the only 4−tuples. Note that exactly one of p, q must be 2, we shall deal with it one at time.

Case 1. $p = 2$.
One may notice, the last digit of $1+5q^b$ is always 6, only $2^4, 2^8, 2^{12}, \ldots$ have this property. More precisely, 2 is a primitive root of 5. Say $a = 4k$ for some natural k then

$$16^k = 1 + 5 \times q^b \equiv 1 \pmod 3 \qquad \text{which implies } q = 3.$$

By LTE we know that $\nu_3(16^k - 1) = 1 + \nu_3(k) = b$. For $b \geq 2$ this clearly ain't true as $16^{3^{b-1}} > 1 + 5 \times 3^b$ and when $b = 1$ we get the only solution as $(2, 3, 4, 1)$.

Case 2. $q = 2$.
Our primary goal is to get the condition that a is even, as it makes life easier while working under $\pmod 3$. As $p^a \equiv 1 \pmod{2^b}$ we know $\text{ord}_{2^b}(p) | \gcd(a, \varphi(2^b))$. If a was odd, $\text{ord}_{2^b}(p) = 1$ but we have an issue with sizes, there exists a natural k satisfying $p = 1 + k2^b$ and

$$p^a = (1 + k2^b)^a \geq k^2 2^{2b} + k2^{b+1} + 1 > 1 + 5 \times 2^b \qquad \text{for sufficiently large } k, b.$$

So a is even, meaning $p^a \equiv 1, 0 \pmod 3$. But $p^a \not\equiv 1 \pmod 3$ as $p^a \equiv 1 + 5 \times 2^b \equiv 0, 2 \pmod 3$ we certainly know $p = 3$. As 3 is a primitive $\pmod 5$ we deduce $a = 4k$ for some natural k.

Theorem 1 (LTE for $p = 2$)**.** Let x, y be odd integers such that $4|x - y$ then

$$\nu_2(x^n - y^n) = \nu_2(x - y) + \nu_2(n).$$

Now by the annoying version of LTE for 2, we have $\nu_2(81^k - 1) = \nu_2(80) + \nu_2(k)$.
But for $b > 4$ we know

$$81^{2^{b-4}} - 1 > 5 \times 2^b.$$

Working through the finite cases of $b \leq 4$ we have a solution only when $b = 4$, so the only solution is $(3, 2, 4, 4)$.

And that altogether proves our initial claim. □

§5 Size And Bounding Arguments

§5.1 Problems

Problem 1 (IMO Shortlist N1, 2016). For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Problem 2 (India P3, 2019). Let m, n be distinct positive integers. Prove that

$$\gcd(m, n) + \gcd(m+1, n+1) + \gcd(m+2, n+2) \leq 2|m-n| + 1.$$

Further, determine when equality holds.

§5.2 Solutions

Problem 1 (IMO Shortlist N1, 2016). For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Solution. We claim that is either $P(x) = x$ or $P(x) = c$ with $c = 1, \ldots, 9$, which are easy to verify.

Now, we shall show that these are the only solutions. The key insight to the problem is the fact that $S(n)$ is very small compared tHere are some of my recent solves from combinatorics which are majorly from olympiads.o n for sufficiently large n (more precisely, it is bounded above by a logarithmic function which grows very slowly) and the entire solution is more or less based on this observation. Firstly let's see what can we tell about P, can $\deg P$ be arbitrary? For convenience let Let $P(x) = \sum_{0 \leq i \leq d} a_i x^i$ where $d = \deg P$.

Claim 1. $d \leq 1$.

Proof. Using the fact that $S(n) \leq 9\lceil \log_{10} n \rceil$, we have the following,

$$9\lceil \log_{10} P(\underbrace{99 \cdots 9}_{\ell \text{ 9s}}) \rceil \geq S(P(99 \cdots 9)) = P(S(99 \cdots 9)) = P(9\ell).$$

For a sufficiently large ℓ. Let's find a very convenient upper bound for P , convenient meaning the one that behaves well inside logarithmic functions. A very natural one is

$$(d+1) \max_{0 \leq i \leq d} \{a_i\} x^d \geq P(x) \qquad \text{when } x \geq 1.$$

Call the messy constant in front of x^d as K. Recall that $9\lceil \log_{10} P(99 \cdots 9) \rceil \geq P(9\ell)$, so the bound simplifies as,

$$\begin{aligned} 9d \underbrace{\lceil \log_{10} 99 \cdots 9 \rceil}_{\ell} + 9\lceil \log_{10} K \rceil &\geq 9\lceil \log_{10} P(10^\ell - 1) \rceil \\ &\geq \sum_{0 \leq i \leq d} a_i (9\ell)^i. \end{aligned}$$

It's not hard to see why the leading coefficient of P must positive so taking sufficiently large ℓ, the above does not hold when $d > 1$ and hence the claim. $\square$

Now we are only left with the cases when $d = 0, 1$. The case when $d = 0$ is trivial, so we are only interested in the other case. Using similar ideas as before, we know that $9\lceil \log_{10} 2n \cdot \max\{a_0, a_1\} \rceil \geq S(n)a_1 + a_0$. Again we put $n = 99 \cdots 9$ and we have,

$$\begin{aligned} 9\lceil \log_{10} 2 \times 99 \cdots 9 \rceil + 9\lceil \log_{10} \cdot \max\{a_0, a_1\} \rceil &= 9\ell + 9\lceil \log_{10} \cdot \max\{a_0, a_1\} \rceil \\ &\geq 9\ell a_1 + a_0. \end{aligned}$$

If $a_1 \neq 1$ we have a contradiction by taking sufficiently large ℓ, therefore $a_1 = 1$. Somehow we need to force that $a_0 = 0$. Suppose $a_0 > 0$ (the case when $a_0 < 0$ is easy to rule out), as

$$S(n + a_0) = S(n) + a_0,$$

take n such that $n + a_0$ is an extremely large power of 10, meanwhile making $S(n)$ arbitrarily large. But this seems a bit off as $S(n + a_0) = 1$ throughout the process whereas $S(n) + a_0$ is just getting bigger and bigger unless $a_0 = 0$ and we are done. $\square$

Problem 2 (India P3, 2019)**.** Let m, n be distinct positive integers. Prove that

$$\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) \leq 2|m - n| + 1.$$

Further, determine when equality holds.

Solution. We claim that the equality holds only when m, n are either consecutive integers or are consecutive even numbers. Easy to check these work and by the end of the solution we shall conclude that these are the only cases. Not to mention we shall also prove the given bound.

By Euclid's algorithm and repeatedly using the fact that $\gcd(a, b) \leq \min(|a|, |b|) \leq \max(|a|, |b|)$ we have,

$$\begin{aligned}\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) &= \gcd(m - n, n) + \gcd(m - n, n + 1) \\ &\quad + \gcd(m - n, n + 2) \\ &\leq |m - n| + \frac{|m - n|}{\gcd(m, n)} \\ &\quad + \frac{2|m - n|}{\gcd(m, n) \cdot \gcd(m + 1, n + 1)}.\end{aligned}$$

We shall now deal with cases $\gcd(m, n) = 1$, $\gcd(m + 1, n + 1) = 1$ and combinations of these one at a time. To be honest, most of our work is done by this bound as the rest it just plugging things into it.

Case 1. $\gcd(m, n) = 1$ and $\gcd(m+1, n+1) > 1$. Clearly from the previously established bound we have,

$$\begin{aligned}\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) &\leq 1 + |m - n| + \frac{2|m - n|}{2} \\ &= 2|m - n| + 1.\end{aligned}$$

One might see that for the above to hold it must be that $\gcd(m - n, n + 1) = |m - n|$ meaning $m - n|(n + 1)$. Similarly $m - n|(n + 2)$ which is only possible when $|m - n| = 1$. In other words, m and n are consecutive.

Case 2. $\gcd(m, n) > 1$ and $\gcd(m + 1, n + 1) = 1$. Following the same idea we used in the previous case we have,

$$\begin{aligned}\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) &\leq |m - n| + 1 + \frac{2|m - n|}{2} \\ &= 2|m - n| + 1.\end{aligned}$$

We know that $n - m|n$ and $n - m|(n + 2)$ which is only true when $|m - n|$. Hence m and n are consecutive even numbers.

Case 3. $\gcd(m, n) > 1$ and $\gcd(m + 1, n + 1) > 1$. At this point its just mechanical oof,

$$\begin{aligned}\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) &\leq |m - n| + \frac{|m - n|}{2} + \frac{2|m - n|}{4} \\ &= 2|m - n|.\end{aligned}$$

Clearly equality does not hold in this case.

Our initial claim holds and we are done. □

§6 Number Theory Flavoured Functional Equations

§6.1 Problems

Problem 1 (USAJMO P1, 2021). Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for positive integers a and b,

$$f(a^2 + b^2) = f(a)f(b) \text{ and } f(a^2) = f(a)^2.$$

Problem 2 (APMO P1, 2019). Let $\mathbb{N}$ be the set of positive integers. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a, b.

Problem 3 (IMO Shortlist N1, 2013). Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

§6.2 Solutions

Problem 1 (USAJMO P1, 2021). Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for positive integers a and b,

$$f(a^2 + b^2) = f(a)f(b) \text{ and } f(a^2) = f(a)^2.$$

Solution. We claim that $f \equiv 1$, easy to check why it works and we shall show that it is the only solution.
Let us evaluate f at first few points to build some intuition. Putting $a = 1$ in $f(a^2) = (f(a))^2$, we get $f(1) = 1$ and put $a = 1$ in $f(2a^2) = (f(a))^2$ to get $f(2) = 1$. Repeating these steps multiple times, we arrive at,

$$f(1) = f(2) = f(3) = f(4) = f(5) = 1.$$

Computing $f(3) = 1$ is slightly tricky, one might have to use the fact that $3^2 + 4^2 = 5^2$ (this gives us the hint that the problem might have something to do with Pythagorean triplets). So, we are somewhat convinced that $f \equiv 1$. Let us setup a strong induction argument, assume $f(n) = 1$ whenever $n \leq N$ for some natural $N \geq 5$. We will show that $f(N + 1) = 1$. To do so we will use a famous result regarding **Primitive Pythagorean Triplets**.

Theorem 1 (Primitive Pythagorean Triplets). A triple of integers (x, y, z) is a primitive Pythagorean triple if and only if $x = r^2 - s^2$, $y = 2rs$ and $z = r^2 + s^2$, where r, s are arbitrary integers of opposite parity $r > s > 0$ and $\gcd(r, s) = 1$.

For further details, refer section 5.3 of An Introduction to the Theory of Numbers authored by Ivan Niven, Herbert S. Zuckerman and Hugh L. Montgomery.

Case 1. When $N + 1$ is odd.
In reference to theorem 1, setting $s = r - 1$ we have have $r^2 - (r - 1)^2 = N + 1$ for some $r \leq N$. By our induction hypothesis, we know that,

$$f((r^2 - s^2)^2 + (2rs)^2) = f(r^2 - s^2)f(2rs) = f(r^2 + s^2) = f(r)f(s) = 1.$$

So $f(r^2 - (r - 1)^2) = f(N + 1) = 1$ as desired.

Case 2. When $N + 1$ is even.
Similar in spirit as that of case 1, there exists r, s with $2rs = N+1$. Again by our induction hypothesis, we have,

$$f((r^2 - s^2)^2 + (2rs)^2) = f(r^2 - s^2)f(2rs) = f(r^2 + s^2) = f(r)f(s) = 1.$$

Clearly, $f(2rs) = f(N + 1) = 1$.

This completes our induction argument, giving us $f \equiv 1$ as desired. □

Problem 2 (APMO P1, 2019). Let $\mathbb{N}$ be the set of positive integers. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers

a, b.

Solution. We claim that $f = \text{id}$ is the only solution and it's easy check the validity.

Now, we proceed to show that no other solution exists. The key idea in these kind of problems is to exploit divisibility constraints with the help of primes along with size arguments. And this intuition yields the following result.

> **Claim** — Given any prime p, it follows that $f(p) = p$.

Proof. Let p denote a prime. Now substituting a and b with p, we have,

$$p^2 + f(p)^2 \equiv 0 \pmod{p + f(p)} \quad \text{implies} \quad 2pf(p) \equiv 0 \pmod{p + f(p)}.$$

Now, we either have $\gcd(f(p), p) = p$ or $\gcd(f(p), p) = 1$ and readily the latter is inferred as ridiculous. Which leaves us with the only possibility,

$$2f(p) \equiv 0 \pmod{1 + \tfrac{f(p)}{p}}.$$

Evidently, $1 + \frac{f(p)}{p}$ shares a common factor with neither $f(p)$ nor p yielding $f(p) = p$ as desired. $\square$

Consider a prime p. From the hypothesis, we have for any a, that $p + a | p^2 + pf(a)$ and by cancellation rule it follows that,

$$p + f(a) + a - a \equiv 0 \pmod{p + a} \quad \text{or} \quad f(a) - a \equiv 0 \pmod{p + a}.$$

But making p sufficiently large contradicts the above unless $f(a) - a = 0$ and we are done. $\square$

Problem 3 (IMO Shortlist N1, 2013). Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Solution. We claim that $f = \text{id}$ is the only solution and can be checked easily.

Now, we proceed to show that this is the only solution. Begin by substituting m with $f(n)$ by which we have,

$$f(n)f \circ f(n) + n \equiv 0 \pmod{f(n)^2 + f(n)}.$$

And as $\gcd(f(n), f(n) + 1) = 1$, it follows that $f(n)|n$ (we have a clear bound, $f(n) \leq n$). This instantly yields $f(1) = 1$. Now we finish off by by substituting n with 1 in the intitial condition, so,

$$mf(m) + 1 \equiv 0 \pmod{m^2 + 1}.$$

But this is only possible when $f(m) \geq m$. Simultaneously, recall that $f(m) \leq m!$ thus yielding us the required. $\square$

Chapter V

Linear Algebra

§1 Characteristic Realm

§1.1 Problem

Problem 1 (Putnam A6, 1988)**.** Let V be an $n-$dimensional vector space. Let $T : V \to V$ be a linear map and suppose there exists $n+1$ eigenvectors, any n of which are linearly independent. Does it follow that T is a scalar multiple of id_V?

§1.2 Solution

Problem 1 (Putnam A6, 1988)**.** Let V be an $n-$dimensional vector space. Let $T : V \to V$ be a linear map and suppose there exists $n+1$ eigenvectors, any n of which are linearly independent. Does it follow that T is a scalar multiple of id_V?

Solution. The answer is yes.

Let the eigen vectors be v_i with corresponding eigen values λ_i for $1 \le i \le n+1$. Our goal is to show that all the eigen values are equal. It is evident that $\{v_1, v_2, \ldots, v_n\}$ forms a basis for V as it is linearly independent and $\dim V = n$. Aha!! this means v_{n+1} can be written as a linear combination of the basis, we then have,

$$\begin{aligned} A \sum_{1\le i\le n} s_i v_i &= \sum_{1\le i\le n} s_i A v_i \\ &= \sum_{1\le i\le n} s_i \lambda_i v_i \\ &= A v_{n+1} \\ &= \lambda_{n+1} v_{n+1} \\ &= \lambda_{n+1} \sum_{1\le i\le n} s_i v_i, \end{aligned}$$

for some scalars $s_1, s_2, \ldots, s_n \in \mathbb{F}$. From the above we know that,

$$\sum_{1\le i\le n} s_i \lambda_i v_i = \lambda_{n+1} \sum_{1\le i\le n} s_i v_i \quad \implies \quad \sum_{1\le i\le n} s_i(\lambda_i - \lambda_{n+1}) v_i = 0.$$

But as $\{v_1, \ldots, v_n\}$ is linearly independent, it follows that all the eigen values must be equal as desired. $\square$

Chapter VI

Real Analysis

§0.1 Problems

Problem 1 (Vojtěch Jarnik IMC Cat I P2, 2015)**.** Consider the infinite chessboard whose rows and columns are indexed by positive integers. Is it possible to put a single positive rational number into each cell of the chessboard so that each positive rational number appears exactly once and the sum of every row and of every column is finite?

§0.2 Solutions

Problem 1 (Vojtěch Jarnik IMC Cat I P2, 2015)**.** Consider the infinite chessboard whose rows and columns are indexed by positive integers. Is it possible to put a single positive rational number into each cell of the chessboard so that each positive rational number appears exactly once and the sum of every row and of every column is finite?

Solution. Yes, there is such a construction and we will see one of them in this solution.

The key idea is to dump all the "unwanted numbers" on the main diagonal.
Let $\phi : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Q}_{>0}$ denote the function that maps any square on the chessboard to the number written on it. Define a set

$$S = \left\{ \frac{1}{2^m 3^n} \mid m \neq n \text{ and } m, n \in \mathbb{Z}_{>0} \right\}.$$

Claim 1. $\mathbb{Z}_{>0} \sim \mathbb{Q}_{>0} \setminus S$.

Proof. Similar to the proof which shows $\mathbb{Z}_{>0} \sim \mathbb{Q}_{>0}$ – refer *Stephen Abbot, Understanding Analysis, Springer, section 1.4.* □

By claim 1 we can place all the members of $\mathbb{Q}_{>0} \setminus S$ on the main diagonal.
Now, we shall deal with the squares not on the main diagonal. Consider the following construction

$$(m, n) \overset{\phi}{\mapsto} \frac{1}{2^m 3^n} \quad \text{where } m \neq n.$$

It's easy to see that the sum in each of the rows and columns converge, and as ϕ is a bijective map, it is a valid construction. □

§1 Sequential Limits

§1.1 Problems

Problem 1 (Romania Grade 11 P1, 2025). Consider the sequence $(a_n)_{n\geq 1}$ given by $a_1 = 1$ and $a_{n+1} = \frac{a_n}{1+\sqrt{1+a_n}}$, for all $n \geq 1$. Show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \sum_{k=1}^{n} \log_2(1+a_k) = \frac{1}{2}.$$

Problem 2 (Putnam A2, 2016). Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate $\lim_{n\to\infty} \frac{M(n)}{n}$.

Problem 3 (Putnam B1, 2016). Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

Problem 4 (ELMO SL 2017). Let $0 < k < \frac{1}{2}$ be a real number and let a_0, b_0 be arbitrary real numbers in $(0, 1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n + 1}{2} \quad \text{and} \quad b_{n+1} = b_n^k$$

for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n.

§1.2 Solutions

Problem 1 (Romania Grade 11 P1, 2025)**.** Consider the sequence $(a_n)_{n\geq 1}$ given by $a_1 = 1$ and $a_{n+1} = \frac{a_n}{1+\sqrt{1+a_n}}$, for all $n \geq 1$. Show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \sum_{k=1}^{n} \log_2(1 + a_k) = \frac{1}{2}.$$

Solution. (Courtesy: `@grupyorum`) We have

$$a_{n+1} = \frac{a_n}{1 + \sqrt{1 + a_n}} = \sqrt{1 + a_n} - 1,$$

implying that $b_{n+1} = \sqrt{b_n}$ for $b_n := a_n + 1$ with $b_1 = 2$. This immediately yields $b_{n+1} = 2^{1/2^n}$. So, $1 = \lim_{n\to\infty} b_n = 1 + \lim_{n\to\infty} a_n$, i.e., $\lim_{n\to\infty} a_n = 0$. This immediately yields:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{1 + \sqrt{1 + a_n}} = \frac{1}{2}.$$

As for the second equality, we have $b_1 := 2$ and

$$\sum_{k=1}^{n} \log_2 b_k = \log_2 b_1 \left(1 + \cdots + \frac{1}{2^{n-1}}\right) \to \frac{1}{2}$$

as $n \to \infty$. □

Problem 2 (Putnam A2, 2016)**.** Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate $\lim_{n\to\infty} \frac{M(n)}{n}$.

Solution. The answer is $\lim_{n\to\infty} \frac{M(n)}{n} = \frac{3+\sqrt{5}}{2}$.

Surprisingly, it is not hard to come up with an explicit formula for $M(n)$. Now, doing the most natural thing that is to simplify the given inequality we get,

$$m^2 - m(3n - 1) + n^2 - n < 0.$$

Treating this as a quadratic expression in m, we can give a bound on possible values of m that satisfy the inequality i.e,

$$\frac{3n - 1 - \sqrt{5n^2 - 2n + 1}}{2} \leq m \leq \frac{3n - 1 + \sqrt{5n^2 - 2n + 1}}{2}.$$

As we are interested in the greatest possible value of m, $M(n)$ can be simply given by the formula,

$$\begin{aligned} M(n) &= \left\lfloor \frac{3n - 1 + \sqrt{5n^2 - 2n + 1}}{2} \right\rfloor \\ &= \frac{3n - 1 + \sqrt{5n^2 - 2n + 1}}{2} - \left\{ \frac{3n - 1 + \sqrt{5n^2 - 2n + 1}}{2} \right\}. \end{aligned}$$

Aha! the limit is easy to compute. As the fractional part is bounded above by 1 and below by 0, when divided by n makes no contribution to the limit and hence,

$$\begin{aligned}\lim_{n\to\infty}\frac{M(n)}{n} &= \lim_{n\to\infty}\frac{3n-1+\sqrt{5n^2-2n+1}}{2n}-\frac{1}{n}\left\{\frac{3n-1+\sqrt{5n^2-2n+1}}{2}\right\}\\ &= \frac{3+\sqrt{5}}{2}.\end{aligned}$$

We are done. □

Problem 3 (Putnam B1, 2016)**.** Let $x_0, x_1, x_2, \dots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

Solution. We claim that $s_n \to e - 1$. Where s_n denotes the partial sum $s_n = x_0 + x_1 + \cdots + x_n$. Bruh this ln seems to annoy us so we rewrite $x_{n+1} = \ln(e^{x_n} - x_n)$ as, $e^{x_{n+1}} = e^{x_n} - x_n$. Let us write few equations in this manner namely,

$$\begin{aligned} e^{x_{n+1}} &= e^{x_n} - x_n.\\ e^{x_n} &= e^{x_{n-1}} - x_{n-1}.\\ &\vdots\\ e^{x_1} &= e^{x_0} - x_0.\end{aligned}$$

Aha! adding all of the above equation yields us,

$$\sum_{0\leq i\leq n} e^{x_i} - \sum_{1\leq i\leq n+1} e^{x_i} = e - e^{x_{n+1}} = s_n.$$

This result gives us a hint that in order to prove the convergence of s_n, we might want to prove the convergence of x_n first.

> **Claim** — $x_n \to 0$.

Proof. Begin by noticing that $(x_n)_{n\geq 0}$ is bounded below by 0 as the function $\ln(e^x - x)$ can take the least value of 0 over $\mathbb{R}$. Also one might see that $\Delta x_n < 0$ hence by **Monotone Convergence Theorem** we say $(x_n)_{n\geq 0}$ converges to a point a say ℓ. Computing the value of ℓ is easy as we have previously seen that $e^{x_{n+1}} = e^{x_n} - x_n$ meaning $e^{\ell} = e^{\ell} - \ell$ which forces $x_n \to 0$. □

Recall that $e - e^{x_{n+1}} = s_n$, which implies $s_n \to e - 1$. □

Problem 4 (ELMO SL 2017)**.** Let $0 < k < \frac{1}{2}$ be a real number and let a_0, b_0 be arbitrary real numbers in $(0, 1)$. The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are then defined recursively by

$$a_{n+1} = \frac{a_n + 1}{2} \quad\text{and}\quad b_{n+1} = b_n^k$$

for $n \geq 0$. Prove that $a_n < b_n$ for all sufficiently large n.
Solution. The key strategy is come up with an explicit formula for a_n and b_n, from then on it's just cake walk.

> **Claim** — Members of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are given by the formulas,
>
> $$a_n = 1 + \frac{a_0 - 1}{2^n} \quad \text{and} \quad b_n = \exp(k^n \log b_0),$$
>
> respectively.

Proof. We first grab the low hanging fruit – one can instantly notice

$$b_n = b_0^{k^n} = \exp(k^n \log b_0).$$

Using the recursive formula i.e, $2a_{n+1} = a_n + 1$, we have,

$$2\sum_{n\geq 0} a_{n+1}X^n = \sum_{n\geq 0} a_n X^n + \sum_{n\geq 0} X^n.$$

Solving for the generating function of $(a_n)_{n\geq 0}$, yields,

$$\sum_{n\geq 0} a_n X^n = \frac{2a_0 - 2}{2 - x} + \frac{1}{1 - x} = (a_0 - 1)\sum_{n\geq 0}\left(\frac{X}{2}\right)^n + \sum_{b\geq 0} X^n.$$

So $a_n = 1 + \frac{a_0-1}{2^n}$ as desired. □

Now, in order to compare a_n and b_n, we establish a suitable lower bound for b_n i.e,

$$b_n = \exp(k^n \log b_0) > 1 + k^n \log(b_0).$$

It suffices to show that $k^n \log b_0 > \frac{a_0-1}{2^n}$ for all sufficiently large n. And this readily follows as $\lim_{n\to\infty}(2k)^n \log b_0 = 0$. □