Handbook on excursion sets

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Contents

1 Priliminary calculations

1.1 Differentiation under the integral sign

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt \tag{1}$$

1.2 Functions

1.2.1 Gamma function: Γ

$$\Gamma : z \to \int_0^\infty t^{z-1} e^{-t} dt \tag{2}$$

1.2.2 Error function: erf

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{3}$$

1.3 Lipschitz-Killing curvatures (LKC)

1.3.1 For a cube in 1,2 and 3 dimensions

$$\text{For } \mathcal{M}_N = \prod_{i=1}^{N} [0\ a], \quad \mathscr{L}_j(\mathcal{M}_N) = \begin{pmatrix} N \\ j \end{pmatrix} a^j \tag{4}$$

N	LKC	Value	Meaning
1	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
1	$\mathscr{L}_1(\mathcal{M})$	a	Length of the segment
2	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
2	$\mathscr{L}_1(\mathcal{M})$	$2a$	Half the boundary length
2	$\mathscr{L}_2(\mathcal{M})$	a^2	Surface area
3	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
3	$\mathscr{L}_1(\mathcal{M})$	$3a$	Twice the caliper diameter
3	$\mathscr{L}_2(\mathcal{M})$	$3a^2$	Half the surface area
3	$\mathscr{L}_3(\mathcal{M})$	a^3	Volume

1.4 Volume of the unit ball

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1+N/2)} \quad (5)$$

$$\left|\begin{array}{l} \omega_0 = 1 \\ \omega_1 = 2 \\ \omega_2 = \pi \\ \omega_3 = 4\pi/3 \\ \ldots \end{array}\right. \quad (6)$$

1.5 Probabilistic Hermite polynomials

$$\mathscr{H}_n = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e{-}x^2/2 \quad (7)$$

$$\left|\begin{array}{l} \mathscr{H}_0(x) = 1 \\ \mathscr{H}_1(x) = x \\ \mathscr{H}_2(x) = x^2 - 1 \\ \mathscr{H}_3(x) = x^3 - 3 \\ \ldots \end{array}\right. \quad (8)$$

1.6 Flag coefficients

$$\left[\begin{array}{c} n \\ j \end{array}\right] = \left(\begin{array}{c} n \\ j \end{array}\right) \frac{\omega_n}{\omega_{n-j}\omega_j} \quad (9)$$

n	j	flag
0	0	1
1	0	1
1	1	1
2	0	1
2	1	$\pi/2$
2	2	1
3	0	1
3	1	2
3	2	2
3	3	1

(10)

1.7 Second spectral moment

$$\lambda_2 = \mathbb{V}\left\{\frac{\partial\, g(\boldsymbol{x})}{\partial\, x_i}\right\} = \frac{\mathrm{d}^2\mathcal{C}(h)}{\mathrm{d}h^2} \quad (11)$$

Covariance	Parameter	λ_2
Gaussian	–	$2\sigma^2/L_c^2$

(12)

2 Gaussian Minkowsky functionals

2.1 Preliminary

2.1.1 Gaussian measure: $\gamma^k(\mathcal{H})$

$$\gamma^k(\mathcal{H}) = P\{\boldsymbol{X} \in \mathcal{H}\} = \frac{1}{\sigma^k(2\pi)^{k/2}} \int_{\mathcal{H}} e^{-\|\boldsymbol{x}-\mu\|^2/2\sigma^2} d\boldsymbol{x} \tag{13}$$

2.1.2 Tube Taylor expansion

$$\gamma^k(\mathrm{tube}(\mathcal{H}, \rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathscr{M}_j^{\gamma^k}(\mathcal{H}) \tag{14}$$

2.2 GMFs for Gaussian distribution

2.2.1 Application to one-dimensional tail hitting set: $k = 1$ and $\mathcal{H} = [\kappa\ \infty)$

Tail probability $\bar{F}$:

$$\gamma^1([\kappa\ \infty)) = P\{X \geq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{\kappa}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = \bar{F}(\kappa) \tag{15}$$

Error function erf:

$$\begin{aligned}
\gamma^1([\kappa\ \infty)) = \bar{F}(\kappa) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} \sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}} \int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2} dt \\
&= \frac{1}{\sqrt{\pi}} \left[\int_0^{\infty} e^{-t^2} dt - \int_0^{\frac{\kappa-\mu}{\sigma\sqrt{2}}} e^{-t^2} dt \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \mathrm{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}} \right) \right] \\
&= \frac{1}{2} \left(1 - \mathrm{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}} \right) \right)
\end{aligned}$$

Tube expansion:

$$\gamma^1(\mathrm{tube}([\kappa\ \infty), \rho)) = \gamma^1([\kappa - \rho\ \infty)) = \bar{F}(\kappa - \rho) = \sum_{j=0}^{\infty} \frac{(-\rho)^j}{j!} \frac{\mathrm{d}^j \bar{F}(\kappa)}{\mathrm{d}\kappa^j} \tag{17}$$

Identification:

- Notations:

  Tail hitting set: $\bar{\mathcal{H}}^{\text{1D}} = [\kappa\ \infty)$ (18a)

  Variable substitution: $\tilde{\kappa} \leftarrow \dfrac{\kappa - \mu}{\sigma}$ (18b)

- For $j = 0$:

$$\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \bar{F}(\kappa) \tag{19}$$

- For $j > 0$:

$$\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^j \bar{F}(\kappa)}{\mathrm{d}\kappa^j} = \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}} \mathscr{H}_{j-1}((\kappa-\mu)/\sigma) \tag{20}$$

- First values:

$$\left|\begin{array}{l}
\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\
\mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2/2} \\
\mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}\ e^{-\tilde{\kappa}^2/2} \\
\mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ (\tilde{\kappa}^2 - 1)\ e^{-\tilde{\kappa}^2/2} \\
\mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^4\sqrt{2\pi}}\ (\tilde{\kappa}^3 - 3\tilde{\kappa})\ e^{-\tilde{\kappa}^2/2} \\
\ldots
\end{array}\right. \tag{21}$$

- Derivatives:

$$\forall j, \quad \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^{j+1}\bar{F}(\kappa)}{\mathrm{d}\kappa^{j+1}} = -\mathscr{M}_{j+1}^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) \tag{22}$$

2.2.2 Application to one-dimensional cumulative hitting set: $k = 1$ and $\mathcal{H} = (-\infty\ \kappa]$

Cumulative probability F:

$$\gamma^1((-\infty\ \kappa]) = P\{X \leq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-(x-\mu)^2/2\sigma^2} dx = F(\kappa) \tag{23}$$

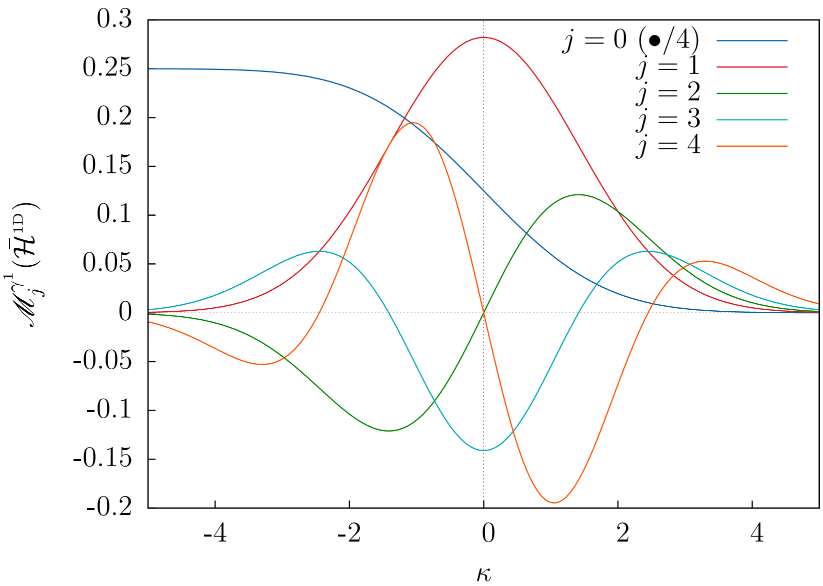


Figure 1: First 4 GMF for $\sigma^2 = 2$ and $\mathcal{H} = \bar{\mathcal{H}}^{1D} = [\kappa\ \infty)$

Error function erf:

$$\gamma^1((-\infty\ \kappa]) = F(\kappa) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\kappa - \mu}{\sigma\sqrt{2}}\right)\right) \tag{24}$$

Tube expension:

$$\gamma^1(\mathrm{tube}((-\infty\ \kappa], \rho)) = \gamma^1((-\infty\ \kappa + \rho]) = F(\kappa + \rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} \tag{25}$$

Identification:

- Notations:

$$\text{Tail hitting set: } \mathcal{H}^{1D} = (-\infty\ \kappa] \tag{26a}$$

$$\text{Variable substitution: } \tilde{\kappa} \leftarrow \frac{\kappa - \mu}{\sigma} \tag{26b}$$

- For $j = 0$:

$$\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{1D}) = F(\kappa) \tag{27}$$

- For $j > 0$:

$$\mathscr{M}_j^{\gamma^1}(\mathcal{H}^{1D}) = \frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} = (-1)^{j+1} \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}} \mathscr{H}_{j-1}((\kappa - \mu)/\sigma) \tag{28}$$

- First values:

$$\left| \begin{aligned} &\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \tfrac{1}{2}\left(1 - \operatorname{erf}\left(\tfrac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\ &\mathscr{M}_1^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \tfrac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2/2} \\ &\mathscr{M}_2^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \tfrac{-1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}\ e^{-\tilde{\kappa}^2/2} \\ &\mathscr{M}_3^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \tfrac{1}{\sigma^3\sqrt{2\pi}}\ \left(\tilde{\kappa}^2 - 1\right)\ e^{-\tilde{\kappa}^2/2} \\ &\mathscr{M}_4^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \tfrac{-1}{\sigma^4\sqrt{2\pi}}\ \left(\tilde{\kappa}^3 - 3\tilde{\kappa}\right)\ e^{-\tilde{\kappa}^2/2} \\ &\ldots \end{aligned} \right. \tag{29}$$

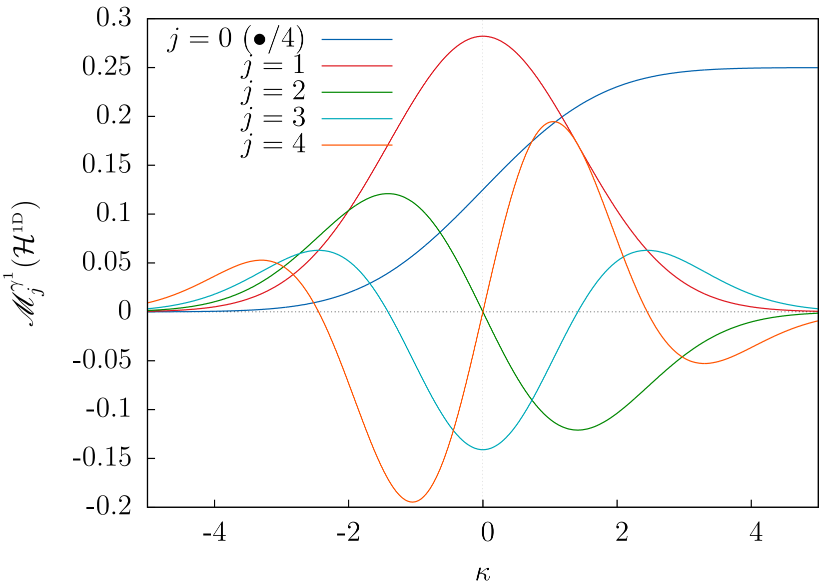


Figure 2: First 4 GMF for $\sigma^2 = 2$ and $\mathcal{H} = \mathcal{H}^{\text{1D}} = (-\infty\ \kappa]$

- Derivatives:

$$\forall j, \quad \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\mathcal{H}^{\text{1D}}) = \frac{\mathrm{d}^{j+1}F(\kappa)}{\mathrm{d}\kappa^{j+1}} = \mathscr{M}_{j+1}^{\gamma^1}(\mathcal{H}^{\text{1D}}) \tag{30}$$

2.3 GMFs for Gaussian related Random Fields

$$g_r : \Omega \times \mathbf{R}^N \xrightarrow{\boldsymbol{g}} \mathbf{R}^k \xrightarrow{S} \mathbf{R} \tag{31}$$

$$P\left\{g_r \in \mathcal{H}\right\} = P\left\{S(g) \in \mathcal{H}\right\} = P\left\{g \in S^{-1}(\mathcal{H})\right\} = \gamma^k(S^{-1}(\mathcal{H})) \tag{32}$$

2.3.1 Log-normal distribution

$$k = 1 \quad , \quad S = \exp \quad \text{and} \quad S^{-1} = \ln \tag{33}$$

$$\left| \begin{array}{lcl} \mu & = & \ln(\mu_{\ln}) - \frac{1}{2}\ln(1 + \sigma_{\ln}^2/\mu_{\ln}^2) \\ \sigma^2 & = & \ln(1 + \sigma_{\ln}^2/\mu_{\ln}^2) \end{array} \right. \tag{34}$$

- Notations:

Log tail hitting set: $\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}} = [\ln(\kappa)\ \infty)$ (35a)
Log cumulative hitting set: $\mathcal{H}_{\ln}^{1\mathrm{D}} = (-\infty\ \ \ln(\kappa)]$ (35b)
Variable substitution: $\tilde{\kappa}_{\ln} \leftarrow \dfrac{\ln(\kappa) - \mu}{\sigma}$ (35c)

- For $\mathcal{H} = \bar{\mathcal{H}}^{1\mathrm{D}} = [\kappa\ \infty) \quad \rightarrow \quad S^{-1}(\mathcal{H}) = \bar{\mathcal{H}}_{\ln}^{1\mathrm{D}} = [\ln(\kappa)\ \infty)$

$$\left| \begin{array}{l} \mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}}) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\ \\ \mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}_{\ln}\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ (\tilde{\kappa}_{\ln}^2 - 1)\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma^4\sqrt{2\pi}}\ (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln})\ e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \ldots \end{array} \right. \tag{36}$$

Derivatives:

$$\left|\begin{array}{l}
\text{Help:} \quad \left(e^{-(\ln(x)-\mu)^2/2\sigma^2}\right)' = -\frac{1}{\sigma^2}\frac{\ln(x)-\mu}{x}\; e^{-(\ln(x)-\mu)^2/2\sigma^2} \\[2ex]
\frac{\mathrm{d}\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma\sqrt{2\pi}}\frac{1}{\kappa}\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2ex]
\frac{\mathrm{d}\mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^2\sqrt{2\pi}}\frac{1}{\kappa}\frac{\ln(\kappa)-\mu}{\sigma}\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2ex]
\frac{\mathrm{d}\mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^3\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^2}{\sigma^2} - 1\right]\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2ex]
\frac{\mathrm{d}\mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^4\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^3}{\sigma^3} - 3\frac{(\ln(\kappa)-\mu)}{\sigma}\right]\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2ex]
\dots \\[2ex]
\text{Guess would be: } \forall j \geq 0, \\
\frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\; e^{-\tilde{\kappa}_{\ln}^2/2}
\end{array}\right. \tag{37}$$

- For $\mathcal{H} = [0\ \kappa] \quad \rightarrow \quad S^{-1}(\mathcal{H}) = \mathcal{H}_{\ln}^{1\mathrm{D}} = (-\infty\ \ \ln(\kappa)]$

$$\left|\begin{array}{l}
\mathscr{M}_0^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\[2ex]
\mathscr{M}_1^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2ex]
\mathscr{M}_2^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{-1}{\sigma^2\sqrt{2\pi}}\; \tilde{\kappa}_{\ln}\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2ex]
\mathscr{M}_3^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\; (\tilde{\kappa}_{\ln}^2 - 1)\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2ex]
\mathscr{M}_4^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{-1}{\sigma^4\sqrt{2\pi}}\; (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln})\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2ex]
\dots
\end{array}\right. \tag{38}$$

Derivatives:

$$\left|\begin{array}{l}
\text{Guess would be: } \forall j \geq 0, \\
\frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{(-1)^j}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\; e^{-\tilde{\kappa}_{\ln}^2/2}
\end{array}\right. \tag{39}$$

3 Expectation Formula

3.1 Gaussian distribution

3.1.1 General case

$$\mathbb{E}\left\{\ \mathscr{L}_j(\mathcal{E})\ \right\} = \sum_{i=0}^{N-j} \begin{bmatrix} i+j \\ i \end{bmatrix} \left(\frac{\lambda_2}{2\pi}\right)^{i/2} \mathscr{L}_{i+j}(\mathcal{M}) \mathscr{M}_i^{\gamma^k}(S^{-1}(\mathcal{H})) \tag{40}$$

Notations:

LKC of $\mathcal{M}$: $\mathscr{L}_j(\mathcal{M}) = \mathscr{L}_j^{\mathcal{M}}$ (41a)

LKC of $\mathcal{E}$: $\mathscr{L}_j(\mathcal{E}) = \mathscr{L}_j^{\mathcal{E}}$ (41b)

GMF of $S^{-1}(\mathcal{H})$: $\mathscr{M}_j^{\gamma^k}(S^{-1}(\mathcal{H})) = \mathscr{M}_j^{\gamma^k}$ (41c)

3.1.2 One dimension

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \left(\frac{\lambda_2}{2\pi}\right)^{1/2} \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{42a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{42b}$$

3.1.3 Two dimensions

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \frac{\lambda_2}{2\pi} \mathscr{L}_2^{\mathcal{M}} \mathscr{M}_2^{\gamma^k} + \sqrt{\frac{\lambda_2}{2\pi}} \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{43a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \sqrt{\frac{\lambda_2 \pi}{8}} \mathscr{L}_2^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{43b}$$

$$\mathbb{E}\left\{\mathscr{L}_2^{\mathcal{E}}\right\} = \mathscr{L}_2^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{43c}$$

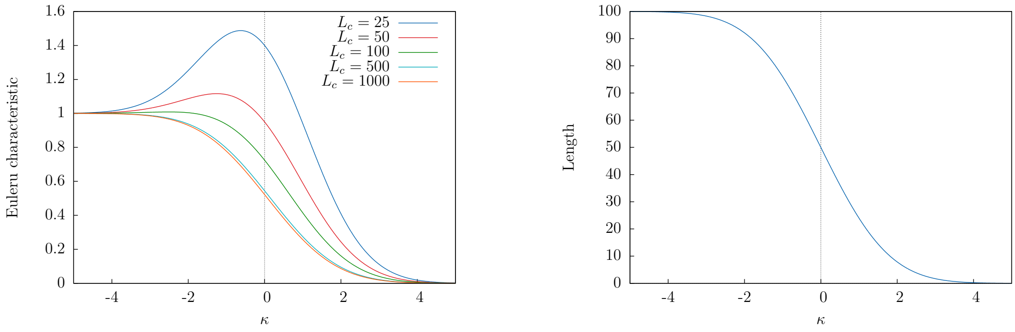
4 Size effect

4.1 One dimensional case

4.2 Two dimensional case

4.2.1 Behavior of the Euler characteristic for different specimen length

It is first important to understand well the different behaviors (regimes) of the Euleur characteristic in terms of the specimen length a.



(a) Euler Characteristic

(b) Length

Figure 3: Expectation of LKC for $a = 100$, $\sigma^2 = 2$ and $\mathcal{H} = [\kappa\ \infty)$ in 1D.

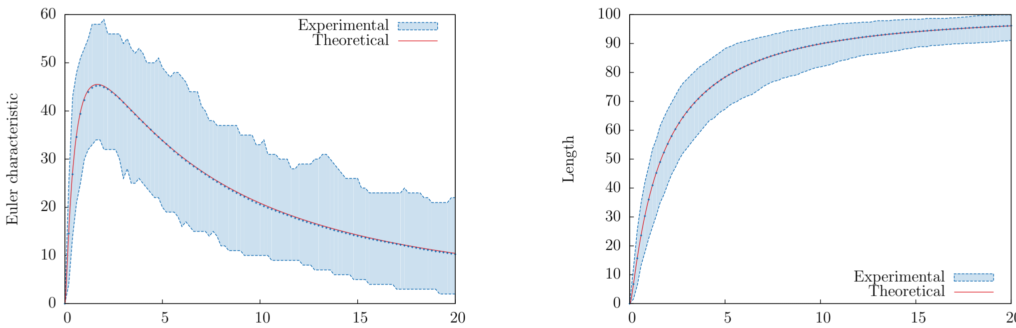


Figure 4: Expectation of LKC for $a = 100$, $\mu = 0.5$, $\sigma^2 = 2$, $L_c = 0.5$, and $\mathcal{H} = [0\ \kappa]$ (log-normal distribution with cumulative hitting set) in 1D. Experimental results are calculated over 1 000 realizations.

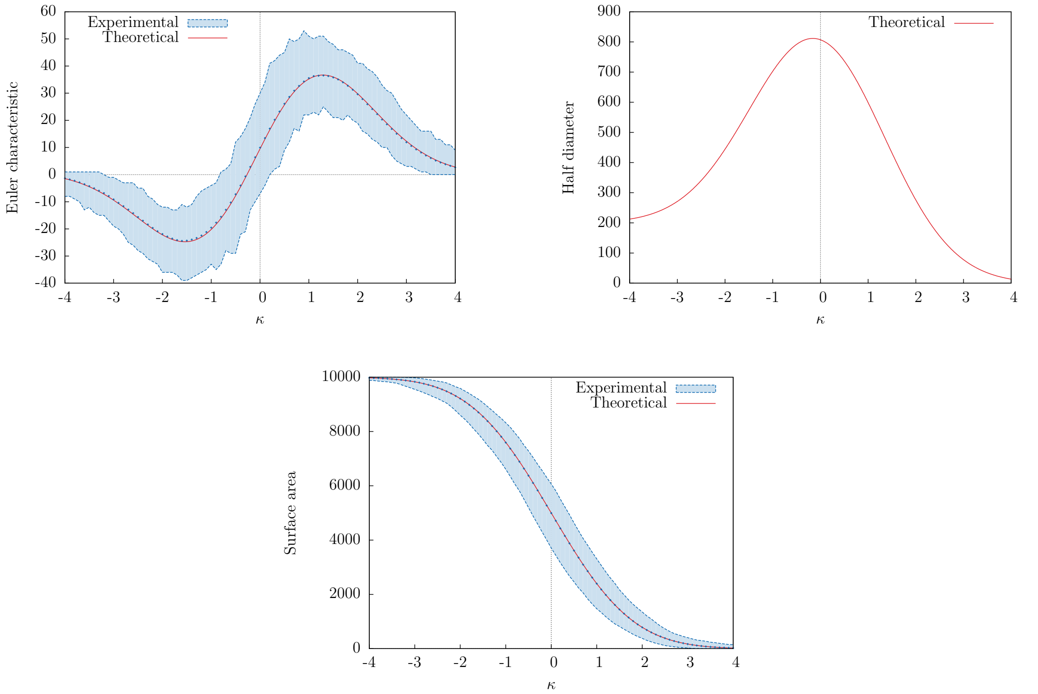


Figure 5: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [\kappa\ \infty)$ (gaussian distribution with tail hitting set) in 2D. Experimental results are calculated over 1 000 realizations.

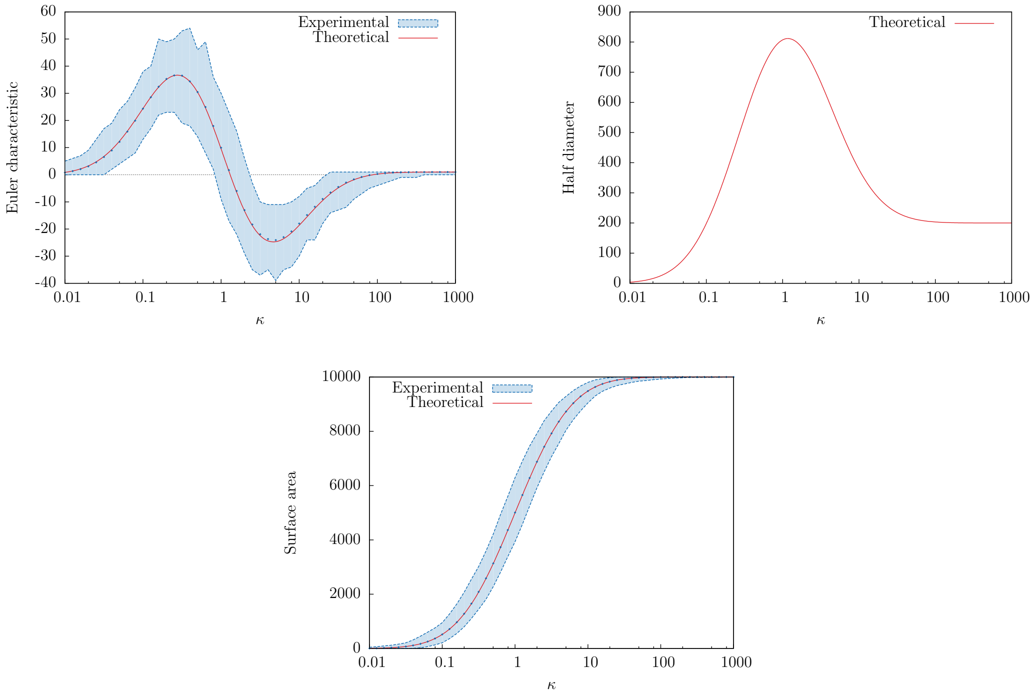
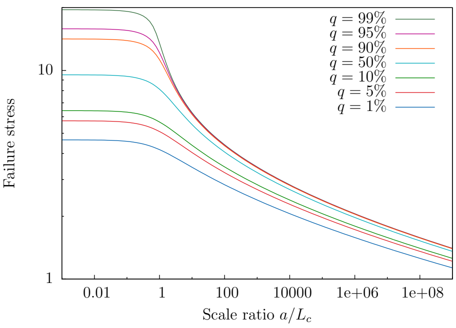
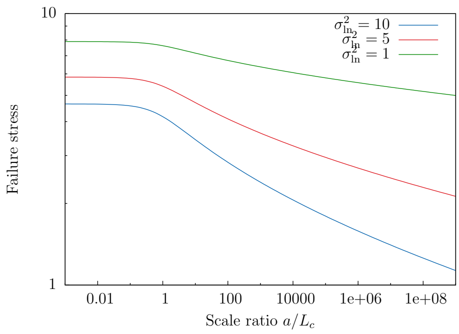


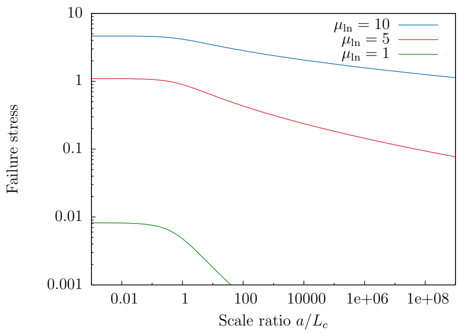
Figure 6: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [0\ \kappa]$ (lognormal distribution with cumulative hitting set) in 2D. Experimental results are calculated over 1 000 realizations.



(a) Variation of the quantile

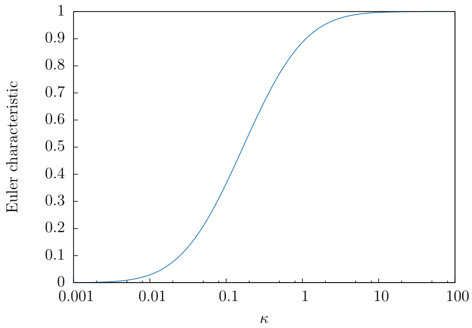


(b) Variation of the variance

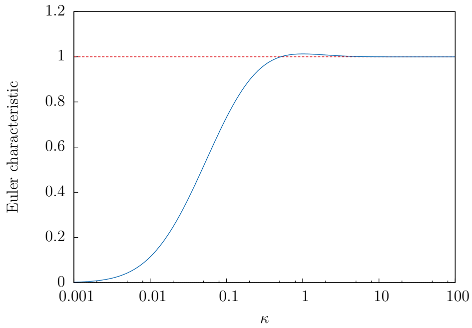


(c) Variation of the mean

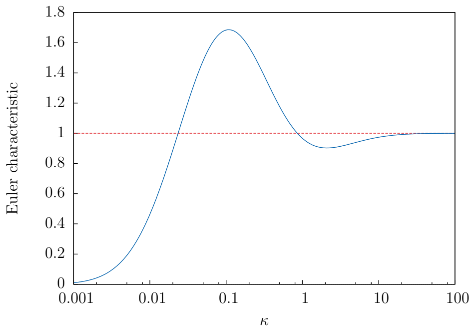
Figure 7: Size Effect: variations around $\mu_{\ln} = 10$, $\sigma^2_{\ln} = 10$ and $q = 1\%$ in 1D.



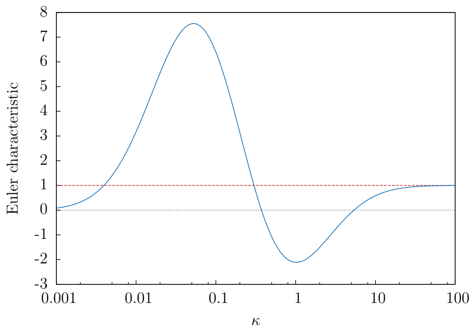
(a) $a = 0.001$



(b) $a = 0.8$



(c) $a = 2.5$



(d) $a = 8$

Figure 8: Size Effect: variations around $\mu_{\ln} = 10$, $\sigma^2_{\ln} = 10$ and $q = 1\%$ in 1D.