My Math Memoir

rotten $\int$ntegral *
Email: anudeepprashantha@gmail.com

September 26, 2025

Contents

Chapter 1

Combinatorics

Here are some of my recent solves from combinatorics which are majorly from olympiads.

§1 Processes

§1.1 Problems

Problem 1 (IMO Shortlist C1, 2022). A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C so that, for any ± 1-sequence, there exists an integer k and indices $1 \le t_1 < \ldots < t_k \le 2022$ so that $t_{i+1} - t_i \le 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \ge C.$$

Problem 2 (India P5, 2025). Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins he could have collected?

Problem 3 (IMO P1, 2022). The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \le 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to ...$
Find all pairs (n, k) with $1 \le k \le 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 4 (India P5, 2018). There are $n \ge 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps. (Assume that, the teacher has an abundant supply of apples.)

§1.2 Solutions

Problem 1 (IMO Shortlist C1, 2022)**.** A ± 1-sequence is a sequence of 2022 numbers $a_1, \ldots, a_{2022}$, each equal to either $+1$ or -1. Determine the largest C so that, for any ± 1-sequence, there exists an integer k and indices $1 \leq t_1 < \ldots < t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all i, and

$$\left| \sum_{i=1}^{k} a_{t_i} \right| \geq C.$$

Solution. We claim that the optimal value of C is 506.
Constructing a sequence that gives an upper bound of 506 on C is very natural. Consider the sequence,

$$1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1, 1, -1, -1, 1, -1.$$

There are 505 copies of $1, 1, -1, -1$ chunks with an anomaly at the end. Say we want to maximise the number of 1s in the list, then we must have at least 507 more 1s than -1s and this is impossible as appending a 1 in our list comes at the cost of appending a -1 unless we are yet to start.

Establishing The Lower bound on C. We will show that no matter what $a_1, a_2, \ldots, a_{2022}$ are, we can always achieve

$$\left| \sum_{1 \leq i \leq k} a_{t_i} \right| \geq 506 \qquad \text{with } t_{i+1} - t_i \leq 2,$$

for some k. Say there are at least 1011 ones in the sequence and we want to include all of it in the list i.e, $a_{t_1}, a_{t_2}, \ldots, a_{t_k}$. We do so according to the following algorithm,

- Begin from the smallest index that contains a 1 and append it to the list.
- If the next nearest 1 is at most two steps away, jump to that index and append it to the list.
- And if you don't find a 1, then always jump to the right by two steps until you find a 1 and of course appending the number you jumped on each time. Eventually if you encounter a 1 which is at most two steps away, follow the above procedure.

Aha! following these steps, the worst case is you would have collected at most 505 (a jump of length 2 is essentially avoiding a -1) number of -1s in the list and we are done. □

Problem 2 (India P5, 2025)**.** Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins he could have collected?
Solution. We claim Griphook can pick a maximum of 1998 coins and a minimum of 668

coins. We prove a more general proposition: that given any n-gon for $n \geq 3$, Griphook can pick at most $n-2$ coins and the minimizing case is a bit tricky given by the following formula,

$$\text{Minimum number of coins Griphook must pick} = \begin{cases} \lceil n/3 \rceil + 1 & \text{if } n \equiv 2 \pmod 3, \\ \\ \lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

(this proposition is simply motivated by checking for small cases). Firstly, we shall deal with the maximizing case and we shall do that by constructing an algorithm for collecting $n-2$ coins and then arguing that it cannot get any better. For convenience we represent the number of coins on each vertex as a string with i^{th} number representing the number of coins on the i^{th} vertex. Consider picking the coins in the following manner, i.e, pick the next vertex each time starting from the second vertex,

$$1\underline{1}1111\ldots111 \to 02\underline{0}111\ldots111 \to 011\underline{0}11\ldots111 \to 0101\underline{0}1\ldots111 \to \cdots \to 010000\ldots010.$$

As you may see there will be exactly two coins left that are two vertices away from each other at the end. Suppose we could do better, clearly we cannot pick all the n coins, so we are left with the only possibility of picking $n-1$ coins and assume we could. By backtracking we have,

$$010000\ldots000 \leftarrow 101000\ldots000 \leftarrow 110100\ldots000 \leftarrow 111010\ldots000 \leftarrow \cdots \leftarrow 111111\ldots101.$$

Note that the above sequence of moves is the only possibility (ahh ignore the cyclic permutations). But, the configuration we get at the end is never achievable hence $n-2$ is the best upper bound.

Now we shall deal with the minimizing case. Divide the string into substrings each of length 3 and for Griphook to stop, it must be that each of them contain at least one 0 at the extremities. Notice, each move can generate only one substring with at least a single 0 at the extremity, hence the lower bound. When $n \equiv 2 \pmod 3$ there will always be a $\overline{202}$ string at the end and we are done. $\square$

Problem 3 (IMO P1, 2022)**.** The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to$ ...
Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.
Solution. We claim that (n, k) is such a pair if and only if it is contained in the following set defined as,

$$S = \left\{ (n, k) \in \mathbb{Z}^2_{\geq 1} \middle| n \leq k \leq n + \left\lceil \frac{n}{2} \right\rceil \right\}$$

Firstly note that for $k < n$ we can construct a counter sequence of coins which shows that the final state cannot be achieved;

$$\underbrace{A\ldots A}_{n-1\ A\text{'s}}\underbrace{C\ldots C}_{n-1\ C\text{'s}}AC.$$

Similarly we show that $k \leq n+\lceil n/2\rceil$ by constructing the sequence which loops to itself every time it completes a cycle. Also, to keep things neat and clean denote the contiguous string of coins of the same metal M of size $\#$ as $M^{\#}$, consider the sequence;

$$A^{\lfloor\frac{n}{2}\rfloor}C^{\lceil\frac{n}{2}\rceil}A^{\lceil\frac{n}{2}\rceil}C^{\lfloor\frac{n}{2}\rfloor} \to C^{\lfloor\frac{n}{2}\rfloor}A^{\lfloor\frac{n}{2}\rfloor}C^{\lceil\frac{n}{2}\rceil}A^{\lceil\frac{n}{2}\rceil} \to \cdots \to A^{\lfloor\frac{n}{2}\rfloor}C^{\lceil\frac{n}{2}\rceil}A^{\lceil\frac{n}{2}\rceil}C^{\lfloor\frac{n}{2}\rfloor} \to \cdots$$

Now, we are only left with the case when $n \leq k \leq n+\lceil n/2\rceil$ and as per our claim this range of values of k works. Assume that for some k in this range, some sequences do not attain the favourable end state. And for this to happen, there has to be a stagnant point after which the size of any contiguous string of coins of the same metal does not increase at all if not it has to reach the end state.

We would like to characterize the sequences that have reached the stagnant point. Note that the right-most contiguous string of the same metal has to have at least $2n+1-k$ coins at each iteration, else, it will contradict the stagnant point. With this argument, we assert that size of every contiguous string of coins of the same metal has to be at least $2n+1-k$ and as $n \leq k \leq n+\lceil n/2\rceil$ we may certainly state that the stagnant point is the end state itself and we are done. $\square$

Problem 4 (India P5, 2018)**.** There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps. (Assume that, the teacher has an abundant supply of apples.)

Solution. Let $\mathcal{G}$ denote the set of girls g_i where $i = 1, 2, \ldots, n$ and let $\alpha_j(g)$ denote the number of apples with g after j^{th} iteration for $j = 0, 1, \ldots, n$.

Something cool pops up when we study the extremal object $\max\limits_{g\in\mathcal{G}} \alpha_j(g)$.

> **Lemma 1.** Monovariant Property $\max\limits_{g\in\mathcal{G}} \alpha_j(g) \geq \max\limits_{g\in\mathcal{G}} \alpha_{j+1}(g)$.

By the above result we have an upper bound for the total number of apples in the class i.e,

$$\sum_{g\in\mathcal{G}} \alpha_j(g) \leq |\mathcal{G}| \max_{g\in\mathcal{G}} \alpha_0(g).$$

But the total number of apples after each iteration seems to be increasing by 1 which forces this process to terminate after a finite number of iterations. $\square$

§2 Constructions

§2.1 Problems

Problem 1 (India P3, 2020). Let S be a subset of $\{0, 1, 2, \dots, 9\}$. Suppose there is a positive integer N such that for any integer $n > N$, one can find positive integers a, b so that $n = a + b$ and all the digits in the decimal representations of a, b (expressed without leading zeros) are in S. Find the smallest possible value of $|S|$.

Problem 2 (Sleepy Students – OTIS). There are n sleepy students working on a morning constest. The contest has six problems, and the score on each problem is a non negative integer less than or equal to 10. Given that no two students got the same score on two or more problems, what is the greatest possible value of n?

Problem 3 (RMM P1, 2015). Does there exist an infinite sequence of postivie integers $a_1, a_2, \dots$ such that $\gcd(a_m, a_n) = 1$ if and only if $|m - n| = 1$?

Problem 4 (Junior Balkan Shortlist C2, 2021). Let n be a positive integer. We are given a $3n \times 3n$ board whose unit squares are colored in black and white in such way that starting with the top left square, every third diagonal is colored in black and the rest of the board is in white. In one move, one can take a 2×2 square and change the color of all its squares in such way that white squares become orange, orange ones become black and black ones become white. Find all n for which, using a finite number of moves, we can make all the squares which were initially black white, and all squares which were initially white black.

Problem 5 (India P4, 2025). Let $n \geq 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

$$\max\left(|a_1 + a_2|, |a_2 + a_3|, \dots, |a_{n-1} + a_n|, |a_n + a_1|\right) \geq t_n \cdot \max(|a_1|, |a_2|, \dots, |a_n|)$$

holds for all real numbers $a_1, a_2, \dots, a_n$.

Problem 6 (Coluring Numbers Efficiently-Kithun). What is the least number required to colour the integers $1, 2, \dots, 2^n - 1$ such that for any set of consecutive integers taken from the given set of integers, there will always be a colour colouring exactly one of them? That is, for all integers i, j such that $1 \leq i \leq j \leq 2^n - 1$, there will be a colour coloring exactly one integer from the set $i, i + 1, \dots, j - 1, j$.

Problem 7 (Ahan Chakraborty-Unknown). Let $n \in \mathbb{N}$. Let $X = \{1, 2, 3, ..., n^2\}$.Let $A \subset X$ with $|A| = n$. Prove that $X \setminus A$ contains an arithmetic progression with n terms.

Problem 8 (IM0 P2, 2014). Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any its k^2 unit squares.

§2.2 Solutions

Problem 1 (India P3, 2020)**.** Let S be a subset of $\{0, 1, 2, \ldots, 9\}$. Suppose there is a positive integer N such that for any integer $n > N$, one can find positive integers a, b so that $n = a + b$ and all the digits in the decimal representations of a, b (expressed without leading zeros) are in S. Find the smallest possible value of $|S|$.
Solution. We claim $\min |S| = 5$, which can be achieved by taking $S = \{0, 1, 2, 3, 7\}$.

Why Does the Construction Work? Note that sum of pairs of elements (not necessarily distinct) of S generate the whole residue class modulo 10,

	0	1	2	3	7
0	**0**	**1**	**2**	3	**7**
1	·	2	**3**	4	**8**
2	·	·	**4**	**5**	**9**
3	·	·	·	**6**	0
7	·	·	·	·	4

The nice thing about the construction is that it makes sure there's no carrying-over stuff, so we may choose each digit of a and b independently. For example, say we want to generate the number 59078294316, we simple choose a, b according to the above table

$$\begin{array}{cccccccccccc} & 2 & 2 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 3 \\ + & 3 & 7 & 0 & 7 & 7 & 2 & 7 & 2 & 2 & 1 & 3 \\ \hline & \mathbf{5} & \mathbf{9} & \mathbf{0} & \mathbf{7} & \mathbf{8} & \mathbf{2} & \mathbf{9} & \mathbf{4} & \mathbf{3} & \mathbf{1} & \mathbf{6} \end{array}$$

Note. In fact we get $N = 1$ which is cool.

Establishing the Lower Bound. We shall show that $|S| > 4$. Suppose there exists a single digit number d such that $s_1 + s_2 \not\equiv d \pmod{10}$ for each $(s_1, s_2) \in S \times S$ then we will not be able to generate

$$\underbrace{\overline{** \ldots *d}}_{\text{take sufficiently large number}}$$

as a sum of a, b. Hence it must be that,

$$\{s_1 + s_2 \pmod{10} \mid s_1, s_2 \in S\} = \mathbb{Z}_{10} \quad \text{meaning,} \quad \binom{|S|}{2} + |S| \geq 10.$$

From the above mentioned bound it is quite clear that $|S| \geq 4$, so we are left to show that $|S| \neq 4$. Suppose $S = \{s_1, s_2, s_3, s_4\}$, notice $2s_i \pmod{10}$ generates four distinct even numbers among $\mathbb{Z}_{10}$. So either exactly one element of S is even or all are odd, the latter implies $s_i + s_j$ is even for $i \neq j$ meaning the number of even numbers generated exceeds $10/2$ which is ridiculous. Similarly, we argue that the former case is also not possible, therefore $|S| > 4$ and we are done. □

Problem 2 (Sleepy Students – OTIS)**.** There are n sleepy students working on a morning

constest. The contest has six problems, and the score on each problem is a non negative integer less than or equal to 10. Given that no two students got the same score on two or more problems, what is the greatest possible value of n?
Solution. The answer is 121.

By pigeonhole principle it is evident that n is atmost 121. And fortunately by the virtue of this artificial setup, there exists a construction for $n = 121$ − consider the scores of each student to be an arithmetic progression of course under modulo 11. There are 11 choices each for the intial term and the common difference hence making $11 \times 11 = 121$ of them.

But how are we sure that no two of them have two or more coinciding terms?
Suppose some two different arithmetic progressions namely, $\{a, a + d, \ldots, a + 5d\}$ and $\{a', a'+d', \ldots, a'+5d'\}$ have two coinciding terms say $a+nd = a'+nd'$ and $a+md = a'+md'$ with $0 \le n \ne m \le 5$. As 11 is prime, by cancellation law we have that $n \equiv m$ which is a contradiction and we are done. □

> **Remark.** I had to look up the ARCH for hints regarding the construction.

Problem 3 (RMM P1, 2015). Does there exist an infinite sequence of postivie integers $a_1, a_2, \ldots$ such that $\gcd(a_m, a_n) = 1$ if and only if $|m - n| = 1$?
Solution. The answer is yes.
Denote the n^{th} prime number as p_n. Consider the following construction −

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
p_1	$\bullet$		$\bullet$		$\bullet$		$\bullet$		$\bullet$	
p_2		$\bullet$		$\bullet$		$\bullet$		$\bullet$		$\bullet$
p_3	$\bullet$			$\bullet$		$\bullet$		$\bullet$		$\bullet$
p_4		$\bullet$			$\bullet$		$\bullet$		$\bullet$	
p_5			$\bullet$			$\bullet$		$\bullet$		$\bullet$
p_6				$\bullet$			$\bullet$		$\bullet$	
p_7					$\bullet$			$\bullet$		$\bullet$
p_8						$\bullet$			$\bullet$	
p_9							$\bullet$			$\bullet$
p_{10}								$\bullet$		

proof with little to no words[1]

Clearly each a_n is finite and satisfy the hypothesis. □

Problem 4 (Junior Balkan Shortlist C2, 2021). Let n be a positive integer. We are given a $3n \times 3n$ board whose unit squares are colored in black and white in such way that

[1] Here a '$\bullet$' is at some position (i, j) if and only if $p_i | a_j$.

starting with the top left square, every third diagonal is colored in black and the rest of the board is in white. In one move, one can take a 2×2 square and change the color of all its squares in such way that white squares become orange, orange ones become black and black ones become white. Find all n for which, using a finite number of moves, we can make all the squares which were initially black white, and all squares which were initially white black.

Solution. The desired end state can be acheived if and only if n is even.

> **Proposition 1.** A square which was initially black should be subjected to $1 \pmod 3$ moves and $2 \pmod 3$ moves if the concerning square was white initially inorder to attain the end state.

Define score of a 2×2 square as the number of times it has been chosen throughout. Observe the top row closely, the rightmost 2×2 square must have a score which is $1 \pmod 3$, so this forces the next square to the left to have a score that is $1 \pmod 3$ and so on till we reach the left most square. Denote black squares as B, white square as W and orange squares as O. For example consider $n = 3$, the scores of squares in the top rows modulo 3 arranged in order must look like,

$$\begin{bmatrix} \_ & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 \\ W & W & B & W & W & B & W & W & B \\ W & B & W & W & B & W & W & B & W \\ B & W & W & B & W & W & B & W & W \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B & W & W & B & W & W & B & W & W \end{bmatrix}.$$

Case 1. When n is odd.

Considering the previous example, note the top-left most square cannot turn black at the end of the process no matter what. This is because the score of the only square it is a part of is $1 \pmod 3$. By induction we can easily show this holds for all odd n. Which means the first row itself can never reach the end state. Not to mention, one can easily show the case $n = 1$ does not work.

Case 2. When n is even.

It would suffice to show that end state can be acheived for 6×6 grid as the successive grids $12 \times 12, 18 \times 18$ and so can be partitioned into 6×6 grids that repeat themselves. The algorithm is, flip the colours of a rows one at a time starting from the top then traversing till the bottom.

$$\begin{bmatrix} W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \\ W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix} \to \begin{bmatrix} B & B & W & B & B & W \\ B & O & O & B & O & O \\ B & W & W & B & W & W \\ W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix} \to \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ B & B & O & B & B & O \\ W & W & B & W & W & B \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \\ O & W & W & O & W & W \\ W & B & W & W & B & W \\ B & W & W & B & W & W \end{bmatrix} \rightarrow \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \\ B & B & W & B & B & W \\ O & O & W & O & O & W \\ B & W & W & B & W & W \end{bmatrix} \rightarrow \begin{bmatrix} B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \\ B & B & W & B & B & W \\ B & W & B & B & W & B \\ W & B & B & W & B & B \end{bmatrix}.$$

Longing turned to having. □

Problem 5 (India P4, 2025)**.** Let $n \geq 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

$$\max\left(|a_1 + a_2|, |a_2 + a_3|, \ldots, |a_{n-1} + a_n|, |a_n + a_1|\right) \geq t_n \cdot \max(|a_1|, |a_2|, \ldots, |a_n|)$$

holds for all real numbers $a_1, a_2, \ldots, a_n$.

Solution. We claim $t_n = 0$ when n is even and $t = 2/n$ otherwise.

The case when n is even is trivial because of the following construction,

$$(a_1, a_2, \ldots, n) = (1, -1, \ldots, -1).$$

Which gives $t_n = 0$.

Now we deal with the case when n is odd. The central theme of the solution is to exploit the triangle inequality, by which we have

$$\begin{aligned} n \cdot \max_{1 \leq i < n} \left\{|a_i + a_{i+1}|, |a_n + a_1|\right\} &\geq |(a_n + a_1)| + \sum_{1 \leq i < n} |(a_i + a_{i+1})| \\ &\geq 2 \max_{1 \leq i \leq n} |a_i| \end{aligned}$$

So we have a nice lower bound for t_n which is $t \geq 2/n$, and fortunate enough this is the upper bound as well. Consider the construction that make sures $\max\limits_{1 \leq i < n} \left\{|a_i + a_{i+1}|, |a_n + a_1|\right\} = 2$,

$$(a_1, a_2, \ldots, a_n) = (1, -3, 5, \ldots, (-1)^{\frac{n-1}{2}} n, \ldots, 5, -3, 1).$$

□

Problem 6 (Coluring Numbers Efficiently-Kithun)**.** What is the least number required to colour the integers $1, 2, \ldots, 2^n - 1$ such that for any set of consecutive integers taken from the given set of integers, there will always be a colour colouring exactly one of them? That is, for all integers i, j such that $1 \leq i \leq j \leq 2^n - 1$, there will be a colour coloring exactly one integer from the set $i, i+1, \ldots, j-1, j$.

Solution. Denote the minimum number of colours required to colour the integers $1, 2, \ldots, 2^n - 1$ such that for any set of consecutive integers taken from the given set of integers, there will always be a colour colouring exactly one of them as χ_n. For convenience call a colouring *minimally-valid* if the colouring is "valid" and uses only χ_n different colours. The key idea is to come up with good enough non-trivial bounds on χ_n. If we are lucky, the lower bound might turn out to be the same as that of the upper bound and we win.

Before jumping into the details of χ_n, let us try colouring the numbers $1, 2, \ldots, 2^4 - 1$ (base case being $n = 3$ is rather trivial giving us little to no insights), maybe we can come up with a nice construction that could be generalised easily.

1 2 3 4 5 6 7 | 8 9 10 11 12 13 14 15

Note that, the above colouring is valid and turns out it is minimally valid as well (proof simply follows by considering a colouring using three or less distinct colours and getting a contradiction). Cool, but how can we proceed for cases $n = 5, 6, 7$ and so on? For $n = 5$ just colour the numbers $1, 2, \ldots, 15$ as shown above and colour 16 with a new colour which is not used to colour the numbers before it. Colour the remaining numbers $17, 18, \ldots, 31$ in the same manner in which we coloured $1, 2, \ldots, 15$. For instance 17 is orange, 18 is blue and so on. This means $\chi_5 \leq 5$. Aha!! we can continue this process indefinitely for bigger n introducing exactly one new colour each time giving us the upper bound $\chi_n \leq n$. We now ask the question, if this upper bound can be refined further and fortunately the answer is a clear no.

Claim 1. $\chi_n \geq n$.

Proof. Partition the given set of integers into three subsets as $A = \{1, 2, \ldots, 2^{n-1} - 1\}$, $B = \{2^{n-1}\}$ and $C = \{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n - 1\}$. Notice, the numbers in A must be coloured with at least χ_{n-1} distinct colours and similarly C must be coloured with at least χ_{n-1} distinct colours. Our goal isto show that $\chi_n = \chi_{n-1} + 1$. Suppose not, then we know that χ_n is at least χ_{n-1}. So the only case we have to deal with is when $\chi_n = \chi_{n-1}$.If you look closely, there must be a number among $1, 2, \ldots, 2^n - 1$ with a unique colour, if not, consider the whole set as $2^n - 1$ consecutive integers giving us a contradiction. In order to minimize χ_n, it must be that numbers in A and C must be coloured with the same set of colours. But note that every colour is used at least two times, which means the number in B that is 2^{n-1} must be coloured with a unique colour and that is a contradiction as we had initially assumed $\chi_n = \chi_{n-1}$. Hence the lower bound. □

It is evident that $n \leq \chi_n \leq n$, hence giving us the required result that is $\chi_n = n$. □

Remark. A further question one may ask is, how many minimally-valid colourings are possible in total given a set of distinct colours $C_1, C_2, \ldots, C_{\chi_n}$?

Problem 7 (Ahan Chakraborty-Unknown). Let $n \in \mathbb{N}$. Let $X = \{1, 2, 3, ..., n^2\}$.Let $A \subset X$ with $|A| = n$. Prove that $X \setminus A$ contains an arithmetic progression with n terms.

Solution. Suppose I could construct a subset A such that $X \setminus A$ does not contain an arithmetic progression with n elements. We shall device an algorithm for constructing it and then arrive at a conclusion that it is not possible. Arrange the elements on X on a $n \times n$ grid as follows,

$$\begin{array}{cccccc}
1 & 2 & 3 & \cdots & n-1 & n \\
n+1 & n+2 & n+3 & \cdots & 2n-1 & 2n \\
2n+1 & 2n+2 & 2n+3 & \cdots & 3n-1 & 3n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^2-2n+1 & n^2-2n+2 & n^2-2n+3 & \cdots & n^2-n-1 & n^2-n \\
n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2-1 & n^2
\end{array}$$

The nice thing about this arrangement is that, several n-element arithmetic progressions are very easy to spot i.e, just walk along the rows, columns, main diagonals. Notice that we must pick exactly one element from each row and column and place it in A, or else the row/column which doesn't share an element with A will be a counter example. If we pick 1, then we cannot pick any other element from the first row and also we cannot pick $n+1$, which means $\{2, 3, \ldots, n+1\} \subseteq X \setminus A$ and that does not sound good to us. Similarly we can argue that none of the elements in the first column can be picked except for the last one (with value $= n^2 - n + 1$). Continuing this argument we can see that the ideal construction is to pick all the diagonal elements (the main diagonal that contains $n^2 - n + 1$). Note that $\{n-1, 2n-2, 3n-3, \ldots, (n-1)^2\} \cup \{(n-1)^2 + n\} \subseteq X \setminus A$, hence contradicting the existence of such an A and we are done. $\square$

> **Remark.** This way of constructing the algorithm resembles placing n non-attacking rooks on a $n \times n$ chessboard, which seem like two totally unrelated topics.

Problem 8 (IM0 P2, 2014). Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any its k^2 unit squares.
Solution. The answer is $\lfloor n - 1 \rfloor$.
Let $f(n)$ denote the greatest such k for a given n.