

Handbook on excursion sets

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1 Preliminary calculations

1.1 Differentiation under the integral sign

$$\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt \quad (1)$$

1.2 Functions

1.2.1 Gamma function: Γ

$$\Gamma : z \rightarrow \int_0^\infty t^{z-1} e^{-t} dt \quad (2)$$

1.2.2 Error function: erf

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (3)$$

1.3 Lipschitz-Killing curvatures (LKC)

1.3.1 For a regular shapes in 1,2 and 3 dimensions

$$\text{For a cube } \mathcal{M}_N = \prod_{i=1}^N [0, a], \quad \mathcal{L}_j(\mathcal{M}_N) = \binom{N}{j} a^j \quad (4)$$

$$\text{For a parallelepiped } \mathcal{P}_N = \prod_{i=1}^N [0, a_i], \quad \mathcal{L}_j(\mathcal{M}_N) = \dots \quad (5)$$

N	LKC	Value	Meaning
1	$\mathcal{L}_0(\mathcal{M})$	1	Euler Characteristic
1	$\mathcal{L}_1(\mathcal{M})$	a	Length of the segment
2	$\mathcal{L}_0(\mathcal{M})$	1	Euler Characteristic
2	$\mathcal{L}_1(\mathcal{M})$	$2a$	Half the boundary length
2	$\mathcal{L}_2(\mathcal{M})$	a^2	Surface area
3	$\mathcal{L}_0(\mathcal{M})$	1	Euler Characteristic
3	$\mathcal{L}_1(\mathcal{M})$	$3a$	Twice the caliper diameter
3	$\mathcal{L}_2(\mathcal{M})$	$3a^2$	Half the surface area
3	$\mathcal{L}_3(\mathcal{M})$	a^3	Volume
3	$\mathcal{L}_0(\mathcal{P})$	1	Euler Characteristic
3	$\mathcal{L}_1(\mathcal{P})$	$a_1 + a_2 + a_3$	Twice the caliper diameter
3	$\mathcal{L}_2(\mathcal{P})$	$a_1 a_2 + a_1 a_3 + a_2 a_3$	Half the surface area
3	$\mathcal{L}_3(\mathcal{P})$	$a_1 a_2 a_3$	Volume

1.4 Volume of the unit ball

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \quad (6) \quad \left| \begin{array}{l} \omega_0 = 1 \\ \omega_1 = 2 \\ \omega_2 = \pi \\ \omega_3 = 4\pi/3 \\ \dots \end{array} \right. \quad (7)$$

1.5 Probabilistic Hermite polynomials

$$\mathcal{H}_n = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad (8) \quad \left| \begin{array}{l} \mathcal{H}_0(x) = 1 \\ \mathcal{H}_1(x) = x \\ \mathcal{H}_2(x) = x^2 - 1 \\ \mathcal{H}_3(x) = x^3 - 3x \\ \dots \end{array} \right. \quad (9)$$

1.6 Flag coefficients

$$\begin{bmatrix} n \\ j \end{bmatrix} = \binom{n}{j} \frac{\omega_n}{\omega_{n-j}\omega_j} \quad (10)$$

n	j	flag
0	0	1
1	0	1
1	1	1
2	0	1
2	1	$\pi/2$
2	2	1
3	0	1
3	1	2
3	2	2
3	3	1

(11)

1.7 Second spectral moment

$$\lambda_2 = \mathbb{V} \left\{ \frac{\partial g(\mathbf{x})}{\partial x_i} \right\} = \frac{d^2 \mathcal{C}(h)}{dh^2} \quad (12)$$

Covariance	Parameter	λ_2
Gaussian	—	$2\sigma^2/L_c^2$

(13)

2 Gaussian Minkowsky functionals

2.1 Preliminary

2.1.1 Gaussian measure: $\gamma^k(\mathcal{H})$

$$\gamma^k(\mathcal{H}) = P\{\mathbf{X} \in \mathcal{H}\} = \frac{1}{\sigma^k (2\pi)^{k/2}} \int_{\mathcal{H}} e^{-\|\mathbf{x}-\mu\|^2/2\sigma^2} d\mathbf{x} \quad (14)$$

2.1.2 Tube Taylor expansion

$$\gamma^k(\text{tube}(\mathcal{H}, \rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma^k}(\mathcal{H}) \quad (15)$$

2.2 GMFs for Gaussian distribution

2.2.1 Application to one-dimensional tail hitting set: $k = 1$ and $\mathcal{H} = [\kappa \infty)$

Tail probability \bar{F} :

$$\gamma^1([\kappa \infty)) = P\{X \geq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{\kappa}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = \bar{F}(\kappa) \quad (16)$$

Error function erf:

$$\begin{aligned} \gamma^1([\kappa \infty)) &= \bar{F}(\kappa) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} \sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} dt \\ &= \frac{1}{\sqrt{\pi}} \left[\int_0^{\infty} e^{-t^2} dt - \int_0^{\frac{\kappa-\mu}{\sigma\sqrt{2}}} e^{-t^2} dt \right] \\ &= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right) \right] \\ &= \frac{1}{2} \left(1 - \operatorname{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right) \right) \end{aligned}$$

Tube expansion:

$$\gamma^1(\text{tube}([\kappa \infty), \rho)) = \gamma^1([\kappa - \rho \infty)) = \bar{F}(\kappa - \rho) = \sum_{j=0}^{\infty} \frac{(-\rho)^j}{j!} \frac{d^j \bar{F}(\kappa)}{d\kappa^j} \quad (18)$$

Identification:

- Notations:

$$\text{Tail hitting set: } \bar{\mathcal{H}}^{1D} = [\kappa \infty) \quad (19a)$$

$$\text{Variable substitution: } \tilde{\kappa} \leftarrow \frac{\kappa - \mu}{\sigma} \quad (19b)$$

- For $j = 0$:

$$\mathcal{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{1D}) = \bar{F}(\kappa) \quad (20)$$

- For $j > 0$:

$$\mathcal{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{1D}) = (-1)^j \frac{d^j \bar{F}(\kappa)}{d\kappa^j} = \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j \sqrt{2\pi}} \mathcal{H}_{j-1}((\kappa - \mu)/\sigma) \quad (21)$$

- First values:

$$\left| \begin{aligned} \mathcal{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{1D}) &= \frac{1}{2} \left(1 - \operatorname{erf} \left(\frac{\tilde{\kappa}}{\sqrt{2}} \right) \right) \\ \mathcal{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{1D}) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{1D}) &= \frac{1}{\sigma^2\sqrt{2\pi}} \tilde{\kappa} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{1D}) &= \frac{1}{\sigma^3\sqrt{2\pi}} (\tilde{\kappa}^2 - 1) e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{1D}) &= \frac{1}{\sigma^4\sqrt{2\pi}} (\tilde{\kappa}^3 - 3\tilde{\kappa}) e^{-\tilde{\kappa}^2/2} \\ &\dots \end{aligned} \right. \quad (22)$$

- Derivatives:

$$\forall j, \quad \frac{d\mathcal{M}_j^{\gamma^1}}{d\kappa}(\bar{\mathcal{H}}^{1D}) = (-1)^j \frac{d^{j+1}\bar{F}(\kappa)}{d\kappa^{j+1}} = -\mathcal{M}_{j+1}^{\gamma^1}(\bar{\mathcal{H}}^{1D}) \quad (23)$$

2.2.2 Application to one-dimensional cumulative hitting set: $k = 1$ and $\mathcal{H} = (-\infty \kappa]$

Cumulative probability F :

$$\gamma^1((-\infty \kappa]) = P\{X \leq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-(x-\mu)^2/2\sigma^2} dx = F(\kappa) \quad (24)$$

Error function erf:

$$\gamma^1((-\infty \kappa]) = F(\kappa) = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{\kappa - \mu}{\sigma\sqrt{2}} \right) \right) \quad (25)$$

Tube expansion:

$$\gamma^1(\text{tube}((-\infty \kappa], \rho)) = \gamma^1((-\infty \kappa + \rho]) = F(\kappa + \rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \frac{d^j F(\kappa)}{d\kappa^j} \quad (26)$$

Identification:

- Notations:

$$\text{Tail hitting set: } \mathcal{H}^{1D} = (-\infty \kappa] \quad (27a)$$

$$\text{Variable substitution: } \tilde{\kappa} \leftarrow \frac{\kappa - \mu}{\sigma} \quad (27b)$$

- For $j = 0$:

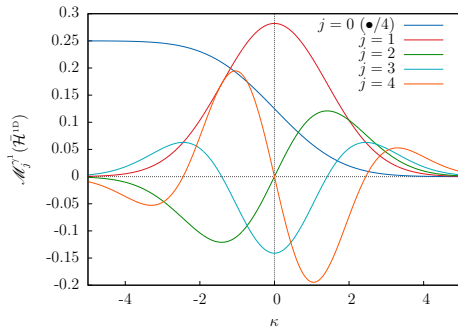
$$\mathcal{M}_0^{\gamma^1}(\mathcal{H}^{1D}) = F(\kappa) \quad (28)$$

- For $j > 0$:

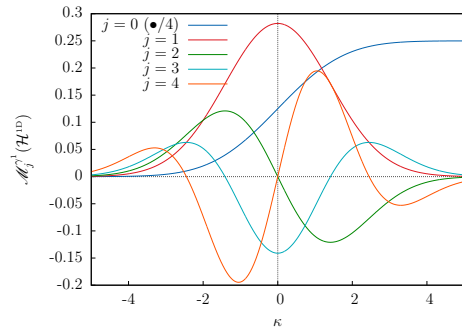
$$\mathcal{M}_j^{\gamma^1}(\mathcal{H}^{1D}) = \frac{d^j F(\kappa)}{d\kappa^j} = (-1)^{j+1} \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j \sqrt{2\pi}} \mathcal{H}_{j-1}((\kappa-\mu)/\sigma) \quad (29)$$

- First values:

$$\left\{ \begin{array}{l} \mathcal{M}_0^{\gamma^1}(\mathcal{H}^{1D}) = \frac{1}{2} \left(1 - \operatorname{erf} \left(\frac{\tilde{\kappa}}{\sqrt{2}} \right) \right) \\ \mathcal{M}_1^{\gamma^1}(\mathcal{H}^{1D}) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_2^{\gamma^1}(\mathcal{H}^{1D}) = \frac{-1}{\sigma^2 \sqrt{2\pi}} \tilde{\kappa} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_3^{\gamma^1}(\mathcal{H}^{1D}) = \frac{1}{\sigma^3 \sqrt{2\pi}} (\tilde{\kappa}^2 - 1) e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_4^{\gamma^1}(\mathcal{H}^{1D}) = \frac{-1}{\sigma^4 \sqrt{2\pi}} (\tilde{\kappa}^3 - 3\tilde{\kappa}) e^{-\tilde{\kappa}^2/2} \\ \dots \end{array} \right. \quad (30)$$



(a) $\mathcal{H} = \mathcal{H}^{1D} = [\kappa, \infty)$



(b) $\mathcal{H} = \mathcal{H}^{1D} = (-\infty, \kappa]$

Figure 1: First 4 GMF for $\sigma^2 = 2$

- Derivatives:

$$\forall j, \quad \frac{d\mathcal{M}_j^{\gamma^1}}{d\kappa}(\mathcal{H}^{1D}) = \frac{d^{j+1} F(\kappa)}{d\kappa^{j+1}} = \mathcal{M}_{j+1}^{\gamma^1}(\mathcal{H}^{1D}) \quad (31)$$

2.3 GMFs for Gaussian related Random Fields

$$g_r : \Omega \times \mathbf{R}^N \xrightarrow{g} \mathbf{R}^k \xrightarrow{S} \mathbf{R} \quad (32)$$

$$P\{g_r \in \mathcal{H}\} = P\{S(g) \in \mathcal{H}\} = P\{g \in S^{-1}(\mathcal{H})\} = \gamma^k(S^{-1}(\mathcal{H})) \quad (33)$$

2.3.1 Log-normal distribution

$$k = 1 \quad , \quad S = \exp \quad \text{and} \quad S^{-1} = \ln \quad (34)$$

$$\left| \begin{array}{l} \mu = \ln(\mu_{\ln}) - \frac{1}{2} \ln(1 + \sigma_{\ln}^2 / \mu_{\ln}^2) \\ \sigma^2 = \ln(1 + \sigma_{\ln}^2 / \mu_{\ln}^2) \end{array} \right. \quad (35)$$

- Notations:

$$\text{Log tail hitting set: } \bar{\mathcal{H}}_{\ln}^{1D} = [\ln(\kappa) \infty) \quad (36a)$$

$$\text{Log cumulative hitting set: } \mathcal{H}_{\ln}^{1D} = (-\infty \ln(\kappa)] \quad (36b)$$

$$\text{Variable substitution: } \tilde{\kappa}_{\ln} \leftarrow \frac{\ln(\kappa) - \mu}{\sigma} \quad (36c)$$

- For $\mathcal{H} = \bar{\mathcal{H}}_{\ln}^{1D} = [\kappa \infty) \rightarrow S^{-1}(\mathcal{H}) = \bar{\mathcal{H}}_{\ln}^{1D} = [\ln(\kappa) \infty)$

$$\left| \begin{array}{l} \mathcal{M}_0^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1D}) = \frac{1}{2} \left(1 - \operatorname{erf} \left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}} \right) \right) \\ \mathcal{M}_1^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1D}) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\tilde{\kappa}_{\ln}^2/2} \\ \mathcal{M}_2^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1D}) = \frac{1}{\sigma^2 \sqrt{2\pi}} \tilde{\kappa}_{\ln} e^{-\tilde{\kappa}_{\ln}^2/2} \\ \mathcal{M}_3^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1D}) = \frac{1}{\sigma^3 \sqrt{2\pi}} (\tilde{\kappa}_{\ln}^2 - 1) e^{-\tilde{\kappa}_{\ln}^2/2} \\ \mathcal{M}_4^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1D}) = \frac{1}{\sigma^4 \sqrt{2\pi}} (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2} \\ \dots \end{array} \right. \quad (37)$$

Derivatives:

$$\begin{aligned}
& \text{Help: } \left(e^{-(\ln(x)-\mu)^2/2\sigma^2} \right)' = -\frac{1}{\sigma^2} \frac{\ln(x)-\mu}{x} e^{-(\ln(x)-\mu)^2/2\sigma^2} \\
& \frac{d\mathcal{M}_0^{\gamma^1}(\tilde{\mathcal{H}}_{\ln}^{\text{1D}})}{d\kappa} = \frac{-1}{\sigma\sqrt{2\pi}} \frac{1}{\kappa} e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
& \frac{d\mathcal{M}_1^{\gamma^1}(\tilde{\mathcal{H}}_{\ln}^{\text{1D}})}{d\kappa} = \frac{-1}{\sigma^2\sqrt{2\pi}} \frac{1}{\kappa} \frac{\ln(\kappa)-\mu}{\sigma} e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
& \frac{d\mathcal{M}_2^{\gamma^1}(\tilde{\mathcal{H}}_{\ln}^{\text{1D}})}{d\kappa} = \frac{-1}{\sigma^3\sqrt{2\pi}} \frac{1}{\kappa} \left[\frac{(\ln(\kappa)-\mu)^2}{\sigma^2} - 1 \right] e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
& \frac{d\mathcal{M}_3^{\gamma^1}(\tilde{\mathcal{H}}_{\ln}^{\text{1D}})}{d\kappa} = \frac{-1}{\sigma^4\sqrt{2\pi}} \frac{1}{\kappa} \left[\frac{(\ln(\kappa)-\mu)^3}{\sigma^3} - 3\frac{(\ln(\kappa)-\mu)}{\sigma} \right] e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
& \dots \\
& \text{Guess would be: } \forall j \geq 0, \\
& \frac{d\mathcal{M}_j^{\gamma^1}(\tilde{\mathcal{H}}_{\ln}^{\text{1D}})}{d\kappa} = \frac{-1}{\sigma^{j+1}\sqrt{2\pi}} \frac{1}{\kappa} \mathcal{H}_j(\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2}
\end{aligned} \tag{38}$$

- For $\mathcal{H} = [0 \ \kappa] \rightarrow S^{-1}(\mathcal{H}) = \mathcal{H}_{\ln}^{\text{1D}} = (-\infty \ \ln(\kappa)]$

$$\begin{aligned}
& \mathcal{M}_0^{\gamma^1}(\mathcal{H}_{\ln}^{\text{1D}}) = \frac{1}{2} \left(1 + \text{erf} \left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}} \right) \right) \\
& \mathcal{M}_1^{\gamma^1}(\mathcal{H}_{\ln}^{\text{1D}}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\tilde{\kappa}_{\ln}^2/2} \\
& \mathcal{M}_2^{\gamma^1}(\mathcal{H}_{\ln}^{\text{1D}}) = \frac{-1}{\sigma^2\sqrt{2\pi}} \tilde{\kappa}_{\ln} e^{-\tilde{\kappa}_{\ln}^2/2} \\
& \mathcal{M}_3^{\gamma^1}(\mathcal{H}_{\ln}^{\text{1D}}) = \frac{1}{\sigma^3\sqrt{2\pi}} (\tilde{\kappa}_{\ln}^2 - 1) e^{-\tilde{\kappa}_{\ln}^2/2} \\
& \mathcal{M}_4^{\gamma^1}(\mathcal{H}_{\ln}^{\text{1D}}) = \frac{-1}{\sigma^4\sqrt{2\pi}} (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2} \\
& \dots
\end{aligned} \tag{39}$$

Derivatives:

$$\begin{aligned}
& \text{Guess would be: } \forall j \geq 0, \\
& \frac{d\mathcal{M}_j^{\gamma^1}(\mathcal{H}_{\ln}^{\text{1D}})}{d\kappa} = \frac{(-1)^j}{\sigma^{j+1}\sqrt{2\pi}} \frac{1}{\kappa} \mathcal{H}_j(\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2}
\end{aligned} \tag{40}$$

3 Expectation Formula

3.1 Gaussian distribution

3.1.1 General case

$$\mathbb{E} \{ \mathcal{L}_j(\mathcal{E}) \} = \sum_{i=0}^{N-j} \begin{bmatrix} i+j \\ i \end{bmatrix} \left(\frac{\lambda_2}{2\pi} \right)^{i/2} \mathcal{L}_{i+j}(\mathcal{M}) \mathcal{M}_i^{\gamma^k}(S^{-1}(\mathcal{H})) \quad (41)$$

Notations:

$$\text{LKC of } \mathcal{M}: \mathcal{L}_j(\mathcal{M}) = \mathcal{L}_j^{\mathcal{M}} \quad (42a)$$

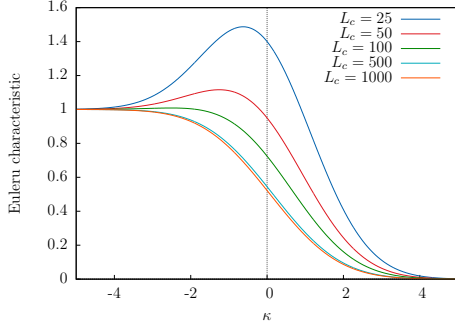
$$\text{LKC of } \mathcal{E}: \mathcal{L}_j(\mathcal{E}) = \mathcal{L}_j^{\mathcal{E}} \quad (42b)$$

$$\text{GMF of } S^{-1}(\mathcal{H}): \mathcal{M}_j^{\gamma^k}(S^{-1}(\mathcal{H})) = \mathcal{M}_j^{\gamma^k} \quad (42c)$$

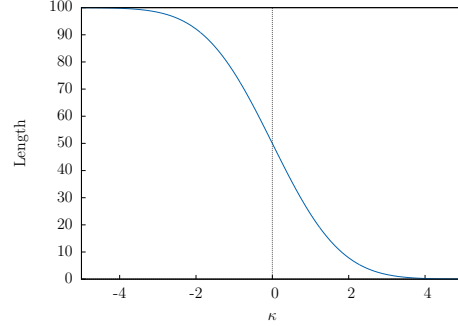
3.1.2 One dimension

$$\mathbb{E} \{ \mathcal{L}_0^{\mathcal{E}} \} = \left(\frac{\lambda_2}{2\pi} \right)^{1/2} \mathcal{L}_1^{\mathcal{M}} \mathcal{M}_1^{\gamma^k} + \mathcal{L}_0^{\mathcal{M}} \mathcal{M}_0^{\gamma^k} \quad (43a)$$

$$\mathbb{E} \{ \mathcal{L}_1^{\mathcal{E}} \} = \mathcal{L}_1^{\mathcal{M}} \mathcal{M}_0^{\gamma^k} \quad (43b)$$



(a) Euler Characteristic



(b) Length

Figure 2: Expectation of LKC for $a = 100$, $\sigma^2 = 2$ and $\mathcal{H} = [\kappa \infty)$ in 1D.

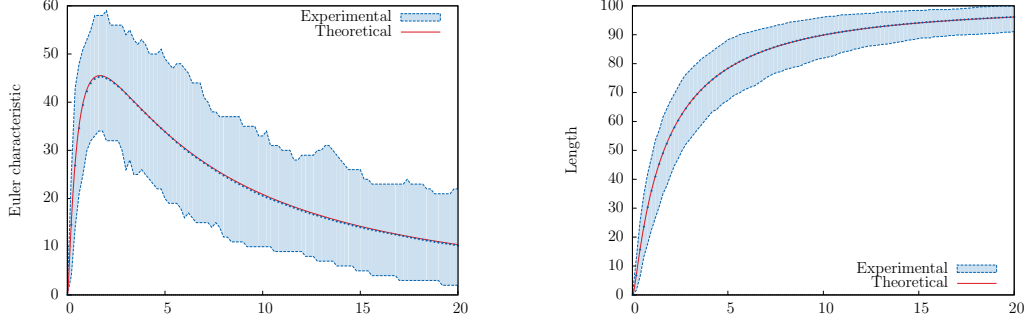


Figure 3: Expectation of LKC for $a = 100$, $\mu = 0.5$, $\sigma^2 = 2$, $L_c = 0.5$, and $\mathcal{H} = [0 \ \kappa]$ (log-normal distribution with cumulative hitting set) in 1D. Experimental results are calculated over 1 000 realizations.

3.1.3 Two dimensions

$$\mathbb{E} \{ \mathcal{L}_0^\mathcal{E} \} = \frac{\lambda_2}{2\pi} \mathcal{L}_2^\mathcal{M} \mathcal{M}_2^{\gamma^k} + \sqrt{\frac{\lambda_2}{2\pi}} \mathcal{L}_1^\mathcal{M} \mathcal{M}_1^{\gamma^k} + \mathcal{L}_0^\mathcal{M} \mathcal{M}_0^{\gamma^k} \quad (44a)$$

$$\mathbb{E} \{ \mathcal{L}_1^\mathcal{E} \} = \sqrt{\frac{\lambda_2 \pi}{8}} \mathcal{L}_2^\mathcal{M} \mathcal{M}_1^{\gamma^k} + \mathcal{L}_1^\mathcal{M} \mathcal{M}_0^{\gamma^k} \quad (44b)$$

$$\mathbb{E} \{ \mathcal{L}_2^\mathcal{E} \} = \mathcal{L}_2^\mathcal{M} \mathcal{M}_0^{\gamma^k} \quad (44c)$$

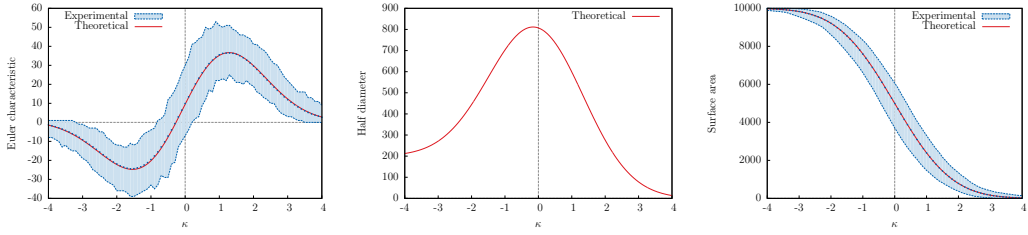


Figure 4: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [\kappa \ \infty)$ (gaussian distribution with tail hitting set) in 2D. Experimental results are calculated over 1 000 realizations.

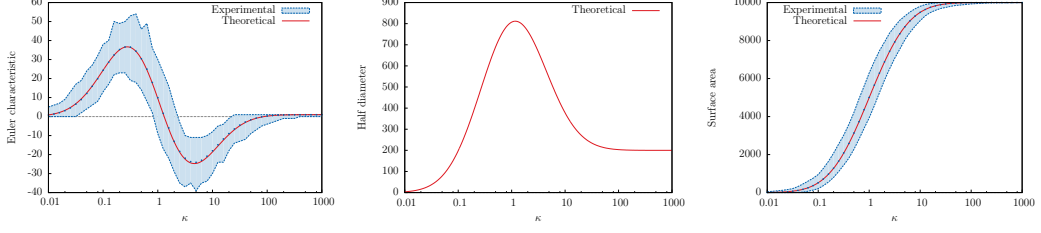


Figure 5: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [0 \ \kappa]$ (lognormal distribution with cumulative hitting set) in 2D. Experimental results are calculated over 1 000 realizations.

3.1.4 Three dimensions

$$\mathbb{E} \{ \mathcal{L}_0^\varepsilon \} = \quad (45a)$$

$$\mathbb{E} \{ \mathcal{L}_1^\varepsilon \} = \quad (45b)$$

$$\mathbb{E} \{ \mathcal{L}_2^\varepsilon \} = \quad (45c)$$

$$\mathbb{E} \{ \mathcal{L}_3^\varepsilon \} = \mathcal{L}_3^M \mathcal{M}_0^{\gamma^k} \quad (45d)$$

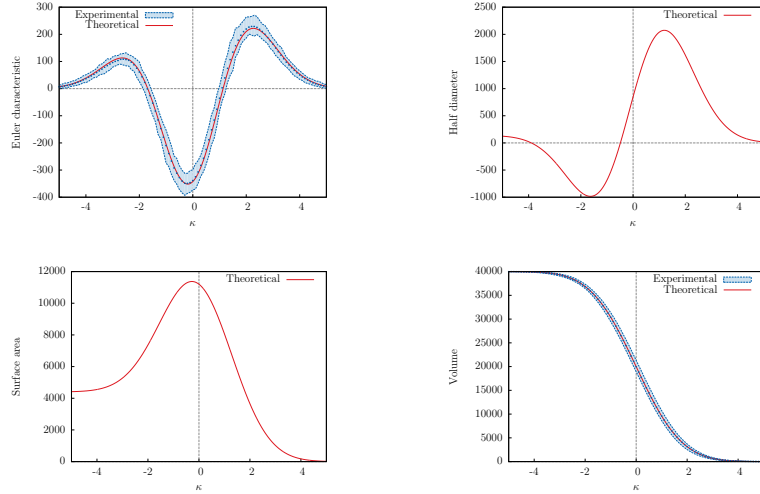


Figure 6: Expectation of LKC for $\mathcal{P} = 100 \times 20 \times 20$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 2$, and $\mathcal{H} = [\kappa \ \infty)$ (gaussian distribution with tail hitting set) in 3D. Experimental results are calculated over 100 realizations.

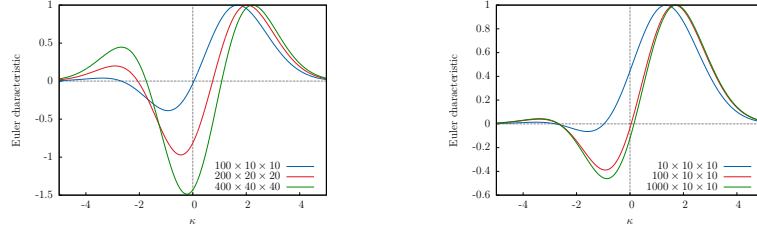
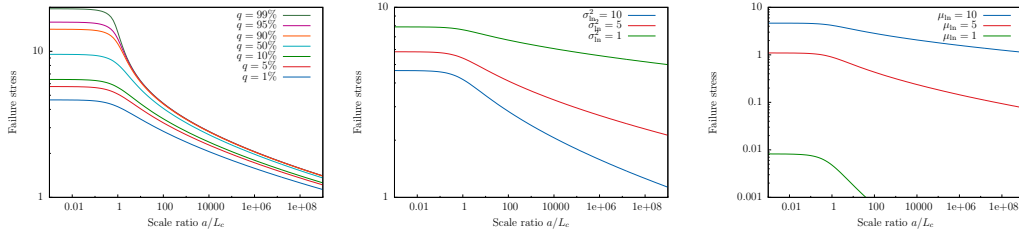


Figure 7: .

4 Size effect

4.1 One dimensional case



(a) Variation of the quantile (b) Variation of the variance (c) Variation of the mean

Figure 8: Size Effect: variations around $\mu_{\ln} = 10$, $\sigma_{\ln}^2 = 10$ and $q = 1\%$ in 1D.

4.2 Two dimensional case

4.2.1 Behavior of the Euler characteristic for different specimen sizes

9(a) Small sizes

Direct percolation. The first (and only) $\kappa(\chi == q)$ correspond to the probability q of percolation. There is only one disconnected element in the excursion set.

9(b) Medium sizes

9(c) Large sizes

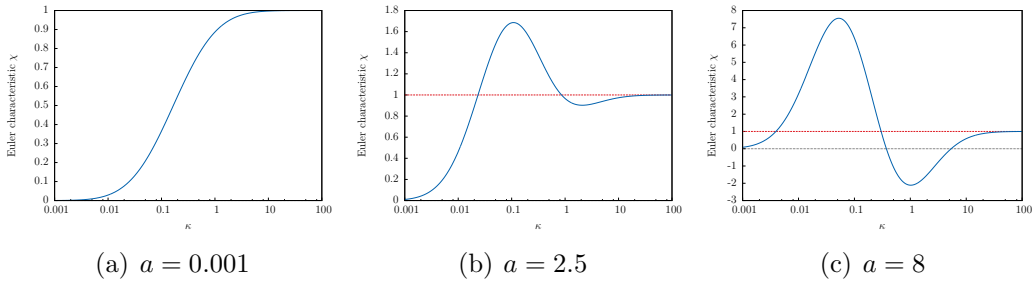


Figure 9: Euler characteristic for different specimen sizes a for $\mu_{\text{ln}} = 0.5$, $\sigma_{\text{ln}}^2 = 2$, $L_c = 1$, and $\mathcal{H} = [0 \ \kappa]$ (lognormal distribution with cumulative hitting set) in a 2D.