HANDBOOK ON EXCURSION SETS

CONTENTS

1 Priliminary calculations

1.1 Differentiation under the integral sign

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt \quad (1)$$

1.2 Functions

1.2.1 Gamma function: Γ

$$\Gamma : z \to \int_0^\infty t^{z-1}e^{-t}dt \quad (2)$$

1.2.2 Error function: erf

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \quad (3)$$

1.3 Lipschitz-Killing curvatures (LKC)

1.3.1 For a regular shapes in 1,2 and 3 dimensions

$$\text{For a cube } \mathcal{M}_N = \prod_{i=1}^{N}[0\ a],\ \ \mathscr{L}_j(\mathcal{M}_N) = \left(\begin{array}{c} N \\ j \end{array}\right) a^j \quad (4)$$

$$\text{For a parallelepiped } \mathcal{P}_N = \prod_{i=1}^{N}[0\ a_i],\ \ \mathscr{L}_j(\mathcal{M}_N) = \dots \quad (5)$$

N	LKC	Value	Meaning
1	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
1	$\mathscr{L}_1(\mathcal{M})$	a	Length of the segment
2	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
2	$\mathscr{L}_1(\mathcal{M})$	$2a$	Half the boundary length
2	$\mathscr{L}_2(\mathcal{M})$	a^2	Surface area
3	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
3	$\mathscr{L}_1(\mathcal{M})$	$3a$	Twice the caliper diameter
3	$\mathscr{L}_2(\mathcal{M})$	$3a^2$	Half the surface area
3	$\mathscr{L}_3(\mathcal{M})$	a^3	Volume
3	$\mathscr{L}_0(\mathcal{P})$	1	Euler Characteristic
3	$\mathscr{L}_1(\mathcal{P})$	$a_1 + a_2 + a_3$	Twice the caliper diameter
3	$\mathscr{L}_2(\mathcal{P})$	$a_1a_2 + a_1a_3 + a_2a_3$	Half the surface area
3	$\mathscr{L}_3(\mathcal{P})$	$a_1a_2a_3$	Volume

1.4 Volume of the unit ball

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1+N/2)} \tag{6}$$

$$\left| \begin{array}{l} \omega_0 = 1 \\ \omega_1 = 2 \\ \omega_2 = \pi \\ \omega_3 = 4\pi/3 \\ \dots \end{array} \right. \tag{7}$$

1.5 Probabilistic Hermite polynomials

$$\mathscr{H}_n = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e{-}x^2/2 \tag{8}$$

$$\left| \begin{array}{l} \mathscr{H}_0(x) = 1 \\ \mathscr{H}_1(x) = x \\ \mathscr{H}_2(x) = x^2 - 1 \\ \mathscr{H}_3(x) = x^3 - 3 \\ \dots \end{array} \right. \tag{9}$$

1.6 Flag coefficients

$$\left[\begin{array}{c} n \\ j \end{array} \right] = \left(\begin{array}{c} n \\ j \end{array} \right) \frac{\omega_n}{\omega_{n-j}\omega_j} \tag{10}$$

(11)

n	j	flag
0	0	1
1	0	1
1	1	1
2	0	1
2	1	$\pi/2$
2	2	1
3	0	1
3	1	2
3	2	2
3	3	1

1.7 Second spectral moment

$$\lambda_2 = \mathbb{V}\left\{ \frac{\partial\, g(\boldsymbol{x})}{\partial\, x_i} \right\} = \frac{\mathrm{d}^2 \mathcal{C}(h)}{\mathrm{d}h^2} \tag{12}$$

(13)

Covariance	Parameter	λ_2
Gaussian	–	$2\sigma^2/L_c^2$

2 Gaussian Minkowsky functionals

2.1 Preliminary

2.1.1 Gaussian measure: $\gamma^k(\mathcal{H})$

$$\gamma^k(\mathcal{H}) = P\{\boldsymbol{X} \in \mathcal{H}\} = \frac{1}{\sigma^k (2\pi)^{k/2}} \int_{\mathcal{H}} e^{-\|\boldsymbol{x}-\mu\|^2/2\sigma^2} d\boldsymbol{x} \tag{14}$$

2.1.2 Tube Taylor expansion

$$\gamma^k(\text{tube}(\mathcal{H},\rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathscr{M}_j^{\gamma^k}(\mathcal{H}) \tag{15}$$

2.2 GMFs for Gaussian distribution

2.2.1 Application to one-dimensional tail hitting set: $k = 1$ and $\mathcal{H} = [\kappa\ \infty)$

Tail probability $\bar{F}$:

$$\gamma^1([\kappa\ \infty)) = P\{X \geq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_\kappa^\infty e^{-(x-\mu)^2/2\sigma^2} dx = \bar{F}(\kappa) \tag{16}$$

Error function erf:

$$\begin{aligned}
\gamma^1([\kappa\ \infty)) = \bar{F}(\kappa) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^\infty e^{-t^2}\sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^\infty e^{-t^2} dt \\
&= \frac{1}{\sqrt{\pi}} \left[\int_0^\infty e^{-t^2} dt - \int_0^{\frac{\kappa-\mu}{\sigma\sqrt{2}}} e^{-t^2} dt \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \text{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}} \right) \right] \\
&= \frac{1}{2} \left(1 - \text{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}} \right) \right)
\end{aligned}$$

Tube expansion:

$$\gamma^1(\text{tube}([\kappa\ \infty),\rho)) = \gamma^1([\kappa-\rho\ \infty)) = \bar{F}(\kappa-\rho) = \sum_{j=0}^{\infty} \frac{(-\rho)^j}{j!} \frac{\mathrm{d}^j \bar{F}(\kappa)}{\mathrm{d}\kappa^j} \tag{18}$$

Identification:

- Notations:

$$\text{Tail hitting set: } \bar{\mathcal{H}}^{\text{1D}} = [\kappa\ \infty) \tag{19a}$$

$$\text{Variable substitution: } \tilde{\kappa} \leftarrow \frac{\kappa-\mu}{\sigma} \tag{19b}$$

- For $j = 0$:

$$\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \bar{F}(\kappa) \tag{20}$$

- For $j > 0$:

$$\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^j \bar{F}(\kappa)}{\mathrm{d}\kappa^j} = \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}} \mathscr{H}_{j-1}((\kappa-\mu)/\sigma) \tag{21}$$

- First values:

$$\left|\begin{array}{l} \mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\ \\ \mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\tilde{\kappa}^2/2} \\ \\ \mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^2\sqrt{2\pi}}\, \tilde{\kappa}\, e^{-\tilde{\kappa}^2/2} \\ \\ \mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\, \left(\tilde{\kappa}^2 - 1\right)\, e^{-\tilde{\kappa}^2/2} \\ \\ \mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^4\sqrt{2\pi}}\, \left(\tilde{\kappa}^3 - 3\tilde{\kappa}\right)\, e^{-\tilde{\kappa}^2/2} \\ \\ \dots \end{array}\right. \tag{22}$$

- Derivatives:

$$\forall j, \quad \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^{j+1}\bar{F}(\kappa)}{\mathrm{d}\kappa^{j+1}} = -\mathscr{M}_{j+1}^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) \tag{23}$$

2.2.2 Application to one-dimensional cumulative hitting set: $k = 1$ and $\mathcal{H} = (-\infty\ \kappa]$

Cumulative probability F:

$$\gamma^1((-\infty\ \kappa]) = P\{X \leq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-(x-\mu)^2/2\sigma^2} dx = F(\kappa) \tag{24}$$

Error function erf:

$$\gamma^1((-\infty\ \kappa]) = F(\kappa) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\kappa - \mu}{\sigma\sqrt{2}}\right)\right) \tag{25}$$

Tube expension:

$$\gamma^1(\text{tube}((-\infty\ \kappa], \rho)) = \gamma^1((-\infty\ \kappa + \rho]) = F(\kappa + \rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} \tag{26}$$

Identification:

- Notations:

Tail hitting set: $\mathcal{H}^{\text{1D}} = (-\infty\ \kappa]$ (27a)

Variable substitution: $\tilde{\kappa} \leftarrow \dfrac{\kappa - \mu}{\sigma}$ (27b)

- For $j = 0$:

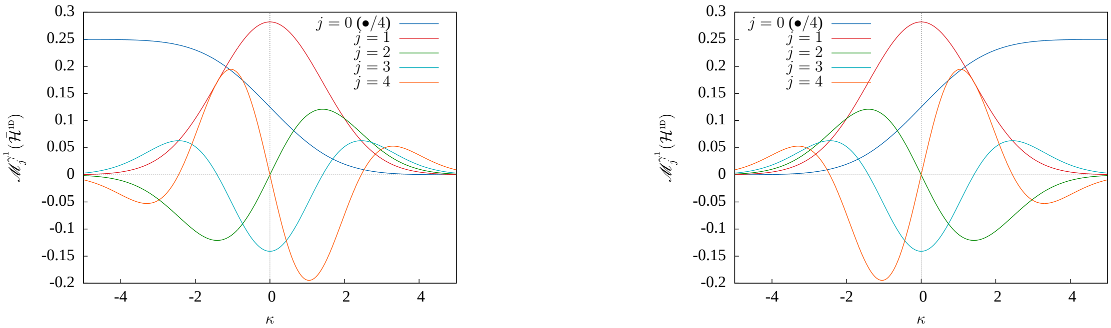
$$\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{\text{1D}}) = F(\kappa) \tag{28}$$

- For $j > 0$:

$$\mathscr{M}_j^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} = (-1)^{j+1} \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}} \mathscr{H}_{j-1}((\kappa - \mu)/\sigma) \tag{29}$$

- First values:

$$\left|\begin{array}{l}
\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\[2ex]
\mathscr{M}_1^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_2^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{-1}{\sigma^2\sqrt{2\pi}}\, \tilde{\kappa}\, e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_3^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\, \left(\tilde{\kappa}^2 - 1\right)\, e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_4^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{-1}{\sigma^4\sqrt{2\pi}}\, \left(\tilde{\kappa}^3 - 3\tilde{\kappa}\right)\, e^{-\tilde{\kappa}^2/2} \\[2ex]
\dots
\end{array}\right. \tag{30}$$



(a) $\mathcal{H} = \mathcal{H}^{\text{1D}} = [\kappa\ \infty)$ (b) $\mathcal{H} = \mathcal{H}^{\text{1D}} = (-\infty\ \kappa]$

Figure 1: First 4 GMF for $\sigma^2 = 2$

- Derivatives:

$$\forall j,\quad \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\mathcal{H}^{\text{1D}}) = \frac{\mathrm{d}^{j+1}F(\kappa)}{\mathrm{d}\kappa^{j+1}} = \mathscr{M}_{j+1}^{\gamma^1}(\mathcal{H}^{\text{1D}}) \tag{31}$$

2.3 GMFs for Gaussian related Random Fields

$$g_r : \Omega \times \mathbf{R}^N \xrightarrow{\boldsymbol{g}} \mathbf{R}^k \xrightarrow{S} \mathbf{R} \tag{32}$$

$$P\left\{g_r \in \mathcal{H}\right\} = P\left\{S(g) \in \mathcal{H}\right\} = P\left\{g \in S^{-1}(\mathcal{H})\right\} = \gamma^k(S^{-1}(\mathcal{H})) \tag{33}$$

2.3.1 Log-normal distribution

$$k = 1\ \ ,\ \ S = \exp \ \text{ and } \ S^{-1} = \ln \tag{34}$$

$$\left| \begin{array}{lcl} \mu & = & \ln(\mu_{\ln}) - \frac{1}{2}\ln(1+\sigma_{\ln}^2/\mu_{\ln}^2) \\ \sigma^2 & = & \ln(1+\sigma_{\ln}^2/\mu_{\ln}^2) \end{array} \right. \tag{35}$$

- Notations:

Log tail hitting set: $\bar{\mathcal{H}}^{\text{1D}}_{\ln} = [\ln(\kappa)\ \infty)$ (36a)

Log cumulative hitting set: $\mathcal{H}^{\text{1D}}_{\ln} = (-\infty\ \ \ln(\kappa)]$ (36b)

Variable substitution: $\tilde{\kappa}_{\ln} \leftarrow \dfrac{\ln(\kappa)-\mu}{\sigma}$ (36c)

- For $\mathcal{H} = \bar{\mathcal{H}}^{\text{1D}} = [\kappa\ \infty) \ \ \rightarrow \ \ S^{-1}(\mathcal{H}) = \bar{\mathcal{H}}^{\text{1D}}_{\ln} = [\ln(\kappa)\ \infty)$

$$\left| \begin{array}{l} \mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\ \\ \mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}_{\ln}\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ (\tilde{\kappa}_{\ln}^2 - 1)\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^4\sqrt{2\pi}}\ (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln})\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \dots \end{array} \right. \tag{37}$$

Derivatives:

$$\left| \begin{array}{l} \text{Help:}\ \ \left(e^{-(\ln(x)-\mu)^2/2\sigma^2}\right)' = -\frac{1}{\sigma^2}\frac{\ln(x)-\mu}{x}\ e^{-(\ln(x)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma\sqrt{2\pi}}\frac{1}{\kappa}\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^2\sqrt{2\pi}}\frac{1}{\kappa}\frac{\ln(\kappa)-\mu}{\sigma}\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^3\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^2}{\sigma^2} - 1\right]\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \frac{\mathrm{d}\mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^4\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^3}{\sigma^3} - 3\frac{(\ln(\kappa)-\mu)}{\sigma}\right]\ e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\ \\ \dots \\ \\ \text{Guess would be:}\ \ \forall j \geq 0, \\ \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\ e^{-\tilde{\kappa}_{\ln}^2/2} \end{array} \right. \tag{38}$$

- For $\mathcal{H} = [0\ \kappa] \quad \rightarrow \quad S^{-1}(\mathcal{H}) = \mathcal{H}^{\text{1D}}_{\ln} = (-\infty\ \ \ln(\kappa)]$

$$\left|\begin{array}{l} \mathscr{M}_0^{\gamma^1}(\mathcal{H}^{\text{1D}}_{\ln}) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\ \mathscr{M}_1^{\gamma^1}(\mathcal{H}^{\text{1D}}_{\ln}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \mathscr{M}_2^{\gamma^1}(\mathcal{H}^{\text{1D}}_{\ln}) = \frac{-1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}_{\ln}\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \mathscr{M}_3^{\gamma^1}(\mathcal{H}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ \left(\tilde{\kappa}^2_{\ln} - 1\right)\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \mathscr{M}_4^{\gamma^1}(\mathcal{H}^{\text{1D}}_{\ln}) = \frac{-1}{\sigma^4\sqrt{2\pi}}\ \left(\tilde{\kappa}^3_{\ln} - 3\tilde{\kappa}_{\ln}\right)\ e^{-\tilde{\kappa}^2_{\ln}/2} \\ \dots \end{array}\right. \tag{39}$$

Derivatives:

$$\left|\begin{array}{l} \text{Guess would be: } \forall j \geq 0, \\ \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}(\mathcal{H}^{\text{1D}}_{\ln})}{\mathrm{d}\kappa} = \frac{(-1)^j}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\ e^{-\tilde{\kappa}^2_{\ln}/2} \end{array}\right. \tag{40}$$

3 Expectation Formula

3.1 Gaussian distribution

3.1.1 General case

$$\mathbb{E}\left\{ \mathscr{L}_j(\mathcal{E}) \right\} = \sum_{i=0}^{N-j} \left[\begin{array}{c} i+j \\ i \end{array} \right] \left(\frac{\lambda_2}{2\pi} \right)^{i/2} \mathscr{L}_{i+j}(\mathcal{M}) \mathscr{M}_i^{\gamma^k}(S^{-1}(\mathcal{H})) \tag{41}$$

Notations:

LKC of $\mathcal{M}$: $\mathscr{L}_j(\mathcal{M}) = \mathscr{L}_j^{\mathcal{M}}$ (42a)

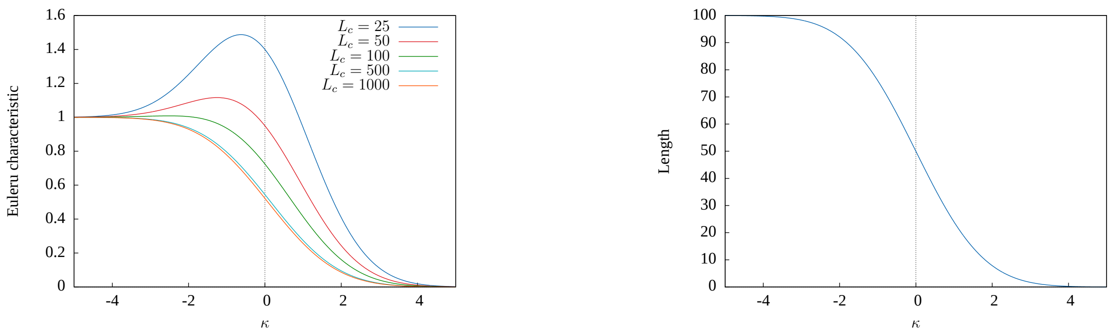
LKC of $\mathcal{E}$: $\mathscr{L}_j(\mathcal{E}) = \mathscr{L}_j^{\mathcal{E}}$ (42b)

GMF of $S^{-1}(\mathcal{H})$: $\mathscr{M}_j^{\gamma^k}(S^{-1}(\mathcal{H})) = \mathscr{M}_j^{\gamma^k}$ (42c)

3.1.2 One dimension

$$\mathbb{E}\left\{ \mathscr{L}_0^{\mathcal{E}} \right\} = \left(\frac{\lambda_2}{2\pi} \right)^{1/2} \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{43a}$$

$$\mathbb{E}\left\{ \mathscr{L}_1^{\mathcal{E}} \right\} = \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{43b}$$



(a) Euler Characteristic (b) Length

Figure 2: Expectation of LKC for $a = 100$, $\sigma^2 = 2$ and $\mathcal{H} = [\kappa\ \infty)$ in 1D.

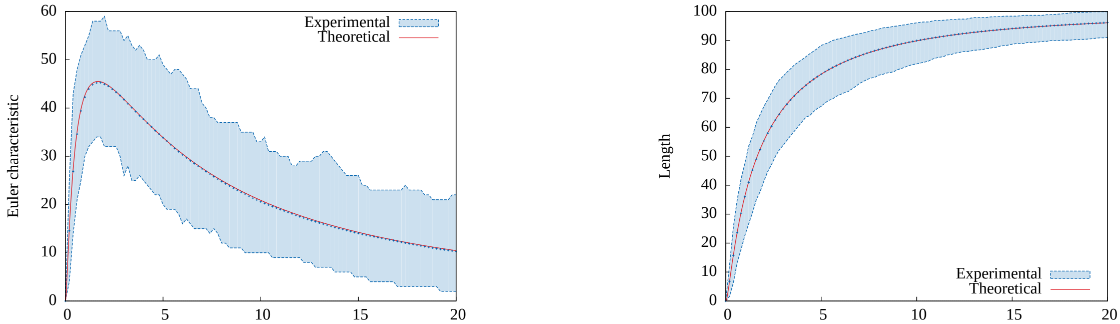


Figure 3: Expectation of LKC for $a = 100$, $\mu = 0.5$, $\sigma^2 = 2$, $L_c = 0.5$, and $\mathcal{H} = [0\ \kappa]$ (log-normal distribution with cumulative hitting set) in 1D. Experimental results are calculated over $1\,000$ realizations.

3.1.3 Two dimensions

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \frac{\lambda_2}{2\pi}\mathscr{L}_2^{\mathcal{M}}\mathscr{M}_2^{\gamma^k} + \sqrt{\frac{\lambda_2}{2\pi}}\mathscr{L}_1^{\mathcal{M}}\mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{44a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \sqrt{\frac{\lambda_2\pi}{8}}\mathscr{L}_2^{\mathcal{M}}\mathscr{M}_1^{\gamma^k} + \mathscr{L}_1^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{44b}$$

$$\mathbb{E}\left\{\mathscr{L}_2^{\mathcal{E}}\right\} = \mathscr{L}_2^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{44c}$$

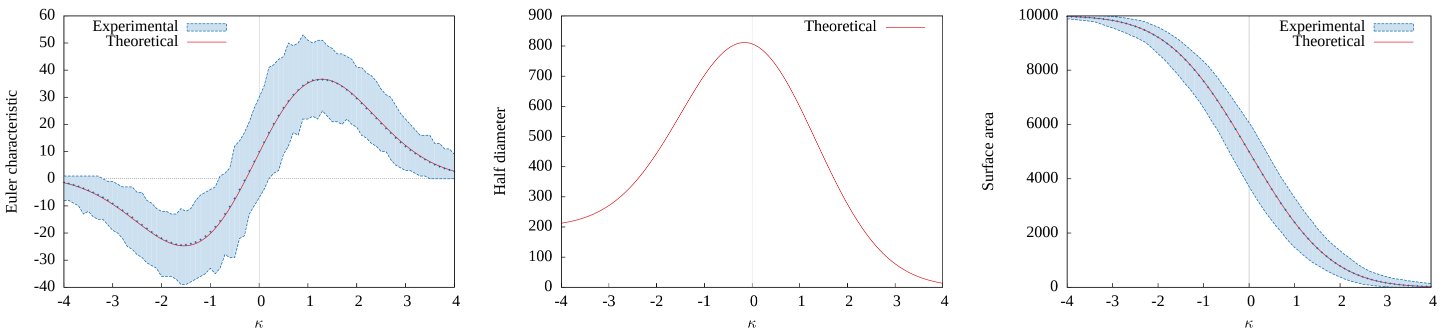


Figure 4: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [\kappa\ \infty)$ (gaussian distribution with tail hitting set) in 2D. Experimental results are calculated over $1\,000$ realizations.

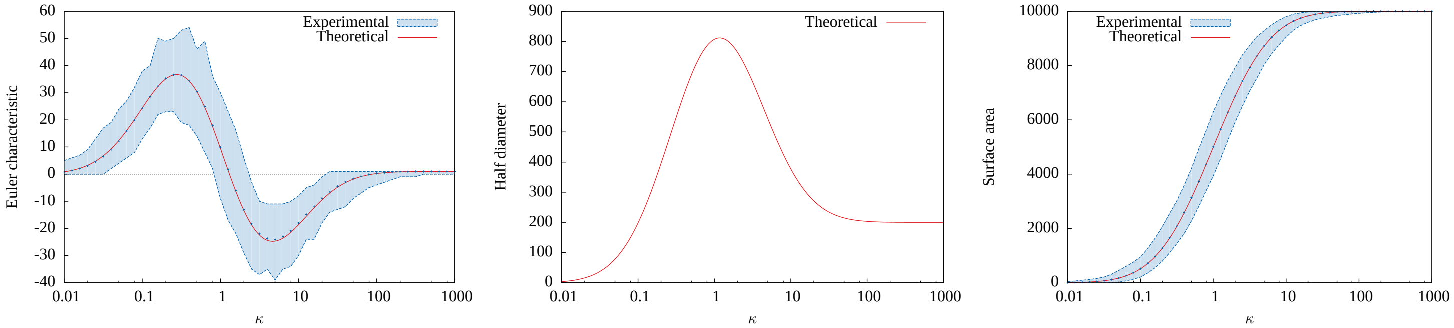


Figure 5: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [0\ \kappa]$ (lognormal distribution with cumulative hitting set) in 2D. Experimental results are calculated over $1\,000$ realizations.

3.1.4 Three dimensions

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \tag{45a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \tag{45b}$$

$$\mathbb{E}\left\{\mathscr{L}_2^{\mathcal{E}}\right\} = \tag{45c}$$

$$\mathbb{E}\left\{\mathscr{L}_3^{\mathcal{E}}\right\} = \mathscr{L}_3^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{45d}$$

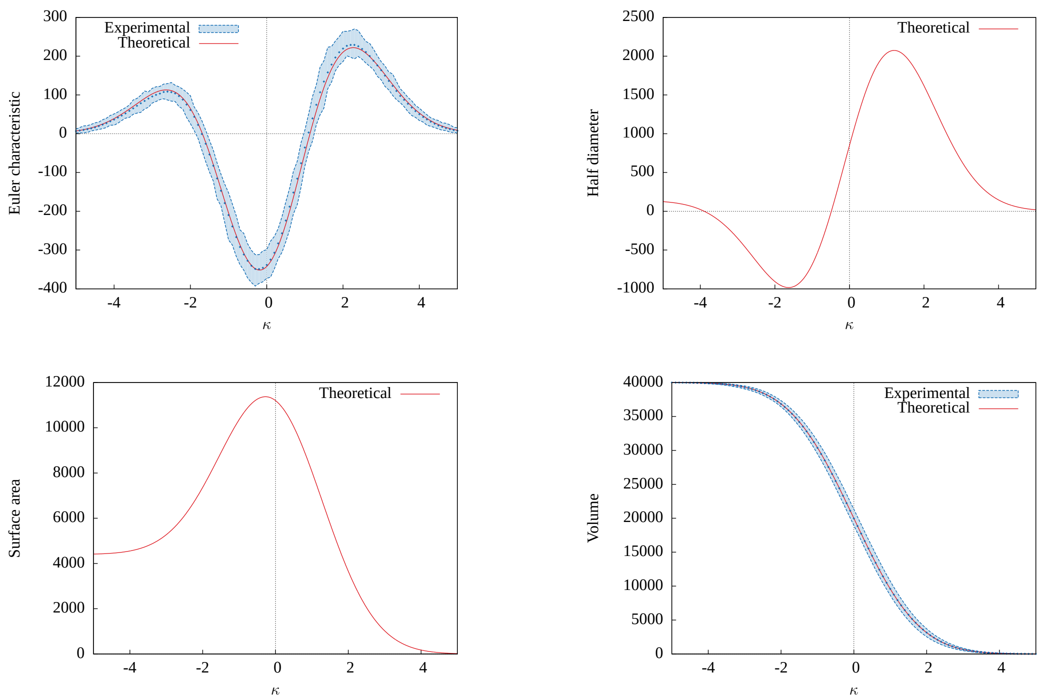


Figure 6: Expectation of LKC for $\mathcal{P} = 100 \times 20 \times 20$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 2$, and $\mathcal{H} = [\kappa\ \infty)$ (gaussian distribution with tail hitting set) in 3D. Experimental results are calculated over 100 realizations.

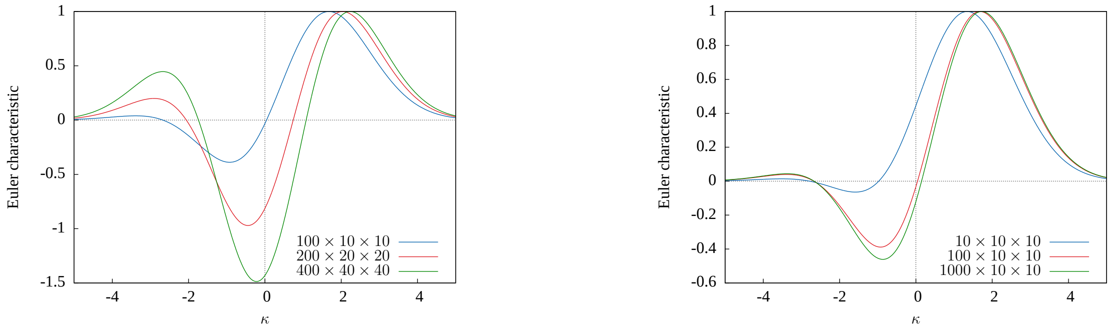
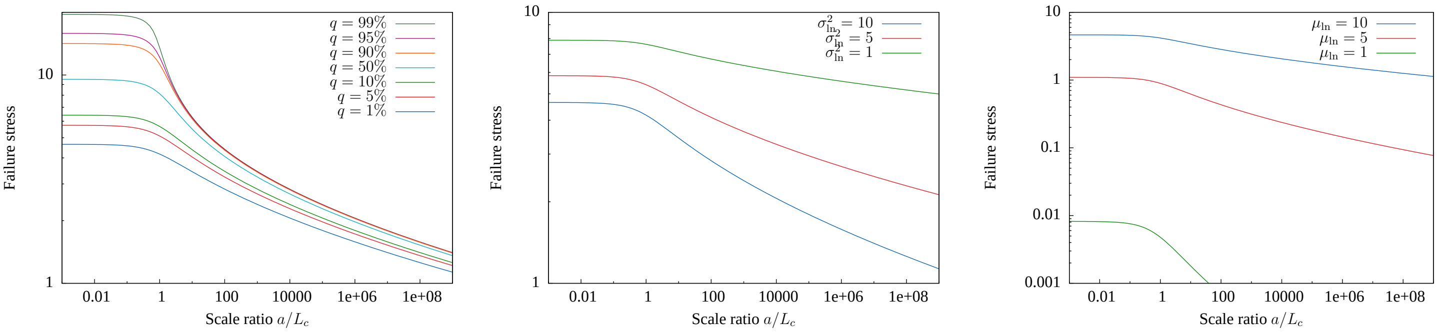


Figure 7: .

4 SIZE EFFECT

4.1 ONE DIMENSIONAL CASE



(a) Variation of the quantile (b) Variation of the variance (c) Variation of the mean

Figure 8: Size Effect: variations around $\mu_{\ln} = 10$, $\sigma^2_{\ln} = 10$ and $q = 1\%$ in 1D.

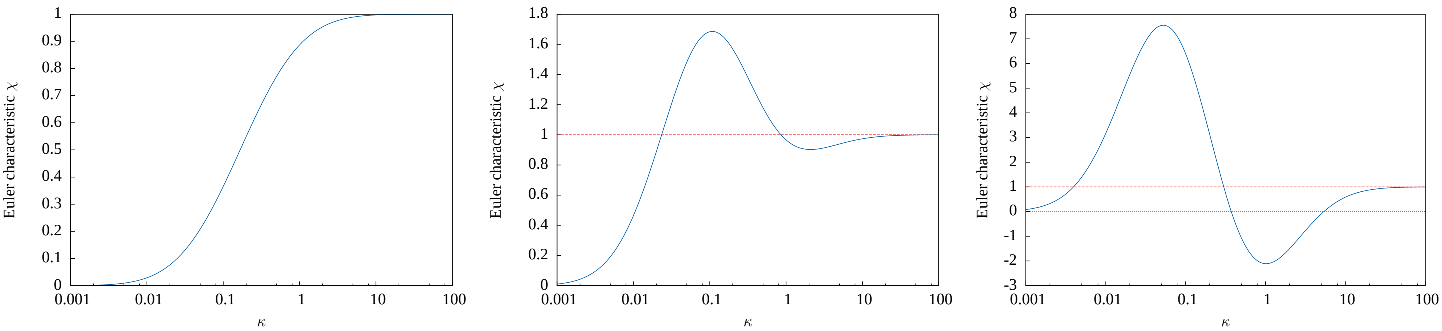
4.2 TWO DIMENSIONAL CASE

4.2.1 Behavior of the Euler characteristic for different specimen sizes

9(a) Small sizes
Direct percolation. The first (and only) $\kappa(\chi == q)$ correspond to the probability q of percolation. There is only one disconnected element in the excursion set.

9(b) Medium sizes

9(c) Large sizes



(a) $a = 0.001$ (b) $a = 2.5$ (c) $a = 8$

Figure 9: Euler characteristic for different specimen sizes a for $\mu_{\mathrm{ln}} = 0.5$, $\sigma^2_{\mathrm{ln}} = 2$, $L_c = 1$, and $\mathcal{H} = [0\ \kappa]$ (lognormal distribution with cumulative hitting set) in a 2D.