

# Handbook on excursion sets

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# 1 Preliminary calculations

## 1.1 Differentiation under the integral sign

$$\frac{d}{dx} \left( \int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt \quad (1)$$

## 1.2 Functions

### 1.2.1 Gamma function: $\Gamma$

$$\Gamma : z \rightarrow \int_0^\infty t^{z-1} e^{-t} dt \quad (2)$$

### 1.2.2 Error function: $\text{erf}$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (3)$$

## 1.3 Lipschitz-Killing curvatures (LKC)

### 1.3.1 For a cube in 1,2 and 3 dimensions

$$\text{For } \mathcal{M}_N = \prod_{i=1}^N [0, a], \quad \mathcal{L}_j(\mathcal{M}_N) = \binom{N}{j} a^j \quad (4)$$

$N$	LKC	Value	Meaning
1	$\mathcal{L}_0(\mathcal{M})$	1	Euler Characteristic
1	$\mathcal{L}_1(\mathcal{M})$	$a$	Length of the segment
2	$\mathcal{L}_0(\mathcal{M})$	1	Euler Characteristic
2	$\mathcal{L}_1(\mathcal{M})$	$2a$	Half the boundary length
2	$\mathcal{L}_2(\mathcal{M})$	$a^2$	Surface area
3	$\mathcal{L}_0(\mathcal{M})$	1	Euler Characteristic
3	$\mathcal{L}_1(\mathcal{M})$	$3a$	Twice the caliper diameter
3	$\mathcal{L}_2(\mathcal{M})$	$3a^2$	Half the surface area
3	$\mathcal{L}_3(\mathcal{M})$	$a^3$	Volume

## 1.4 Volume of the unit ball

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \quad (5) \quad \left| \begin{array}{l} \omega_0 = 1 \\ \omega_1 = 2 \\ \omega_2 = \pi \\ \omega_3 = 4\pi/3 \\ \dots \end{array} \right. \quad (6)$$

## 1.5 Probabilistic Hermite polynomials

$$\mathcal{H}_n = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad (7) \quad \left| \begin{array}{l} \mathcal{H}_0(x) = 1 \\ \mathcal{H}_1(x) = x \\ \mathcal{H}_2(x) = x^2 - 1 \\ \mathcal{H}_3(x) = x^3 - 3x \\ \dots \end{array} \right. \quad (8)$$

## 1.6 Flag coefficients

$$\left[ \begin{array}{c} n \\ j \end{array} \right] = \binom{n}{j} \frac{\omega_n}{\omega_{n-j}\omega_j} \quad (9) \quad \begin{array}{c|c|c} n & j & \text{flag} \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & \pi/2 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 2 \\ 3 & 3 & 1 \end{array} \quad (10)$$

## 1.7 Second spectral moment

$$\lambda_2 = \mathbb{V} \left\{ \frac{\partial g(\mathbf{x})}{\partial x_i} \right\} = \frac{d^2 \mathcal{C}(h)}{dh^2} \quad (11)$$

Covariance	Parameter	$\lambda_2$
Gaussian	—	$2\sigma^2/L_c^2$

(12)

## 2 Gaussian Minkowsky functionals

### 2.1 Preliminary

#### 2.1.1 Gaussian measure: $\gamma^k(\mathcal{H})$

$$\gamma^k(\mathcal{H}) = P\{\mathbf{X} \in \mathcal{H}\} = \frac{1}{\sigma^k (2\pi)^{k/2}} \int_{\mathcal{H}} e^{-\|\mathbf{x}-\mu\|^2/2\sigma^2} d\mathbf{x} \quad (13)$$

#### 2.1.2 Tube Taylor expansion

$$\gamma^k(\text{tube}(\mathcal{H}, \rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma^k}(\mathcal{H}) \quad (14)$$

### 2.2 GMFs for Gaussian distribution

#### 2.2.1 Application to one-dimensional tail hitting set: $k = 1$ and $\mathcal{H} = [\kappa \infty)$

Tail probability  $\bar{F}$ :

$$\gamma^1([\kappa \infty)) = P\{X \geq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{\kappa}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = \bar{F}(\kappa) \quad (15)$$

Error function erf:

$$\begin{aligned} \gamma^1([\kappa \infty)) &= \bar{F}(\kappa) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} \sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} dt \\ &= \frac{1}{\sqrt{\pi}} \left[ \int_0^{\infty} e^{-t^2} dt - \int_0^{\frac{\kappa-\mu}{\sigma\sqrt{2}}} e^{-t^2} dt \right] \\ &= \frac{1}{\sqrt{\pi}} \left[ \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \text{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right) \right] \\ &= \frac{1}{2} \left( 1 - \text{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right) \right) \end{aligned}$$

Tube expansion:

$$\gamma^1(\text{tube}([\kappa \infty), \rho)) = \gamma^1([\kappa - \rho \infty)) = \bar{F}(\kappa - \rho) = \sum_{j=0}^{\infty} \frac{(-\rho)^j}{j!} \frac{d^j \bar{F}(\kappa)}{d\kappa^j} \quad (17)$$

Identification:

- Notations:

$$\text{Tail hitting set: } \bar{\mathcal{H}}^{1\text{D}} = [\kappa \infty) \quad (18a)$$

$$\text{Variable substitution: } \tilde{\kappa} \leftarrow \frac{\kappa - \mu}{\sigma} \quad (18b)$$

- For  $j = 0$ :

$$\mathcal{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) = \bar{F}(\kappa) \quad (19)$$

- For  $j > 0$ :

$$\mathcal{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) = (-1)^j \frac{d^j \bar{F}(\kappa)}{d\kappa^j} = \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j \sqrt{2\pi}} \mathcal{H}_{j-1}((\kappa - \mu)/\sigma) \quad (20)$$

- First values:

$$\left| \begin{array}{l} \mathcal{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) = \frac{1}{2} \left( 1 - \text{erf} \left( \frac{\tilde{\kappa}}{\sqrt{2}} \right) \right) \\ \mathcal{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) = \frac{1}{\sigma^2 \sqrt{2\pi}} \tilde{\kappa} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) = \frac{1}{\sigma^3 \sqrt{2\pi}} (\tilde{\kappa}^2 - 1) e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) = \frac{1}{\sigma^4 \sqrt{2\pi}} (\tilde{\kappa}^3 - 3\tilde{\kappa}) e^{-\tilde{\kappa}^2/2} \\ \dots \end{array} \right. \quad (21)$$

- Derivatives:

$$\forall j, \quad \frac{d\mathcal{M}_j^{\gamma^1}}{d\kappa}(\bar{\mathcal{H}}^{1\text{D}}) = (-1)^j \frac{d^{j+1} \bar{F}(\kappa)}{d\kappa^{j+1}} = -\mathcal{M}_{j+1}^{\gamma^1}(\bar{\mathcal{H}}^{1\text{D}}) \quad (22)$$

### 2.2.2 Application to one-dimensional cumulative hitting set: $k = 1$ and $\mathcal{H} = (-\infty \kappa]$

Cumulative probability  $F$ :

$$\gamma^1((-\infty \kappa]) = P\{X \leq \kappa\} = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-(x-\mu)^2/2\sigma^2} dx = F(\kappa) \quad (23)$$

Error function erf:

$$\gamma^1((-\infty \ \kappa]) = F(\kappa) = \frac{1}{2} \left( 1 + \operatorname{erf} \left( \frac{\kappa - \mu}{\sigma\sqrt{2}} \right) \right) \quad (24)$$

Tube expansion:

$$\gamma^1(\text{tube}((-\infty \ \kappa], \rho)) = \gamma^1((-\infty \ \kappa + \rho]) = F(\kappa + \rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \frac{d^j F(\kappa)}{d\kappa^j} \quad (25)$$

Identification:

- Notations:

$$\text{Tail hitting set: } \mathcal{H}^{1D} = (-\infty \ \kappa] \quad (26a)$$

$$\text{Variable substitution: } \tilde{\kappa} \leftarrow \frac{\kappa - \mu}{\sigma} \quad (26b)$$

- For  $j = 0$ :

$$\mathcal{M}_0^{\gamma^1}(\mathcal{H}^{1D}) = F(\kappa) \quad (27)$$

- For  $j > 0$ :

$$\mathcal{M}_j^{\gamma^1}(\mathcal{H}^{1D}) = \frac{d^j F(\kappa)}{d\kappa^j} = (-1)^{j+1} \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j \sqrt{2\pi}} \mathcal{H}_{j-1}((\kappa - \mu)/\sigma) \quad (28)$$

- First values:

$$\left| \begin{array}{l} \mathcal{M}_0^{\gamma^1}(\mathcal{H}^{1D}) = \frac{1}{2} \left( 1 - \operatorname{erf} \left( \frac{\tilde{\kappa}}{\sqrt{2}} \right) \right) \\ \mathcal{M}_1^{\gamma^1}(\mathcal{H}^{1D}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_2^{\gamma^1}(\mathcal{H}^{1D}) = \frac{-1}{\sigma^2\sqrt{2\pi}} \tilde{\kappa} e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_3^{\gamma^1}(\mathcal{H}^{1D}) = \frac{1}{\sigma^3\sqrt{2\pi}} (\tilde{\kappa}^2 - 1) e^{-\tilde{\kappa}^2/2} \\ \mathcal{M}_4^{\gamma^1}(\mathcal{H}^{1D}) = \frac{-1}{\sigma^4\sqrt{2\pi}} (\tilde{\kappa}^3 - 3\tilde{\kappa}) e^{-\tilde{\kappa}^2/2} \\ \dots \end{array} \right. \quad (29)$$

- Derivatives:

$$\forall j, \quad \frac{d\mathcal{M}_j^{\gamma^1}}{d\kappa}(\mathcal{H}^{1D}) = \frac{d^{j+1}F(\kappa)}{d\kappa^{j+1}} = \mathcal{M}_{j+1}^{\gamma^1}(\mathcal{H}^{1D}) \quad (30)$$

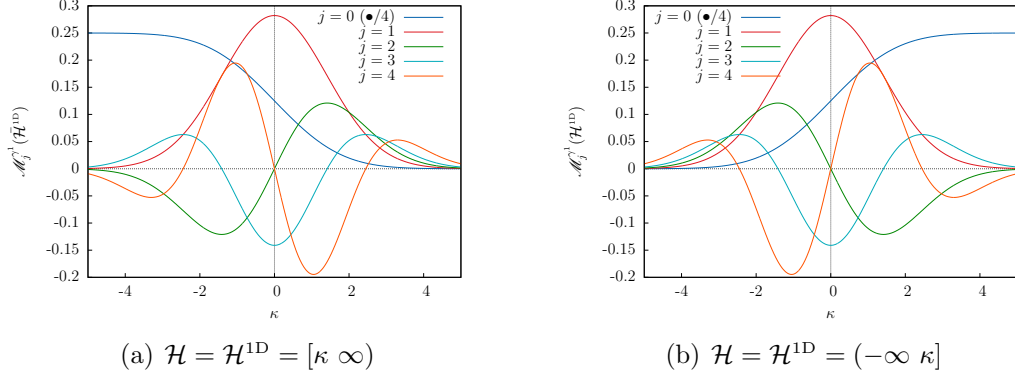


Figure 1: First 4 GMF for  $\sigma^2 = 2$

## 2.3 GMFs for Gaussian related Random Fields

$$g_r : \Omega \times \mathbf{R}^N \xrightarrow{g} \mathbf{R}^k \xrightarrow{S} \mathbf{R} \quad (31)$$

$$P\{g_r \in \mathcal{H}\} = P\{S(g) \in \mathcal{H}\} = P\{g \in S^{-1}(\mathcal{H})\} = \gamma^k(S^{-1}(\mathcal{H})) \quad (32)$$

### 2.3.1 Log-normal distribution

$$k = 1, \quad S = \exp \quad \text{and} \quad S^{-1} = \ln \quad (33)$$

$$\begin{cases} \mu &= \ln(\mu_{\ln}) - \frac{1}{2} \ln(1 + \sigma_{\ln}^2 / \mu_{\ln}^2) \\ \sigma^2 &= \ln(1 + \sigma_{\ln}^2 / \mu_{\ln}^2) \end{cases} \quad (34)$$

- Notations:

$$\text{Log tail hitting set: } \bar{\mathcal{H}}_{\ln}^{1D} = [\ln(\kappa), \infty) \quad (35a)$$

$$\text{Log cumulative hitting set: } \mathcal{H}_{\ln}^{1D} = (-\infty, \ln(\kappa)] \quad (35b)$$

$$\text{Variable substitution: } \tilde{\kappa}_{\ln} \leftarrow \frac{\ln(\kappa) - \mu}{\sigma} \quad (35c)$$



- For  $\mathcal{H} = \bar{\mathcal{H}}^{1\text{D}} = [\kappa \infty) \rightarrow S^{-1}(\mathcal{H}) = \bar{\mathcal{H}}_{\ln}^{1\text{D}} = [\ln(\kappa) \infty)$

$$\left| \begin{aligned}
 \mathcal{M}_0^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}}) &= \frac{1}{2} \left( 1 - \operatorname{erf} \left( \frac{\tilde{\kappa}_{\ln}}{\sqrt{2}} \right) \right) \\
 \mathcal{M}_1^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}}) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\tilde{\kappa}_{\ln}^2/2} \\
 \mathcal{M}_2^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}}) &= \frac{1}{\sigma^2\sqrt{2\pi}} \tilde{\kappa}_{\ln} e^{-\tilde{\kappa}_{\ln}^2/2} \\
 \mathcal{M}_3^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}}) &= \frac{1}{\sigma^3\sqrt{2\pi}} (\tilde{\kappa}_{\ln}^2 - 1) e^{-\tilde{\kappa}_{\ln}^2/2} \\
 \mathcal{M}_4^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}}) &= \frac{1}{\sigma^4\sqrt{2\pi}} (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2} \\
 &\dots
 \end{aligned} \right. \quad (36)$$

Derivatives:

$$\left| \begin{aligned}
 \text{Help: } \left( e^{-(\ln(x)-\mu)^2/2\sigma^2} \right)' &= -\frac{1}{\sigma^2} \frac{\ln(x)-\mu}{x} e^{-(\ln(x)-\mu)^2/2\sigma^2} \\
 \frac{d\mathcal{M}_0^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}})}{d\kappa} &= \frac{-1}{\sigma\sqrt{2\pi}} \frac{1}{\kappa} e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
 \frac{d\mathcal{M}_1^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}})}{d\kappa} &= \frac{-1}{\sigma^2\sqrt{2\pi}} \frac{1}{\kappa} \frac{\ln(\kappa)-\mu}{\sigma} e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
 \frac{d\mathcal{M}_2^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}})}{d\kappa} &= \frac{-1}{\sigma^3\sqrt{2\pi}} \frac{1}{\kappa} \left[ \frac{(\ln(\kappa)-\mu)^2}{\sigma^2} - 1 \right] e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
 \frac{d\mathcal{M}_3^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}})}{d\kappa} &= \frac{-1}{\sigma^4\sqrt{2\pi}} \frac{1}{\kappa} \left[ \frac{(\ln(\kappa)-\mu)^3}{\sigma^3} - 3\frac{(\ln(\kappa)-\mu)}{\sigma} \right] e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\
 &\dots \\
 \text{Guess would be: } \forall j \geq 0, \\
 \frac{d\mathcal{M}_j^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\text{D}})}{d\kappa} &= \frac{-1}{\sigma^{j+1}\sqrt{2\pi}} \frac{1}{\kappa} \mathcal{H}_j(\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2}
 \end{aligned} \right. \quad (37)$$

- For  $\mathcal{H} = [0 \ \kappa] \rightarrow S^{-1}(\mathcal{H}) = \mathcal{H}_{\ln}^{1D} = (-\infty \ \ln(\kappa)]$

$$\left| \begin{aligned}
\mathcal{M}_0^{\gamma^1}(\mathcal{H}_{\ln}^{1D}) &= \frac{1}{2} \left( 1 + \operatorname{erf} \left( \frac{\tilde{\kappa}_{\ln}}{\sqrt{2}} \right) \right) \\
\mathcal{M}_1^{\gamma^1}(\mathcal{H}_{\ln}^{1D}) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\tilde{\kappa}_{\ln}^2/2} \\
\mathcal{M}_2^{\gamma^1}(\mathcal{H}_{\ln}^{1D}) &= \frac{-1}{\sigma^2\sqrt{2\pi}} \tilde{\kappa}_{\ln} e^{-\tilde{\kappa}_{\ln}^2/2} \\
\mathcal{M}_3^{\gamma^1}(\mathcal{H}_{\ln}^{1D}) &= \frac{1}{\sigma^3\sqrt{2\pi}} (\tilde{\kappa}_{\ln}^2 - 1) e^{-\tilde{\kappa}_{\ln}^2/2} \\
\mathcal{M}_4^{\gamma^1}(\mathcal{H}_{\ln}^{1D}) &= \frac{-1}{\sigma^4\sqrt{2\pi}} (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2} \\
&\dots
\end{aligned} \right. \quad (38)$$

Derivatives:

$$\left| \begin{aligned}
&\text{Guess would be: } \forall j \geq 0, \\
\frac{d\mathcal{M}_j^{\gamma^1}(\mathcal{H}_{\ln}^{1D})}{d\kappa} &= \frac{(-1)^j}{\sigma^{j+1}\sqrt{2\pi}} \frac{1}{\kappa} \mathcal{H}_j(\tilde{\kappa}_{\ln}) e^{-\tilde{\kappa}_{\ln}^2/2}
\end{aligned} \right. \quad (39)$$

## 3 Expectation Formula

### 3.1 Gaussian distribution

#### 3.1.1 General case

$$\mathbb{E} \{ \mathcal{L}_j(\mathcal{E}) \} = \sum_{i=0}^{N-j} \left[ \begin{matrix} i+j \\ i \end{matrix} \right] \left( \frac{\lambda_2}{2\pi} \right)^{i/2} \mathcal{L}_{i+j}(\mathcal{M}) \mathcal{M}_i^{\gamma^k}(S^{-1}(\mathcal{H})) \quad (40)$$

Notations:

$$\text{LKC of } \mathcal{M}: \mathcal{L}_j(\mathcal{M}) = \mathcal{L}_j^{\mathcal{M}} \quad (41a)$$

$$\text{LKC of } \mathcal{E}: \mathcal{L}_j(\mathcal{E}) = \mathcal{L}_j^{\mathcal{E}} \quad (41b)$$

$$\text{GMF of } S^{-1}(\mathcal{H}): \mathcal{M}_j^{\gamma^k}(S^{-1}(\mathcal{H})) = \mathcal{M}_j^{\gamma^k} \quad (41c)$$

#### 3.1.2 One dimension

$$\mathbb{E} \{ \mathcal{L}_0^{\mathcal{E}} \} = \left( \frac{\lambda_2}{2\pi} \right)^{1/2} \mathcal{L}_1^{\mathcal{M}} \mathcal{M}_1^{\gamma^k} + \mathcal{L}_0^{\mathcal{M}} \mathcal{M}_0^{\gamma^k} \quad (42a)$$

$$\mathbb{E} \{ \mathcal{L}_1^{\mathcal{E}} \} = \mathcal{L}_1^{\mathcal{M}} \mathcal{M}_0^{\gamma^k} \quad (42b)$$

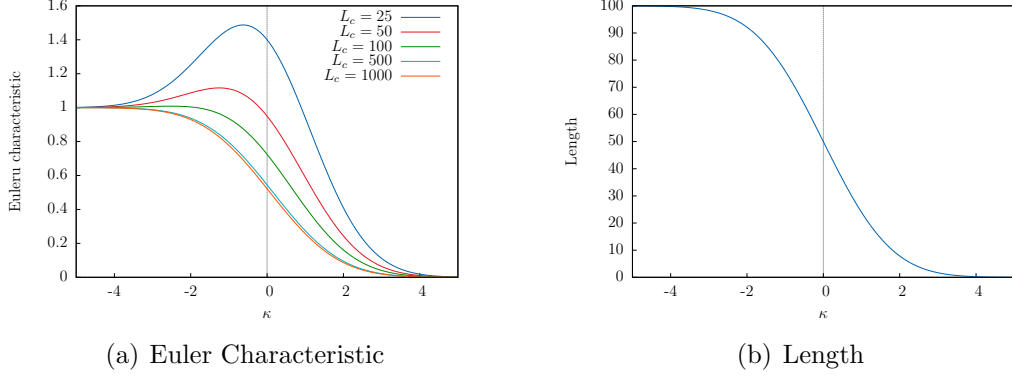


Figure 2: Expectation of LKC for  $a = 100$ ,  $\sigma^2 = 2$  and  $\mathcal{H} = [\kappa, \infty)$  in 1D.

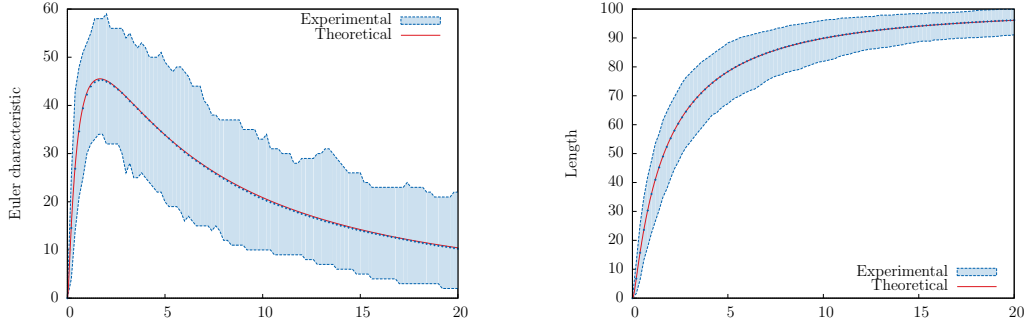


Figure 3: Expectation of LKC for  $a = 100$ ,  $\mu = 0.5$ ,  $\sigma^2 = 2$ ,  $L_c = 0.5$ , and  $\mathcal{H} = [0, \kappa]$  (log-normal distribution with cumulative hitting set) in 1D. Experimental results are calculated over 1 000 realizations.

### 3.1.3 Two dimensions

$$\mathbb{E} \{ \mathcal{L}_0^\varepsilon \} = \frac{\lambda_2}{2\pi} \mathcal{L}_2^\mathcal{M} \mathcal{M}_2^{\gamma^k} + \sqrt{\frac{\lambda_2}{2\pi}} \mathcal{L}_1^\mathcal{M} \mathcal{M}_1^{\gamma^k} + \mathcal{L}_0^\mathcal{M} \mathcal{M}_0^{\gamma^k} \quad (43a)$$

$$\mathbb{E} \{ \mathcal{L}_1^\varepsilon \} = \sqrt{\frac{\lambda_2 \pi}{8}} \mathcal{L}_2^\mathcal{M} \mathcal{M}_1^{\gamma^k} + \mathcal{L}_1^\mathcal{M} \mathcal{M}_0^{\gamma^k} \quad (43b)$$

$$\mathbb{E} \{ \mathcal{L}_2^\varepsilon \} = \mathcal{L}_2^\mathcal{M} \mathcal{M}_0^{\gamma^k} \quad (43c)$$

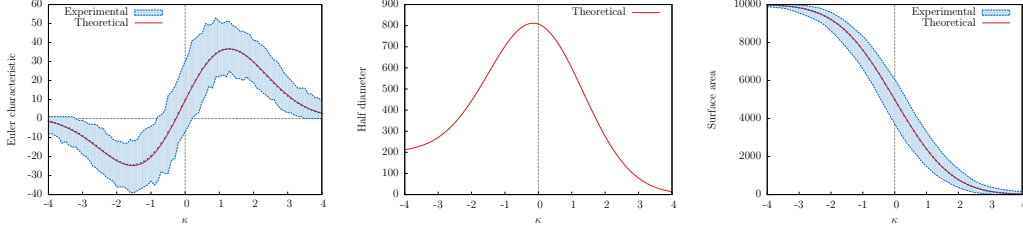


Figure 4: Expectation of LKC for  $a = 100$ ,  $\mu = 0.0$ ,  $\sigma^2 = 2$ ,  $L_c = 5$ , and  $\mathcal{H} = [\kappa \infty)$  (gaussian distribution with tail hitting set) in 2D. Experimental results are calculated over 1 000 realizations.

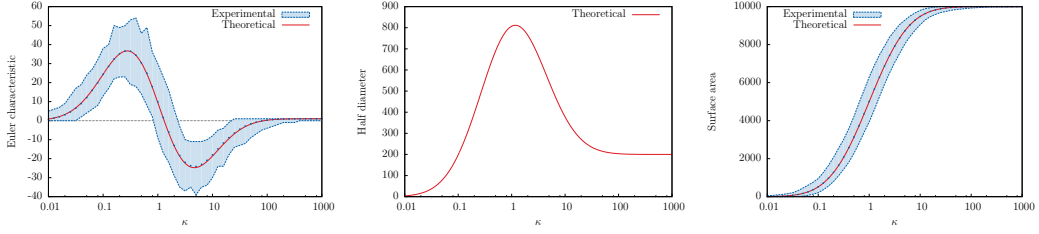
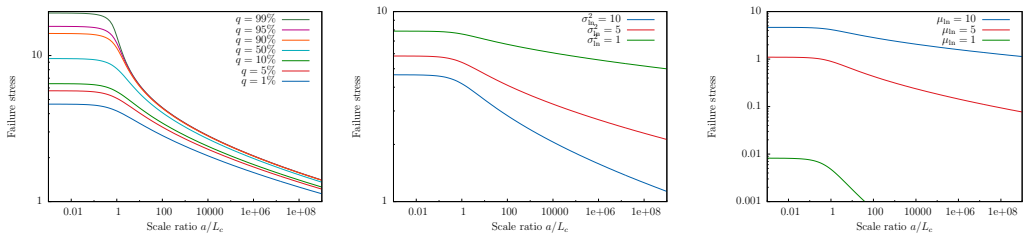


Figure 5: Expectation of LKC for  $a = 100$ ,  $\mu = 0.0$ ,  $\sigma^2 = 2$ ,  $L_c = 5$ , and  $\mathcal{H} = [0 \kappa]$  (lognormal distribution with cumulative hitting set) in 2D. Experimental results are calculated over 1 000 realizations.

## 4 Size effect

### 4.1 One dimensional case



(a) Variation of the quantile (b) Variation of the variance (c) Variation of the mean

Figure 6: Size Effect: variations around  $\mu_{\ln} = 10$ ,  $\sigma_{\ln}^2 = 10$  and  $q = 1\%$  in 1D.

## 4.2 Two dimensional case

### 4.2.1 Behavior of the Euler characteristic for different specimen sizes

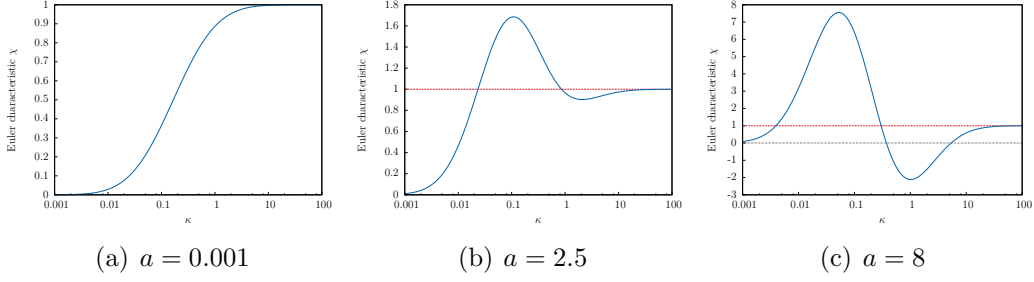


Figure 7: Euler characteristic for different specimen sizes  $a$  for  $\mu_{\ln} = 0.5$ ,  $\sigma_{\ln}^2 = 2$ ,  $L_c = 1$ , and  $\mathcal{H} = [0, \kappa]$  (lognormal distribution with cumulative hitting set) in a 2D.

7(a) Small sizes

**Direct percolation.** The first (and only)  $\chi = q$  correspond to the probability  $q$  of percolation.

- Medium sizes (Fig. 7(b)): Direct percolation.