HANDBOOK ON EXCURSION SETS

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CONTENTS

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1 Priliminary calculations

1.1 Differentiation under the integral sign

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt \tag{1}$$

1.2 Functions

- Gamma function: $\Gamma(z)$

$$\Gamma : z \mapsto \int_0^\infty t^{z-1}e^{-t}dt \tag{2}$$

- Incomplete Gamma function: $\Gamma(z,x)$

$$\Gamma : z \times x \mapsto \int_x^\infty t^{z-1}e^{-t}dt \tag{3}$$

- Normalized incomplete Gamma function: $\bar{\Gamma}(z,x)$

$$\bar{\Gamma} : z \times x \mapsto \Gamma(z,x)/\Gamma(z) \tag{4}$$

- Error function: erf(x)

$$\text{erf} : x \mapsto \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{5}$$

1.3 Lipschitz-Killing curvatures (LKC)

- For a cube:

$$\mathcal{M}_N = \prod_{i=1}^{N}[0\ a],\ \ \mathscr{L}_j(\mathcal{M}_N) = \left(\begin{array}{c} N \\ j \end{array}\right) a^j \tag{6}$$

- For a parallelepiped

$$\mathcal{P}_N = \prod_{i=1}^{N}[0\ a_i],\ \ \mathscr{L}_j(\mathcal{M}_N) = \ldots \tag{7}$$

N	LKC	Value	Meaning
1	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
1	$\mathscr{L}_1(\mathcal{M})$	a	Length of the segment
2	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
2	$\mathscr{L}_1(\mathcal{M})$	$2a$	Half the boundary length
2	$\mathscr{L}_2(\mathcal{M})$	a^2	Surface area
3	$\mathscr{L}_0(\mathcal{M})$	1	Euler Characteristic
3	$\mathscr{L}_1(\mathcal{M})$	$3a$	Twice the caliper diameter
3	$\mathscr{L}_2(\mathcal{M})$	$3a^2$	Half the surface area
3	$\mathscr{L}_3(\mathcal{M})$	a^3	Volume
3	$\mathscr{L}_0(\mathcal{P})$	1	Euler Characteristic
3	$\mathscr{L}_1(\mathcal{P})$	$a_1 + a_2 + a_3$	Twice the caliper diameter
3	$\mathscr{L}_2(\mathcal{P})$	$a_1a_2 + a_1a_3 + a_2a_3$	Half the surface area
3	$\mathscr{L}_3(\mathcal{P})$	$a_1a_2a_3$	Volume

1.4 Volume of the unit ball

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1+N/2)} \tag{8}$$

$$\left|\begin{array}{l} \omega_0 = 1 \\ \omega_1 = 2 \\ \omega_2 = \pi \\ \omega_3 = 4\pi/3 \\ \dots \end{array}\right. \tag{9}$$

1.5 Probabilistic Hermite polynomials

$$\mathscr{H}_n = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e{-}x^2/2 \tag{10}$$

$$\left|\begin{array}{l} \mathscr{H}_0(x) = 1 \\ \mathscr{H}_1(x) = x \\ \mathscr{H}_2(x) = x^2 - 1 \\ \mathscr{H}_3(x) = x^3 - 3 \\ \dots \end{array}\right. \tag{11}$$

1.6 Flag coefficients

$$\left[\begin{array}{c} n \\ j \end{array}\right] = \left(\begin{array}{c} n \\ j \end{array}\right) \frac{\omega_n}{\omega_{n-j}\omega_j} \tag{12}$$

(13)

n	j	flag
0	0	1
1	0	1
1	1	1
2	0	1
2	1	$\pi/2$
2	2	1
3	0	1
3	1	2
3	2	2
3	3	1

1.7 Second spectral moment

$$\lambda_2 = \mathbb{V}\left\{\frac{\partial\, g(\boldsymbol{x})}{\partial\, x_i}\right\} = \frac{\mathrm{d}^2\mathcal{C}(h)}{\mathrm{d}h^2} \tag{14}$$

(15)

Covariance	Parameter	λ_2
Gaussian	–	$2\sigma^2/L_c^2$

2 Gaussian Minkowsky functionals

2.1 Preliminary

2.1.1 Gaussian measure: $\gamma^k(\mathcal{H})$

$$\gamma^k(\mathcal{H}) = P\{\boldsymbol{X} \in \mathcal{H}\} = \frac{1}{\sigma^k(2\pi)^{k/2}}\int_{\mathcal{H}} e^{-\|\boldsymbol{x}-\mu\|^2/2\sigma^2}d\boldsymbol{x} \tag{16}$$

2.1.2 Tube Taylor expansion

$$\gamma^k(\text{tube}(\mathcal{H},\rho)) = \sum_{j=0}^{\infty}\frac{\rho^j}{j!}\mathscr{M}_j^{\gamma^k}(\mathcal{H}) \tag{17}$$

2.2 GMFs for Gaussian distribution

2.2.1 Application to: $k=1$ and $\mathcal{H} = [\kappa\ \infty)$

Tail probability $\bar{F}$:

$$\gamma^1([\kappa\ \infty)) = P\{X \geq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}}\int_{\kappa}^{\infty} e^{-(x-\mu)^2/2\sigma^2}dx = \bar{F}(\kappa) \tag{18}$$

Error function erf:

$$\begin{aligned}\gamma^1([\kappa\ \infty)) = \bar{F}(\kappa) &= \frac{1}{\sigma\sqrt{2\pi}}\int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2}\sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}}\int_{\frac{\kappa-\mu}{\sigma\sqrt{2}}}^{\infty} e^{-t^2}dt \\ &= \frac{1}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-t^2}dt - \int_0^{\frac{\kappa-\mu}{\sigma\sqrt{2}}} e^{-t^2}dt\right] \\ &= \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\text{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right)\right] \\ &= \frac{1}{2}\left(1 - \text{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right)\right)\end{aligned}$$

Tube expansion:

$$\gamma^1(\text{tube}([\kappa\ \infty),\rho)) = \gamma^1([\kappa-\rho\ \infty)) = \bar{F}(\kappa-\rho) = \sum_{j=0}^{\infty}\frac{(-\rho)^j}{j!}\frac{\mathrm{d}^j\bar{F}(\kappa)}{\mathrm{d}\kappa^j} \tag{20}$$

Identification:

- Notations:

$$\text{Tail hitting set: } \bar{\mathcal{H}}^{\text{1D}} = [\kappa\ \infty) \tag{21a}$$

$$\text{Variable substitution: } \tilde{\kappa} \leftarrow \frac{\kappa - \mu}{\sigma} \tag{21b}$$

- For $j = 0$:

$$\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \bar{F}(\kappa) \tag{22}$$

- For $j > 0$:

$$\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^j \bar{F}(\kappa)}{\mathrm{d}\kappa^j} = \frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}} \mathscr{H}_{j-1}((\kappa-\mu)/\sigma) \tag{23}$$

- First values:

$$\left|\begin{array}{l} \mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\ \mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\tilde{\kappa}^2/2} \\ \mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^2\sqrt{2\pi}}\, \tilde{\kappa}\, e^{-\tilde{\kappa}^2/2} \\ \mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\, \left(\tilde{\kappa}^2 - 1\right)\, e^{-\tilde{\kappa}^2/2} \\ \mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) = \frac{1}{\sigma^4\sqrt{2\pi}}\, \left(\tilde{\kappa}^3 - 3\tilde{\kappa}\right)\, e^{-\tilde{\kappa}^2/2} \\ \ldots \end{array}\right. \tag{24}$$

- Derivatives:

$$\forall j,\quad \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\bar{\mathcal{H}}^{\text{1D}}) = (-1)^j \frac{\mathrm{d}^{j+1}\bar{F}(\kappa)}{\mathrm{d}\kappa^{j+1}} = -\mathscr{M}_{j+1}^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}) \tag{25}$$

2.2.2 Application to: $k = 1$ and $\mathcal{H} = (-\infty\ \kappa]$

Cumulative probability F:

$$\gamma^1((-\infty\ \kappa]) = P\{X \leq \kappa\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-(x-\mu)^2/2\sigma^2} dx = F(\kappa) \tag{26}$$

Error function erf:

$$\gamma^1((-\infty\ \kappa]) = F(\kappa) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\kappa-\mu}{\sigma\sqrt{2}}\right)\right) \tag{27}$$

Tube expension:

$$\gamma^1(\text{tube}((-\infty\ \kappa], \rho)) = \gamma^1((-\infty\ \kappa+\rho]) = F(\kappa+\rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} \tag{28}$$

Identification:

- Notations:

Tail hitting set: $\mathcal{H}^{\text{1D}} = (-\infty\ \kappa]$ (29a)

Variable substitution: $\tilde{\kappa} \leftarrow \dfrac{\kappa - \mu}{\sigma}$ (29b)

- For $j = 0$:

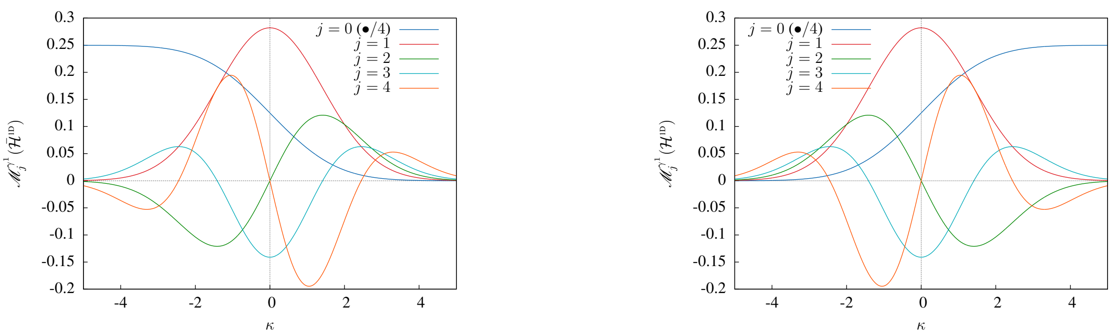
$$\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{\text{1D}}) = F(\kappa) \tag{30}$$

- For $j > 0$:

$$\mathscr{M}_j^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{\mathrm{d}^j F(\kappa)}{\mathrm{d}\kappa^j} = (-1)^{j+1}\frac{e^{-(\kappa-\mu)^2/2\sigma^2}}{\sigma^j\sqrt{2\pi}}\mathscr{H}_{j-1}((\kappa-\mu)/\sigma) \tag{31}$$

- First values:

$$\left|\begin{array}{l}
\mathscr{M}_0^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{\tilde{\kappa}}{\sqrt{2}}\right)\right) \\[2ex]
\mathscr{M}_1^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_2^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{-1}{\sigma^2\sqrt{2\pi}}\ \tilde{\kappa}\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_3^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\ \left(\tilde{\kappa}^2 - 1\right)\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\mathscr{M}_4^{\gamma^1}(\mathcal{H}^{\text{1D}}) = \frac{-1}{\sigma^4\sqrt{2\pi}}\ \left(\tilde{\kappa}^3 - 3\tilde{\kappa}\right)\ e^{-\tilde{\kappa}^2/2} \\[2ex]
\ldots
\end{array}\right. \tag{32}$$



(a) $\mathcal{H} = \mathcal{H}^{\text{1D}} = [\kappa\ \infty)$

(b) $\mathcal{H} = \mathcal{H}^{\text{1D}} = (-\infty\ \kappa]$

Figure 1: First 4 GMF for $\sigma^2 = 2$

- Derivatives:

$$\forall j,\ \ \frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}}{\mathrm{d}\kappa}(\mathcal{H}^{\text{1D}}) = \frac{\mathrm{d}^{j+1}F(\kappa)}{\mathrm{d}\kappa^{j+1}} = \mathscr{M}_{j+1}^{\gamma^1}(\mathcal{H}^{\text{1D}}) \tag{33}$$

2.2.3 Application to: $k \in \mathbb{N}^*$ and $\mathcal{H} = \mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0, \kappa)$

Simplification $\mu = 0$ gives Gaussian measure:

$$\gamma_k(\mathcal{H}) = \frac{1}{\sigma^k (2\pi)^{k/2}} \int_{\mathcal{H}} e^{-\|\boldsymbol{x}\|^2/2\sigma^2} d\boldsymbol{x} \tag{34}$$

Spherical coordinates in $\mathbb{R}^k$:

$$\begin{cases} \|\boldsymbol{x}\| &= r \\ x_1 &= r\cos(\theta_1) \\ x_2 &= r\sin(\theta_1)\cos(\theta_2) \\ x_2 &= r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\ \vdots \\ x_{k-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{k-2})\cos(\theta_{k-1}) \\ x_k &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{k-2})\sin(\theta_{k-1}) \end{cases} \tag{35}$$

Volume element in $\mathbb{R}^k$:

$$\begin{aligned} dV &= \left|\det \frac{\partial x_i}{\partial(r, \theta_j)}\right| dr d\theta_1 \cdots d\theta_{k-1} \\ &= r^{k-1}\sin^{k-2}(\theta_1)\sin^{k-3}(\theta_2)\cdots\sin(\theta_{k-2}) dr d\theta_1 \cdots d\theta_{k-1} \end{aligned}$$

Gaussian volume using $(r, \{\theta_i\})$:

$$\begin{aligned} \gamma_k(\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa)) &= \frac{1}{\sigma^k(2\pi)^{k/2}} \int_{r=\kappa}^{\infty} r^{k-1} e^{-r^2/2\sigma^2} dr \\ &\times \underbrace{\int_0^{\pi} \sin^{k-2}(\theta_1) d\theta_1 \int_0^{2\pi} \sin^{k-3}(\theta_2) d\theta_2 \cdots \int_0^{2\pi} d\theta_{k-1}}_{2\pi^{k/2}/\Gamma(k/2) \quad (k-1\text{-dimensional volume of unit sphere})} \\ &= \frac{1}{\sigma^k 2^{k/2-1}\Gamma(k/2)} \int_{r=\kappa}^{\infty} r^{k-1} e^{-r^2/2\sigma^2} dr \end{aligned} \tag{37}$$

Tube expansion for $\mathcal{H} = \mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa)$:

$$\gamma^k(\text{tube}(\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa), \rho)) = \gamma^k(\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0, \kappa - \rho)) = \sum_{j=0}^{\infty} \frac{(-\rho)^j}{j!} \frac{\mathrm{d}^j \gamma_k(\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa))}{\mathrm{d}\kappa^j} \tag{38}$$

Simplification of $\gamma_k(\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa))$ with variable substitution:

$$\begin{cases} t = r^2/2\sigma^2 & \rightarrow \quad r = \sqrt{2t}\sigma \\ dt = r dr/\sigma^2 & \rightarrow \quad dr = \sqrt{1/2t}\sigma dt \end{cases} \tag{39}$$

$$\begin{aligned} \gamma_k(\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa)) &= \frac{1}{\sigma^k \Gamma(k/2) 2^{k/2-1}} \int_{t=\kappa^2/2\sigma^2}^{\infty} (\sqrt{2t}\sigma)^{k-1} e^{-t} \frac{\sigma}{\sqrt{2t}} dt \\ &= \frac{1}{\Gamma(k/2)} \int_{t=\kappa^2/2\sigma^2}^{\infty} t^{k/2-1} e^{-t} dt \\ &= \overline{\Gamma}(k/2, \kappa^2/2\sigma^2) \end{aligned} \tag{40}$$

2.2.4 Application to: $k \in \mathbb{N}^*$ for various $\mathcal{H}$

$$\begin{array}{lcl} \gamma_k(\mathcal{B}_{\mathbb{R}^k}(0,\kappa)) & = & \overline{\gamma}(k/2, \kappa^2/2\sigma^2) \\ \gamma_k(\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa)) & = & \overline{\Gamma}(k/2, \kappa^2/2\sigma^2) \\ \gamma_k(\mathcal{B}_{\mathbb{R}^k}(0,\kappa_1) \cup (\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa_2)))) & = & \overline{\gamma}(k/2, \kappa_1^2/2\sigma^2) + \overline{\Gamma}(k/2, \kappa_2^2/2\sigma^2) \\ \gamma_k(\mathbb{R}^k \backslash (\mathcal{B}_{\mathbb{R}^k}(0,\kappa_1) \cup (\mathbb{R}^k \backslash \mathcal{B}_{\mathbb{R}^k}(0,\kappa_2))))) & = & 1 - \overline{\gamma}(k/2, \kappa_1^2/2\sigma^2) - \overline{\Gamma}(k/2, \kappa_2^2/2\sigma^2) \end{array} \tag{41}$$

2.3 GMFs for Gaussian related Random Fields

$$g_r : \Omega \times \mathbb{R}^N \xrightarrow{\boldsymbol{g}} \mathbb{R}^k \xrightarrow{S} \mathbb{R} \tag{42}$$

$$P\left\{g_r \in \mathcal{H}\right\} = P\left\{S(g) \in \mathcal{H}\right\} = P\left\{g \in S^{-1}(\mathcal{H})\right\} = \gamma^k(S^{-1}(\mathcal{H})) \tag{43}$$

2.3.1 Log-normal distribution

$$k = 1, \;\; S : \mathbb{R} \mapsto \mathbb{R} = \exp \;\; \text{and} \;\; S^{-1} = \ln \tag{44}$$

$$\left| \begin{array}{lcl} \mu & = & \ln(\mu_{\ln}) - \frac{1}{2}\ln(1 + \sigma_{\ln}^2/\mu_{\ln}^2) \\ \sigma^2 & = & \ln(1 + \sigma_{\ln}^2/\mu_{\ln}^2) \end{array} \right. \tag{45}$$

- Notations:

Log tail hitting set: $\bar{\mathcal{H}}^{\text{1D}}_{\ln} = [\ln(\kappa) \ \infty)$ (46a)

Log cumulative hitting set: $\mathcal{H}^{\text{1D}}_{\ln} = (-\infty \ \ln(\kappa)]$ (46b)

Variable substitution: $\tilde{\kappa}_{\ln} \leftarrow \dfrac{\ln(\kappa) - \mu}{\sigma}$ (46c)

- For $\mathcal{H} = \bar{\mathcal{H}}^{\text{1D}} = [\kappa \ \infty) \;\; \rightarrow \;\; S^{-1}(\mathcal{H}) = \bar{\mathcal{H}}^{\text{1D}}_{\ln} = [\ln(\kappa) \ \infty)$

$$\left| \begin{array}{l} \mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{2}\left(1 - \text{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\ \\ \mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^2\sqrt{2\pi}} \, \tilde{\kappa}_{\ln} \, e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^3\sqrt{2\pi}} \, (\tilde{\kappa}_{\ln}^2 - 1) \, e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \mathscr{M}_4^{\gamma^1}(\bar{\mathcal{H}}^{\text{1D}}_{\ln}) = \frac{1}{\sigma^4\sqrt{2\pi}} \, (\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln}) \, e^{-\tilde{\kappa}_{\ln}^2/2} \\ \\ \ldots \end{array} \right. \tag{47}$$

Derivatives:

$$
\left|
\begin{array}{l}
\text{Help:}\quad \left(e^{-(\ln(x)-\mu)^2/2\sigma^2}\right)' = -\frac{1}{\sigma^2}\frac{\ln(x)-\mu}{x}\; e^{-(\ln(x)-\mu)^2/2\sigma^2} \\[2em]
\frac{\mathrm{d}\mathscr{M}_0^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma\sqrt{2\pi}}\frac{1}{\kappa}\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2em]
\frac{\mathrm{d}\mathscr{M}_1^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^2\sqrt{2\pi}}\frac{1}{\kappa}\frac{\ln(\kappa)-\mu}{\sigma}\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2em]
\frac{\mathrm{d}\mathscr{M}_2^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^3\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^2}{\sigma^2} - 1\right]\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2em]
\frac{\mathrm{d}\mathscr{M}_3^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^4\sqrt{2\pi}}\frac{1}{\kappa}\left[\frac{(\ln(\kappa)-\mu)^3}{\sigma^3} - 3\frac{(\ln(\kappa)-\mu)}{\sigma}\right]\; e^{-(\ln(\kappa)-\mu)^2/2\sigma^2} \\[2em]
\ldots \\[2em]
\text{Guess would be: } \forall j \geq 0, \\
\frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}(\bar{\mathcal{H}}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{-1}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\; e^{-\tilde{\kappa}_{\ln}^2/2}
\end{array}
\right.
\tag{48}
$$

- For $\mathcal{H} = [0\ \kappa] \quad \rightarrow \quad S^{-1}(\mathcal{H}) = \mathcal{H}_{\ln}^{1\mathrm{D}} = (-\infty\ \ \ln(\kappa)]$

$$
\left|
\begin{array}{l}
\mathscr{M}_0^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\tilde{\kappa}_{\ln}}{\sqrt{2}}\right)\right) \\[2em]
\mathscr{M}_1^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2em]
\mathscr{M}_2^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{-1}{\sigma^2\sqrt{2\pi}}\; \tilde{\kappa}_{\ln}\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2em]
\mathscr{M}_3^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{1}{\sigma^3\sqrt{2\pi}}\; \left(\tilde{\kappa}_{\ln}^2 - 1\right)\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2em]
\mathscr{M}_4^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}}) = \frac{-1}{\sigma^4\sqrt{2\pi}}\; \left(\tilde{\kappa}_{\ln}^3 - 3\tilde{\kappa}_{\ln}\right)\; e^{-\tilde{\kappa}_{\ln}^2/2} \\[2em]
\ldots
\end{array}
\right.
\tag{49}
$$

Derivatives:

$$
\left|
\begin{array}{l}
\text{Guess would be: } \forall j \geq 0, \\
\frac{\mathrm{d}\mathscr{M}_j^{\gamma^1}(\mathcal{H}_{\ln}^{1\mathrm{D}})}{\mathrm{d}\kappa} = \frac{(-1)^j}{\sigma^{j+1}\sqrt{2\pi}}\frac{1}{\kappa}\mathscr{H}_j(\tilde{\kappa}_{\ln})\; e^{-\tilde{\kappa}_{\ln}^2/2}
\end{array}
\right.
\tag{50}
$$

2.3.2 χ_k^2 distribution

$$k \in \mathbb{N}^*, \quad S : \mathbb{R}^k \mapsto \mathbb{R} = \| \bullet \|_2 \tag{51}$$

Identification with tail hitting set:

$$\bar{\mathcal{H}}_{\chi}^{k\mathrm{D}} = S^{-1}([\kappa\ \infty)) = \mathcal{B}_{\mathbb{R}^k}\left(0, \sqrt{\kappa}\right) \tag{52}$$

- For $j = 0$:

$$\mathscr{M}_0^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) = \overline{\Gamma}(k/2, \kappa/2\sigma^2) \tag{53}$$

- For $j = 1$:

$$\mathscr{M}_1^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) = -\frac{\mathrm{d}\overline{\Gamma}(k/2, \kappa/2\sigma^2)}{\mathrm{d}\kappa} = \left(\frac{\kappa}{2\sigma^2}\right)^{k/2-1} e^{-\kappa/2\sigma^2} \tag{54}$$

- For $j > 1$:

$$\begin{aligned}\mathscr{M}_j^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) &= (-1)^j \frac{\mathrm{d}^j\overline{\Gamma}(k/2, \kappa/2\sigma^2)}{\mathrm{d}\kappa^j} \\ &= \frac{1}{(2\sigma^2)^{2j-2}} \prod_{l=1}^{j-1} \left(\frac{k}{2} - l\right) \left(\frac{\kappa}{2\sigma^2}\right)^{k/2-j} e^{-\kappa/2\sigma^2}\end{aligned} \tag{55}$$

- First values:

$$\begin{cases} \mathscr{M}_0^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) = \displaystyle\int_{\frac{\kappa}{2\sigma^2}}^{\infty} t^{k/2-1} e^{-t} dt \\ \mathscr{M}_1^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) = \left(\dfrac{\kappa}{2\sigma^2}\right)^{k/2-1} e^{-\kappa/2\sigma^2} \\ \mathscr{M}_2^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) = \dfrac{1}{(2\sigma^2)^2} \left[\dfrac{k}{2} - 1\right] \left(\dfrac{\kappa}{2\sigma^2}\right)^{k/2-2} e^{-\kappa/2\sigma^2} \\ \mathscr{M}_3^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) = \dfrac{1}{(2\sigma^2)^4} \left[\left(\dfrac{k}{2} - 1\right)\left(\dfrac{k}{2} - 2\right)\right] \left(\dfrac{\kappa}{2\sigma^2}\right)^{k/2-3} e^{-\kappa/2\sigma^2} \\ \mathscr{M}_4^{\gamma^k}(\bar{\mathcal{H}}_\chi^{\text{kD}}) = \dfrac{1}{(2\sigma^2)^6} \left[\left(\dfrac{k}{2} - 1\right)\left(\dfrac{k}{2} - 2\right)\left(\dfrac{k}{2} - 3\right)\right] \left(\dfrac{\kappa}{2\sigma^2}\right)^{k/2-4} e^{-\kappa/2\sigma^2} \\ \dots \end{cases} \tag{56}$$

3 Expectation Formula

3.1 General case

$$\mathbb{E}\left\{\, \mathscr{L}_j(\mathcal{E}) \,\right\} = \sum_{i=0}^{N-j} \left[\begin{array}{c} i+j \\ i \end{array} \right] \left(\frac{\lambda_2}{2\pi} \right)^{i/2} \mathscr{L}_{i+j}(\mathcal{M}) \mathscr{M}_i^{\gamma^k}(S^{-1}(\mathcal{H})) \tag{57}$$

Notations:

LKC of $\mathcal{M}$: $\mathscr{L}_j(\mathcal{M}) = \mathscr{L}_j^{\mathcal{M}}$ (58a)

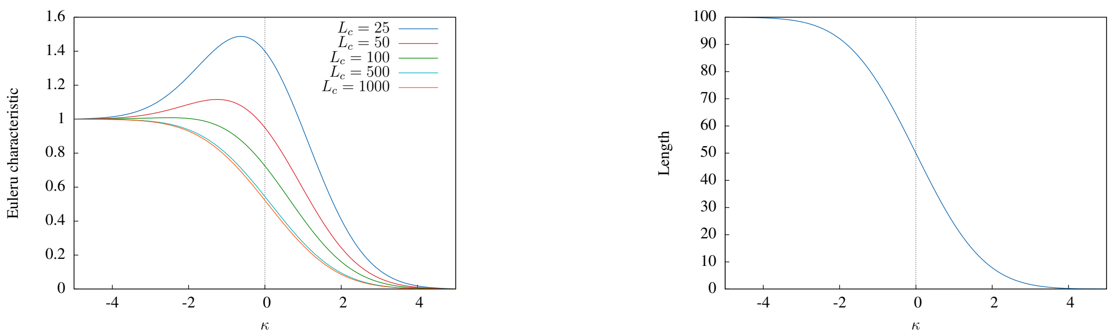
LKC of $\mathcal{E}$: $\mathscr{L}_j(\mathcal{E}) = \mathscr{L}_j^{\mathcal{E}}$ (58b)

GMF of $S^{-1}(\mathcal{H})$: $\mathscr{M}_j^{\gamma^k}(S^{-1}(\mathcal{H})) = \mathscr{M}_j^{\gamma^k}$ (58c)

3.2 One dimension

$$\mathbb{E}\left\{ \mathscr{L}_0^{\mathcal{E}} \right\} = \left(\frac{\lambda_2}{2\pi} \right)^{1/2} \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{59a}$$

$$\mathbb{E}\left\{ \mathscr{L}_1^{\mathcal{E}} \right\} = \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{59b}$$



(a) Euler Characteristic (b) Length

Figure 2: Expectation of LKC for $a = 100$, $\sigma^2 = 2$ and $\mathcal{H} = [\kappa\ \infty)$ in 1D.

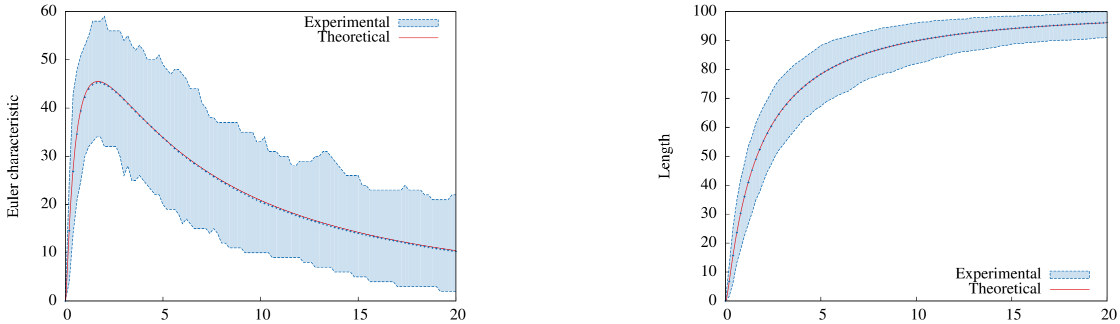


Figure 3: Expectation of LKC for $a = 100$, $\mu = 0.5$, $\sigma^2 = 2$, $L_c = 0.5$, and $\mathcal{H} = [0\ \kappa]$ (log-normal distribution with cumulative hitting set) in 1D. Experimental results are calculated over $1\,000$ realizations.

3.3 Two dimensions

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \frac{\lambda_2}{2\pi}\mathscr{L}_2^{\mathcal{M}}\mathscr{M}_2^{\gamma^k} + \sqrt{\frac{\lambda_2}{2\pi}}\mathscr{L}_1^{\mathcal{M}}\mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{60a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \sqrt{\frac{\lambda_2\pi}{8}}\mathscr{L}_2^{\mathcal{M}}\mathscr{M}_1^{\gamma^k} + \mathscr{L}_1^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{60b}$$

$$\mathbb{E}\left\{\mathscr{L}_2^{\mathcal{E}}\right\} = \mathscr{L}_2^{\mathcal{M}}\mathscr{M}_0^{\gamma^k} \tag{60c}$$

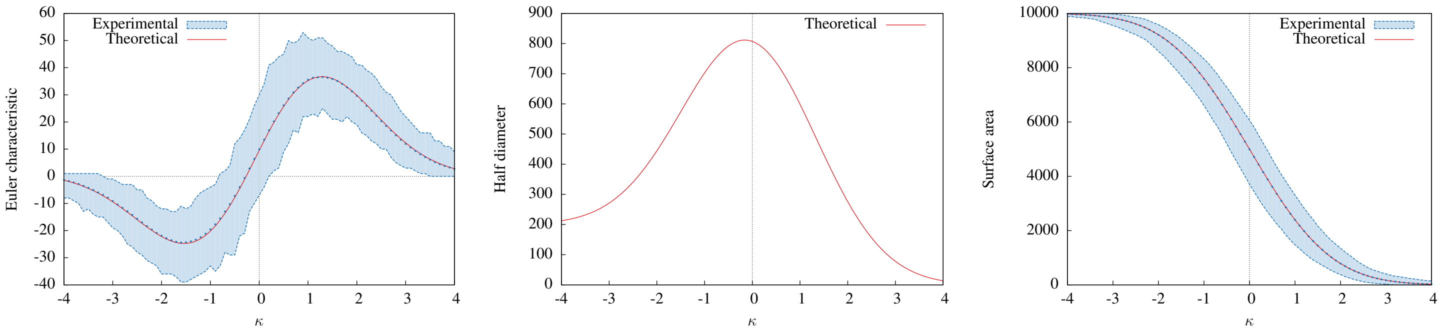


Figure 4: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [\kappa\ \infty)$ (gaussian distribution with tail hitting set) in 2D. Experimental results are calculated over $1\,000$ realizations.

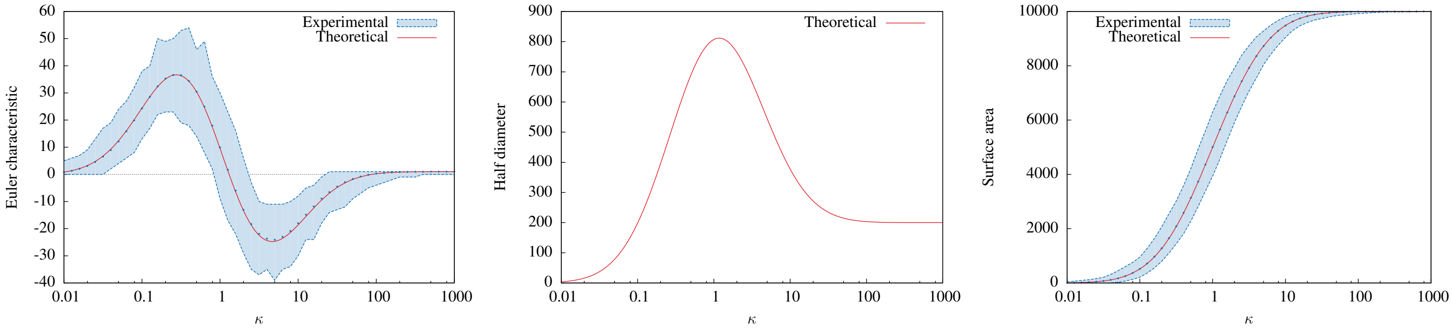


Figure 5: Expectation of LKC for $a = 100$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 5$, and $\mathcal{H} = [0\ \kappa]$ (lognormal distribution with cumulative hitting set) in 2D. Experimental results are calculated over $1\,000$ realizations.

3.3.1 Three dimensions

$$\mathbb{E}\left\{\mathscr{L}_0^{\mathcal{E}}\right\} = \left(\frac{\lambda_2}{2\pi}\right)^{\frac{3}{2}} \mathscr{L}_3^{\mathcal{M}} \mathscr{M}_3^{\gamma^k} + \frac{\lambda_2}{2\pi} \mathscr{L}_2^{\mathcal{M}} \mathscr{M}_2^{\gamma^k} + \left(\frac{\lambda_2}{2\pi}\right)^{\frac{1}{2}} \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_0^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{61a}$$

$$\mathbb{E}\left\{\mathscr{L}_1^{\mathcal{E}}\right\} = \frac{\lambda_2}{\pi} \mathscr{L}_3^{\mathcal{M}} \mathscr{M}_2^{\gamma^k} + \frac{\sqrt{\lambda_2 \pi}}{2\sqrt{2}} \mathscr{L}_2^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_1^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{61b}$$

$$\mathbb{E}\left\{\mathscr{L}_2^{\mathcal{E}}\right\} = \sqrt{\frac{2\lambda_2}{\pi}} \mathscr{L}_3^{\mathcal{M}} \mathscr{M}_1^{\gamma^k} + \mathscr{L}_2^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{61c}$$

$$\mathbb{E}\left\{\mathscr{L}_3^{\mathcal{E}}\right\} = \mathscr{L}_3^{\mathcal{M}} \mathscr{M}_0^{\gamma^k} \tag{61d}$$

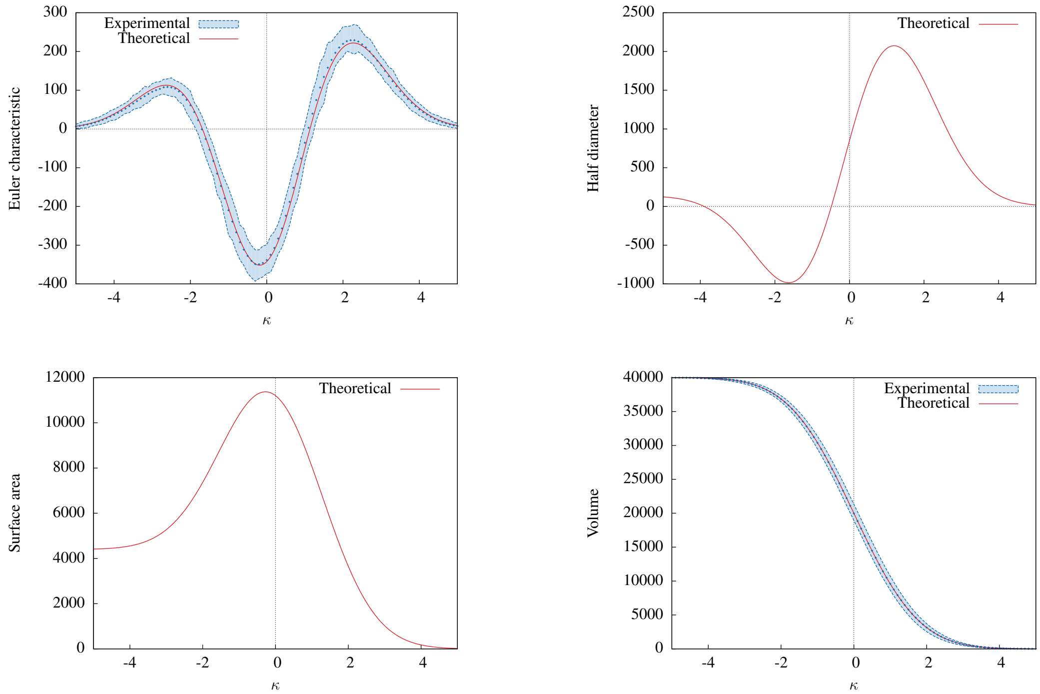


Figure 6: Expectation of LKC for $\mathcal{P} = 100 \times 20 \times 20$, $\mu = 0.0$, $\sigma^2 = 2$, $L_c = 2$, and $\mathcal{H} = [\kappa\ \infty)$ (gaussian distribution with tail hitting set) in 3D. Experimental results are calculated over 100 realizations.