Handbook on excursion sets

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Version: December 16, 2014

Contents

1 Priliminary calculations

1.1 Differentiation under the integral sign

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt \tag{1}$$

1.2 Functions

1.2.1 Gamma function: Γ

$$\Gamma : z \to \int_0^\infty t^{z-1} e^{-t} dt \tag{2}$$

1.2.2 Error function: erf

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{3}$$

1.3 Lipschitz-Killing curvatures (LKC)

1.3.1 For a cube in 1,2 and 3 dimensions

$$\text{For } \mathcal{M}_N = \prod_{i=1}^{N} [0\ a], \quad \mathscr{L}_j(\mathcal{M}_N) = \begin{pmatrix} N \\ j \end{pmatrix} a^j \tag{4}$$

N	LKC	Value	Meaning
1	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
1	$\mathscr{L}_1(\mathcal{M})$	a	Length of the segment
2	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
2	$\mathscr{L}_1(\mathcal{M})$	$2a$	Half the boundary length
2	$\mathscr{L}_2(\mathcal{M})$	a^2	Surface area
3	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
3	$\mathscr{L}_1(\mathcal{M})$	$3a$	Twice the caliper diameter
3	$\mathscr{L}_2(\mathcal{M})$	$3a^2$	Half the surface area
3	$\mathscr{L}_3(\mathcal{M})$	a^3	Volume

1.4 Volume of the unit ball

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \tag{5}$$

$$\left|\begin{array}{l} \omega_0 = 1 \\ \omega_1 = 2 \\ \omega_2 = \pi \\ \omega_3 = 4\pi/3 \\ \ldots \end{array}\right. \tag{6}$$

1.5 Probabilistic Hermite polynomials

$$\mathscr{H}_n = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e{-}x^2/2 \tag{7}$$

$$\left|\begin{array}{l} \mathscr{H}_0(x) = 1 \\ \mathscr{H}_1(x) = x \\ \mathscr{H}_2(x) = x^2 - 1 \\ \mathscr{H}_3(x) = x^3 - 3 \\ \ldots \end{array}\right. \tag{8}$$

1.6 Flag coefficients

$$\left[\begin{array}{c} n \\ j \end{array}\right] = \left(\begin{array}{c} n \\ j \end{array}\right) \frac{\omega_n}{\omega_{n-j}\omega_j} \tag{9}$$

(10)

n	j	flag
0	0	1
1	0	1
1	1	1
2	0	1
2	1	$\pi/2$
2	2	1
3	0	1
3	1	2
3	2	2
3	3	1

1.7 Second spectral moment

$$\lambda_2 = \mathbb{V}\left\{\frac{\partial\, g(\boldsymbol{x})}{\partial\, x_i}\right\} = \frac{\mathrm{d}^2\mathcal{C}(h)}{\mathrm{d}h^2} \tag{11}$$

(12)

Covariance	Parameter	λ_2
Gaussian	–	$2\sigma^2/L_c^2$

2 Gaussian Minkowsky functionals

2.1 Preliminary

2.1.1 Gaussian measure: $\gamma^k(\mathcal{H})$

$$\gamma^k(\mathcal{H}) = P\{\boldsymbol{X} \in \mathcal{H}\} = \frac{1}{\sigma^k (2\pi)^{k/2}} \int_{\mathcal{H}} e^{-\|\boldsymbol{x}-\mu\|^2/2\sigma^2} d\boldsymbol{x} \tag{13}$$

2.1.2 Tube Taylor expansion

$$\gamma^k(\mathrm{tube}(\mathcal{H}, \rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathscr{M}_j^{\gamma^k}(\mathcal{H}) \tag{14}$$

2.2 GMFs for Gaussian distribution

2.2.1 Application to one-dimensional tail hitting set: $k = 1$ and $\mathcal{H} = [\kappa\ \infty)$

Tail probability $\bar{F}$:

$$\gamma^1([\kappa\ \infty)) = P\{X \geq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{\kappa}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = \bar{F}(\kappa) \tag{15}$$

Error function erf:

$$\begin{aligned}
\gamma^1([\kappa\ \infty)) = \bar{F}(\kappa) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} \sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} dt \\
&= \frac{1}{\sqrt{\pi}} \left[\int_0^{\infty} e^{-t^2} dt - \int_0^{\frac{\kappa-\mu}{\sigma\sqrt{2}}} e^{-t^2} dt \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \mathrm{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}} \right) \right] \\
&= \frac{1}{2} \left(1 - \mathrm{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}} \right) \right)
\end{aligned}$$

Tube expansion:

$$\gamma^1(\mathrm{tube}([\kappa\ \infty), \rho)) = \gamma^1([\kappa - \rho\ \infty)) = \bar{F}(\kappa - \rho) = \sum_{j=0}^{\infty} \frac{(-\rho)^j}{j!} \frac{\mathrm{d}^j \bar{F}(\kappa)}{\mathrm{d}\kappa^j} \tag{17}$$

Identification:

- Notations:

  Tail hitting set: $\bar{\mathcal{H}}^{\text{1D}} = [\kappa\ \infty)$ (18a)

  Variable substitution: $\tilde{\kappa} \leftarrow \dfrac{\kappa - \mu}{\sigma}$ (18b)

- For $j = 0$:
$$\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \bar{F}(\kappa) \tag{19}$$

- For $j > 0$:
$$\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^j \bar{F}(\kappa)}{\mathrm{d}\kappa^j} = \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}} \mathscr{H}_{j-1}((\kappa-\mu)/\sigma) \tag{20}$$

- First values:
$$\left|\begin{array}{l} \mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\ \mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2/2} \\ \mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}\ e^{-\tilde{\kappa}^2/2} \\ \mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ (\tilde{\kappa}^2 - 1)\ e^{-\tilde{\kappa}^2/2} \\ \mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^4\sqrt{2\pi}}\ (\tilde{\kappa}^3 - 3\tilde{\kappa})\ e^{-\tilde{\kappa}^2/2} \\ \dots \end{array}\right. \tag{21}$$

- Derivatives:
$$\forall j, \quad \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^{j+1}\bar{F}(\kappa)}{\mathrm{d}\kappa^{j+1}} = -\mathscr{M}_{j+1}^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) \tag{22}$$

2.2.2 Application to one-dimensional cumulative hitting set: $k = 1$ and $\mathcal{H} = (-\infty\ \kappa]$

Cumulative probability F:
$$\gamma^1((-\infty\ \kappa]) = P\{X \leq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-(x-\mu)^2/2\sigma^2} dx = F(\kappa) \tag{23}$$

Error function erf:

$$\gamma^1((-\infty\ \kappa]) = F(\kappa) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right)\right) \tag{24}$$

Tube expension:

$$\gamma^1(\mathrm{tube}((-\infty\ \kappa],\rho)) = \gamma^1((-\infty\ \kappa+\rho]) = F(\kappa+\rho) = \sum_{j=0}^{\infty}\frac{\rho^j}{j!}\frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} \tag{25}$$

Identification:

- Notations:

  Tail hitting set: $\mathcal{H}^{1\mathrm{D}} = (-\infty\ \kappa]$ (26a)

  Variable substitution: $\tilde{\kappa} \leftarrow \dfrac{\kappa-\mu}{\sigma}$ (26b)

- For $j = 0$:

$$\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) = F(\kappa) \tag{27}$$

- For $j > 0$:

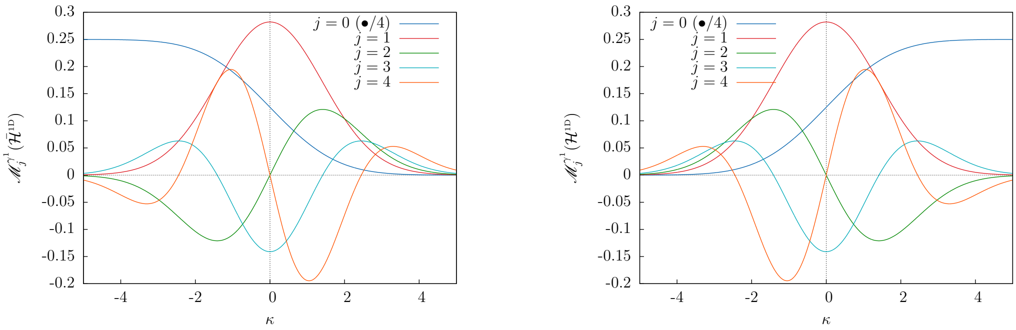
$$\mathscr{M}_j^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) = \frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} = (-1)^{j+1}\frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}}\mathscr{H}_{j-1}((\kappa-\mu)/\sigma) \tag{28}$$

- First values:

$$\left|\begin{array}{l}
\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) = \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\[2ex]
\mathscr{M}_1^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_2^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) = \frac{-1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_3^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ (\tilde{\kappa}^2-1)\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_4^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) = \frac{-1}{\sigma^4\sqrt{2\pi}}\ (\tilde{\kappa}^3-3\tilde{\kappa})\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\dots
\end{array}\right. \tag{29}$$

- Derivatives:

$$\forall j,\quad \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\mathcal{H}^{1\mathrm{D}}) = \frac{\mathrm{d}^{j+1}F(\kappa)}{\mathrm{d}\kappa^{j+1}} = \mathscr{M}_{j+1}^{\gamma^1}(\mathcal{H}^{1\mathrm{D}}) \tag{30}$$



(a) $\mathcal{H} = \mathcal{H}^{1\mathrm{D}} = [\kappa\ \infty)$ (b) $\mathcal{H} = \mathcal{H}^{1\mathrm{D}} = (-\infty\ \kappa]$

Figure 1: First 4 GMF for $\sigma^2 = 2$

2.3 GMFs for Gaussian related Random Fields

$$g_r : \Omega \times \mathbf{R}^N \xrightarrow{\boldsymbol{g}} \mathbf{R}^k \xrightarrow{S} \mathbf{R} \tag{31}$$

$$P\left\{g_r \in \mathcal{H}\right\} = P\left\{S(g) \in \mathcal{H}\right\} = P\left\{g \in S^{-1}(\mathcal{H})\right\} = \gamma^k(S^{-1}(\mathcal{H})) \tag{32}$$

2.3.1 Log-normal distribution

$$k = 1 \quad , \quad S = \exp \quad \text{and} \quad S^{-1} = \ln \tag{33}$$

$$\left| \begin{array}{lcl} \mu & = & \ln(\mu_{\ln}) - \frac{1}{2}\ln(1 + \sigma_{\ln}^2/\mu_{\ln}^2) \\ \sigma^2 & = & \ln(1 + \sigma_{\ln}^2/\mu_{\ln}^2) \end{array} \right. \tag{34}$$

- Notations:

$$\text{Log tail hitting set:} \quad \bar{\mathcal{H}}_{\ln}^{1\mathrm{D}} = [\ln(\kappa)\ \infty) \tag{35a}$$

$$\text{Log cumulative hitting set:} \quad \mathcal{H}_{\ln}^{1\mathrm{D}} = (-\infty\ \ \ln(\kappa)] \tag{35b}$$

$$\text{Variable substitution:} \quad \tilde{\kappa}_{\ln} \leftarrow \frac{\ln(\kappa) - \mu}{\sigma} \tag{35c}$$

- For $\mathcal{H} = \bar{\mathcal{H}}^{\text{1D}} = [\kappa\ \infty) \quad \rightarrow \quad S^{-1}(\mathcal{H}) = \bar{\mathcal{H}}^{\text{1D}}_{\ln} = [\ln(\kappa)\ \infty)$

$$\left| \begin{array}{l} \mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\ \\ \mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \\ \mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}_{\ln}\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \\ \mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ \left(\tilde{\kappa}^2_{\ln} - 1\right)\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \\ \mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^4\sqrt{2\pi}}\ \left(\tilde{\kappa}^3_{\ln} - 3\tilde{\kappa}_{\ln}\right)\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \\ \dots \end{array} \right. \tag{36}$$

Derivatives:

$$\left| \begin{array}{l} \text{Help:} \quad \left(e^{-(\ln(x)-\mu)^2/2\sigma^2}\right)' = -\frac{1}{\sigma^2}\frac{\ln(x)-\mu}{x}\ e^{-(\ln(x)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma\sqrt{2\pi}}\frac{1}{\kappa}\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^2\sqrt{2\pi}}\frac{1}{\kappa}\frac{\ln(\kappa)-\mu}{\sigma}\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^3\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^2}{\sigma^2} - 1\right]\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^4\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^3}{\sigma^3} - 3\frac{(\ln(\kappa)-\mu)}{\sigma}\right]\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \dots \\ \\ \text{Guess would be: } \forall j \geq 0, \\ \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\ e^{-\tilde{\kappa}^2_{\ln}/2} \end{array} \right. \tag{37}$$

- For $\mathcal{H} = [0\ \kappa] \quad \rightarrow \quad S^{-1}(\mathcal{H}) = \mathcal{H}^{\mathrm{1D}}_{\ln} = (-\infty\ \ \ln(\kappa)]$

$$\left|\begin{array}{l}
\mathscr{M}^{\gamma^1}_0(\mathcal{H}^{\mathrm{1D}}_{\ln}) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\
\\
\mathscr{M}^{\gamma^1}_1(\mathcal{H}^{\mathrm{1D}}_{\ln}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2_{\ln}/2} \\
\\
\mathscr{M}^{\gamma^1}_2(\mathcal{H}^{\mathrm{1D}}_{\ln}) = \frac{-1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}_{\ln}\ e^{-\tilde{\kappa}^2_{\ln}/2} \\
\\
\mathscr{M}^{\gamma^1}_3(\mathcal{H}^{\mathrm{1D}}_{\ln}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ \left(\tilde{\kappa}^2_{\ln} - 1\right)\ e^{-\tilde{\kappa}^2_{\ln}/2} \\
\\
\mathscr{M}^{\gamma^1}_4(\mathcal{H}^{\mathrm{1D}}_{\ln}) = \frac{-1}{\sigma^4\sqrt{2\pi}}\ \left(\tilde{\kappa}^3_{\ln} - 3\tilde{\kappa}_{\ln}\right)\ e^{-\tilde{\kappa}^2_{\ln}/2} \\
\\
\dots
\end{array}\right. \tag{38}$$

Derivatives:

$$\left|\begin{array}{l}
\text{Guess would be: } \forall j \geq 0, \\
\frac{\mathrm{d}\mathscr{M}^{\gamma^1}_j(\mathcal{H}^{\mathrm{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{(-1)^j}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\ e^{-\tilde{\kappa}^2_{\ln}/2}
\end{array}\right. \tag{39}$$

3 Expectation Formula

3.1 Gaussian distribution

3.1.1 General case

$$\mathbb{E}\{\ \mathscr{L}_j(\mathcal{E})\ \} = \sum_{i=0}^{N-j} \begin{bmatrix} i+j \\ i \end{bmatrix} \left(\frac{\lambda_2}{2\pi}\right)^{i/2} \mathscr{L}_{i+j}(\mathcal{M})\mathscr{M}_i^{\gamma^k}(S^{-1}(\mathcal{H})) \tag{40}$$

Notations:

LKC of $\mathcal{M}$: $\mathscr{L}_j(\mathcal{M}) = \mathscr{L}_j^{\mathcal{M}}$ (41a)

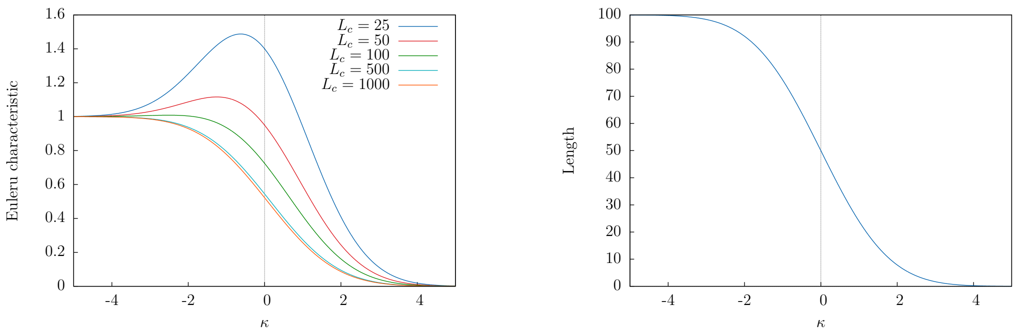
LKC of $\mathcal{E}$: $\mathscr{L}_j(\mathcal{E}) = \mathscr{L}_j^{\mathcal{E}}$ (41b)

GMF of $S^{-1}(\mathcal{H})$: $\mathscr{M}_j^{\gamma^k}(S^{-1}(\mathcal{H})) = \mathscr{M}_j^{\gamma^k}$ (41c)

3.1.2 One dimension

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \left(\frac{\lambda_2}{2\pi}\right)^{1/2} \mathscr{L}_1^{\mathcal{M}}\mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{42a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \mathscr{L}_1^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{42b}$$



(a) Euler Characteristic

(b) Length

Figure 2: Expectation of LKC for $a = 100$, $\sigma^2 = 2$ and $\mathcal{H} = [\kappa\ \infty)$ in 1D.

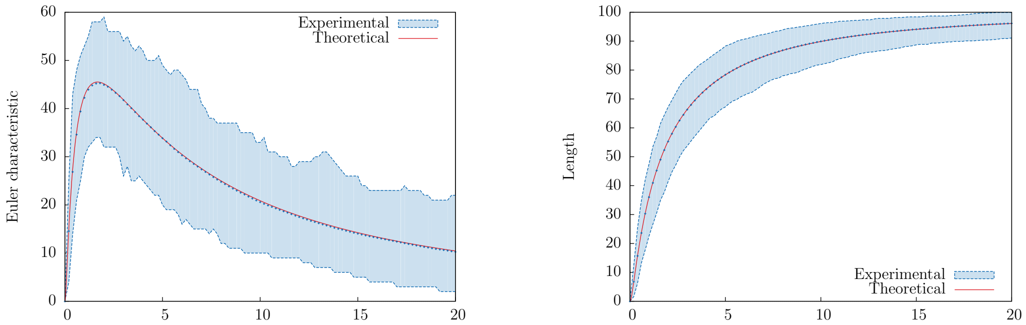


Figure 3: Expectation of LKC for $a = 100$, $\mu = 0.5$, $\sigma^2 = 2$, $L_c = 0.5$, and $\mathcal{H} = [0\ \kappa]$ (log-normal distribution with cumulative hitting set) in 1D. Experimental results are calculated over 1 000 realizations.

3.1.3 Two dimensions

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \frac{\lambda_2}{2\pi}\mathscr{L}_2^{\mathcal{M}}\mathscr{M}_2^{\gamma^k} + \sqrt{\frac{\lambda_2}{2\pi}}\mathscr{L}_1^{\mathcal{M}}\mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{43a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \sqrt{\frac{\lambda_2\pi}{8}}\mathscr{L}_2^{\mathcal{M}}\mathscr{M}_1^{\gamma^k} + \mathscr{L}_1^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{43b}$$

$$\mathbb{E}\left\{\mathscr{L}_2^{\mathcal{E}}\right\} = \mathscr{L}_2^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{43c}$$

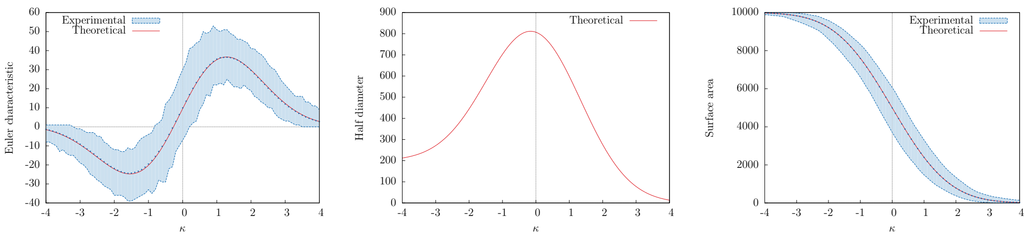


Figure 4: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [\kappa\ \infty)$ (gaussian distribution with tail hitting set) in 2D. Experimental results are calculated over 1 000 realizations.

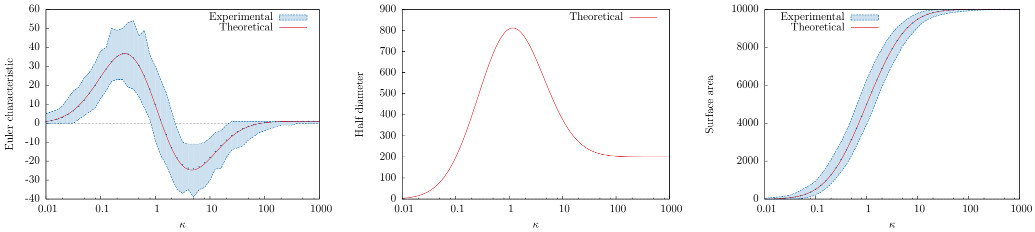


Figure 5: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [0\ \kappa]$ (lognormal distribution with cumulative hitting set) in 2D. Experimental results are calculated over 1 000 realizations.

3.1.4 Three dimensions

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \tag{44a}$$
$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \tag{44b}$$
$$\mathbb{E}\left\{\mathscr{L}_2^{\mathcal{E}}\right\} = \tag{44c}$$
$$\mathbb{E}\left\{\mathscr{L}_3^{\mathcal{E}}\right\} = \tag{44d}$$
$$\tag{44e}$$

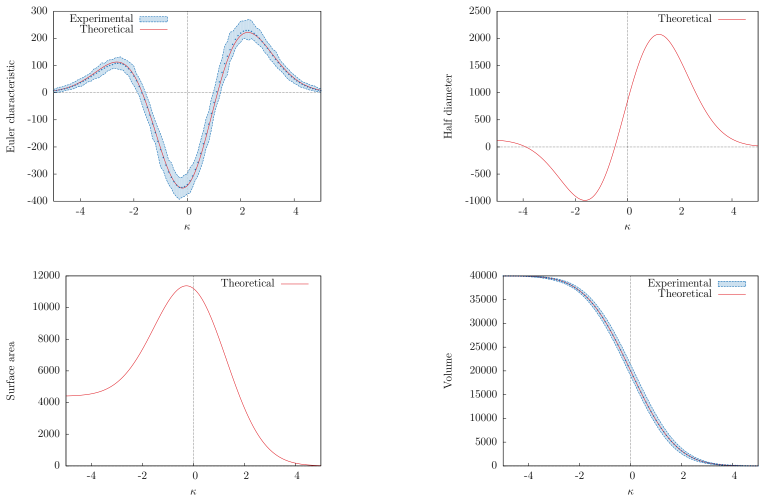
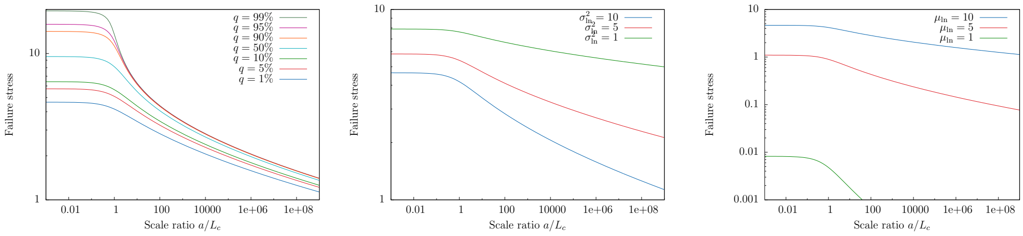


Figure 6: Expectation of LKC for $a = 100 \times 20 \times 20$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 2$, and $\mathcal{H} = [\kappa\ \infty)$ (gaussian distribution with tail hitting set) in 3D. Experimental results are calculated over 100 realizations.

4 Size effect

4.1 One dimensional case

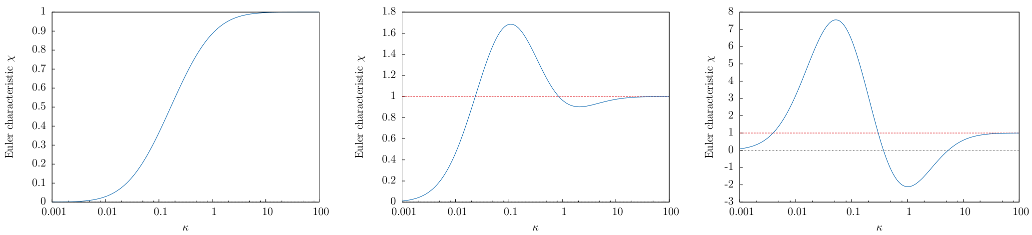


(a) Variation of the quantile (b) Variation of the variance (c) Variation of the mean

Figure 7: Size Effect: variations around $\mu_{\text{ln}} = 10$, $\sigma^2_{\text{ln}} = 10$ and $q = 1\%$ in 1D.

4.2 Two dimensional case

4.2.1 Behavior of the Euler characteristic for different specimen sizes



(a) $a = 0.001$ (b) $a = 2.5$ (c) $a = 8$

Figure 8: Euler characteristic for different specimen sizes a for $\mu_{\text{ln}} = 0.5$, $\sigma^2_{\text{ln}} = 2$, $L_c = 1$, and $\mathcal{H} = [0\ \kappa]$ (lognormal distribution with cumulative hitting set) in a 2D.

8(a) Small sizes
Direct percolation. The first (and only) $\kappa(\chi == q)$ correspond to the probability q of percolation. There is only one disconnected element in the excursion set.

8(b) Medium sizes

8(c) Large sizes